Applicable Analysis and Discrete Mathematics - Accepted manuscript
Received: July 11, 2011.
Revised: 10-novembar-2011
Please cite this article as: Jozef Kratica, Vera Kovačević-Vujčić, Mirjana Čangalović, Milica Stojanović: Minimal doubly resolving sets and the strong metric dimension of Hamming graphs, Appl. Anal. Discrete Math. (to appear)
DOI: DOI:10.2298/AADM111116023K

Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **x** (xxxx), xxx–xxx. doi:10.2298/AADMxxxxxxxx

MINIMAL DOUBLY RESOLVING SETS AND THE STRONG METRIC DIMENSION OF HAMMING GRAPHS

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We consider the problem of determining the cardinality $\psi(H_{2,k})$ of minimal doubly resolving sets of Hamming graphs $H_{2,k}$. We prove that for $k \geq 6$ every minimal resolving set of $H_{2,k}$ is also a doubly resolving set, and, consequently, $\psi(H_{2,k})$ is equal to the metric dimension of $H_{2,k}$, which is known from the literature. Moreover, we find an explicit expression for the strong metric dimension of all Hamming graphs $H_{n,k}$.

1. INTRODUCTION

The metric dimension problem, introduced independently by Slater [22] and Harary [8], has been widely investigated [1,3,5-7,9-13]. It arises in many diverse areas including network discovery and verification [2], geographical routing protocols [18], the robot navigation, connected joints in graphs, chemistry, etc.

Given a simple connected undirected graph $G = (V_G,E_G)$, where $V_G = \{1, 2, ..., n\}$, $|E_G| = m$, $d(u, v)$ denotes the distance between vertices u and v, i.e. the length of a shortest $u - v$ path. A vertex x of graph G is said to resolve two vertices u and v of G if $d(u, x) \neq d(v, x)$. A subset $B = \{x_1, x_2, ..., x_k\}$ of V_G is a *resolving set* of G if every two distinct vertices of G are resolved by some vertex of B. Given a vertex t, the k-tuple $r(t, B) = (d(t, x_1), d(t, x_2), ..., d(t, x_k))$ is called the *vector of metric coordinates* of t with respect to B. A *metric basis* of G is a resolving set of the minimum cardinality. The *metric dimension* of G, denoted by $\beta(G)$, is the cardinality of its metric basis.

2010 Mathematics Subject Classification: 05C75, 05C12

Keywords and Phrases: Hamming graphs, metric dimension, minimal doubly resolving set, strong metric dimension, graph theory.

Caceres et al. [3] define the notion of a doubly resolving set as follows. Vertices x, y of graph G ($n \geq 2$) are said to doubly resolve vertices u, v of G if $d(u,x) - d(u,y) \neq d(v,x) - d(v,y)$. A subset $D = \{x_1, x_2, ..., x_l\}$ of V_G is a *doubly resolving set* of G if every two distinct vertices of G are doubly resolved by some two vertices of D. The minimal doubly resolving set problem consists of finding a doubly resolving set of G with the minimum cardinality, denoted by $\psi(G)$. Note that if x, y doubly resolve u, v then $d(u,x) - d(v,x) \neq 0$ or $d(u,y) - d(v,y) \neq 0$, and hence x or y resolves u, v. Therefore, a doubly resolving set is also a resolving set and $\beta(G) \leq \psi(G)$.

The strong metric dimension problem was introduced by A. Sebo and E. Tannier [21] and further investigated by O. R. Oellermann and J. Peters-Fransen [20]. Recently, the strong metric dimension of distance hereditary graphs has been studied by T. May and O. R. Oellermann [19]. A vertex w strongly resolves two vertices u and v if u belongs to a shortest $v - w$ path or v belongs to a shortest $u - w$ path. A subset S of V_G is a *strong resolving set* of G if every two distinct vertices of G are strongly resolved by some vertex of S. A *strong metric basis* of G is a strong resolving set of the minimum cardinality. Now, the *strong metric dimension* of G, denoted by $sdim(G)$ is defined as the cardinality of its strong metric basis. It is easy to see that if a vertex w strongly resolves vertices u and v then w also resolves these vertices. Hence every strong resolving set is a resolving set and $\beta(G) \leq sdim(G)$.

All three previously defined problems are NP-hard in general case. The proofs of NP-hardness are given for the metric dimension problem in [13], for the minimal doubly resolving set problem in [16] and for the strong metric dimension problem in [20]. The first metaheuristic approaches to all three problems have been proposed in [14-16].

If S is a strong resolving set of G, then set $\{r(v,S) | v \in V\}$ uniquely determines graph G in the following sense. If G' is a graph with $V_{G'} = V_G$ such that S strongly resolves G', and if for all vertices v we have $r_G(v,S) = r_{G'}(v,S)$, then $G = G'$. If S is a resolving set or a doubly resolving set, then set S need not uniquely determine G.

The next example illustrates previously defined invariants $\beta(G)$, $\psi(G)$ and $sdim(G)$.

Example 1. Consider graph G_1 of Figure 1. It is easy to check that $\beta(G_1)$, $\psi(G_1)$ and $sdim(G_1)$ are all different. Namely, $\{A, B\}$ is a minimal resolving set, $\{A, B, E, F\}$ is a minimal doubly resolving set, while $\{A, B, E\}$ is a minimal strong resolving set of G_1. Therefore, $\beta(G_1) = 2$, $\psi(G_1) = 4$ and $sdim(G_1) = 3$. $\square$

For some simple classes of graphs it is possible to determine $\beta(G)$, $\psi(G)$ and $sdim(G)$ explicitly: the complete graph K_n with n vertices has $\beta(K_n) = \psi(K_n) = sdim(K_n) = n - 1$, the cycle C_n with n vertices has $\beta(C_n) = 2$, $\psi(C_n) = 3$, while $sdim(C_n) = \lceil n/2 \rceil$.

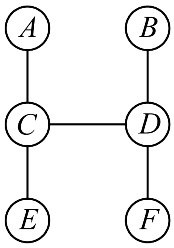

FIGURE 1. Graph G_1

The *Cartesian product* $G\square H$ of graphs $G{=}(V_G, E_G)$ and $H{=}(V_H, E_H)$, is the graph with the vertex set $V_G \times V_H = \{(a, v) |\ a \in V_G, v \in V_H\}$, where (a, v) is adjacent to (b, w) whenever $a = b$ and $\{v, w\} \in E_H$, or $v = w$ and $\{a, b\} \in E_G$.

We consider here a special class of Cartesian products of graphs, the so called Hamming graphs. The *Hamming graph* $H_{n,k}$ is defined as:

(1) $$H_{n,k} = \underbrace{K_k\square K_k\square ...\square K_k}_{n}$$

where K_k denotes the complete graph with k vertices. The vertices of Hamming graphs can be considered also as n-dimensional vectors, with coordinates which take values from set $\{0, 1, ..., k-1\}$. Two vertices are adjacent if they differ in exactly one coordinate. Specially, for k=2, graph $H_{n,2}$ can be interpreted as the *hypercube* Q_n of order n, whose vertices are n-dimensional binary vectors which are adjacent if they have only one different coordinate.

It is obvious that $H_{n,k}$ has k^n vertices and every vertex has the n-dimensional neighborhood with $k-1$ neighbors with respect to each coordinate, so the overall number of edges is $k^n \cdot n \cdot (k-1)/2$.

Example 2. Hamming graph $H_{4,3}$ has $3^4 = 81$ vertices, which are all ternary vectors of length 4. For example, vertex (0,1,1,0) has adjacent vertices (0,1,1,1), (0,1,1,2), (0,1,0,0), (0,1,2,0), (0,0,1,0), (0,2,1,0), (1,1,1,0), (2,1,1,0). □

Chvátal in [4] has obtained an upper bound on the number of guesses for the game *Mastermind*, which also gives an upper bound for metric dimension of Hamming graphs. This is an asymptotic result which can be formulated as follows:

For each $\epsilon > 0$, there exists an integer $n(\epsilon)$ such that if $n > n(\epsilon)$ and $k < n^{1-\epsilon}$, then

$$\beta(H_{n,k}) \leq (2+\epsilon) \cdot n \cdot \frac{1 + 2 \cdot log_2 k}{log_2 n - log_2 k}$$

It has been shown in [3] that the metric dimension of Hamming graphs $H_{2,k}$ for all $k \geq 1$, is equal to

$$\beta(H_{2,k}) = \lfloor \frac{4k-2}{3} \rfloor \tag{2}$$

In this paper we study the minimal cardinality $\psi(H_{2,k})$ of doubly resolving sets of Hamming graphs $H_{2,k}$ and prove that for $k \geq 5$ it is equal to $\beta(H_{2,k})$. Moreover, we show that the strong metric dimension for all Hamming graphs $H_{n,k}$ is $sdim(H_{n,k}) = (k-1) \cdot k^{n-1}$.

The paper is organized as follows. Section 2 is devoted to minimal doubly resolving sets of Hamming graphs $H_{2,k}$, while in Section 3 we find the exact value of $sdim(H_{n,k})$. Finally, Section 4 summarizes conclusions of the paper.

2. MINIMAL DOUBLY RESOLVING SETS OF $H_{2,k}$

In order to find the cardinality of minimal doubly resolving sets in Hamming graphs $H_{2,k}$ we will use here the following definitions and results from Caceres et al. [3], which are related to the Cartesian product $G \Box H$ of arbitrary graphs G and H.

Let S be a subset of $V_{G \Box H}$. The *projection of S onto G* is the set of all vertices $u \in V_G$ such that there exists a vertex $(u, v) \in S$. Similarly, *the projection of S onto H* contains all vertices $v \in V_H$ for which there exists a vertex $(u, v) \in S$. A *column* of $G \Box H$ is the set of vertices $\{(u, v) | v \in V_H\}$ for a fixed vertex $u \in V_G$ and a *row* of $G \Box H$ is the set $\{(u, v) | u \in V_G\}$ for a fixed vertex $v \in V_H$.

LEMMA 1. *([3]) For arbitrary graphs G and H and for every resolving set S of $G \Box H$, the projection of S onto G is a resolving set in G and the projection of S onto H is a resolving set in H.*

Let us consider the Hamming graph $H_{2,k} = K_k \Box K_k$. Its vertices can be represented as all two-dimensional vectors with coordinates from set $\{0, 1, ..., k-1\}$, where two vertices are adjacent if they differ in exactly one coordinate. For a subset S of $V_{H_{2,k}}$, the projection of S onto the first K_k is the is the set of the first coordinates of all vertices from S, while the projection of S onto the second K_k is the set of the second coordinates of these vertices. A row of $H_{2,k}$ contains k vertices with the same second coordinate, and a column of $H_{2,k}$ consists of k vertices with the same first coordinate. Therefore, each row or column of $H_{2,k}$ represents a clique. Moreover, two vertices of $H_{2,k}$ are adjacent if and only if they belong to the same row or the same column. Otherwise, they are at distance two. Vertices of $H_{2,k}$ can be represented by the points of the $k \times k$ grid, where rows and columns of the grid correspond to rows and columns of $H_{2,k}$.

Now we will prove the following lemma.

LEMMA 2. *For any Hamming graph $H_{2,k}$, where $k \geq 6$, every minimal resolving set is also a doubly resolving set, and consequently $\psi(H_{2,k}) \leq \beta(H_{2,k})$.*

Proof. Let B be a minimal resolving set of $H_{2,k}$, and let us suppose that it is not a doubly resolving set. Then, there exist two distinct vertices $u, v \in V_{H_{2,k}}$, $u \neq v$ and some integer $q \geq 0$ such that

$$r(u,B) - r(v,B) = (q, q, ..., q) \tag{3}$$

Since $Diam(H_{2,k}) = 2$, the vectors $r(u,B)$ and $r(v,B)$ can contain only values 0,1 or 2, and therefore, $q \in \{0,1,2\}$.

Value q cannot be equal to 0, because in that case there would not exist a vertex from B resolving u and v, which is in contradiction with the fact that B is a resolving set.

For $q = 2$ vectors $r(u,B)$ and $r(v,B)$ could take only values $r(u,B) = (2,2,...,2)$ and $r(v,B) = (0,0,...,0)$. But, the vector of metric coordinates of a vertex can contain at most one value 0, which appears only when this vertex belongs to B. As $r(v,B)$ contains more than one 0, the case $q = 2$ is also impossible.

Let us suppose that $q = 1$. Without loss of generality, we can assume that $r(v,B)$ has coordinates in non-decreasing order. Taking into account that $r(v,B)$ has at most one zero coordinate and cannot have any coordinate greater than 1, there exist only two possibilities:

Case 1: $r(u,B) = (2,2,2,...,2)$ and $r(v,B) = (1,1,1,...,1)$. As all coordinates of $r(v,B)$ are equal to 1, then $v \notin B$ and each vertex from B belongs to the same row $r(v)$ or the same column $c(v)$ as vertex v. Let us consider projection $P_1(B)$ onto the first set K_k from the Cartesian product $H_{2,k} = K_k \Box K_k$, which contains the first coordinates of all vertices from B. According to Lemma 1, $P_1(B)$ is a resolving set of K_k and, therefore, $|P_1(B)| \geq \beta(K_k) = k-1$. As the first coordinates of vertices from $B \cap r(v)$ are all different, while the first coordinates of vertices from $B \cap c(v)$ are all equal to the first coordinate of vertex v, which does not belong to B, then $|P_1(B)| = |B \cap r(v)| + 1$. Therefore, there should be at least $k-2$ vertices from B in row $r(v)$. Considering projection $P_2(B)$ onto the second set K_k from $H_{2,k} = K_k \Box K_k$, we can conclude in a similar way that $|P_2(B)| = |B \cap c(v)| + 1 \geq k-1$, and there should be at least $k-2$ vertices from B in column $c(v)$. As $v \notin B$, it follows that $|B| \geq 2k-4$. Since B is a minimal resolving set, then, according to (2), $|B| = \lfloor \frac{4k-2}{3} \rfloor$. But, for $k \geq 6$ it holds that $\lfloor \frac{4k-2}{3} \rfloor < 2k-4$, which is a contradiction.

Case 2: $r(u,B) = (1,2,2,...,2)$ and $r(v,B) = (0,1,1,...,1)$. As $r(v,B)$ contains value 0 and all other its coordinates are equal to 1, then $v \in B$ and each other vertex from B belongs to row $r(v)$ or column $c(v)$. Using similar arguments as in Case 1, we can conclude that $|B| \geq 2n-3$. But, for $k \geq 4$ it holds that $\lfloor \frac{4k-2}{3} \rfloor < 2k-3$, which is a contradiction. Furthermore, it can be shown that in this case B cannot be even a resolving set. Actually, from the additional fact that $r(u,B) = (1,2,2,...,2)$ it follows that $u \in r(v)$ or $u \in c(v)$, and vertex u does not belong to the row and the column of any vertex from $B \setminus \{v\}$. Let us suppose, without loss of generality, that

$u \in r(v)$. Then, consequently, all vertices of B belong to $c(v)$. But, for such a B and $k \geq 3$, each two vertices from $V_{H_{2,k}} \setminus c(v)$ which are at the same row, are not resolved by any vertex from B. Therefore, B is not a resolving set.

Since for all possible values of q we have obtained contradictions, there are no two vertices $u, v \in V_{H_{2,k}}$ which satisfy condition (3), and, consequently, B is a doubly resolving set. $\square$

Now the minimal cardinality $\psi(H_{2,k})$ of doubly resolving sets in $H_{2,k}$ can be obtained by the following theorem.

THEOREM 3. $\psi(H_{2,k}) = \begin{cases} 3, & k = 2, 3 \\ 5, & k = 4 \\ \lfloor \frac{4k-2}{3} \rfloor, & k \geq 5 \end{cases}$.

Proof. As $\psi(H_{2,k}) \geq \beta(H_{2,k})$ and $\beta(H_{2,k}) = \lfloor \frac{4k-2}{3} \rfloor$, then it follows directly from Lemma 2 that $\psi(H_{2,k}) = \lfloor \frac{4k-2}{3} \rfloor$ for $k \geq 6$. Values $\psi(H_{2,2}) = 3$, $\psi(H_{2,3}) = 3$, $\psi(H_{2,4}) = 5$ and $\psi(H_{2,5}) = 6$ are obtained by total enumeration. $\square$

Let us mention that condition $k \geq 6$ in the statement of Lemma 2 cannot be omitted. Namely, as $\beta(H_{2,2}) < \psi(H_{2,2}))$ and $\beta(H_{2,4}) < \psi(H_{2,4}))$, then, obviously, this statement does not hold for $k = 2$ and $k = 4$. It also does not hold for $k = 5$, which is shown in the following example.

Example 3. Let us consider graph $H_{2,5}$ and subset B of its vertices, indicated by bolded points in the grid representation of $H_{2,5}$ on Figure 2. It is easy to see that B is a resolving set. As $\lfloor \frac{4k-2}{3} \rfloor = 6$ for $k = 5$, B represents a metric basis of $H_{2,5}$. But, for vertices u and v, indicated in Figure 2, $r(u, B) = (2, 2, 2, 2, 2, 2)$ and $r(v, B) = (1, 1, 1, 1, 1, 1)$ and, consequently, B is not a doubly resolving set. $\square$

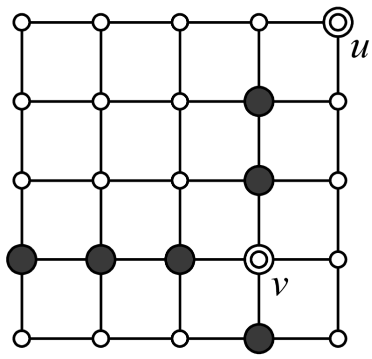

FIGURE 2. Graph $H_{2,5}$

3. STRONG METRIC DIMENSION OF $H_{n,k}$

In this section we obtain the exact value of the strong metric dimension for any Hamming graph $H_{n,k}$.

THEOREM 4. *For any Hamming graph $H_{n,k}$, $sdim(H_{n,k}) = (k-1)\cdot k^{n-1}$.*

In order to prove Theorem 4 we use the following lemma from [17], which is related to a strong resolving set of an arbitrary graph G.

LEMMA 5. *([17]) If S is a strong resolving set of graph G, then, for every two vertices $u, v \in V_G$ such that $d(u,v) = Diam(G)$, $u \in S$ or $v \in S$.*

Next, we state and prove several lemmas which support the proof of Theorem 4.

LEMMA 6. *For any set $S \subseteq V_{H_{n,k}}$, where $|S| > k^{n-1}$, there exist two vertices u and v from S such that $d(u,v) = Diam(H_{n,k})$.*

Proof. Let us define

$$S(x_2, x_3, ..., x_n) = \{(i, (i+x_2)\ mod\ k, ..., (i+x_n)\ mod\ k) \mid i = 0, ..., k-1\}$$

Since each two distinct vertices $u, v \in S(x_2, x_3, ..., x_n)$ have translated coordinates, different at all positions, then $d(u,v) = n = Diam(G)$.

Let us consider the family $F = \{S(x_2, x_3, ..., x_n) \mid 0 \le x_i < k, 1 \le i \le n\}$. It can be proved that F is a partition of set $V_{H_{n,k}}$. Each pair of sets in F are disjoint. Indeed, if $S(x_2, x_3, ..., x_n) \cap S(y_2, y_3, ..., y_n) \ne \emptyset$ for some distinct $(x_2, x_3, ..., x_n)$ and $(y_2, y_3, ..., y_n)$, then there exist integers a and b from $\{0, 1, ..., k-1\}$ such that

$$(a, (a+x_2)\ mod\ k, ..., (a+x_n)\ mod\ k) = (b, (b+y_2)\ mod\ k, ..., (b+y_n)\ mod\ k).$$

It follows that $a = b$. Moreover, from $(a+x_i)\ mod\ k = (b+y_i)\ mod\ k$, $a = b$ and $x_i, y_i \in \{0, 1, ..., k-1\}$, it follows that $x_i = y_i$ for all $i = 2, ..., n$. Therefore, $S(x_2, x_3, ..., x_n) \cap S(y_2, y_3, ..., y_n) = \emptyset$. Also, if $u = (a, a_2, ..., a_n)$ is a vertex from $V_{H_{n,k}}$, then there is a set from family F which contains u. Indeed, for $i = 2, ...n$ the equality $(a+x)\ mod\ k = a_i$ with unknown x has the unique solution x_i^* in the set $\{0, 1, ..., k-1\}$. Namely, $x_i^* = \begin{cases} a_i - a, & a_i \ge a \\ k + a_i - a, & a_i < a \end{cases}$. Therefore, $u \in S(x_2^*, x_3^*, ..., x_n^*)$.

Let S be an arbitrary subset of $V_{H_{n,k}}$, where $|S| > k^{n-1}$. As F is a partition of $V_{H_{n,k}}$ and $|F| = k^{n-1}$, then there exists a set $S(x_2, x_3, ..., x_n)$ from F which has at least two common members with S. If $u, v \in S \cap S(x_2, x_3, ..., x_n)$, then, from $u, v \in S(x_2, x_3, ..., x_n)$, it follows that $d(u,v) = Diam(H_{n,k})$. Since $u, v \in S$, the proof is completed. □

LEMMA 7. *If S is a strong resolving set of $H_{n,k}$ then $|S| \ge (k-1)\cdot k^{n-1}$.*

Proof. Let us assume that $|S| < (k-1)\cdot k^{n-1}$. Then, for set $\bar{S} = V_{H_{n,k}} \setminus S$, it follows that $|\bar{S}| > k^{n-1}$, and due to Lemma 6, there exist $u, v \in \bar{S}$ such that $d(u,v) = Diam(H_{n,k})$. But, according to Lemma 5, $u \in S$ or $v \in S$, which is a contradiction. □

Lemma 8. *The set* $S = \{(x_1, x_2, ..., x_n) \mid 1 \le x_1 < k,\ 0 \le x_i < k \ for\ 2 \le i \le n\}$ *is a strong resolving set of* $H_{n,k}$.

Proof. According to the definition of S, set $V_{H_{n,k}} \setminus S$ contains all vertices of $H_{n,k}$ with the first coordinate equal to 0. Let us consider an arbitrary pair of vertices $u, v \in V_{H_{n,k}} \setminus S$. If $v = (0, x_2, ..., x_n)$, then there exists a vertex $w \in S$ such that $w = (1, x_2, ..., x_n)$. As $d(v, w) = 1$ and $d(u, w) = d(u, v) + 1$, then $d(u, w) = d(u, v) + d(v, w)$, i.e. vertex v belongs to a shortest $u - w$ path, and, therefore, w strongly resolves vertices u and v. Also, for $u \in S$ or $v \in S$, vertex u or vertex v obviously strongly resolves pair u, v. Therefore, S is a strong resolving set. □

Proof of Theorem 4. As the set S from Lemma 8 has cardinality equal to $(k - 1) \cdot k^{n-1}$, then it follows directly from Lemma 7 and Lemma 8 that S is a strong metric basis of $H_{n,k}$, and consequently $sdim(H_{n,k}) = (k - 1) \cdot k^{n-1}$. □

Corollary 9. *For all* n, *the strong metric dimension of the hypercube* Q_n *is equal to* 2^{n-1}.

Proof. Since $Q_n = H_{n,2}$, the statement follows directly from Theorem 4. □

4. CONCLUSIONS

In this paper we have studied minimal doubly resolving sets and the strong metric dimension of Hamming graphs. We have proved that for $k \ge 5$ the cardinality of minimal doubly resolving sets of $H_{2,k}$ is equal to the metric dimension of $H_{2,k}$. In addition, the explicit expression for the strong metric dimension of $H_{n,k}$ is conjectured and proved.

Interesting open problems are to find the metric dimension and/or the cardinality of minimal doubly resolving sets (or a tight upper bound) for some other classes of Hamming graphs $H_{n,k},\ n \ge 3$. Future work can also be directed towards obtaining the cardinality of minimal doubly resolving sets and/or the strong metric dimension of some other challenging classes of graphs.

ACKNOWLEDGEMENT

This research was partially supported by Serbian Ministry of Education and Science under the grant no. 174033.

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