Applicable Analysis and Discrete Mathematics - Accepted manuscript
Received: January 15, 2013.
Revised: August 27, 2013
Please cite this article as: M. Gionfriddo, E. Guardo, L. Milazzo: EXTENDING BICOLORINGS FOR STEINER TRIPLE SYSTEMS, Appl. Anal. Discrete Math. (to appear)
DOI: 10.2298/AADM130827019G

EXTENDING BICOLORINGS FOR STEINER TRIPLE SYSTEMS

M. Gionfriddo, E. Guardo, L. Milazzo

In memory of Lucia Gionfriddo

Abstract. We initiate the study of extended bicolorings of Steiner triple systems (STS) which start with a k-bicoloring of an STS(v) and end up with a k-bicoloring of an STS($2v+1$) obtained by a doubling construction, using only the original colors used in coloring the subsystem STS(v). By producing many such extended bicolorings, we obtain several infinite classes of orders for which there exist STSs with different lower and upper chromatic number.

1. Introduction

A *Steiner triple system* (STS) is a pair $(V, \mathcal{B})$ where V is a v-set and $\mathcal{B}$ is a collection of 3-subsets of V called *triples* such that every 2-subset of V is contained in exactly one triple, see [4]. A *coloring* of an STS $(V, \mathcal{B})$ is a mapping $\phi : V \to C$; the elements of C are called colors. If $|C| = k$, we have a k-coloring. For each $c \in C$, the set $\phi^{-1}(c) = \{x : \phi(x) = c\}$ is a *color class.* A coloring ϕ of $(V, \mathcal{B})$ is a *bicoloring* if $|\phi(B)| = 2$ for all $B \in \mathcal{B}$. Here $\phi(B) = \bigcup_{x \in B} \phi(x)$. Thus in a bicoloring of $(V, \mathcal{B})$, every triple has two elements in one color class and one in another class, so there are no monochromatic triples nor polychromatic triples (i.e. triples receiving three colors). A *strict* k-bicoloring is one in which exactly k colors are used. From now on we assume that all our bicolorings are strict, unless the contrary is explicitly stated.

Considerations of bicolorings of Steiner triple systems arose from the theory of mixed hypergraphs pioneered by Voloshin [22,23]. In a mixed hypergraph setting, there are two kinds of edges: C-edges which must contain two vertices colored with the *same* color, and D-edges which must contain two vertices of *different* colors. Requiring all edges of a Steiner system to be both, C-triples and D-triples leads to the concept of bicolorings. In the literature, often the terms BSTS, BSQS, or bi-STS coloring are used instead of bicoloring (cf. [6,14,15,16,17,18,19]). We can also find results related to particular color patterns for different designs in [1,5,7,9,10,11,12,13,20].

The minimum (maximum) possible number k in a strict k-bicoloring of an STS is called the *lower* (*upper*) chromatic number of the STS. However, not every STS has a bicoloring. The smallest such example occurs for STSs of order 15: of the 80 nonisomorphic systems, 57 are uncolorable. In fact, every STS(v) whose independence number is at most $\frac{v}{3}$ is uncolorable. It is likely that almost all STSs have this property although to best of our knowledge this remains unproved (cf. [4]).

Given a k-bicoloring C, if the cardinalities of the color classes are $n_1, n_2, \ldots, n_k$, we will write for brevity $C = C(n_1, n_2, \ldots, n_k)$, and assume, unless stated to the contrary, that $n_1 \leq n_2 \leq \cdots \leq n_k$.

In this paper, we want to initiate a study of *extended* bicolorings, i.e. bicolorings of an STS(w) which start with a bicoloring of a sub-STS(v). Essential for us in this endeavor will be a well-known recursive construction known as a *doubling construction* (other names: $v \to 2v+1$ rule, doubling plus one construction etc.) which starts with an STS(v) and ends with an STS($2v+1$).

To obtain such a construction, all that is needed, apart from the subsystem, is a 1-factorization of the complete graph K_{v+1}. Indeed, let $(X, \mathcal{F})$ where $\mathcal{F} = \{F_1, \ldots, F_v\}$ is a 1-factorization of K_{v+1} (where $|X| = v+1$ must be even). If $(V, \mathcal{B})$, $V = \{a_1, \ldots, a_v\}$, is an STS(v), form the set of triples $\mathcal{C} = \{\{a_i, x, y\} : a_i \in V, \{x, y\} \in F_i\}$. Then $(V \cup X, \mathcal{B} \cup \mathcal{C})$ is an STS($2v+1$) (cf. [4]).

An easy observation is that if a given STS(v), $(V, \mathcal{B})$, admits a k-bicoloring $C = C(n_1, \ldots, n_k)$, then *any* STS($2v+1$) obtained from $(V, \mathcal{B})$ by a doubling construction admits a $(k+1)$-bicoloring $C(n_1, \ldots, n_k, n_{k+1})$ where the $v+1$ vertices of X are colored with a new color, and so $n_{k+1} = v+1$. Another such $(k+1)$-bicoloring that can be always obtained is $C' = C'(n'_1, \ldots, n'_k, n'_{k+1})$ where $n'_i = 2n_i$ for $i = 1, \ldots, k$, and $n'_{k+1} = 1$ (see [3]).

The question that we want to address is the following. Given an STS(v) with a bicoloring $C = C(n_1, \ldots, n_k)$, when does there exist an STS($2v+1$) obtained by a doubling construction which admits a bicoloring $C' = C'(n'_1, \ldots, n'_k)$? In other words, when can we color the elements of X with the original k colors of the k-bicoloring C *without* introducing an extra color as above? If such a coloring exists, we call it an *extended* bicoloring of C. Thus extended bicolorings may exist only for orders $2v+1 \equiv 3$ or 7 (mod 12) as $v \equiv 1$ or 3 (mod 6).

The importance of extended bicolorings lies in the fact that they enable one to construct STSs with different lower and upper chromatic numbers; there are only scarce results in the literature on the latter (cf., e.g., [14]).

The extendibility of partial colorings is a relevant issue in graph theory (see [2,21]), both for its theoretical interest and practical applications. Here we initiate the study of extending bicolorings in Steiner triple systems, with the aim to derive consequences in the coloring theory of mixed hypergraphs. In this way the present work relates several intensively studied areas.

2. Extended bicolorings

Let $S = (V, \mathcal{B})$ be an STS(v) which is k-bicolorable with $C = C(n_1, \ldots, n_k)$ and let $S' = (X, \mathcal{C})$ be an STS($2v + 1$) obtained from S by a doubling construction. We are trying to investigate the conditions under which there exists an *extended* bicoloring of S', say $C' = C'(n'_1, \ldots, n'_k)$ where the elements of the subsystem $(V, \mathcal{B})$ are colored as in C, and the elements of $Y = X \setminus V$ are colored with the same colors as those used in C. If $c_i = n'_i - n_i$, $1 \leq i \leq k$ are the numbers of vertices in Y colored with the color $i \in C$ then clearly, $\sum_{i=1}^{k} c_i = v + 1$ (it may happen that $c_j = 0$ for some $j \in \{1, \ldots, k\}$). Beside this obvious condition, the following is a necessary condition for the existence of an extended k-bicoloring of S'.

Theorem 1. *Let $S = (V, \mathcal{B})$ be an STS(v) which is k-bicolorable with $C = C(n_1, \ldots, n_k)$ and let $S' = (X, \mathcal{C})$ be an STS($2v + 1$) obtained from S by a doubling construction. With the notation as above,*

$$\sum_{i=1}^{k} c_i^2 + 2 \sum_{i=1}^{k} n_i c_i = (v+1)^2. \qquad (1)$$

Proof. The number of pairs of elements of Y equals $\binom{v+1}{2}$. Clearly, the number of monochromatic pairs among these is $\sum_{i=1}^{k} \binom{c_i}{2}$. On the other hand, the number of two-colored pairs among these is $\sum_{i=1}^{k} n_i c_i$; indeed, if $a_i \in V$ is colored with color j, then any pair $\{x, y\}$ in the 1-factor F_i is either monochromatic or else one of x, y is colored with color j. Consequently, the number of two-colored pairs in F_i is n_j. Thus we have

$$\sum_{i=1}^{k} \binom{c_i}{2} \; + \; \sum_{i=1}^{k} n_i c_i = \binom{v+1}{2}$$

whence (1) follows easily. □

Solutions of (1) will be called *solutions with respect to C*. We stress that condition (1) given by Theorem 1 is only necessary for the existence of an extended bicoloring. It certainly is not sufficient: in [6] condition (1) was determined for $v = 2^h - 1$ and all of its solutions were determined for $h \leq 10$, nevertheless these solutions do not lead to any extended bicolorings.

Corollary 2. *Let S' be a k-bicolorable STS($2v+1$) obtained by a doubling construction from a k-bicolorable STS(v) with the coloring $C = C(n_1, \ldots,$*

n_k), and let $(c_1, \ldots, c_k)$ be a solution to (1) with respect to C.
1. If there is a $c_j = 0$, then all c_i's are even.
2. If there is j such that $c_j > \frac{v+1}{2}$, then there exists no extended bicoloring of C.

Proof. 1. If there is a j such that $c_j = 0$, then in any factor corresponding to an element $a_l \in V$ colored with the color j, all pairs must be monochromatic which implies that every c_i has to be even.
2. If $c_j > \frac{v+1}{2}$ for some j, then in all factors associated with elements of V colored with color j, there must exist monochromatic pairs of color j, and thus monochromatic triples, which is a contradiction. □

We illustrate the use of Corollary 2 on the example of a (potential) extended bicoloring of an STS(19). First notice that no extended bicolorings of STS(v) exist for $v = 7$ or $v = 15$, as shown in [6]. The unique STS(9) admits a bicoloring $C = C(1, 4, 4)$, and no other bicolorings (see [3] or [17]). The following are all solutions with respect to C: (a) $(3, 2, 5)$, (b) $(3, 5, 2)$, (c) $(5, 0, 5)$, (d) $(5, 5, 0)$, (e) $(8, 0, 2)$, and (f) $(8, 2, 0)$. Corollary 2.1 eliminates solutions (c), (d), (e) and (f) from contention. Concerning (a), since $c_1 = 3$ and $c_3 = 5$, it must be that in the four 1-factors associated with elements colored with color 2, there are exactly two 2-colored pairs colored with colors 1 and 2 and with 2 and 3, one monochromatic pair of color 1, and two monochromatic pairs of color 3. Since $c_1 = 3$, this is easily seen to be impossible, so solution (a) cannot lead to an extended bicoloring. The same reasoning applies to the solution (b). Thus there exist no extended bicolorings of any STS(19) (obtained from an STS(9) by a doubling construction, of course). Thus the smallest w for which an STS(w) may admit an extended bicoloring is $w = 27$ (where the STS(27) is obtained from an STS(13) by a doubling construction).

We remark that due to the above, any uniquely 3-colorable STS(19), or any 3- and 4-colorable STS(19) cannot contain a sub-STS(9). It was shown in [14] that there exist uniquely 3-bicolorable, uniquely 4-bicolorable, and also 3- and 4-bicolorable STS(19).

Theorem 3. *Let S be a k-bicolorable STS(v) with the k-bicoloring $C = C(n_1, \ldots, n_k)$. and suppose there exist $i, j, i \neq j$, such that $n_i + n_j = \frac{v+1}{2} \equiv 0 \ (mod\ 2)$. Then there exists an STS($2v + 1$), S', obtained by a doubling construction from S such that S' has an extended k-bicoloring C'.*

Proof. Since $v + 1 \equiv 0 \ (mod\ 4)$, we may use as the 1-factorization $\mathcal{F} = \{F_1, \ldots, F_v\}$ in the doubling construction the following 1-factorization. Write $Y = Y_1 \cup Y_2$ where $|Y_i| = \frac{v+1}{2}$; take $F_1, \ldots, F_{\frac{v+1}{2}}$ to be the 1-factors of any 1-factorization of the complete bipartite graph $K_{\frac{v+1}{2}, \frac{v+1}{2}}$

with bipartition (Y_1, Y_2); for the remaining $\frac{v-1}{2}$ 1-factors $F_{\frac{v+3}{2}}, \ldots, F_v$, take $F_i = G_i \cup H_i$, $i = \frac{v+3}{2}, \ldots, v$, where G_i, H_i are the 1-factors of any 1-factorization of $K_{\frac{v+1}{2}}$ on Y_1, and Y_2, respectively. Color now the $\frac{v+1}{2}$ vertices of Y_1 with color i and the $\frac{v+1}{2}$ vertices of Y_2 with color j. Associate the 1-factors $F_1, \ldots, F_{\frac{v+1}{2}}$ with the vertices of V colored in the coloring C with either color i or color j, and associate the remaining 1-factors with the elements of V colored in C with colors other than i or j. We obtain in this way an extended k-bicoloring of the resulting STS$(2v+1)$. Indeed, if a_q is an element of V which is colored with i or j, then any triple T containing a_q is two-colored: one of the two elements of T other than a_q is colored with color i, and the other with color j. On the other hand, if a_r is an element of V colored in C with a color other than i or j, then any triple T containing a_r is also two-colored since the two elements of T other than a_r are *both* colored with i or both colored with j. $\square$

A more general version of Theorem 3 is the following.

Theorem 4. *Let S be a k-bicolorable STS(v) with the k-bicoloring $C = C(n_1, \ldots, n_k)$. Suppose that there exist p integers n_{k_i}, $1 \le i \le p < k$ such that $n_{k_1} + n_{k_2} = \frac{v+1}{2^{p-1}}$ is an even integer, and further $n_{k_i} = \frac{v+1}{2^{p-i+1}}$ for $3 \le i \le p$ are all even. Then there exists an STS($2v+1$) obtained by a doubling construction from S which has an extended k-bicoloring.*

The proof of this theorem is more technical than that of Theorem 3, especially in the description of the 1-factorization $\mathcal{F}$ involved in the doubling construction. Since in what follows we do not make use of this more general version, with one exception, this proof is omitted (see Appendix [8]).

3. Small extended bicolorings

As shown earlier, there exist no extended bicolorings of STS(w) for $w \le 19$. Since $w \equiv 3$ or 7 $(mod\ 12)$, the smallest w for which there might exist an extended bicoloring is $w = 27$. Such an extended bicoloring does indeed exist.

Theorem 5. *There exists an STS(27), $(W, \mathcal{C})$ obtained by a doubling construction from an STS(13), $(V, \mathcal{B})$, which has an extended 3-bicoloring $C = C(2, 5, 6)$. For this system, $\chi = 3$ and $\bar{\chi} = 4$.*

Proof. All solutions (c_1, c_2, c_3) with respect to the coloring C (cf. Theorem 1) are as follows: (a) $(4, 4, 6)$, (b) $(7, 1, 6)$, (c) $(4, 7, 3)$, (d) $(7, 7, 0)$, (e) $(10, 1, 3)$, (f) $(10, 4, 0)$. By Corollary 2, solutions (d), (e), and (f) cannot lead to an extended bicoloring of C. Concerning solution (c), there are three monochromatic pairs of elements of color 3. Two of these pairs may occur in the two 1-factors corresponding to the vertices of V of color 1 but the third pair cannot occur in a 1-factor corresponding to a vertex of V of color 2 (as there are 7 vertices of $W \setminus V$ of color 2, and that would force a monochromatic triple of color 2), nor clearly in a 1-factor corresponding to a vertex of color 3. Thus solution (c) does not lead to an extended bicoloring of C either.

On the other hand, each of the first two solutions, namely $(4,4,6)$ and $(7,1,6)$, lead to an extended bicoloring $C' = C'(6,9,12)$. The 1-factorizations $\mathcal{F}$ used in the respective doubling constructions are given in the Appendix [8]. Our STS(27), $(W, \mathcal{C})$, besides having an extended 3-bicoloring with respect to C, is also 4-bicolorable with the coloring $C" = C"(2,5,6,14)$. At the same time, a 5-bicoloring of $(W, \mathcal{C})$ is impossible due to [20], since $27 < 2^5 - 1$. Thus $\chi = 3, \bar{\chi} = 4$, as claimed. □

Concerning order 31, an inspection of the tables in [3] shows that there exists no extended bicoloring for this order: there exists no 3-bicoloring of an STS(15), and no 4-bicoloring of STS(31) whatsoever. However, the next admissible order 39 shows a quite different behaviour.

Theorem 6. *There exist STS(*39*) admitting extended bicolorings obtained from extended bicolorings of STS(*19*) of type* $C_1 = C(4,6,9)$ *and* $C_2 = C(1,2,8,8)$*. More specifically, there exist STS(*39*) with* $(\chi, \bar{\chi})$ *equal to either* $(3,4)$*, or* $(4,5)$*, or* $(3,5)$*.*

Proof. It was shown in [14] that there exist STS(19) (a) admitting only the 3-bicoloring $C(4,6,9)$, (b) admitting only the 4-bicoloring $C(1,2,8,8)$, and (c) admitting both, the 3-bicoloring $C(4,6,9)$ and the 4-bicoloring $C(1,2,8,8)$. By Theorem 3, both C_1 and C_2 are extendable bicolorings (since we have $4+6=10$ and $2+8=10$, respectively). Starting with an STS(19) of type (a), (b), or (c), we obtain an STS(39) of the respective kind as claimed. □

In what follows we discuss in somewhat less detailed manner the existence of extended bicolorings for those STS(v) of orders $43 \leq v \leq 99$ which can be obtained by a doubling construction.

Theorem 7. *For an STS(*v*),* $v \in \{51, 63, 67, 75\}$*, there exists no extended bicoloring.*

Proof. (i) For an STS(25), the only types of bicoloring that are possible are $C_1 = C(5,10,10)$ and $C_2 = C(1,4,8,12)$. While there exist 12 solutions with respect to C_1, none satisfies the condition of Corollary 2 and thus cannot lead to an extended bicoloring. There exist no solutions with respect to C_2, and so no STS(51) can have an extended bicoloring.
(ii) By [6], no STS(63) obtained by doubling from an STS(31) can have an extended bicoloring.

(iii) None of the bicolorings of any STS(33) or STS(37) (cf. [3]) yields a solution with respect to such a coloring, thus there is no extended bicoloring of any STS(67) or STS(75). □

Theorem 8. *For an STS(v), $v \in \{43, 55, 79, 87, 91, 99\}$, there exist extended bicolorings. More specifically, there exists an extended 3-bicoloring of an STS(43), extended 4-bicolorings of an STS(v) for $v \in \{55, 87, 91\}$, and extended 4- and 5-bicolorings of an STS(v) for $v \in \{79, 99\}$.*

Proof. (i) A bicolorable STS(21) can only be 3-bicolorable, with colorings $C_1 = C(5, 6, 10)$ or $C_2 = C(4, 8, 9)$ (see [3] or [14]). Both are extendable to a 3-bicoloring $C = C(10, 16, 17)$ of an STS(43) for which we have $\chi = 3$ and $\bar{\chi} = 4$ (there exists no 5-bicolorable STS(43), cf. [3]). The 1-factorization $\mathcal{F}$ in the corresponding doubling construction is given in the Appendix [8].
(ii) By Theorem 3, the 4-bicoloring $C(1, \mathbf{4}, \mathbf{10}, 12)$ of an STS(27) is extendable to a 4-bicoloring $C(1, 12, 18, 24)$ of an STS(55), further the 4-bicoloring $C(1, \mathbf{8}, \mathbf{12}, 18)$, and the 4-bicoloring $C(\mathbf{2}, 6, 13, \mathbf{18})$, respectively, of an STS(39) is extendible to a 4-bicoloring $C(1, 18, 28, 32)$, and to a 4-bicoloring $C(6, 13, 22, 38)$ of an STS(79), respectively; finally, the 4-bicoloring $C(1, \mathbf{10}, \mathbf{12}, 20)$, and the 4-bicoloring $C(\mathbf{4}, 4, 17, \mathbf{18})$, respectively, of an STS(43) is extendable to a 4-bicoloring $C(1, 20, 32, 34)$, and to a 4-bicoloring $C(4, 17, 26, 40)$, respectively. (The two essential colors are indicated in bold.)
(iii) Extendability of the 5-bicolorings $C(1, 2, 8, 8, 20)$ and $C(1, 4, 4, 10, 20)$ of an STS(43) follows from the more general Theorem 4.
(iv) There are only two possible types of a 4-bicoloring of an STS(45), namely $C_1 = C(2, 8, 14, 21)$ and $C_2 = C(4, 6, 13, 22)$. There are 12 solutions with respect to C_1 but none of them leads to an extended bicoloring. Similarly, there are 12 solutions with respect to C_2 but only one of them, namely $(c_1, c_2, c_3, c_4) = (4, 8, 12, 22)$ leads to an extended 4-bicoloring. The corresponding 1-factorization $\mathcal{F}$ in the doubling construction that leads to this extended bicoloring is given in the Appendix [8].
(v) Although there exist 3-, 4-, and 5-bicolorable STS(49), none of the 3-bicolorings is extendable. On the other hand, 4-bicolorings $C(2, 8, 18, 21)$ and $C(5, 6, 14, 24)$ as well as the 5-bicoloring $C(1, 4, 4, 20, 20)$ are all extendable. This is shown by examining all solutions with respect to the particular bicoloring C. Due to the considerable number of these solutions (84, 29 and 27, respectively), we omit the details. The 1-factorizations occurring in the doubling constructions leading to the respective extended bicolorings are given in the Appendix [8].

Theorem 9. *There exist extended 4-bicolorings for each order $w \in \{127, 151, 159, 175\}$; there exist extended 5-bicolorings for each order $w \in \{103, 111, 127, 135, 151, 159, 175\}$.*

Proof. Below we list 4- and 5-bicolorings (known to exist by [3]) to which it is possible to apply Theorem 3; the two essential colors are in bold.

order $2v+1$	extendable colorings of an STS(v)
103	$(1, \mathbf{2}, 8, 16, \mathbf{24})$
111	$(1, 2, \mathbf{8}, \mathbf{20}, 24)$
127	$(2, \mathbf{14}, \mathbf{18}, 29)$, $(\mathbf{4}, 9, 22, \mathbf{28})$, $(\mathbf{2}, 5, 6, 20, \mathbf{30})$
135	$(1, \mathbf{2}, 16, 16, \mathbf{32})$
151	$(4, \mathbf{12}, \mathbf{26}, 33)$, $(1, 4, \mathbf{10}, \mathbf{28}, 32)$
159	$(\mathbf{4}, 14, 25, \mathbf{36})$, $(\mathbf{6}, 10, 29, \mathbf{34})$, $(1, \mathbf{4}, 12, 26, \mathbf{36})$,
	$(1, 2, \mathbf{16}, \mathbf{24}, 36)$
175	$(\mathbf{4}, 17, 26, \mathbf{40}$, $(2, 5, \mathbf{10}, \mathbf{34}, 36)$.

We summarize our results as follows.

Theorem 10. *Let* $\Omega = \{27, 39, 43, 55, 79, 87, 91, 99, 103, 111, 127, 135, 151, 159, 175\}$. *For each* $v \in \Omega$, *there exists an STS(v) with an extended bicoloring, and thus for all* $v \in \Omega$, *we have* $\chi \neq \bar{\chi}$.

Proof. For each $v \in \Omega$ we have an extended k-coloring for some k, and also (at least) a $(k+1)$-bicoloring (with $n_{k+1} = v+1$). □

Corollary 11. *For each* $v \in \Omega' = \{27, 39, 43, 91, 99, 103, 127, 135, 151\}$, *there exists an infinite class of STS(w), where* $w = 2^t(v+1) - 1$, $t \geq 1$, *such that* $\chi \neq \bar{\chi}$.

Proof. Apply repeatedly the doubling construction to the appropriate STS(v). □

4. Conclusion

In this paper, we have investigated extended bicolorings with the explicit aim to prove the existence of STSs with $\chi \neq \bar{\chi}$, that is, with different lower and upper chromatic number. We established the existence of extended bicolorings and of such STSs for several infinite classes of orders $2v+1 \equiv 3$ or 7 $(mod\ 12)$, by utilizing the doubling construction. The problem of determining for which orders $v \equiv 1$ or 3 $(mod\ 6)$ does there exist an STS(v) with different lower and upper chromatic number is certainly worthwhile. Another interesting question is, how large can the difference $\bar{\chi} - \chi$ be? It is also a legitimate question to ask whether an analogue of extended bicolorings may exist for other recursive constructions, such as the known $v \to 2v+t$ rules where $t > 1$ (cf. [4]). For example, is it possible to use the $v \to 2v+5$ rule starting with an STS(7) and ending up with an STS(19) to show that the 3-bicoloring $(1, 2, 4)$ for STS(7) can be extended to a 3-bicoloring $(4, 6, 9)$ for an STS(19)? The next example answers this question.

Example 12. The following STS(19) with a sub-STS(7) has a 3-bicoloring and these triples: $\{0,1,9\},\{2,3,9\},\{0,2,10\},\{1,3,10\},\{0,3,15\}$, $\{1,2,15\},\{9,10,15\},\{0,4,11\},\{0,5,12\},\{0,6,13\},\{0,7,14\},\{0,8,16\}$, $\{1,4,12\},\{1,5,11\},\{1,6,14\},\{1,7,13\},\{1,8,17\},\{2,4,13\},\{2,5,14\}$, $\{2,6,11\},\{2,7,12\},\{2,8,18\},\{3,4,14\},\{3,5,16\},\{3,6,17\},\{3,7,18\}$, $\{3,8,11\},\{4,5,9\},\{4,6,18\},\{4,7,16\},\{4,8,10\},\{5,6,10\},\{5,7,17\}$, $\{5,8,13\},\{6,7,9\},\{6,8,12\},\{7,8,15\},\{9,11,16\},\{9,12,17\},\{9,13,18\}$, $\{8,9,14\},\{7,10,11\},\{10,12,16\},\{10,13,17\},\{10,14,18\},\{11,12,18\}$, $\{11,13,15\},\{11,14,17\},\{3,12,13\},\{12,14,15\},\{13,14,16\},\{6,15,16\}$, $\{4,15,17\},\{5,15,18\},\{2,16,17\},\{1,16,18\},\{0,17,18\}$.
The first seven triples are those of an STS(7) on $\{0,1,2,3,9,10,15\}$; the three color classes are $\{0,1,2,3,4,5,6,7,8\}$, $\{9,10,11,12,13,14\}$ and $\{15,16,17,18\}$.

Even if the answer in this case proved to be affirmative, and may proved so in similar cases, it is not immediately clear that this will have as a consequence the existence of STSs with $\chi \neq \bar{\chi}$. Thus the doubling construction appears to offer most benefits from the stated applications point of view. Nevertheless, it seems to us worthwhile to study "extended" bicolorings for recursive rules for STSs other than doubling.

Acknowledgements

Thanks to Alex Rosa for valuable comments, and to Mariusz Meszka for providing us with Example 12. The authors also thank the reviewers and editors for their assistance.

References

[1] A.Amato, M.Gionfriddo, L.Milazzo, *2-Regular Equicolourings for P_4-designs*, Discrete Math. **312** (2012), 2252–2261.

[2] M. Biró, M. Hujter, Zs. Tuza, *Precoloring extension. I. Interval graphs*, Discrete Math. **100** (1992), 267–279.

[3] C.J.Colbourn, J.H.Dinitz, A.Rosa, *Bicoloring Steiner triple systems*, Electron. J. Combin. **6** (1999, No.R25).

[4] C.J.Colbourn, A. Rosa, *Triple Systems*, Oxford Univ. Press, 1999.

[5] L.Gionfriddo, *Voloshin colorings for P_3-designs*, Discrete Math. **275** (2004), 137–149.

[6] M.Gionfriddo, M.Buratti, L.Milazzo, V.Voloshin, *Lower and upper chromatic numbers for BSTSs(2^h-1)*, Computer Sci. J. Moldova **9** (2001), 259–272.

[7] M.Gionfriddo, L.Gionfriddo, G.Ragusa, *Equitable specialized block-colourings for 4-cycle systems - I*, Discrete Math. **310** (2010), 3126–3131.

[8] M.Gionfriddo, E.Guardo, L.Milazzo, *Appendix of Extending bicolorings for Steiner triple systems*, arXiv:1308.4793 (2013).

[9] M.Gionfriddo, P.Horák, L.Milazzo, A.Rosa, *Equitable specialized block-colourings for Steiner triple systems*, Graphs Combin. **24** (2008), 313–326.

[10] M.Gionfriddo, L.Milazzo, A.Rosa, V.Voloshin, *Bicoloring Steiner systems S(2,4,v)*, Discrete Math. **283** (2004), 249–253.

[11] M.Gionfriddo, G.Quattrocchi, *Colouring 4-cycle systems with equitably coloured blocks*, Discrete Math. **284** (2004), 137–148.

[12] M.Gionfriddo, G.Quattrocchi, *Embedding balanced P_3-designs into (balanced) P_4-designs*, Discrete Math. **308** (2008), 155–160.

[13] M.Gionfriddo, G.Ragusa, *Equitable specialized block-colourings for* 4*-cycle systems - II*, Discrete Math. **310** (2010), 1986–1994.

[14] G.Lo Faro, L.Milazzo, A.Tripodi, *The first BSTS with different upper and lower chromatic numbers*, Australas. J. Combin. **22** (2000), 123–133.

[15] G.Lo Faro, L.Milazzo, A.Tripodi, *On the upper and lower chromatic numbers of BSQS(16)*, Electron. J. Combin **8 R6**.

[16] L.Milazzo, *The monochromatic block number*, Discrete Math. **165/166** (1997), 487-496.

[17] L.Milazzo, Zs.Tuza, *Strict colourings for classes of Steiner triple systems*, Discrete Math. **182** (1998), 233–243.

[18] L.Milazzo, Zs.Tuza, *Upper chromatic number of Steiner triple and quadruple systems*, Discrete Math. **174** (1997), 247–259.

[19] L.Milazzo, Zs.Tuza, V.Voloshin, *Strict colouring for triple and quadruple Steiner systems: a survey*, Discrete Math. **261** (2003), 399–411.

[20] L.Milazzo, Zs.Tuza, *A class of Steiner systems S(2,4,v) with arcs of extremal size*, Tatra Mt. Math. Publ. **36** (2007), 153–162.

[21] Zs.Tuza, *Graph colorings with local constraints – A survey.*, Discuss. Math. Graph Theory **17** (1997), 161–228.

[22] V.Voloshin, *The mixed hypergraphs*, Comput. Sci. J. Moldova, **1** (1993), 45–52.

[23] V.Voloshin, *On the upper chromatic number of hypergraphs*, Australas. J. Combin. **11** (2001), 25–45.