On the metric dimension and fractional metric dimension for hierarchical product of graphs

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Abstract

A set of vertices $W$ resolves a graph $G$ if every vertex of $G$ is uniquely determined by its vector of distances to the vertices in $W$. The metric dimension for $G$, denoted by $\dim(G)$, is the minimum cardinality of a resolving set of $G$. In order to study the metric dimension for the hierarchical product $G_2 \sqcup G_1$, of two rooted graphs $G_2$ and $G_1$, we first introduce a new parameter, the rooted metric dimension $\rdim(G_1)$ for a rooted graph $G_1$. If $G_1$ is not a path with an end-vertex $u_1$, we show that $\dim(G_2 \sqcup G_1) = |V(G_2)| \cdot \rdim(G_1)$, where $|V(G_2)|$ is the order of $G_2$. If $G_1$ is a path with an end-vertex $u_1$, we obtain some tight inequalities for $\dim(G_2 \sqcup G_1)$. Finally, we show that similar results hold for the fractional metric dimension.

Key words: resolving set; metric dimension; resolving function; fractional metric dimension; hierarchical product; binomial tree.

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1 Introduction

All graphs considered in this paper are nontrivial and connected. For a graph $G$, we often denote by $V(G)$ and $E(G)$ the vertex set and the edge set of $G$, respectively. For any two vertices $u$ and $v$ of $G$, denote by $d_G(u, v)$ the distance between $u$ and $v$ in $G$, and write $R_G\{u, v\} = \{w \mid w \in V(G), d_G(u, w) \neq d_G(v, w)\}$. If the graph $G$ is clear from the context, the notations $d_G(u, v)$ and $R_G\{u, v\}$ will be written $d(u, v)$ and $R\{u, v\}$, respectively. A subset $W$ of $V(G)$ is a resolving set of $G$ if $W \cap R\{u, v\} \neq \emptyset$ for any two distinct vertices $u$ and $v$. A metric basis of $G$ is a resolving set of $G$ with minimum cardinality. The cardinality of a metric basis of $G$ is the metric dimension for $G$, denoted by $\dim(G)$.

Metric dimension was introduced independently by Harary and Melter [15], and by Slater [24]. As a graph parameter it has numerous applications, among them are computer science and robotics [18], network discovery and verification [5], strategies
for the Mastermind game [8] and combinatorial optimization [23]. Metric dimension has been heavily studied, see [3] for a number of references on this topic.

The problem of finding the metric dimension for a graph was formulated as an integer programming problem independently by Chartrand et al. [7], and by Currie and Oellermann [10]. In graph theory, fractionalization of integer-valued graph theoretic concepts is an interesting area of research (see [22]). Currie and Oellermann [10] and Fehr et al. [11] defined fractional metric dimension as the optimal solution of the linear relaxation of the integer programming problem. Arumugam and Mathew [1] initiated the study of the fractional metric dimension for graphs. For more information, see [2, 12, 13].

Let \( g : V(G) \rightarrow [0, 1] \) be a real value function. For \( W \subseteq V(G) \), denote \( g(W) = \sum_{v \in W} g(v) \). The weight of \( g \) is defined by \( |g| = g(V(G)) \). We call \( g \) a resolving function of \( G \) if \( g(R\{u, v\}) \geq 1 \) for any two distinct vertices \( u \) and \( v \). The minimum weight of a resolving function of \( G \) is called the fractional metric dimension for \( G \), denoted by \( \text{fdim}(G) \).

It was noted in [14, p. 204] and [18] that determining the metric dimension for a graph is an NP-complete problem. So it is desirable to reduce the computation for the metric dimension for product graphs to the computation for some parameters of the factor graphs; see [6] for cartesian products, [16] for lexicographic products, and [25] for corona products. Recently, the fractional metric dimension for the above three products was studied in [2, 12, 13].

In order to model some real-life complex networks, Barrière et al. [4] introduced the hierarchical product of graphs and showed that it is associative. A rooted graph \( G^u \) is the graph \( G \) in which one vertex \( u \), called root vertex, is labeled in a special way to distinguish it from other vertices. Let \( G_1^{u_1} \) and \( G_2^{u_2} \) be two rooted graphs. The hierarchical product \( G_2^{u_2} \sqcap G_1^{u_1} \) is the rooted graph with the vertex set \( \{x_2x_1 \mid x_1 \in V(G_1^u), i = 1, 2\} \), having the root vertex \( u_2u_1 \), where \( x_2x_1 \) is adjacent to \( y_2y_1 \) whenever \( x_2 = y_2 \) and \( \{x_1, y_1\} \in E(G_1) \), or \( x_1 = y_1 = u_1 \) and \( \{x_2, y_2\} \in E(G_2) \). See [17, 19, 20, 21] for more information.

In this paper, we study the (fractional) metric dimension for the hierarchical product \( G_2^{u_2} \sqcap G_1^{u_1} \) of rooted graphs \( G_2^{u_2} \) and \( G_1^{u_1} \). In Section 2, we introduce a new parameter, the rooted metric dimension \( \text{rdim}(G^u) \) for a rooted graph \( G^u \). If \( G_1 \) is not a path with an end-vertex \( u_1 \), we show that \( \dim(G_2^{u_2} \sqcap G_1^{u_1}) = |V(G_2)| \cdot \text{rdim}(G_1^{u_1}) \). If \( G_1 \) is a path with an end-vertex \( u_1 \), we obtain some tight inequalities for \( \dim(G_2^{u_2} \sqcap G_1^{u_1}) \). In Section 3, we show that similar results hold for the fractional metric dimension.

## 2 Metric dimension

In order to study the metric dimension for the hierarchical product of graphs, we first introduce the rooted metric dimension for a rooted graph.

A rooted resolving set of a rooted graph \( G^u \) is a subset \( W \) of \( V(G) \) such that \( W \cup \{u\} \) is a resolving set of \( G \). A rooted metric basis of \( G^u \) is a rooted resolving set of \( G^u \) with the minimum cardinality. The cardinality of a rooted metric basis of \( G^u \) will be here called rooted metric dimension of \( G^u \) and denoted by \( \text{rdim}(G^u) \).
The following observation is obvious.

**Observation 2.1** If there exists a metric basis of $G$ containing $u$, then $\text{rdim}(G^u) = \dim(G) - 1$. If any metric basis of $G$ does not contain $u$, then $\text{rdim}(G^u) = \dim(G)$.

For graphs $H_1$ and $H_2$ we use $H_1 \cup H_2$ to denote the disjoint union of $H_1$ and $H_2$ and $H_1 + H_2$ to denote the graph obtained from the disjoint union of $H_1$ and $H_2$ by joining every vertex of $H_1$ with every vertex of $H_2$.

**Observation 2.2** ([7]) Let $G$ be a graph of order $n$. Then $1 \leq \dim(G) \leq n - 1$. Moreover,

(i) $\dim(G) = 1$ if and only if $G$ is the path $P_n$ of length $n$.

(ii) $\dim(G) = n - 1$ if and only if $G$ is the complete graph $K_n$ on $n$ vertices.

**Proposition 2.3** ([7, Theorem 4]) Let $G$ be a graph of order $n \geq 4$. Then $\dim(G) = n - 2$ if and only if $G = K_{s,t}$ $(s, t \geq 1)$, $G = K_s + K_t$ $(s \geq 1, t \geq 2)$, or $G = K_s + (K_1 \cup K_t)$ $(s, t \geq 1)$, where $K_t$ is a null graph and $K_{s,t}$ is a complete bipartite graph.

**Proposition 2.4** Let $G^u$ be a rooted graph of order $n$. Then $0 \leq \text{rdim}(G^u) \leq n - 2$.

Moreover,

(i) $\text{rdim}(G^u) = 0$ if and only if $G = P_n$ and $u$ is one of its end-vertices.

(ii) $\text{rdim}(G^u) = n - 2$ if and only if $G = K_{n,1}$ and $u$ is the centre.

**Proof.** If $G$ is a complete graph, by Observation 2.2 (ii) we have $\dim(G) = n - 1$, so Observation 2.1 implies that $\text{rdim}(G^u) = n - 2$. If $G$ is not a complete graph, then $1 \leq \dim(G) \leq n - 2$, which implies that $0 \leq \text{rdim}(G^u) \leq n - 2$ by Observation 2.1.

(i) Since $\text{rdim}(G^u) = 0$ if and only if $\{u\}$ is a metric basis of $G$, by Observation 2.2 (i), (ii) holds.

(ii) Suppose $\text{rdim}(G^u) = n - 2$. Then $\dim(G) = n - 1$ or $n - 2$. If $\dim(G) = n - 1$, then $G = K_n$. Now we consider $\dim(G) = n - 2$. If $n = 3$, then $\dim(G) = 1$, which implies that $G = K_{1,2}$ and $u$ is the centre. Now suppose $n \geq 4$. Then $G$ is one of the graphs listed in Proposition 2.3. If $s, t \geq 2$ or $G = K_s + (K_1 \cup K_t)$, then there exists a metric basis containing $u$, which implies that $\text{rdim}(G^u) = n - 3$, a contradiction. Hence $G = K_{1,n-1}$. Since any metric basis of $K_{1,n-1}$ does not contain the centre, the vertex $u$ is the centre of $K_{1,n-1}$. The converse is routine. \hfill \Box

Next, we shall express the metric dimension for the hierarchical product $G^u_2 \sqcap G^u_1$ in terms of $\text{rdim}(G^u_1)$.

Let $G^u_1$ and $G^u_2$ be two rooted graphs. For any two vertices $x_2x_1$ and $y_2y_1$ of $G^u_2 \sqcap G^u_1$, observe that

\[
d(x_2x_1, y_2y_1) = \begin{cases} 
  d_{G_1}(x_1, y_1), & \text{if } x_2 = y_2, \\
  d_{G_2}(x_2, y_2) + d_{G_1}(x_1, u_1) + d_{G_1}(y_1, u_1), & \text{if } x_2 \neq y_2.
\end{cases}
\]

**Lemma 2.5** Let $x_2x_1$ and $y_2y_1$ be two distinct vertices of $G^u_2 \sqcap G^u_1$.

(i) If $x_2 = y_2$, then

$$R\{x_2x_1, y_2y_1\} = \begin{cases} 
  \{x_2z \mid z \in R_{G_1}\{x_1, y_1\}\}, & \text{if } u_1 \notin R_{G_1}\{x_1, y_1\}, \\
  V(G^u_2 \sqcap G^u_1) \setminus \{x_2z \mid z \notin R_{G_1}\{x_1, y_1\}\}, & \text{if } u_1 \in R_{G_1}\{x_1, y_1\}.
\end{cases}$$

(ii) If $x_2 \neq y_2$, then $\{x_2z, y_2z\} \cap R\{x_2x_1, y_2y_1\} \neq \emptyset$ for any $z \in V(G_1)$.
Proof. (i) If $u_1 \notin R_{G_1} \{x_1, y_1\}$, then $d_{G_1}(x_1, u_1) = d_{G_1}(y_1, u_1)$. By (1), the inequal-
ity $d(x_2 x_1, x_2 z_1) \neq d(y_2 y_1, z_2 z_1)$ holds if and only if $z_2 = x_2$ and $d_{G_1}(x_1, z_1) \neq d_{G_1}(x_1, z_1)$.
It follows that $R\{x_2 x_1, y_2 y_1\} = \{x_2 z \mid z \in R_{G_1} \{x_1, y_1\}\}$. If $u_1 \in R_{G_1} \{x_1, y_1\}$, then $d_{G_1}(x_1, u_1) \neq d_{G_1}(y_1, u_1)$. By (1), the equality $d(x_2 x_1, x_2 z_1) = d(y_2 y_1, z_2 z_1)$ holds if and only if $z_2 = x_2$ and $d_{G_1}(x_1, z_1) = d_{G_1}(y_1, z_1)$.
It follows that $R\{x_2 x_1, y_2 y_1\} = V(G_{22}^R \cap G_{11}^u) \setminus \{x_2 z \mid z \notin R_{G_1} \{x_1, y_1\}\}$.

(ii) Suppose $x_2 z \notin R\{x_2 x_1, y_2 y_1\}$. Then $d(x_2 x_1, x_2 z) = d(y_2 y_1, x_2 z)$.
By (1),
$$d_{G_1}(x_1, z) = d_{G_2}(x_2, y_2) + d_{G_1}(y_1, u_1) + d_{G_1}(z, u_1) \geq d_{G_2}(x_2, y_2) + d_{G_1}(y_1, z),$$
which implies that
$$d_{G_2}(x_2, y_2) + d_{G_1}(x_1, u_1) + d_{G_1}(z, u_1) \geq 2d_{G_2}(x_2, y_2) + d_{G_1}(y_1, z) > d(y_2 y_1, x_2 z).$$
Hence, $y_2 z \in R\{x_2 x_1, y_2 y_1\}$, as desired. $\Box$

Lemma 2.6 Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. Then
$$\text{rdim}(G_{22}^R \cap G_{11}^u) \geq |V(G_2)| \cdot \text{rdim}(G_{11}^u).$$
Proof. Let $\overline{W}$ be a rooted metric basis of $G_{22}^R \cap G_{11}^u$. For $v \in V(G_2)$, write $\overline{W}_v = \{z \mid vz \in \overline{W}\}$. For any two distinct vertices $x, y$ of $G_1$, there exists a vertex $wz$ in $\overline{W} \cup \{u_2 y_1\}$ such that $d(vw, wz) \neq d(vy, wz)$. If $w = v$, by (1) we get $d_{G_1}(x, z) \neq d_{G_1}(y, z)$, which implies that $z \in (\overline{W}_v \cup \{u_1\}) \cap R_{G_1} \{x, y\}$. If $w \neq v$, by (1) we have $d_{G_1}(x, u_1) \neq d_{G_1}(y, u_1)$, which implies that $u_1 \in R_{G_1} \{x, y\}$. Therefore, we have $(\overline{W}_v \cup \{u_1\}) \cap R_{G_1} \{x, y\} \neq \emptyset$, which implies that $\overline{W}_v$ is a rooted resolving set of $G_{11}^u$. Since $\overline{W}$ is the disjoint union of all $\overline{W}_v$’s, one gets
$$\text{rdim}(G_{22}^R \cap G_{11}^u) = |\overline{W}| = \sum_{v \in V(G_2)} |\overline{W}_v| \geq |V(G_2)| \cdot \text{rdim}(G_{11}^u),$$
as desired. $\Box$

Theorem 2.7 Let $G_1^{u_1}$ and $G_2^{u_2}$ be two rooted graphs. If $G_1$ is not a path with an
end-vertex $u_1$, then
$$\text{dim}(G_{22}^R \cap G_{11}^u) = |V(G_2)| \cdot \text{rdim}(G_{11}^u).$$
Proof. By Lemma 2.6, we only need to prove that
$$\text{dim}(G_{22}^R \cap G_{11}^u) \leq |V(G_2)| \cdot \text{rdim}(G_{11}^u). \quad (2)$$
Let $W$ be a rooted metric basis of $G_{11}^u$. Then $W \neq \emptyset$. Write $\overline{W} = \{vw \mid v \in V(G_2), w \in W\}$. Note that $|\overline{W}| = |V(G_2)| \cdot \text{rdim}(G_{11}^u)$. In order to prove (2), we only need to show that $\overline{W}$ is a resolving set of $G_{22}^R \cap G_{11}^u$. It suffices to show that, for any two distinct vertices $x_2 y_1$ and $y_2 y_1$ of $G_{22}^R \cap G_{11}^u$,
$$\overline{W} \cap R\{x_2 x_1, y_2 y_1\} \neq \emptyset. \quad (3)$$
If \( x_2 = y_2 \) and \( u_1 \not\in R_{G_1}\{x_1, y_1\} \), then \( W \cap R_{G_1}\{x_1, y_1\} \neq \emptyset \), by Lemma 2.5 (i) we obtain (3). If \( x_2 = y_2 \) and \( u_1 \in R_{G_1}\{x_1, y_1\} \), by Lemma 2.5 (i) we have \( vw \in W \cap R\{x_2x_1, y_2y_1\} \) for any \( v \neq x_2 \) and any \( w \in W \), which implies that (3) holds. If \( x_2 \neq y_2 \), since \( \{x_2w, y_2w\} \subseteq \overline{W} \) for any \( w \in W \), the inequality (3) holds by Lemma 2.5 (ii). We accomplish our proof. \( \square \)

Combining Observation 2.1 and Theorem 2.7, we have the following result.

**Corollary 2.8** Let \( G_n^{u_1} \) and \( G_n^{u_2} \) be two rooted graphs.

(i) If there exists a metric basis of \( G_1 \) containing \( u_1 \) and \( G_1 \) is not a path, then

\[
\dim(G_n^{u_2} \cap G_n^{u_1}) = |V(G_2)|(|\dim(G_1)| - 1).
\]

(ii) If any metric basis of \( G_1 \) does not contain \( u_1 \), then

\[
\dim(G_n^{u_2} \cap G_n^{u_1}) = |V(G_2)|\dim(G_1).
\]

The binomial tree \( T_n \) is the hierarchical product of \( n \) copies of the complete graph on two vertices, which is a useful data structure in the context of algorithm analysis and designs [9]. It was proved that the metric dimension for a tree can be expressed in terms of its parameters in [7, 15, 24].

**Corollary 2.9** Let \( n \geq 2 \). Then \( \dim(T_n) = 2^{n-2} \).

**Proof.** Note that \( \dim(T_2) = 1 \). Now suppose \( n \geq 3 \). Since \( T_n = (K_2^0 \sqcap \cdots \sqcap K_2^0) \sqcap (K_2^0 \sqcap K_2^0) \) and \( \rdim(K_2^0 \sqcap K_2^0) = 1 \), the desired result follows by Theorem 2.7. \( \square \)

We always assume that \( 0 \) is one end-vertex of \( P_n \). In the remaining of this section, we shall prove some tight inequalities for \( \dim(G^n \cap P_n^0) \).

**Proposition 2.10** Let \( G^n \) be a rooted graph with diameter \( d \). Then

\[
\begin{align*}
\dim(G^n \cap P_n^0) &\leq \dim(G^n \cap P_{n+1}^0) \text{ for } 1 \leq n \leq d - 1, \\
\dim(G^n \cap P_n^0) &\equiv \dim(G^n \cap P_{n+1}^0) \text{ for } n \geq d.
\end{align*}
\]

**Proof.** If \( G = K_2 \), then \( G^n \cap P_n^0 \) is the path, which implies that (5) holds. Now we only consider \( |V(G)| \geq 3 \). Suppose that \( \overline{W}_{n+1} \) is a metric basis of \( G^n \cap P_{n+1} \). Let \( P_n = (z_0 = 0, z_1, \ldots, z_{n-1}) \). Define \( \pi_n : V(G^n \cap P_{n+1}^0) \longrightarrow V(G^n \cap P_n^0) \) by

\[
\pi_n(vz_i) = \begin{cases} 
vz_{n-1}, & \text{if } i = n, \\
vz_i, & \text{if } i \leq n - 1.
\end{cases}
\]

Then \( \pi_n(\overline{W}_{n+1}) \) is a resolving set of \( G^n \cap P_{n+1}^0 \), which implies that \( \dim(G^n \cap P_n^0) \leq \dim(G^n \cap P_{n+1}^0) \) for any positive integer \( n \). So (4) holds.

In order to prove (5), we only need to show that \( \overline{W}_n \) is a resolving set of \( G^n \cap P_{n+1}^0 \) for \( n \geq d \). Pick any two distinct vertices \( vz_i \) and \( vz_j \) of \( G^n \cap P_{n+1}^0 \). It suffices to prove that

\[
\overline{W}_n \cap R_{G^n \cap P_{n+1}^0}\{vz_i, vz_j\} \neq \emptyset.
\]
Without loss of generality, we may assume that $0 \leq i \leq j \leq n$. If $j \leq n-1$, then $R_{G\cap P_n^m} \{v_1 z_i, v_2 z_j\} \supseteq R_{G\cap P_n^m} \{v_1 z_i, v_2 z_j\}$; and so (6) holds. Now suppose $j = n$.

Claim: There exist two distinct vertices $w_1$ and $w_2$ of $G$ such that

$$\overline{W}_n \cap \{w_1 z_k \mid 0 \leq k \leq n-1\} \neq \emptyset \text{ and } \overline{W}_n \cap \{w_2 z_k \mid 0 \leq k \leq n-1\} \neq \emptyset. \quad (7)$$

Suppose for the contradiction that there exists a vertex $w \in V(G)$ such that $\overline{W}_n \subseteq \{w z_k \mid 0 \leq k \leq n-1\}$. If the degree of $w$ in $G$ is one, then there exists an induced path $(w, x, y) \in G$. For any $w z_k \in \overline{W}_n$, we have $d(x z_1, w z_k) = k + 2 = d(y z_0, w z_k)$, contrary to the fact that $\overline{W}_n$ is a metric basis of $G^n \cap P_n^0$. If the degree of $w$ in $G$ is at least two, pick two distinct neighbors $x$ and $y$ of $w$ in $G$. Then $d(x z_0, w z_k) = k + 1 = d(y z_0, w z_k)$ for any $w z_k \in \overline{W}_n$, a contradiction. Hence our claim is valid.

Now we prove (6) for $j = n$. By the claim, we may pick two distinct vertices $w_1$ and $w_2$ satisfying (7).

Case 1. $v_1 = v_2$. Since $\{w_1 z_k \mid 0 \leq k \leq n-1\}$ or $\{w_2 z_k \mid 0 \leq k \leq n-1\}$ is a subset of $R_{G\cap P_n^m} \{v_1 z_i, v_1 z_n\}$, the inequality (6) holds.

Case 2. $v_1 \neq v_2$.

Case 2.1. $i = 0$. Without loss of generality, we may assume that $w_1 \neq v_2$. Pick $z_k$ satisfying $w_1 z_k \in \overline{W}_n$. Then

$$d(v_1 z_0, w_1 z_k) = d_G(v_1, w_1) + k \leq d + k \leq n + k < d_G(v_2, w_1) + n + k = d(v_2 z_n, w_1 z_k),$$

which implies that $w_1 z_k \in R_{G\cap P_n^m} \{v_1 z_0, v_2 z_n\}$. So (6) holds.

Case 2.2. $i \geq 1$. Note that

$$R_{G\cap P_n^m} \{v_1 z_i, v_2 z_n\} = R_{G\cap P_n^m} \{v_1 z_{i-1}, v_2 z_{n-1}\} \supseteq R_{G\cap P_n^m} \{v_1 z_{i-1}, v_2 z_{n-1}\}.$$

Then (6) holds. \qed

**Proposition 2.11** For any rooted graph $G^n$, we have

$$\dim(G) \leq \dim(G^n \cap P_n^0) \leq |V(G)| - 1. \quad (8)$$

**Proof.** Let $z$ be the other end-vertex of $P_n$. Fix a vertex $v_0 \in V(G)$ and write $\overline{S} = \{v z \mid v \in V(G) \setminus \{v_0\}\}$. Since $\{z\}$ is a resolving set of $P_n$, the set $\overline{S}$ resolves $G \cap P_n$ by (1). Hence $\dim(G^n \cap P_n^0) \leq |\overline{S}| = |V(G)| - 1$. Since $G^n$ is isomorphic to $G^n \cap P_n^0$, Proposition 2.10 implies that $\dim(G) \leq \dim(G^n \cap P_n^0)$. \qed

For $m \geq 2$, we have $\dim(K_m^u \cap P_n^0) = m - 1$. This shows that the inequalities (4) and (8) are tight.

**Example 2.12** For $m \geq 3$ and $n \geq 2$, we have $\dim(P_m^u \cap P_n^0) = 2$. In fact, write $P_k = (z_0 = 0, z_1, \ldots, z_{k-1})$, then $\{z_0 z_{n-1}, z_{m-1} z_{n-1}\}$ is a resolving set of $P_m^u \cap P_n^0$.

**Example 2.13** Let $C_m$ be the cycle with length $m$. Then $\dim(C_m^u \cap P_n^0) = 2$. In fact, write $P_k = (z_0 = 0, z_1, \ldots, z_{m-1})$ and $C_m = (c_0, c_1, \ldots, c_{m-1}, c_0)$, then $\{c_0 z_{n-1}, c_1 z_{n-1}\}$ is a resolving set of $C_m^u \cap P_n^0$. 

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3 Fractional metric dimension

In order to study the fractional metric dimension for the hierarchical product of graphs, we first introduce the fractional rooted metric dimension for a rooted graph. Similar to the fractionalization of metric dimension, we give a fractional version of the rooted metric dimension for a rooted graph. Let $G^u$ be a rooted graph of order $n$. Write

$$\mathcal{P}^u = \{ \{v, w\} \mid v, w \in V(G), v \neq w, d(v, u) = d(w, u)\}.$$  

Suppose $\mathcal{P}^u \neq \emptyset$. Write $V(G) \setminus \{u\} = \{v_1, \ldots, v_{n-1}\}$ and $\mathcal{P}^u = \{\alpha_1, \ldots, \alpha_m\}$. Let $A^u$ be the $m \times (n-1)$ matrix with

$$(A^u)_{ij} = \begin{cases} 1, & \text{if } v_j \text{ resolves } \alpha_i, \\ 0, & \text{otherwise}. \end{cases}$$

The integer programming formulation of the rooted metric dimension for $G^u$ is given by

Minimize $f(x_1, \ldots, x_{n-1}) = x_1 + \cdots + x_{n-1}$

Subject to $A^u x \geq 1$

where $x = (x_1, \ldots, x_{n-1})^T, x_i \in \{0, 1\}$ and $1$ is the $m \times 1$ column vector all of whose entries are 1. The optimal solution of the linear programming relaxation of the above integer programming problem, where we replace $x_i \in \{0, 1\}$ by $x_i \in [0, 1]$, gives the fractional rooted metric dimension for $G^u$, which we denote by $rdim_f(G^u)$.

Let $G^u$ be a rooted graph which is not a path with an end-vertex $u$. A rooted resolving function of a rooted graph $G^u$ is a real value function $g : V(G) \rightarrow [0, 1]$ such that $g(R\{v, w\}) \geq 1$ for each $\{v, w\} \in \mathcal{P}^u$. The fractional rooted metric dimension for $G^u$ is the minimum weight of a rooted resolving function of $G^u$.

Proposition 3.1 Let $G^u$ be a rooted graph which is not a path with an end-vertex $u$. Then

(i) $rdim_f(G^u) \leq rdim(G^u)$.

(ii) $rdim_f(G^u) \leq \frac{|V(G)|-1}{2}$.

(iii) $\dim_f(G) - 1 \leq rdim_f(G^u) \leq \dim_f(G)$.

Proof. (i) Let $W$ be a rooted metric basis of $G^u$. Define $g : V(G) \rightarrow [0, 1]$ by

$$g(v) = \begin{cases} 1, & \text{if } v \in W, \\ 0, & \text{if } v \notin W. \end{cases}$$

For any $\{x, y\} \in \mathcal{P}^u$, there exists a vertex $v \in W$ such that $d(x, v) \neq d(y, v)$. Then $g(R\{x, y\}) \geq g(v) = 1$, which implies that $g$ is a rooted resolving function of $G^u$. Hence $rdim_f(G^u) \leq |g| = |W| = rdim(G^u)$.

(ii) The function $g : V(G) \rightarrow [0, 1]$ defined by

$$g(v) = \begin{cases} 0, & \text{if } v = u, \\ \frac{1}{2}, & \text{if } v \neq u. \end{cases}$$
is a rooted resolving function of $G_u$. Hence $\text{rdim}_f(G_u) \leq \frac{|V(G)| - 1}{2}$.

(iii) It is clear that $\text{rdim}_f(G_u) \leq \text{dim}_f(G)$. Let $g$ be a rooted resolving function of $G_u$. Then the function $h : V(G) \rightarrow [0, 1]$ defined by

$$h(v) = \begin{cases} 1, & \text{if } v = u, \\ g(v), & \text{if } v \neq u \end{cases}$$

is a resolving function of $G$. Hence $\text{dim}_f(G) \leq \text{rdim}_f(G_u) + 1$, as desired. \hfill \Box

If $u$ is not an end-vertex of the path $P_n$, then $\text{rdim}_f(P^n_u) = \text{rdim}(P^n_u) = \text{dim}_f(P_n) = 1$, which implies that the upper bounds in Proposition 3.1 (i) and (iii) are tight. The fact that $\text{rdim}_f(K^n_u) = \frac{n-1}{2}$ shows that the inequality in Proposition 3.1 (ii) is tight.

Next, we study the fractional metric dimension for the hierarchical product of graphs.

For two rooted graphs $G^u_1$ and $G^u_2$, write

$$\mathcal{P}^u_1 = \{\{x, y\} \subseteq V(G_1) \mid x \neq y, d_{G_1}(x, u_1) = d_{G_1}(y, u_1)\},$$

$$\mathcal{P}^{u_2 \cap u_1}_2 = \{\{x_2x_1, y_2y_1\} \subseteq V(G^u_2 \cap G^u_1) \mid x_2x_1 \neq y_2y_1, d(x_2x_1, u_2u_1) = d(y_2y_1, u_2u_1)\}.$$

**Lemma 3.2** Let $G^u_1$ and $G^u_2$ be two rooted graphs. If $G_1$ is not a path with an end-vertex $u_1$, then

$$\text{rdim}_f(G^u_2 \cap G^u_1) \geq |V(G_2)| \cdot \text{rdim}_f(G^u_1).$$

**Proof.** Suppose that $g$ is a rooted resolving function of $G^u_2 \cap G^u_1$ with weight $\text{rdim}_f(G^u_2 \cap G^u_1)$. For each $z \in V(G_2)$, define

$$\overline{g}_z : V(G_1) \rightarrow [0, 1], \quad x \mapsto g(zz).$$

Write $\overline{\mathcal{P}}^{u_1} = \{\{zx, zy\} \mid z \in V(G_2), \{x, y\} \in \mathcal{P}^{u_1}\}$. By (1), we have $\overline{\mathcal{P}}^{u_1} \subseteq \overline{\mathcal{P}}^{u_2 \cap u_1}$. Hence $\overline{g}_z(R_{G_1}\{x, y\}) \geq 1$ for any $\{x, y\} \in \mathcal{P}^{u_1}$, which implies that $|\overline{g}_z| \geq \text{rdim}_f(G^u_1)$. Consequently,

$$\text{rdim}_f(G^u_2 \cap G^u_1) = |\overline{g}| = \sum_{z \in V(G_2)} |\overline{g}_z| \geq |V(G_2)| \cdot \text{rdim}_f(G^u_1),$$

as desired. \hfill \Box

**Theorem 3.3** Let $G^u_1$ and $G^u_2$ be two rooted graphs. If $G_1$ is not a path with an end-vertex $u_1$, then

$$\text{dim}_f(G^u_2 \cap G^u_1) = |V(G_2)| \cdot \text{rdim}_f(G^u_1).$$

**Proof.** Combining Proposition 3.1 and Lemma 3.2, we only need to prove that

$$\text{dim}_f(G^u_2 \cap G^u_1) \leq |V(G_2)| \cdot \text{rdim}_f(G^u_1). \quad (9)$$

By Proposition 2.4 we have $\mathcal{P}^{u_1} \neq \emptyset$. Let $g$ be a rooted resolving function of $G_1$ with weight $\text{rdim}_f(G^u_1)$. Define

$$\overline{g} : V(G^u_2 \cap G^u_1) \rightarrow [0, 1], \quad x_2x_1 \mapsto g(x_1).$$

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We shall show that, for any two distinct vertices \( x_2x_1 \) and \( y_2y_1 \) of \( G_2^{u_2} \cap G_1^{u_1} \),
\[
\overline{g}(R\{x_2x_1, y_2y_1\}) \geq 1.
\] (10)

**Case 1.** \( x_2 = y_2 \). If \( u_1 \not\in R_G\{x_1, y_1\} \), by Lemma 2.5 we get \( R\{x_2x_1, y_2y_1\} = \{x_2z \mid z \in R_G\{x_1, y_1\}\} \), which implies that \( \overline{g}(R\{x_2x_1, y_2y_1\}) = g(R_G\{x_1, y_1\}) \cdot \dim(1) \).
Since \( \{x_1, y_1\} \in P^{u_1} \), we obtain (10). If \( u_1 \in R_G\{x_1, y_1\} \), by Lemma 2.5 we have \( R\{x_2x_1, y_2y_1\} \supseteq \{vz \mid z \in V(G_1)\} \) for any \( v \in V(G_2) \setminus \{x_2\} \), which implies that \( \overline{g}(R\{x_2x_1, y_2y_1\}) \geq |g| \), so (10) holds.

**Case 2.** \( x_2 \neq y_2 \). Write \( W = \{z \mid x_2z \in R\{x_2x_1, y_2y_1\}\} \) and \( S = \{z \mid y_2z \in R\{x_2x_1, y_2y_1\}\} \). By Lemma 2.5 we have \( W \cup S = V(G_1) \). Then
\[
\overline{g}(R\{x_2x_1, y_2y_1\}) \geq \sum_{z \in W} \overline{g}(x_2z) + \sum_{z \in S} g(W) + g(S) \overline{g}(y_2z) \geq |g|,
\]
which implies that (10) holds.

Therefore, \( \overline{g} \) is a resolving function of \( G_2^{u_2} \cap G_1^{u_1} \), which implies that \( \dimf(G_2^{u_2} \cap G_1^{u_1}) \leq |\overline{g}| \). Since \( |\overline{g}| = |V(G_2)| \cdot \dimf(G_1^{u_1}) \), we obtain (9). Our proof is accomplished.

By Theorem 3.3, we obtain the following corollary immediately.

**Corollary 3.4** Let \( n \geq 2 \). Then \( \dimf(T_n) = 2^{n-2} \).

Arunagam and Mathew [1] proposed a natural problem: Characterize graphs for which \( \dimf(G) = \dim(G) \). By Corollaries 2.9 and 3.4, the binomial tree \( T_n \) satisfies \( \dimf(T_n) = \dim(T_n) \). But this problem is still open.

It seems that there is a gap to determine \( \dimf(G^{u} \cap P_n^0) \). We conclude this paper by giving some tight inequalities involving it.

**Proposition 3.5** For any rooted graph \( G^u \), we have
\[
\dimf(G) \leq \dimf(G^u \cap P_n^0) \leq \dimf(G^u \cap P_{n+1}^0) \leq \frac{|V(G)|}{2}.
\]

**Proof.** Write \( P_n = (z_0 = 0, z_1, \ldots, z_{n-1}) \). For a resolving function \( \overline{g}_{n+1}^0 \) of \( G^u \cap P_{n+1}^0 \), we define \( \overline{g}_{n+1}^0 : V(G^u \cap P_{n}^0) \rightarrow [0, 1] \) by
\[
\overline{g}_{n+1}^0(x_2x_1) = \begin{cases} \overline{g}_{n+1}^0(x_2z_{n-1}) + \overline{g}_{n+1}^0(x_2z_n), & \text{if } x_1 = z_{n-1}, \\ \overline{g}_{n+1}^0(x_2x_1), & \text{if } x_1 \neq z_{n-1}. \end{cases}
\]
Then \( \overline{g}_{n+1}^0 \) is a resolving function of \( G^u \cap P_{n+1}^0 \). Since \( |\overline{g}_{n+1}^0| = |\overline{g}_{n+1}| \), we have
\[
\dimf(G) = \dimf(G^u \cap P_n^0) \leq \dimf(G^u \cap P_{n+1}^0) \leq \dimf(G^u \cap P_n^0).
\]

For proving the last inequality, define \( \overline{h} : V(G^u \cap P_{n+1}^0) \rightarrow [0, 1] \) by
\[
\overline{h}(x_2x_1) = \begin{cases} \frac{1}{2}, & \text{if } x_1 = z_n, \\ 0, & \text{if } x_1 \neq z_n. \end{cases}
\]
Then \( \overline{h} \) is a resolving function of \( G^u \cap P_{n+1}^0 \) with weight \( \frac{|V(G)|}{2} \). Hence \( \dimf(G^u \cap P_{n+1}^0) \leq \frac{|V(G)|}{2} \). \( \Box \)

For \( m \geq 2 \), we have \( \dimf(K_m^u \cap P_n^0) = \frac{m}{2} \). This shows that all the inequalities in Proposition 3.5 are tight.
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References


