

Received: December 09, 2013.

Revised: June 21, 2014.

Please cite this article as: Zsolt Tuza, V. Halasz:

Asymptotically optimal induced decompositions

Appl. Anal. Discrete Math. Vol. 8 (2014) - to appear

DOI: 10.2298/AADM140718009H

## APPLICABLE ANALYSIS AND DISCRETE MATHEMATICS

available online at <http://pefmath.etf.rs>

APPL. ANAL. DISCRETE MATH. 8 (2014), xxx-xxx.

doi:10.2298/AADM140718009H

ASYMPTOTICALLY OPTIMAL INDUCED  
DECOMPOSITIONS*Veronika Halász, Zsolt Tuza*

Solving a problem raised by Bondy and Szwarcfiter [J. Graph Theory, 72 (2013), 462–477] we prove that if the edge set of a graph  $G$  of order  $n$  can be decomposed into edge-disjoint *induced* copies of the path  $P_4$  or of the paw  $K_4 - P_3$ , then the complement of  $G$  has at least  $cn^{3/2}$  edges. This lower bound is tight apart from the actual value of  $c$ , and completes the determination of asymptotic growth for the graphs with at most four vertices. More generally the lower bound  $cn^{3/2}$  holds for any graph without isolated vertices which is not a complete multipartite graph; but a linear upper bound is valid for any complete tripartite graph.

## 1. INTRODUCTION

The subject of this note lies in the intersection of two areas of high importance: extremal graph theory and edge decompositions of graphs. This issue was addressed in the recent paper of Bondy and Szwarcfiter:

**Problem 1** ([1]). *Given a graph  $F$ , determine the maximum number  $\text{ex}^*(n, F)$  of edges in a graph  $G$  of order  $n$  such that the edge set of  $G$  can be decomposed into edge-disjoint induced subgraphs isomorphic to  $F$ .*

Nearly three decades earlier, with a very different approach, Frankl and Füredi [4] considered a closely related problem on hypergraph packing. They introduced a function  $f(n, F)$  whose definition is more technical but always satisfies the inequality  $f(n, F) \leq \text{ex}^*(n, F)$ . Hence, lower bounds on their problem are also

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2010 Mathematics Subject Classification. 05C35, 05C70.

Keywords and Phrases. Induced subgraph, decomposition, complete multipartite graph, extremal graph theory.

lower bounds on  $\text{ex}^*(n, F)$ , while upper bounds on  $\text{ex}^*(n, F)$  are also upper bounds on the problem of [4].

For some good reasons, to be explained below, here we study the complementary function

$$\overline{\text{ex}^*}(n, F) := \binom{n}{2} - \text{ex}^*(n, F).$$

We solve an open problem raised in [1], prove a general asymptotic lower bound on  $\overline{\text{ex}^*}(n, F)$ , and give constructive upper bounds for some types of graphs.

**Earlier results** The fundamental result of Wilson [5] states that every sufficiently large complete graph admits an edge decomposition into complete subgraphs of given order whenever two obvious necessary divisibility conditions hold. Since every complete subgraph necessarily is induced, this shows that  $\overline{\text{ex}^*}(n, K_p) = O(n)$  holds for every fixed  $p \geq 3$ , and that  $\overline{\text{ex}^*}(n, K_p)$  oscillates between 0 and  $cn + O(1)$  for some  $c = c(p)$ . (Of course,  $\overline{\text{ex}^*}(n, K_2) = 0$  holds for all  $n$ .)

It is also easily observed (as noted first in [1]) that if  $F'$  is obtained from  $F$  by adding an isolated vertex, then  $\overline{\text{ex}^*}(n, F) \leq \overline{\text{ex}^*}(n, F') \leq \overline{\text{ex}^*}(n-1, F) + n-1$ , thus we lose at most a linear additive term if any fixed number of isolates are added to  $F$ . For this reason we assume throughout this paper that  $F$  does not have isolated vertices.

In general, in the recent paper [3], Cohen and Tuza proved that

$$(1) \quad \overline{\text{ex}^*}(n, F) = o(n^2)$$

holds for all non-edgeless graphs  $F$  as  $n$  gets large. Due to the connection between the problems of [1] and [4], the same asymptotic upper bound can be deduced from the results of Frankl and Füredi, too. In comparison, the methods in [4] are probabilistic, whilst the results of [3] are partly constructive, applying the properties of various classes of Kneser graphs.

Because of (1), the main problem is to determine the order of magnitude of  $\overline{\text{ex}^*}(n, F)$  for a given  $F$  as a function of  $n$ . Several estimates have been proved in [1] and [3]:

- $\overline{\text{ex}^*}(n, F) = \Theta(n)$  if  $F$  is a complete equipartite non-complete graph (and in particular if  $F = C_4$ ) or  $F$  is a star or  $F = K_4 - e$  ([1]);
- $\overline{\text{ex}^*}(n, F) = \Theta(n^{3/2})$  if  $F = 2K_2$  or  $F = C_6$  (lower bounds in [1], constructive upper bounds in [3]).

In some cases, more precise or even exact results are known, but here we prefer to emphasize growth order.

**Our results** One of the main results in this paper is the following general lower bound, which is proved in the next section.

**Theorem 2.** *If  $F$  is a graph without isolated vertices, and  $F$  is not a complete multipartite graph, then there exists a constant  $c = c(F) > 0$  such that  $\overline{\text{ex}}^*(n, F) \geq cn^{3/2}$ .*

In particular, concentrating on 4-vertex subgraphs, we obtain:

**Corollary 3.** *Every graph  $F$  containing the path  $P_4$  or the matching  $2K_2$  or the paw  $K_4 - P_3$  as an induced subgraph satisfies  $\overline{\text{ex}}^*(n, F) \geq cn^{3/2}$  for some constant  $c > 0$ .*

Combining these lower bounds with the constructions of [3], the following cases solve Problem 1 of Bondy and Szwarcfiter [1].

**Corollary 4.** *We have  $\overline{\text{ex}}^*(n, P_4) = \Theta(n^{3/2})$  and  $\overline{\text{ex}}^*(n, K_4 - P_3) = \Theta(n^{3/2})$ .*

For graphs containing an induced  $K_4 - P_3$ , the lower bound  $cn^{3/2}$  for a slightly different problem was proved in [4, Proposition 2.4]. For paths, until now only a linear lower bound was known in general, and  $\Theta(n^{3/2})$  was proved for *regular* graphs decomposable into induced copies of  $F$  (see [1]).

We conjecture that no other growth function occurs as  $\overline{\text{ex}}^*(n, F)$  which would lie strictly between  $\Theta(n)$  and  $\Theta(n^{3/2})$ .

**Conjecture 5.** *If  $F$  is a complete multipartite graph, then  $\overline{\text{ex}}^*(n, F) = O(n)$ .*

As mentioned above, linear upper bound was known previously for complete *equipartite* graphs, for stars, and for  $K_{2,1,1}$ . We prove the following further cases. The first one is very simple, while the other one is the second main result of this paper.

**Proposition 6.** *If  $F = K_{a,b}$ , with  $a \geq 2$  and  $b \geq 1$ , then  $\overline{\text{ex}}^*(n, F) = O(n)$ .*

**Theorem 7.** *If  $F = K_{a,b,c}$  is a complete tripartite graph, then  $\overline{\text{ex}}^*(n, F) = O(n)$ .*

Our proof in Section 4 applies an incremental construction. Further graph classes supporting Conjecture 5 are presented in the forthcoming paper [2], applying some design-theoretic tools. Nevertheless, while our theorem completely solves the tripartite case, so far there remain infinite unsettled families in general, even among the 4-partite graphs.

We also have a general lower bound.

**Proposition 8.** *If  $F$  is a complete multipartite graph, but  $F$  is not complete, then  $\overline{\text{ex}}^*(n, F) = \Omega(n)$ .*

This means that one can replace  $O(n)$  with  $\Theta(n)$  in the previous results.

## 2. GRAPHS WHICH ARE NOT COMPLETE MULTIPARTITE

Here we prove the general lower bound of order  $n^{3/2}$  for graphs other than complete multipartite graphs plus any fixed number of isolated vertices. (As we noted in the introduction, an isolated vertex can add at most a linear term to  $\overline{\text{ex}}^*(n, F)$ .)

**Proof of Theorem 2.** Let  $F$  be any isolate-free graph satisfying the assumptions of the theorem. Denote by  $p$  the number of vertices and by  $q$  the number of edges in  $F$ . Since  $F$  is not complete multipartite, it contains some vertex  $w$  and edge  $yz$  such that  $wy$  and  $wz$  are non-edges. Indeed, the complement of  $F$  contains some connected component of order at least 3 which is not a complete graph, and then this component contains an induced path  $yzw \cong P_3$ , a proper choice for the three vertices named above in  $F$ .

Let  $G = (V, E)$  be a graph of order  $n$ , which is extremal for  $\text{ex}^*(n, F)$ ; and let  $H = \overline{G}$  be its complement. By (1), for every  $\epsilon > 0$  there exists  $n_0 = n_0(\epsilon, F)$  such that, for every  $n > n_0$ ,  $G$  has more than  $(1/2 - \epsilon)n^2$  edges. As a consequence, the edge set of  $G$  is decomposed into more than  $\frac{1-2\epsilon}{2q}n^2$  copies of  $F$ . In each copy, vertex  $w$  is mapped to some vertex of  $G$ . Let  $k_v$  denote the number of copies of  $F$  in which  $w$  is mapped to vertex  $v \in V$ . Then we have

$$\sum_{v \in V} k_v > \frac{1-2\epsilon}{2q} n^2.$$

Choosing now  $\epsilon = 1/10$ , it follows that at least  $\frac{n}{5q}$  among the  $n$  terms on the left side are not smaller than  $\frac{n}{5q}$ . This specifies a set  $X \subset V$  such that

$$|X| \geq \frac{n}{5q} \quad \text{and} \quad k_x \geq \frac{n}{5q} \quad \text{for all } x \in X.$$

The copies of the edge  $yz$  appear in the non-neighborhoods of the copies of  $w$ . This requires at least  $k_x$  distinct edges in the complementary neighborhood  $N_H(x)$ , implying

$$\binom{d_H(x)}{2} \geq k_x,$$

$$d_H(x) > \sqrt{2k_x} \geq \sqrt{0.4n/q}$$

for every  $x \in X$ . Consequently,

$$\overline{\text{ex}^*}(n, F) = |E(H)| = \frac{1}{2} \sum_{v \in V} d_H(v) \geq \frac{1}{2} \sum_{x \in X} d_H(x) > \frac{1}{5q\sqrt{10q}} n^{3/2}.$$

This inequality proves the theorem.

### 3. LINEAR LOWER BOUND

In this short section we prove that every non-complete graph  $F$  without isolates has a linear lower bound on  $\overline{\text{ex}^*}(n, F)$ .

**Proof of Proposition 8.** Let  $F$  be a complete multipartite graph, say with  $q$  edges. If  $F = K_{1,q}$  is a star, then the exact value of  $\overline{\text{ex}^*}(n, F)$  is known by [1, Theorem 3]; the complement of the extremal graph is the union of complete graphs  $K_q$ , with one additional  $K_r$  if  $n$  is not a multiple of  $q$  and  $n \equiv r \pmod{q}$ . In this case we have  $\overline{\text{ex}^*}(n, F) = \frac{q-1}{2}n + O(1)$ .

If  $F$  is not a star, then either it has more than two vertex classes or it is bipartite with at least two vertices in each class. Thus, in either case,  $F$  contains a vertex pair, say  $\{v_0, w_0\}$ , such that  $v_0w_0$  is a non-edge, moreover  $v_0$  and  $w_0$  have at least two common neighbors in  $F$ . Also,  $q \geq 4$  holds.

Let  $G = (V, E)$  be a graph of order  $n$ , which is extremal for  $\text{ex}^*(n, F)$ ; and let  $H = \overline{G}$  be its complement. Then  $H$  has  $\overline{\text{ex}^*}(n, F)$  edges; let us denote  $m := \text{ex}^*(n, F) = |E(G)|$  and  $\overline{m} := \overline{\text{ex}^*}(n, F) = |E(H)| = \binom{n}{2} - m$ . Each of the  $\overline{m}$  edges in  $H$  is the image of  $\{v_0, w_0\}$  in at most  $(n-2)/2$  copies of  $F$ . Consequently, using also the fact that  $q > 2$ , we obtain:

$$m \leq q \frac{n-2}{2} \overline{m},$$

$$\binom{n}{2} = m + \overline{m} < q \frac{n-1}{2} \overline{m},$$

$$\overline{m} > n/q.$$

This completes the proof.

#### 4. GRAPHS SUPPORTING CONJECTURE 5

Here we prove linear upper bounds on  $\overline{\text{ex}^*}(n, F)$  for complete bipartite and tripartite graphs  $F$ . The graph  $K_4 - e$  and the stars (both treated in [1]) occur as particular cases.

**Proof of Proposition 6.** It is easy to decompose  $K_{ab,ab}$  into induced copies of  $K_{a,b}$ , as follows. We partition the first vertex class into  $b$  disjoint sets of size  $a$ , and the second vertex class into  $a$  disjoint sets of size  $b$ . The combinations of those sets yield  $ab$  copies of  $K_{a,b}$ , which together partition the edge set of  $K_{ab,ab}$ .

Suppose next that  $n$  is of the form  $n = kab$  for some integer  $k \geq 2$ . We replace each edge of  $K_{n/ab}$  with an independent set of cardinality  $ab$ , and substitute the above decomposition of  $K_{ab,ab}$  into the image of each edge of  $K_{n/ab}$ . In this way a graph of order  $n$  is obtained, which admits a decomposition into induced copies of  $K_{a,b}$ , and its complement has as few as  $\frac{ab-1}{2}n$  edges.

Finally, if  $n \equiv r \pmod{ab}$ , then we make the same construction on  $n' := n - r$  vertices and insert  $r$  isolates. This graph is decomposable into induced copies of  $K_{a,b}$ , and its complement has fewer than  $\frac{3}{2}abn$  edges.

**Proof of Theorem 7.** We first carry out a construction in several steps which will yield the complete tripartite graph  $K_{a^2bc, ab^2c, abc^2}$ ; and then use it in a second phase to create a large dense graph decomposable into induced copies of  $K_{a,b,c}$ . Not alone the graph  $K_{a^2bc, ab^2c, abc^2}$ , but also the steps leading to it, will be essential in the sense that they simultaneously maintain two edge partitions: one into copies of  $K_{a,b,c}$  and the other into copies of  $K_{abc, abc}$ , with a strong interrelation between the two. More formally, the vertex set of  $K_{a^2bc, ab^2c, abc^2}$  will be written in the form  $A^* \cup B^* \cup C^*$  with

$$A^* = A^1 \cup \dots \cup A^a, \quad B^* = B^1 \cup \dots \cup B^b, \quad C^* = C^1 \cup \dots \cup C^c,$$

with  $|A^{i'}| = |B^{j'}| = |C^{k'}| = abc$  for all  $1 \leq i' \leq a$ ,  $1 \leq j' \leq b$ ,  $1 \leq k' \leq c$ , and each copy of  $K_{a,b,c}$  in the edge decomposition of this  $K_{a^2bc, ab^2c, abc^2}$  will be contained in some  $A^{i'} \cup B^{j'} \cup C^{k'}$ .

Our approach is to start with an initial construction and extend it incrementally, making it denser in each step. At the beginning we take three disjoint sets  $A, B, C$  of equal cardinality  $|A| = |B| = |C| = abc$ , partitioned into sets of cardinalities  $a, b$ , and  $c$ , respectively:

$$A = \bigcup_{j=1}^b \bigcup_{k=1}^c A_{j,k}, \quad B = \bigcup_{i=1}^a \bigcup_{k=1}^c B_{i,k}, \quad C = \bigcup_{i=1}^a \bigcup_{j=1}^b C_{i,j}.$$

- Packing some copies of  $K_{a,b,c}$  as subgraphs of  $K_{abc, abc, abc}$  on the vertex set  $A \cup B \cup C$ .

For every triplet  $(i, j, k)$  with  $1 \leq i \leq a$ ,  $1 \leq j \leq b$ ,  $1 \leq k \leq c$ , define the vertex set

$$V_{i,j,k} = A_{j,k} \cup B_{i,k} \cup C_{i,j}.$$

We use each  $V_{i,j,k}$  to insert a copy of  $K_{a,b,c}$  with vertex classes  $A_{j,k}, B_{i,k}, C_{i,j}$  inside  $A \cup B \cup C$ . It is immediate to verify that the copies determined by  $V_{i,j,k}$  and  $V_{i',j',k'}$  are edge-disjoint for any two ordered triplets  $(i, j, k) \neq (i', j', k')$  because they share vertices in at most one vertex class. Indeed, changing for instance the value of  $i$  to another  $i'$  modifies  $V_{i,j,k}$  in both  $B$  and  $C$ , and therefore  $V_{i,j,k} \cap V_{i',j',k'} = A_{j,k}$ ; and changing more than one of  $i, j$  and  $k$  yields  $V_{i,j,k} \cap V_{i',j',k'} = \emptyset$ .

If we fix the first subscript  $i$  for the moment, and let  $j$  run from 1 to  $b$  and also let  $k$  run from 1 to  $c$ , then the union  $B_{i,1} \cup \dots \cup B_{i,c}$  of the corresponding  $b$ -element sets in  $B$  has cardinality  $bc$ ; and similarly, the union  $C_{i,1} \cup \dots \cup C_{i,b}$  of the corresponding  $c$ -element sets in  $C$  has cardinality  $bc$ . Consequently, the subgraph between  $B$  and  $C$ , whose edges are covered with the copies of  $K_{a,b,c}$ , is the vertex-disjoint union of  $a$  copies of  $K_{bc, bc}$ .

Analogously, fixing the second subscript  $j$ , and letting  $i, k$  run over their range, we see that the subgraph composed from the copies of  $K_{a,b,c}$  between  $A$  and  $C$  is the vertex-disjoint union of  $b$  copies of  $K_{ac, ac}$ . In the same way, the edges which are covered so far between  $A$  and  $B$  form the union of  $c$  vertex-disjoint copies of  $K_{ab, ab}$ .

For reference in the next step, we denote this construction by  $G[A, B, C]$ . Observe that the vertex classes in the copies of  $K_{bc, bc}$  mentioned above induce partitions of both  $B$  and  $C$  into  $a$  subsets in each; and similarly the copies of  $K_{ac, ac}$  partition  $A$  and  $C$  into  $b$  subsets and the copies of  $K_{ab, ab}$  partition  $A$  and  $B$  into  $c$  subsets.

- Saturation of edges between  $A$  and  $B$  in a star-like extension.

We use copies of  $G[A, B, C]$  as building blocks in the following way. We take  $c$  graphs isomorphic to  $G[A, B, C]$ , denoted as

$$G[A, B, C^{k'}] \quad (1 \leq k' \leq c),$$

where the sets  $C^1, \dots, C^c$  are mutually disjoint but  $A$  and  $B$  are common in all those copies of  $G[A, B, C]$ . Moreover, the vertices of  $A$  occur in a different order in each  $G[A, B, C^{k'}]$ , in such a way that the corresponding vertex sets determining the copies of  $K_{a,b,c}$  are

$$V_{i,j,k}^{k'} = A_{j+k'-1,k} \cup B_{i,k} \cup C_{j,k}^{k'},$$

where  $j + k' - 1$  in the subscript of  $A$  is meant cyclically modulo  $c$ . Between  $A$  and  $B$  this cyclic shift yields that the  $c$  copies of  $K_{ab, ab}$  together compose the complete bipartite graph  $K_{abc, abc}$ .

Since each  $G[A, B, C^{k'}]$  is isomorphic to  $G[A, B, C]$ , the edges from  $A$  to any  $C^{k'}$  form  $b$  disjoint copies of  $K_{ac, ac}$ . This generates a partition of  $C^{k'}$  into  $b$  classes in a natural way; and similarly, from  $B$  to each  $C^{k'}$  we have  $a$  disjoint copies of  $K_{bc, bc}$ , generating a partition of  $C^{k'}$  into  $a$  classes.

It should be emphasized that the second subscripts in the sets  $A_{j,k}$  have not been permuted. As a consequence, the copies of  $K_{ac, ac}$  between  $A$  and any  $C^{k'}$  define the same vertex partition of  $A$  into  $b$  sets of cardinality  $ac$ ; i.e., the partition is independent of the value of  $k'$ . This property will be essential for the next step.

For reference in the next step, we denote this construction by  $G[A, B, C^*]$ .

- Saturation of edges between  $A$  and  $C^*$ .

Here we use copies of  $G[A, B, C^*]$  as building blocks. We stick them together on the set  $A \cup C^*$ , creating  $b$  copies  $B^1, \dots, B^b$  of  $B$ , so that the next graph is built from the subgraphs

$$G[A, B^{j'}, C^*] \quad (1 \leq j' \leq b),$$

where the sets  $B^1, \dots, B^b$  are mutually disjoint. We again take a cyclic permutation inside  $A$ , now for the  $b$  classes of cardinality  $ac$  which belong to the copies of  $K_{ac, ac}$  between  $A$  and each  $C^{k'}$ . Here we recall that all  $C^{k'}$  generate the same partition of  $A$ , therefore the cyclic shift of the  $b$  classes in  $A$  is feasible.

There are  $b$  different values of  $j'$ , thus this step makes each  $(A, C^{k'})$  a complete bipartite graph isomorphic to  $K_{abc, abc}$ . Hence, the  $c$  pairs  $(A, C^{k'})$  together induce

$K_{abc,abc^2}$ . Also, the  $b$  copies of the subgraph  $K_{abc,abc}$  inherited from  $G[A, B, C^*]$  now form  $K_{abc,ab^2c}$  with vertex classes  $A$  and  $B^1 \cup \dots \cup B^b$ .

When performing this step, we require that the vertices inside all  $B^{j'}$  and also inside all  $C^{k'}$  be indexed in the same order, so that for every fixed  $j'$  the  $K_{bc,bc}$  subgraphs in  $B^{j'} \cup C^{k'}$  generate the same partition of  $B^{j'}$  for all  $k'$ , and also for every fixed  $k'$  the  $K_{bc,bc}$  subgraphs in  $B^{j'} \cup C^{k'}$  generate the same partition of  $C^{k'}$  for all  $j'$ . Altogether the edge sets in the  $bc$  pairs  $(B^{j'}, C^{k'})$  compose  $a$  vertex-disjoint copies of  $K_{b^2c,bc^2}$ .

For reference in the next step, we denote this construction by  $G[A, B^*, C^*]$ .

- Saturation of edges between  $B^*$  and  $C^*$ .

Here we use copies of  $G[A, B^*, C^*]$  as building blocks. Recall that  $A \cup B^*$  and  $A \cup C^*$  induce complete bipartite graphs, while  $B^* \cup C^*$  induces the graph  $aK_{b^2c,bc^2}$ . So, we may view the vertex classes of the latter as  $B^* = B_1 \cup \dots \cup B_a$  and  $C^* = C_1 \cup \dots \cup C_a$ , where each  $B_i \cup C_i$  induces  $K_{b^2c,bc^2}$  ( $1 \leq i \leq a$ ), and no more edges occur inside  $B^* \cup C^*$ .

We now take  $a$  graphs isomorphic to  $G[A, B^*, C^*]$ , denoted as

$$G[A^{i'}, B^*, C^*] \quad (1 \leq i' \leq a),$$

where the sets  $A^1, \dots, A^a$  are mutually disjoint but  $B^*$  and  $C^*$  are common in all those copies. Similarly to the previous steps, we apply cyclic shift inside  $B^* \cup C^*$  to ensure that each  $G[A^{i'}, B^*, C^*]$  contains precisely the edges between  $B_i$  and  $C_{i+i'-1}$  for all  $1 \leq i \leq a$ .

In this way we obtain a graph, which we denote by  $G[A^*, B^*, C^*]$ . As we indicated at the very beginning of the proof already, this graph is isomorphic to  $K_{a^2bc,ab^2c,abc^2}$ ; but in fact it is more than that. The procedure above describes edge decompositions of  $G[A^*, B^*, C^*]$  into induced subgraphs isomorphic to  $K_{abc,abc}$ , as well as into induced subgraphs isomorphic to  $K_{a,b,c}$ . Moreover, in this decomposition each copy of  $K_{a,b,c}$  is embedded into a copy of  $K_{abc,abc,abc}$  induced by some  $A^{i'} \cup B^{j'} \cup C^{k'}$  in  $G[A^*, B^*, C^*]$ .

- Construction of dense induced packing.

If we contract each of the sets  $A^{i'}$ ,  $B^{j'}$ , and  $C^{k'}$  in  $G[A^*, B^*, C^*] \cong K_{a^2bc,ab^2c,abc^2}$  to a distinct single vertex, we obtain a graph  $G^* \cong K_{a,b,c}$ . Wilson's theorem on general graphs implies that if  $n'$  is sufficiently large with respect to  $a$ ,  $b$  and  $c$ , moreover  $\binom{n'}{2}$  is a multiple of  $ab + bc + ca$  (the number of edges in  $G^*$ ) and  $n' - 1$  is a multiple of  $\gcd(a + b, b + c, c + a)$ , the greatest common divisor of the vertex degrees, then the complete graph of order  $n'$  admits an edge decomposition into copies of  $G^*$ .

Now, for any given  $n$  (large enough), let  $n'$  be the largest integer such that  $n' \leq \frac{n}{abc}$  and  $n'$  satisfies the two divisibility conditions listed above. By what has been said,  $K_{n'}$  admits an edge decomposition into subgraphs  $G_1^*, G_2^*, \dots$  isomorphic

to  $G^*$ . Replace each vertex of  $K_{n'}$  with an independent set of  $abc$  vertices, and add further  $n - abc n'$  isolates. In this way we obtain a graph  $G$  of order  $n$ . Then each  $G_\ell^*$  becomes an induced subgraph  $G_\ell$  of  $G$ , isomorphic to  $G[A^*, B^*, C^*]$ . Based on the procedure of constructing  $G[A^*, B^*, C^*]$ , every  $G_\ell^*$  is decomposable into induced copies of  $K_{a,b,c}$ , and this is an induced  $K_{a,b,c}$ -decomposition of  $G$ , too, because the ‘induced subgraph’ relation is transitive.

Disregarding small values of  $n$ , the number of isolated vertices in  $G$  is less than  $abc$  times the gap occurring between two consecutive values of  $n'$  which are feasible for  $G^*$ -decomposition. Moreover, omitting the isolates from  $G$ , the complementary degree of each vertex becomes precisely  $abc - 1$ . Thus, the overall number of non-edges in  $G$  is at most  $K \cdot n$ , for some constant  $K = K(a, b, c)$ . This completes the proof of the theorem.

## 5. CONCLUDING REMARKS

In this paper we studied the growth of the number of edges in graphs which admit edge decompositions into induced subgraphs isomorphic to a given graph. Our intuition says that we have identified a gap in the exponent of  $n$ ; this is expressed as Conjecture 5, in comparison with Theorem 2.

The questions raised above would be interesting for uniform hypergraphs  $\mathcal{F}$ , too, similarly to the problem studied in [4]. The determination of the analogous function  $\overline{\text{ex}}^*(n, \mathcal{F})$ , however, is probably rather hard and seems to be beyond reach with current methods, unless  $\mathcal{F}$  is rather simple.

**Acknowledgements.** Research of the second author was supported in part by the Hungarian Scientific Research Fund, OTKA grant T-81493, and by the Hungarian State and the European Union under the grant TAMOP-4.2.2.A-11/1/KONV-2012-0072. The authors thank the referee for his/her comments.

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