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A DIAGONAL RECURRENCE RELATION FOR THE STIRLING NUMBERS OF THE FIRST KIND

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In the paper, the authors present an explicit form for a family of inhomogeneous linear ordinary differential equations, find a more significant expression for all derivatives of a function related to the solution to the family of inhomogeneous linear ordinary differential equations in terms of the Lerch transcendent, establish an explicit formula for computing all derivatives of the solution to the family of inhomogeneous linear ordinary differential equations, acquire the absolute monotonicity, complete monotonicity, the Bernstein function property of several functions related to the solution to the family of inhomogeneous linear ordinary differential equations, discover a diagonal recurrence relation of the Stirling numbers of the first kind, and derive an inequality for bounding the logarithmic function.

1. MAIN RESULTS

In [**7**, Section 3, Theorem 1], by inductive argument, it was proved that the family of inhomogeneous linear ordinary differential equations

$$\big(\sqrt{1-4t}\,-1\big)F^{(n)}(t)+\sum_{i=1}^{n}\frac{a_{n,i}}{(1-4t)^{n-i+1/2}}F^{(i-1)}(t)=\frac{a_{n,0}}{(1-4t)^{n}} \tag{1}$$

for $n\in\mathbb{N}$ have a solution

$$F(t)=\begin{cases}\dfrac{1}{2}\dfrac{\ln(1-4t)}{\sqrt{1-4t}\,-1}, & 0\ne t<\dfrac{1}{4};\\ 1, & t=0,\end{cases} \tag{2}$$

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where

$$a_{n,0}=-2^{2n-1}(n-1)!,\quad a_{n,1}=-2^{n}(2n-3)!!,\quad a_{n,n}=-2n,$$
$$a_{n,i}=-S_{n-i+2,i-2}2^{n-i+1}(2n-2i-1)!!,\quad 2\le i\le n-1,$$
$$S_{n,0}=n,\quad S_{n,j}=\sum_{\ell=1}^{n}S_{\ell,j-1},\quad j\ge 1.$$

This is a core conclusion in the paper [**7**].

It is obvious that the above sequence $a_{n,i}$ for $n\in\mathbb{N}$ and $0\le i\le n$ is given recursively. It is natural to ask for a question: can one find an explicit expression of the above sequence $a_{n,i}$ for $n\in\mathbb{N}$ and $0\le i\le n$? In other words, can one find an explicit form for the family of inhomogeneous linear ordinary differential equations (1)? Our answer to this question can be stated as Theorem 1 below.

Theorem 1. *For $n\in\mathbb{N}$, the family of inhomogeneous linear ordinary differential equations*

$$\bigl(1-\sqrt{1-4t}\,\bigr)(1-4t)^{n}F^{(n)}(t)+2n(1-4t)^{n-1/2}F^{(n-1)}(t)\\ +2^{n}\sum_{r=0}^{n-2}\binom{n}{r}\frac{(2n-2r+1)!!}{2^{r}}(1-4t)^{r+1/2}F^{(r)}(t)=2^{2n-1}(n-1)!$$

has the solution (2).

We observe that the function $F(t)$ is a composite of the functions

$$f(x)=\begin{cases}\dfrac{\ln x}{x-1}, & 0<x\ne 1\\ 1, & x=1\end{cases}$$

and $x=h(t)=\sqrt{1-4t}\,$. By the Leibniz theorem for differentiation of a product, it is not difficult to obtain

$$f^{(n)}(x)=\frac{(-1)^{n}n!}{(x-1)^{n+1}}\left[\ln x-\sum_{k=1}^{n}\frac{1}{k}\left(\frac{x-1}{x}\right)^{k}\right],\quad n\ge 0. \tag{3}$$

Recall from [**3**, Chapter 14], [**8**, Chapter XIII], [**28**, Chapter 1], and [**30**, Chapter IV] that a function f is said to be completely monotonic on an interval $I\subseteq\mathbb{R}$ if f has derivatives of all orders on I and $(-1)^{n}f^{(n)}(t)\ge 0$ for all $t\in I$ and $n\in\{0\}\cup\mathbb{N}$. Recall from [**22**, p. 1161] and [**28**, Chapter 3] that a function $f:I\subseteq(-\infty,\infty)\to[0,\infty)$ is called a Bernstein function on I if $f(t)$ has derivatives of all orders and $f'(t)$ is completely monotonic on I.

The second main result of this paper is to present a more significant expression than (3) for the nth derivative of the function $f(x)$ in terms of the Lerch transcendent

$$\Phi(z,s,a)=\sum_{k=0}^{\infty}\frac{z^{k}}{(a+k)^{s}},\quad a\ne 0,-1,\dots.$$

From the more significant expression, we obtain complete monotonicity of the functions $f(x)$ and $x^{n+1}f^{(n)}(x)$ and derive an inequality of the logarithmic function $\ln x$.

Theorem 2. *For $n \ge 0$, the nth derivative of the function $f(x)$ can be expressed by*

$$f^{(n)}(x) = \begin{cases} (-1)^n \dfrac{n!}{x^{n+1}} \Phi\left(\dfrac{x-1}{x}, 1, n+1\right), & 0 < x \ne 1; \\ (-1)^n \dfrac{n!}{n+1}, & x = 1. \end{cases} \tag{4}$$

Consequently,

1. *the function $f(x)$ is completely monotonic on $(0,\infty)$;*
2. *the functions $x^{n+1}f^{(n)}(x)$ for all $n \ge 0$ are Bernstein functions on $(0,\infty)$;*
3. *the inequality*

$$\ln x > \sum_{k=1}^{n} \frac{1}{k}\left(\frac{x-1}{x}\right)^k, \quad n \in \mathbb{N} \tag{5}$$

   (a) *holds*
      i. *either for $x > 1$ and all $n \in \mathbb{N}$,*
      ii. *or for $0 < x < 1$ and odd n;*

   (b) *reverses for $0 < x < 1$ and even n.*

Recall from [**5**] and [**30**, Chapter IV] that a function f is said to be absolutely monotonic on an interval I if it has derivatives of all orders and $f^{(k-1)}(t) \ge 0$ for $t \in I$ and $k \in \mathbb{N}$.

The third main result of this paper is an explicit formula for the nth derivative of $F(t)$. From the explicit formula, we deduce the absolute monotonicity of the function $F(t)$.

Theorem 3. *The nth derivative of the function $F(t)$ can be expressed by*

$$F^{(n)}(t) = \frac{2^n}{(1-4t)^{n+1/2}} \sum_{\ell=0}^{n} \ell![2(n-\ell)-1]!! \binom{2n-\ell-1}{\ell-1} \Phi\left(\frac{\sqrt{1-4t}-1}{\sqrt{1-4t}}, 1, \ell+1\right).$$

Consequently, the function $F(t)$ is absolutely monotonic on $\left(-\infty, \frac{1}{4}\right)$ and, equivalently, the function $F(-t)$ is completely monotonic on $\left(-\frac{1}{4}, \infty\right)$.

In combinatorics [**4, 11, 13**], the Stirling number of the first kind $s(n,k)$ can be defined such that the number of permutations of n elements which contain exactly k permutation cycles is the nonnegative number $|s(n,k)| = (-1)^{n-k}s(n,k)$.

In [**11, 12**], diagonal recurrence relations for the Stirling numbers of the first and second kinds were discovered.

The fourth main result of this paper is a diagonal recurrence relation of the Stirling numbers of the first kind $s(n,k)$.

Theorem 4. *The Stirling numbers of the first kind $s(n,k)$ satisfy the diagonal recurrence relation*

$$\frac{s(n+k,k)}{\binom{n+k}{k}}=\sum_{\ell=0}^{n}(-1)^\ell\frac{\langle k\rangle_\ell}{\ell!}\sum_{m=1}^{\ell}(-1)^m\binom{\ell}{m}\frac{s(n+m,m)}{\binom{n+m}{m}}, \tag{6}$$

where

$$\langle x\rangle_n=\prod_{k=0}^{n-1}(x-k)=\begin{cases}x(x-1)\cdots(x-n+1), & n\ge1\\ 1, & n=0\end{cases}$$

is the falling factorial.

2. LEMMAS

In order to prove our main results, we recall several lemmas below.

Lemma 1 ([**4**, p. 134, Theorem A] and [**4**, p. 139, Theorem C])**.** *For $n\ge k\ge0$, the Bell polynomials of the second kind, or say, partial Bell polynomials, denoted by $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$, are defined by*

$$\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})=\sum_{\substack{1\le i\le n,\ell_i\in\{0\}\cup\mathbb{N}\\ \sum_{i=1}^n i\ell_i=n\\ \sum_{i=1}^n \ell_i=k}}\frac{n!}{\prod_{i=1}^{n-k+1}\ell_i!}\prod_{i=1}^{n-k+1}\left(\frac{x_i}{i!}\right)^{\ell_i}.$$

The Faà di Bruno formula can be described in terms of the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ by

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n}f\circ h(t)=\sum_{k=0}^{n}f^{(k)}(h(t))\mathrm{B}_{n,k}\big(h'(t),h''(t),\dots,h^{(n-k+1)}(t)\big). \tag{7}$$

The Faà di Bruno formula (7) and the Bell polynomials of the second kind $\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1})$ are useful tools in combinatorics and mathematical analysis and they were often used to compute the higher-order derivatives for a composite function. For example, see [**31, 32**].

Lemma 2 ([**4**, p. 135])**.** *For complex numbers a and b, we have*

$$\mathrm{B}_{n,k}\big(abx_1,ab^2x_2,\dots,ab^{n-k+1}x_{n-k+1}\big)=a^kb^n\mathrm{B}_{n,k}(x_1,x_2,\dots,x_{n-k+1}). \tag{8}$$

Lemma 3 ([**4**, p. 135, Theorem B] and [**11**, Theorem 1.1])**.** *For $n\ge k\ge0$, we have*

$$\mathrm{B}_{n,k}(1!,2!,\dots,(n-k+1)!)=\binom{n-1}{k-1}\frac{n!}{k!} \tag{9}$$

and

$$\mathrm{B}_{n,k}\left(\frac{1!}{2},\frac{2!}{3},\dots,\frac{(n-k+1)!}{n-k+2}\right)=(-1)^{n-k}\frac{1}{k!}\sum_{m=0}^{k}(-1)^m\binom{k}{m}\frac{s(n+m,m)}{\binom{n+m}{m}}. \tag{10}$$

Lemma 4 ([**1**, p. 40])**.** *Let $p=p(x)$ and $q=q(x)\ne0$ be two differentiable functions. Then*

$$
\left[\frac{p(x)}{q(x)}\right]^{(k)}=\frac{(-1)^k}{q^{k+1}}\begin{vmatrix}
p & q & 0 & \cdots & 0 & 0\\
p' & q' & q & \cdots & 0 & 0\\
p'' & q'' & \binom{2}{1}q' & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
p^{(k-2)} & q^{(k-2)} & \binom{k-2}{1}q^{(k-3)} & \cdots & q & 0\\
p^{(k-1)} & q^{(k-1)} & \binom{k-1}{1}q^{(k-2)} & \cdots & \binom{k-1}{k-2}q' & q\\
p^{(k)} & q^{(k)} & \binom{k}{1}q^{(k-1)} & \cdots & \binom{k}{k-2}q'' & \binom{k}{k-1}q'
\end{vmatrix}
\tag{11}
$$

for $k\ge0$. In other words, the formula (11) *can be rewritten as*

$$
\frac{\mathrm{d}^k}{\mathrm{d}\,x^k}\left[\frac{p(x)}{q(x)}\right]=\frac{(-1)^k}{q^{k+1}(x)}\big|W_{(k+1)\times(k+1)}(x)\big|,
\tag{12}
$$

where $|W_{(k+1)\times(k+1)}(x)|$ denotes the determinant of the $(k+1)\times(k+1)$ matrix

$$
W_{(k+1)\times(k+1)}(x)=\begin{pmatrix}U_{(k+1)\times1}(x) & V_{(k+1)\times k}(x)\end{pmatrix},
$$

the quantity $U_{(k+1)\times1}(x)$ is a $(k+1)\times1$ matrix whose elements $u_{\ell,1}(x)=p^{(\ell-1)}(x)$ for $1\le\ell\le k+1$, and $V_{(k+1)\times k}(x)$ is a $(k+1)\times k$ matrix whose elements

$$
v_{i,j}(x)=\begin{cases}\dbinom{i-1}{j-1}q^{(i-j)}(x), & i-j\ge0\\ 0, & i-j<0\end{cases}
$$

for $1\le i\le k+1$ and $1\le j\le k$.

Remark 1. The formula (12) in Lemma 4 has been applied in the papers [**6, 10, 14, 15, 16, 17, 18, 19, 21, 23, 24, 25, 26, 27, 29**] to express the Apostol–Bernoulli polynomials, the Cauchy product of central Delannoy numbers, the Bernoulli polynomials, the Schröder numbers, the (generalized) Fibonacci polynomials, the Catalan numbers, derangement numbers, and the Euler numbers and polynomials in terms of the Hessenberg and tridiagonal determinants. This implies that Lemma 4 is an effectual tool to express some mathematical quantities in terms of the Hessenberg and tridiagonal determinants.

Lemma 5 ([**2**, p. 222, Theorem] and [**27**, Remark 3])**.** *Let $M_0=1$ and*

$$
M_n=\begin{vmatrix}
m_{1,1} & m_{1,2} & 0 & \cdots & 0 & 0\\
m_{2,1} & m_{2,2} & m_{2,3} & \cdots & 0 & 0\\
m_{3,1} & m_{3,2} & m_{3,3} & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
m_{n-2,1} & m_{n-2,2} & m_{n-2,3} & \cdots & m_{n-2,n-1} & 0\\
m_{n-1,1} & m_{n-1,2} & m_{n-1,3} & \cdots & m_{n-1,n-1} & m_{n-1,n}\\
m_{n,1} & m_{n,2} & m_{n,3} & \cdots & m_{n,n-1} & m_{n,n}
\end{vmatrix}
$$

for $n\in\mathbb{N}$. Then the sequence M_n for $n\ge0$ satisfies $M_1=m_{1,1}$ and

$$M_n=m_{n,n}M_{n-1}+\sum_{r=1}^{n-1}(-1)^{n-r}m_{n,r}\left(\prod_{j=r}^{n-1}m_{j,j+1}\right)M_{r-1},\quad n\ge2. \tag{13}$$

Lemma 6. *The Lerch transcendent $\Phi(z,s,a)$ satisfies*

$$\Phi(z,s,a)=\frac{1}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}e^{-ax}}{1-ze^{-x}}\,\mathrm{d}\,x,\quad \Re(s),\Re(a)>0,\quad z\in\mathbb{C}\setminus[1,\infty) \tag{14}$$

and

$$\Phi(t,1,n+1)=-\frac{\ln(1-t)}{t^{n+1}}-\sum_{k=1}^{n}\frac{1}{n-k+1}\frac{1}{t^k},\quad n\ge0, \tag{15}$$

where

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

denotes the classical Euler gamma function. Consequently, the Lerch transcendent $\Phi(x,s,a)$ for $s,a>0$ is an absolutely monotonic function of $x\in(-\infty,1)$ and the function $\Phi\bigl(\frac{x-1}{x},1,n+1\bigr)$ is a Bernstein function of $x\in(0,\infty)$.

Proof. The integral representation (14) can be found in [**9**, p. 612, Entry 25.14.5].

From (14), it follows that

$$\frac{\partial^n}{\partial z^n}\Phi(z,s,a)=\frac{n!}{\Gamma(s)}\int_0^\infty\frac{x^{s-1}e^{-(a+n)x}}{(1-ze^{-x})^{n+1}}\,\mathrm{d}\,x$$

for $\Re(s),\Re(a)>0$ and $z\in\mathbb{C}\setminus[1,\infty)$. Hence, the Lerch transcendent $\Phi(x,s,a)$ for $s,a>0$ is an absolutely monotonic function of $x\in(-\infty,1)$.

Since

$$\ln(1-t)=-\sum_{k=1}^\infty\frac{t^k}{k},\quad |t|<1,$$

we have

$$\begin{aligned}-\frac{\ln(1-t)}{t^{n+1}}-\sum_{k=1}^{n}\frac{1}{n-k+1}\frac{1}{t^k}&=\sum_{k=1}^{\infty}\frac{1}{k}\frac{1}{t^{n-k+1}}-\sum_{\ell=1}^{n}\frac{1}{\ell}\frac{1}{t^{n-\ell+1}}\\&=\sum_{k=n+1}^{\infty}\frac{1}{k}\frac{1}{t^{n-k+1}}=\sum_{k=0}^{\infty}\frac{1}{n+k+1}\frac{1}{t^k}=\Phi(t,1,n+1).\end{aligned}$$

The identity (15) is thus proved.

A direct computation by employing (7), (8), and (9) gives

$$\frac{\partial^k}{\partial x^k}\Phi\left(\frac{x-1}{x},1,n+1\right)=\sum_{\ell=0}^{k}\frac{\partial^\ell}{\partial u^\ell}\Phi(u,1,n+1)$$

$$
\begin{aligned}
&\times \mathrm{B}_{k,\ell}\biggl(\frac{1!}{x^2},-\frac{2!}{x^3},\dots,(-1)^{k-\ell}\frac{(k-\ell+1)!}{x^{k-\ell+2}}\biggr)\\
&=\sum_{\ell=0}^{k}\frac{\partial^\ell}{\partial u^\ell}\Phi(u,1,n+1)\frac{(-1)^{k+\ell}}{x^{k+\ell}}\mathrm{B}_{k,\ell}(1!,2!,\dots,(k-\ell+1)!)\\
&=\frac{(-1)^k}{x^k}\sum_{\ell=0}^{k}\frac{\ell!}{\Gamma(1)}\int_0^\infty\frac{e^{-(n+\ell+1)t}}{(1-ue^{-t})^{\ell+1}}\,\mathrm{d}\,t\frac{(-1)^\ell}{x^\ell}\binom{k-1}{\ell-1}\frac{k!}{\ell!}\\
&=\frac{(-1)^k k!}{x^k}\int_0^\infty e^{-(n+1)t}\sum_{\ell=0}^{k}\frac{e^{-\ell t}}{(1-ue^{-t})^{\ell+1}}\frac{(-1)^\ell}{x^\ell}\binom{k-1}{\ell-1}\,\mathrm{d}\,t\\
&=\frac{(-1)^{k+1}k!}{x^k}\int_0^\infty e^{-(n+1)t}\frac{e^t}{(e^t-1)(e^tx-x+1)}\biggl[\frac{(e^t-1)x}{(e^t-1)x+1}\biggr]^k\,\mathrm{d}\,t\\
&=(-1)^{k+1}k!\int_0^\infty\frac{(e^t-1)^{k-1}}{[(e^t-1)x+1]^{k+1}e^{nt}}\,\mathrm{d}\,t
\end{aligned}
$$

for $k\in\mathbb{N}$, where $u=1-\frac{1}{x}$. Thus, the function $\Phi\bigl(\frac{x-1}{x},1,n+1\bigr)$ is a Bernstein function on $(0,\infty)$. The proof of Lemma 6 is complete. □

Lemma 7 ([**25**, Theorem 4])**.** *Let* $h_{a,b}(x)=\sqrt{a+bx}$ *for* $a,b\in\mathbb{R}$ *and* $b\ne0$ *and let* $n\in\mathbb{N}$. *Then the Bell polynomials of the second kind* $\mathrm{B}_{n,k}$ *satisfy*

$$
\begin{aligned}
\text{(16)}\quad &\mathrm{B}_{n,k}\bigl(h_{a,b}'(x),h_{a,b}''(x),\dots,h_{a,b}^{(n-k+1)}(x)\bigr)\\
&=(-1)^{n+k}[2(n-k)-1]!!\biggl(\frac{b}{2}\biggr)^n\binom{2n-k-1}{2(n-k)}\frac{1}{(a+bx)^{n-k/2}}.
\end{aligned}
$$

3. PROOFS OF MAIN RESULTS

We now start out to prove our main results.

Proof of Theorem 1. By virtue of Lemma 4, we have

$$
2F^{(n)}(t)=\biggl[\frac{\ln(1-4t)}{\sqrt{1-4t}\,-1}\biggr]^{(n)}=\frac{(-1)^n}{(\sqrt{1-4t}\,-1)^{n+1}}M_{n+1}\triangleq\frac{(-1)^n}{(\sqrt{1-4t}\,-1)^{n+1}}
$$

$$
\times\begin{vmatrix}
\ln(1-4t) & \sqrt{1-4t}-1 & 0 & \cdots & 0\\
\frac{-4}{1-4t} & \frac{-2}{(1-4t)^{1/2}} & \sqrt{1-4t}-1 & \cdots & 0\\
\frac{-4^2 1!}{(1-4t)^2} & \frac{-2^2 1!!}{(1-4t)^{3/2}} & \binom{2}{1}\frac{-2}{(1-4t)^{1/2}} & \cdots & 0\\
\frac{-4^3 2!}{(1-4t)^3} & \frac{-2^3 3!!}{(1-4t)^{5/2}} & \binom{3}{1}\frac{-2^2 1!!}{(1-4t)^{3/2}} & \cdots & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots\\
\frac{-4^{n-2}(n-3)!}{(1-4t)^{n-2}} & \frac{-2^{n-2}(2n-7)!!}{(1-4t)^{(2n-5)/2}} & \binom{n-2}{1}\frac{-2^{n-3}(2n-9)!!}{(1-4t)^{(2n-7)/2}} & \cdots & 0\\
\frac{-4^{n-1}(n-2)!}{(1-4t)^{n-1}} & \frac{-2^{n-1}(2n-5)!!}{(1-4t)^{(2n-3)/2}} & \binom{n-1}{1}\frac{-2^{n-2}(2n-7)!!}{(1-4t)^{(2n-5)/2}} & \cdots & \sqrt{1-4t}-1\\
\frac{-4^{n}(n-1)!}{(1-4t)^{n}} & \frac{-2^{n}(2n-3)!!}{(1-4t)^{(2n-1)/2}} & \binom{n}{1}\frac{-2^{n-1}(2n-5)!!}{(1-4t)^{(2n-3)/2}} & \cdots & \binom{n}{n-1}\frac{-2}{(1-4t)^{1/2}}
\end{vmatrix}.
$$

Further applying the recurrence relation (13) to the above determinant yields

$$\begin{aligned}M_{n+1}&=\binom{n}{n-1}\frac{-2}{(1-4t)^{1/2}}M_n+(-1)^n\frac{-4^n(n-1)!}{(1-4t)^n}\bigl(\sqrt{1-4t}\,-1\bigr)^n\\&\quad+\sum_{r=2}^{n}(-1)^{n-r+1}\binom{n}{r-2}\frac{-2^{n-r+2}[2(n-r)+1]!!}{(1-4t)^{[2(n-r)+3]/2}}\bigl(\sqrt{1-4t}\,-1\bigr)^{n-r+1}M_{r-1}\end{aligned}$$

for $n\in\mathbb{N}$. This equality can be rewritten as

$$\begin{aligned}\frac{(-1)^n}{(\sqrt{1-4t}\,-1)^{n+1}}M_{n+1}&=\frac{1}{\sqrt{1-4t}\,-1}\biggl[\binom{n}{n-1}\frac{2}{(1-4t)^{1/2}}\frac{(-1)^{n-1}}{(\sqrt{1-4t}\,-1)^n}M_n\\&\quad-\frac{4^n(n-1)!}{(1-4t)^n}+\sum_{r=2}^{n}\binom{n}{r-2}\frac{2^{n-r+2}[2(n-r)+1]!!}{(1-4t)^{[2(n-r)+3]/2}}\frac{(-1)^{r-2}}{(\sqrt{1-4t}\,-1)^{r-1}}M_{r-1}\biggr]\end{aligned}$$

for $n\in\mathbb{N}$. In a word, we obtain

$$\begin{aligned}2F^{(n)}(t)&=\frac{1}{\sqrt{1-4t}\,-1}\biggl[\binom{n}{n-1}\frac{2}{(1-4t)^{1/2}}2F^{(n-1)}(t)\\&\quad+\sum_{r=2}^{n}\binom{n}{r-2}\frac{2^{n-r+2}[2(n-r)+1]!!}{(1-4t)^{[2(n-r)+3]/2}}2F^{(r-2)}(t)-\frac{4^n(n-1)!}{(1-4t)^n}\biggr],\quad n\in\mathbb{N}.\end{aligned}$$

Theorem 1 is thus proved. □

Proof of Theorem 2. Applying $p(x)=\ln x$ and $q(x)=x-1$ to Lemma 4, expanding the obtained determinant according to the last row consecutively, and making use of (15) yield

$$\begin{aligned}\left(\frac{\ln x}{x-1}\right)^{(n)}&=\frac{(-1)^n}{(x-1)^{n+1}}\\&\quad\times\begin{vmatrix}\ln x & x-1 & 0 & 0 & \cdots & 0 & 0\\ \frac{1}{x} & 1 & x-1 & 0 & \cdots & 0 & 0\\ -\frac{1}{x^2} & 0 & 2 & x-1 & \cdots & 0 & 0\\ \frac{2}{x^3} & 0 & 0 & 3 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (-1)^{n-2}\frac{(n-2)!}{x^{n-1}} & 0 & 0 & 0 & \cdots & n-1 & x-1\\ (-1)^{n-1}\frac{(n-1)!}{x^n} & 0 & 0 & 0 & \cdots & 0 & n\end{vmatrix}\\&=\frac{(-1)^n n}{(x-1)^{n+1}}\begin{vmatrix}\ln x & x-1 & 0 & 0 & \cdots & 0 & 0\\ \frac{1}{x} & 1 & x-1 & 0 & \cdots & 0 & 0\\ -\frac{1}{x^2} & 0 & 2 & x-1 & \cdots & 0 & 0\\ \frac{2}{x^3} & 0 & 0 & 3 & \cdots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\ (-1)^{n-3}\frac{(n-3)!}{x^{n-2}} & 0 & 0 & 0 & \cdots & n-2 & x-1\\ (-1)^{n-2}\frac{(n-2)!}{x^{n-1}} & 0 & 0 & 0 & \cdots & 0 & n-1\end{vmatrix}\end{aligned}$$

$$
\begin{aligned}
&+(-1)^{n-1}\frac{(n-1)!}{x^n(x-1)}\\
&=\frac{(-1)^n n(n-1)}{(x-1)^{n+1}}\begin{vmatrix}
\ln x & x-1 & 0 & 0 & \cdots & 0 & 0\\
\frac{1}{x} & 1 & x-1 & 0 & \cdots & 0 & 0\\
-\frac{1}{x^2} & 0 & 2 & x-1 & \cdots & 0 & 0\\
\frac{2}{x^3} & 0 & 0 & 3 & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots\\
(-1)^{n-4}\frac{(n-4)!}{x^{n-3}} & 0 & 0 & 0 & \cdots & n-3 & x-1\\
(-1)^{n-3}\frac{(n-3)!}{x^{n-2}} & 0 & 0 & 0 & \cdots & 0 & n-2
\end{vmatrix}\\
&+(-1)^{n-1}\frac{(n-2)!n}{x^{n-1}(x-1)^2}+(-1)^{n-1}\frac{(n-1)!}{x^n(x-1)}\\
&=\frac{(-1)^n\langle n\rangle_{n-1}}{(x-1)^{n+1}}\begin{vmatrix}\ln x & x-1\\ \frac{1}{x} & 1\end{vmatrix}+(-1)^{n-1}\sum_{k=1}^{n-1}\frac{(n-k)!\langle n\rangle_{k-1}}{x^{n-k+1}(x-1)^k}\\
&=\frac{(-1)^n\langle n\rangle_{n-1}}{(x-1)^{n+1}}\left(\ln x-\frac{x-1}{x}\right)+(-1)^{n-1}\sum_{k=1}^{n-1}\frac{(n-k)!\langle n\rangle_{k-1}}{x^{n-k+1}(x-1)^k}\\
&=\frac{(-1)^n\langle n\rangle_{n}}{(x-1)^{n+1}}\ln x+(-1)^{n-1}\sum_{k=1}^{n}\frac{(n-k)!\langle n\rangle_{k-1}}{x^{n-k+1}(x-1)^k}\\
&=(-1)^{n-1}n!\left[\sum_{k=1}^{n}\frac{1}{(n-k+1)x^{n-k+1}(x-1)^k}-\frac{1}{(x-1)^{n+1}}\ln x\right]\\
&=(-1)^n\frac{n!}{x^{n+1}}\Phi\left(\frac{x-1}{x},1,n+1\right).
\end{aligned}
$$

The formula (4) is thus proved.

From absolute monotonicity and Bernstein function property in Lemma 6, it is immediate to obtain that the function $f(x)$ is completely monotonic on $(0,\infty)$ and the functions $x^{n+1}f^{(n)}(x)$ for all $n\ge 0$ are Bernstein functions on $(0,\infty)$.

The inequality (5) follows from the complete monotonicity of $f(x)$ and the formula (3). The proof of Theorem 2 is complete. □

Proof of Theorem 3. By (7), (16) in Lemma 7, and (4) in sequence, we obtain

$$
\begin{aligned}
&F^{(m)}(t)=[f(h_{1,-4}(t))]^{(m)}\\
&=\sum_{\ell=0}^{m}f^{(\ell)}(x)\mathrm{B}_{m,\ell}\big(h_{1,-4}'(t),h_{1,-4}''(t),\dots,h_{1,-4}^{(m-\ell+1)}(t)\big)\\
&=2^m\sum_{\ell=0}^{m}\frac{\ell!}{x^{\ell+1}}\Phi\left(\frac{x-1}{x},1,\ell+1\right)[2(m-\ell)-1]!!\binom{2m-\ell-1}{2(m-\ell)}\frac{1}{(1-4t)^{m-\ell/2}}\\
&=\frac{2^m}{(1-4t)^{m+1/2}}\sum_{\ell=0}^{m}\ell![2(m-\ell)-1]!!\binom{2m-\ell-1}{\ell-1}\Phi\left(\frac{\sqrt{1-4t}-1}{\sqrt{1-4t}},1,\ell+1\right).
\end{aligned}
$$

The absolute monotonicity of $F(t)$ on $\left(-\infty, \frac{1}{4}\right)$ follows from the absolute monotonicity of the Lerch transcendent $\Phi(x,s,a)$ for $s,a>0$ with respect to $x\in(-\infty,1)$. $\square$

Proof of Theorem 4. It is well known that the Stirling numbers of the first kind $s(n,k)$ for $n\ge k\ge 1$ can be generated [**4, 11, 13**] by

$$\frac{[\ln(1+x)]^k}{k!}=\sum_{n=k}^{\infty}s(n,k)\frac{x^n}{n!},\quad |x|<1.$$

It can be rewritten as

$$\left[\frac{\ln(1+x)}{x}\right]^k=\sum_{n=0}^{\infty}\frac{s(n+k,k)}{\binom{n+k}{k}}\frac{x^n}{n!},\quad |x|<1.$$

This means that

$$\begin{aligned}
\frac{s(n+k,k)}{\binom{n+k}{k}}&=\lim_{x\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}\left[\frac{\ln(1+x)}{x}\right]^k=\lim_{x\to0}\frac{\mathrm{d}^n}{\mathrm{d}\,x^n}[f(x+1)]^k\\
&=\lim_{x\to0}\sum_{\ell=0}^{n}\left(u^k\right)^{(\ell)}\mathrm{B}_{n,\ell}\bigl(f'(x+1),f''(x+1),\dots,f^{(n+\ell-1)}(x+1)\bigr)\\
&=\lim_{x\to0}\sum_{\ell=0}^{n}\langle k\rangle_\ell u^{k-\ell}\mathrm{B}_{n,\ell}\bigl(f'(x+1),f''(x+1),\dots,f^{(n+\ell-1)}(x+1)\bigr)\\
&=\sum_{\ell=0}^{n}\langle k\rangle_\ell f^{k-\ell}(1)\mathrm{B}_{n,\ell}\bigl(f'(1),f''(1),\dots,f^{(n+\ell-1)}(1)\bigr)\\
&=\sum_{\ell=0}^{n}\langle k\rangle_\ell\mathrm{B}_{n,\ell}\left(-\frac{1!}{2},\frac{2!}{3},\dots,(-1)^{n+\ell-1}\frac{(n+\ell-1)!}{n-\ell+2}\right)\\
&=(-1)^n\sum_{\ell=0}^{n}\langle k\rangle_\ell\mathrm{B}_{n,\ell}\left(\frac{1!}{2},\frac{2!}{3},\dots,\frac{(n+\ell-1)!}{n-\ell+2}\right)\\
&=\sum_{\ell=0}^{n}(-1)^\ell\frac{\langle k\rangle_\ell}{\ell!}\sum_{m=1}^{\ell}(-1)^m\binom{\ell}{m}\frac{s(n+m,m)}{\binom{n+m}{m}},
\end{aligned}$$

where $u=f(x+1)$ and we used the identity (10) in the last step. The recurrence relation (6) is thus proved. $\square$

Remark 2. This paper is a slightly revised version of the preprint [**20**].

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