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ON THE LAPLACIAN SPECTRA OF PRODUCT GRAPHS

S. Barik, R. B. Bapat, S. Pati

Graph products and their structural properties have been studied extensively by many researchers. We investigate the Laplacian eigenvalues and eigenvectors of the product graphs for the four standard products, namely, the Cartesian product, the direct product, the strong product and the lexicographic product. A complete characterization of Laplacian spectrum of the Cartesian product of two graphs has been done by Merris. We give an explicit complete characterization of the Laplacian spectrum of the lexicographic product of two graphs using the Laplacian spectra of the factors. For the other two products, we describe the complete spectrum of the product graphs in some particular cases. We supply some new results relating to the algebraic connectivity of the product graphs. We describe the characteristic sets for the Cartesian product and for the lexicographic product of two graphs. As an application we construct new classes of Laplacian integral graphs.

1. Introduction

We consider simple graphs with a finite vertex set. If u and v are two adjacent vertices in a graph G, we use uv or $[u, v]$ to denote the edge. We write $u \sim v$ to mean that u and v are adjacent; see Godsil and Royle [**14**]. For a graph G we use $V(G)$ and $E(G)$ to denote its vertex set and its edge set, respectively. Given a graph G on n vertices, its *Laplacian matrix* is the $n \times n$ matrix $L(G) = D(G) - A(G)$, where $A(G)$ is the $(0, 1)$ adjacency matrix of G and $D(G)$ is the diagonal matrix of vertex degrees of G. It is known that $L(G)$ is a positive semidefinite matrix and the smallest eigenvalue of $L(G)$ is 0 with a corresponding eigenvector $\mathbf{1}$, the vector of all

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ones. By $S(G)$, we denote the Laplacian spectrum of G. For a graph G of order n, we write $S(G) = (\lambda_1, \ldots, \lambda_n)$ with the understanding that $\lambda_1 \leq \cdots \leq \lambda_n$. Fiedler [**11**] observed that, for a graph G the value $\lambda_2 > 0$ if and only if G is connected. This eigenvalue provides a measure of the connectivity of the graph and is known as the *algebraic connectivity* of G. We denote it by $a(G)$. The study of algebraic connectivity has attracted a lot of researchers recently. For an overall background, we refer the reader to [**2, 7, 26, 27**]. An eigenvector Y of $L(G)$ corresponding to $a(G)$ is known as a *Fiedler vector*. By $Y(v)$, we denote the coordinate of the vector Y corresponding to the vertex v. A graph G is called regular if every vertex of G has equal degree. A bipartite graph is called semiregular if each vertex in the same part of a bipartition has the same degree.

Let F and H be two graphs with disjoint vertex sets $\{u_1, \ldots, u_m\}$ and $\{v_1, \ldots, v_n\}$, respectively. A *graph product* G of F and H is a new graph, whose vertex set is $V(G) = V(F) \times V(H)$, the Cartesian product of $V(F)$ and $V(H)$. The adjacency of two distinct vertices (u_i, v_j) and (u_r, v_s) in $V(G)$ is determined entirely by the *adjacency/equality/non adjacency* of u_i and u_r in F and that of v_j and v_s in H. That is, we may define the vertices $(u_i, v_j) \sim (u_r, v_s)$ if a subset of the following alternate 8 conditions are satisfied.

1. $u_i \sim u_r$ and $v_j = v_s$,
2. $u_i \not\sim u_r$ and $v_j = v_s$,
3. $u_i \sim u_r$ and $v_j \sim v_s$,
4. $u_i \not\sim u_r$ and $v_j \sim v_s$,
5. $u_i \sim u_r$ and $v_j \not\sim v_s$,
6. $u_i \not\sim u_r$ and $v_j \not\sim v_s$,
7. $u_i = u_r$ and $v_j \sim v_s$, and
8. $u_i = u_r$ and $v_j \not\sim v_s$.

For example, one could define a product if either condition 1 or condition 2 or condition 5 is satisfied. Thus, one can define 256 different types of graph products. The graphs obtained by taking the products of two graphs are called the *product graphs*, and the two graphs are called the *factors*.

Out of these, four graph products namely, the Cartesian product (condition 1 or condition 7), the direct product (condition 3), the strong product (condition 1 or condition 3 or condition 7) and the lexicographic product (condition 1 or condition 3 or condition 5 or condition 7) are known as the standard graph products and have been studied by many researchers. We refer the reader to the book by Imrich and Klavžar [**19**] for a study of graph products and their structural properties. Note that, these four graph products are associative and they have a common property that 'if we take the product of two simple graphs, then we will produce a simple graph'. Also, except for the lexicographic product, the other three products are commutative in the sense of isomorphism.

Like the graph operations such as union, join, corona etc., the graph products are also used in constructing many important classes of graphs. Any graph invariant can be studied on graph products. A standard problem is that of describing some properties of a graph invariant of a product graph knowing the corresponding invariants of the factors. Investigation of the adjacency spectra and Laplacian

spectra of product graphs is also an interesting topic for researchers. Results describing the adjacency matrix and its spectra (also called the ordinary spectra) of the product graphs can be found in Brouwer and Haemers [**7**] and Cvetković, Doob and Sachs [**10**]. In [**20**], Kaveh and Alinejad have provided a general expression for Laplacian matrix of product graphs using the Laplacian matrices of the factors. They have characterized the Laplacian eigenvalues of the product graphs in some particular cases and have noted that the product graphs find their applications in structural mechanics, configuration processing, parallel computing.

Let us denote the cardinality of a set S by $|S|$. We remind our readers that, as $L(G)$ is positive semidefinite, its eigenvalues are nonnegative and we have a set of real orthogonal eigenvectors. It now follows that a Fiedler vector has at least one positive entry and at least one negative entry. Following Bapat and Pati [**4**], let us call a vertex v a *characteristic vertex* if $Y(v) = 0$ and there is a vertex $w \sim v$ such that $Y(w) \neq 0$. Let us call an edge $[u, v]$ a *characteristic edge* if $Y(u)Y(v) < 0$. By $\mathcal{C}(G, Y)$ let us denote the set of characteristic elements of G with respect to Y. In [**4**, Corollary 13], the authors showed that for a connected graph and a Fiedler vector Y, we have $1 \leq |\mathcal{C}(G, Y)| \leq |E(G)| - |V(G)| + 2$. In particular for a tree, the cardinality of a characteristic set is 1. Fiedler [**12**, Item 2.5] showed that if T is a tree and Y is a Fiedler vector with no zero entry, then $a(T)$ is simple. Merris [**25**] proved that if $\mathcal{C}(T, Y)$ is just a vertex then it remains the same vertex for any other Fielder vector. The characteristic set has been able to gain some attention; see, for example, [**3, 15, 16, 23**]. The algebraic connectivity and Fiedler vectors find their applications in graph clustering.

Let G be a connected graph and W be a subset of vertices of G. Following [**5**], we define a *branch* at W to be a component of $G \setminus W$. A branch at W is called a *Perron branch* if the smallest eigenvalue of the principal submatrix of $L(G)$, corresponding to the branch, is less than or equal to $a(G)$. If T is a tree with a characteristic vertex v it is called Type I (recall that for a tree it is the same vertex for any Fiedler vector). It is known that if T is a Type I tree with characteristic vertex v, then there are at least two Perron branches at v; see Kirkland, Neumann and Shader [**23**, Theorem 2]. It is known that there are infinitely many Type I trees with nonisomorphic Perron branches; see Grone and Merris [**17**], Kirkland [**21**] and [**6**, Corollary 3.6].

A graph is called a *Laplacian integral graph* if the Laplacian spectrum consists of integers. The study of Laplacian integral graphs has attracted many researchers; see Kirkland et al. [**22**] and the references therein. Using graph products, one can construct new classes of Laplacian integral graphs.

In this article, we aim is to study the Laplacian spectrum, algebraic connectivity and the characteristic set of the product graphs under the four standard graph products.

Results describing the Laplacian eigenvalues and eigenvectors of the Cartesian product of graphs have already been derived; see Merris [**26**]. We supply a new result relating the characteristic sets; see Theorem 5. As an application we construct an infinite class of graphs with nonisomorphic Perron branches, whose characteristic set consists of vertices only. A result similar to that of Fiedler's monotonicity theorem is also supplied. These are done in Section 2.

Results describing the adjacency spectra of the direct (Kronecker) product graph from those of the factors are available in the literature, when the factors are regular graphs; see Cvetkovic [**10**, Section 2.5]. As the direct product of regular

graphs is regular, one can easily compute the Laplacian spectrum of the direct product, in this case. First we supply a minor result which gives partial information of the Laplacian spectrum of the direct product when only one of the factors is known to be regular. Using that we give some conditions under which the direct product of two Laplacian integral graphs is Laplacian integral. We then provide a sharp upper bound for the algebraic connectivity of the direct product of two graphs and provide a class on which the bound is attained. These are done in Section 3.

For the strong product of two regular graphs, the Laplacian eigenvalues and eigenvectors can be described from those of the factors easily using the corresponding results of the Cartesian and the direct product. Observing that it is not possible to describe the complete Laplacian spectrum of the strong product of graphs in general, we mention a small observation giving partial information of the Laplacian spectrum of the strong product when only one of the factors is known to be regular. Using that we give some conditions under which the strong product of two Laplacian integral graphs is Laplacian integral. These are done in Section 4.

The Laplacian eigenvalues and eigenvectors of the lexicographic product of graphs have been described in general, though not explicitly. In [**24**], Neumann and Pati have characterized the Laplacian spectra of graphs $G[T, G_1, \ldots, G_n]$, where T is a tree of order n and $G_1, \ldots G_n$ are any graphs; see Lemmas 1 and 3. In [**8**] Cardoso, Freitas, Martins and Robbiano proved a similar characterization for the graphs $G[G, G_1, \ldots, G_n]$, where G was any graph of order n, however they have used the notation $G[G_1, \ldots, G_n]$. Note that the graph operation $G[G_1, \ldots, G_n]$ was introduced by Schwenk in [**28**] viewing it as a generalization of the join operation. Observe that, if we take $G_1 = G_2 = \cdots = G_n = H$, then $G[G_1, \ldots, G_n]$ is nothing but the lexicographic product $G[H]$. Interestingly, both the results in [**8, 24**] make use of the normalized version of the vertex weighted Laplacians introduced by Chung and Langlands in [**9**], which is actually a perturbed Laplacian as remarked in [**24**]. We have supplied a description of the Laplacian eigenvalues of $G[H]$ explicitly in terms of the Laplacian eigenvalues of G and H. We also have described the characteristic set of the lexicographic product of two graphs; see Theorem 32. These are done in Section 5.

Recall that the *Kronecker product* $A \otimes B$ of two matrices $A = [a_{ij}]$ and B is the partitioned matrix $[a_{ij}B]$ and that $(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$. Let us use K_n, P_n and C_n, to denote the complete graph, the path, and the cycle, on n vertices, respectively. By $\mathbf{e}_i$ we denote the vector with ith entry equal to 1 and all other entries 0. By J_n we denote the matrix of all ones of size n. By I_n we denote the identity matrix of size n.

2. Graph Cartesian product

The Cartesian product $F \square H$ of two graphs F and H is a graph with vertex set $V(F) \times V(H)$ where the adjacency of vertices is determined by the following rule: (u_i, v_j) and (u_r, v_s) are adjacent if either $\big(u_i = u_r$ and $v_j \sim v_s\big)$ or $\big(u_i \sim u_r$ and $v_j = v_s\big)$. One may also view $F \square H$ as the graph obtained from F by replacing each of its vertices with a copy of H and each of its edges with $|V(H)|$ edges joining corresponding vertices of H in the two copies. It is known that the graph $F \square H$ is isomorphic to the graph $H \square F$; see [**19**].

Example 1. Let $F = P_2$ and $H = C_3$. The graph $F \square H$, obtained by taking

Cartesian product of F and H, is shown in Figure 1.

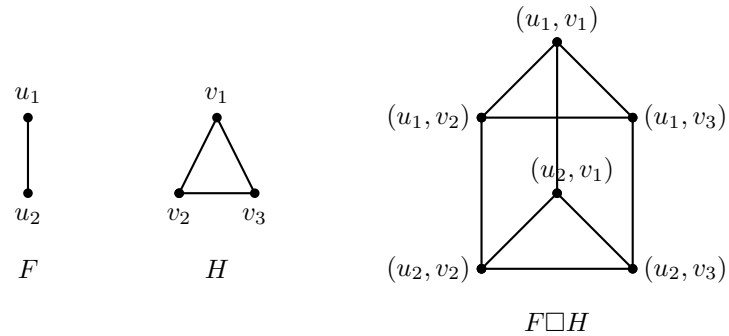


Figure 1: Cartesian product

Remark 2. If $|V(F)| = m$, $|V(H)| = n$, then $|V(F\Box H)| = mn$ and $|E(F\Box H)| = m|E(H)| + n|E(F)|$.

The graph Cartesian product has been widely investigated and is arguably the most interesting one among all products; see Mohar [**27**]. For example, by taking the n-fold Cartesian product of K_2 one can get the *n-cube* Q_n on 2^n vertices. It is known that $F\Box H$ is connected if and only if the both the graphs F and H are connected; see [**19**].

Let F and H be graphs on m and n vertices. Fiedler [**11**, Item 3.4] has observed that

$$L(F\Box H) = L(F) \otimes I_n + I_m \otimes L(H). \tag{1}$$

The next result is essentially contained in [**11**, Item 3.4]; see also Merris [**26**].

Theorem 3. (Fiedler [**11**]; Merris [**26**]) *Let F and H be graphs with $S(F) = (\lambda_1, \ldots, \lambda_m)$ and $S(H) = (\mu_1, \ldots, \mu_n)$. Then the eigenvalues of $L(F\Box H)$ are*

$$\lambda_i + \mu_j, \qquad 1 \le i \le m,\ 1 \le j \le n.$$

Moreover, if X_i is an eigenvector of $L(F)$ affording λ_i and Y_j is an eigenvector of $L(H)$ affording μ_j, then $X_i \otimes Y_j$ is an eigenvector of $L(F\Box H)$ affording $\lambda_i + \mu_j$. In particular, $a(F\Box H) = \min(a(F), a(H))$. Furthermore, $a(F\Box H)$ is simple if and only if either $a(F) < a(H)$ with $a(F)$ simple or $a(H) < a(F)$ with $a(H)$ simple.

The following is a direct application of Theorem 3. It describes a construction of new Laplacian integral graphs from the known ones.

Corollary 4. *Let F and H be two graphs on disjoint vertex sets. Then $F\Box H$ is Laplacian integral if and only if both F and H are Laplacian integral.*

The next result which describes the characteristic set of the Cartesian product may be seen as continuation of Theorem 3. It is our main result of the section. It will be followed by many applications. If X is a Fiedler vector of a graph F, then by $\tilde{C}(F, X)$ denote the graph whose vertex set is the set of all vertices involved in $\mathcal{C}(F, X)$ (including the endvertices of the characteristic edges) and whose edge set is the set of all characteristic edges in $\mathcal{C}(F, X)$. Let us refer to this as the *characteristic graph* of F with respect to X.

Theorem 5. *Let F and H be connected graphs on vertices $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively. Let F^* and H^* be the empty graphs on $V(F)$ and $V(H)$, respectively. Let X and Y be Fiedler vectors of F and H, respectively. If $X \otimes \mathbb{1}$ is a Fiedler vector of $F\Box H$ (this happens when $a(F) \leq a(H)$), then*

$$\tilde{C}(F\Box H, X \otimes \mathbb{1}) = \tilde{C}(F, X)\Box H^*$$

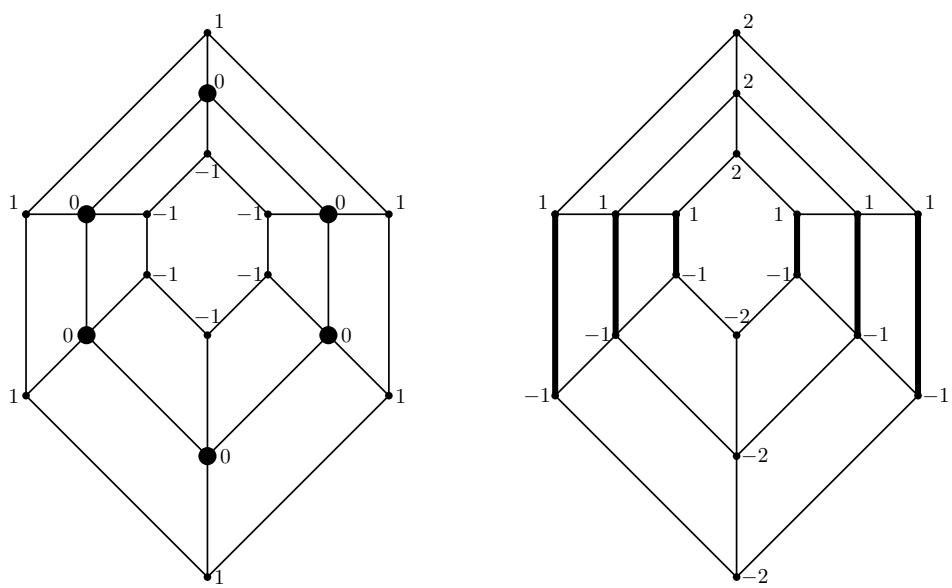
and if $\mathbb{1} \otimes Y$ is a Fiedler vector of $F\Box H$ (this happens when $a(F) \geq a(H)$), then

$$\tilde{C}(F\Box H, \mathbb{1} \otimes Y) = F^*\Box\tilde{C}(H, Y).$$

Proof. Assume that $X \otimes \mathbb{1}$ is a Fiedler vector of $F\Box H$. Observe that $u_i \in \tilde{C}(F, X)$ is a characteristic vertex if and only if $(u_i, v_j) \in \tilde{C}(F\Box H, X \otimes \mathbb{1})$, $j = 1, 2, \ldots, n$, are characteristic vertices. Similarly, $u_i \in \tilde{C}(F, X)$ is an endvertex of a characteristic edge if and only if $(u_i, v_j) \in \tilde{C}(F\Box H, X \otimes \mathbb{1})$, $j = 1, 2, \ldots, n$, are endvertices of characteristic edges.

Note also that $u_i \sim u_k$ in F if and only if $(u_i, v_j) \sim (u_k, v_j)$ in $F\Box H$, for each $v_j \in H$. Hence the edge $[u_i, u_k] \in \tilde{C}(F, X)$ if and only if the edge $[(u_i, v_j), (u_k, v_j)] \in \tilde{C}(F\Box H, X \otimes \mathbb{1})$, $j = 1, 2, \ldots, n$. It follows that $\tilde{C}\Big(F\Box H, X \otimes \mathbb{1}\Big) = \tilde{C}(F, X)\Box H^*$. The other equality is proved similarly. ∎

We now give an illustration of Theorem 5.



$\tilde{C}(P_3\Box C_6, X \otimes \mathbb{1})$ $\tilde{C}(P_3\Box C_6, \mathbb{1} \otimes Y)$

Figure 2: Characteristic graphs of $P_3\Box C_6$ with respect to $X \otimes \mathbb{1}$ and $\mathbb{1} \times Y$

Example 6. Take the path P_3 with vertex set $\{u_1, u_2, u_3\}$ and the cycle C_6 with vertex set $\{v_1, v_2, v_3, v_4, v_5, v_6\}$. Note that $X = [-1, 0, 1]^T$ is a Fiedler vector of P_3

and $Y = [1, 2, 1, -1, -2, -1]^T$ is a Fiedler vector of C_6. As $a(P_3) = a(C_6) = 1$, we see by Theorem 3, that $X \otimes \mathbb{1}$ and $\mathbb{1} \otimes Y$ are Fiedler vectors of $P_3 \square C_6$. Notice that $\tilde{C}(P_3, X)$ is the graph with vertex set $\{u_2\}$ and no edges. Observe that

$$X \otimes \mathbb{1} = \begin{bmatrix} -1 & -1 & -1 & -1 & -1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T,$$

so that the characteristic graph $\tilde{C}(P_3 \square C_6, X \otimes \mathbb{1})$ is the graph with vertices $\{(u_2, v_i) : i = 1, \ldots, 6\}$ and no edges. It can be seen (in Figure 2) that $\tilde{C}(P_3 \square C_6, X \otimes \mathbb{1}) = \tilde{C}(P_3, X) \square C_6^*$. Similarly, the characteristic graph $\tilde{C}(P_3 \square C_6, Y)$ is

$$\Big(\{u_1, u_2, u_3\}, \emptyset\Big) \square \Big(\{v_3, v_4, v_6, v_1\}, \big\{[v_3, v_4], [v_1, v_6]\big\}\Big) =$$

$$\Big(\{u_1, u_2, u_3\} \times \{v_3, v_4, v_6, v_1\}, \big\{[(u_i, v_3), (u_i, v_4)], [(u_i, v_1), (u_i, v_6)] : i = 1, 2, 3\big\}\Big).$$

As an immediate application of Theorem 3, we see that the ladder of length at least 2 has a simple algebraic connectivity. Applying Theorem 5 we give a description of the Fiedler vector and the characteristic set of a ladder of length k ($k \geq 2$).

Proposition 7. *Let $G = P_{k+1} \square P_2$, be a ladder of length k ($k \geq 2$) and vertex set $\{1, 2, \ldots, 2k+2\}$; see Figure 3. Then $a(G)$ is simple. Let X be a Fiedler vector of P_{k+1}. Then $X \otimes \mathbb{1}$ is a Fiedler vector of G. If k is even, then the characteristic set of G consists of two zero vertices $k+1$ and $k+2$ only. If k is odd, then the characteristic set of G consists of exactly two edges, $[k, k+2]$ and $[k+1, k+3]$.*

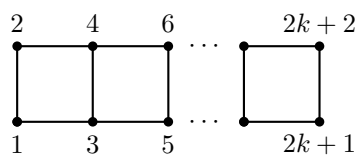


Figure 3: $P_{k+1} \square P_2$, the ladder of length k

Proof. Since $a(P_{k+1}) < a(P_2)$ for $k \geq 2$, using Theorem 3, we see that $a(G) = a(P_{k+1})$ and it is simple. By Theorem 5, $X \otimes \mathbb{1}$ is a Fiedler vector of G and its characteristic set is $\tilde{C}(P_{k+1}, X) \square P_2^*$. The rest follows routinely. ∎

Consider the grid $G = P_7 \square P_3$. We know that $a(G)$ is simple. Note that, $X = [-.3009, -.2413, -.1339, 0, .1339, .2413, .3009]^T$ is a Fiedler vector of P_7. Hence, the vector $X \otimes \mathbb{1}$ is a Fiedler vector of G. See Figure 4, where values near each vertex are the values of the Fiedler vector. The vertices with valuations 0 are the characteristic vertices. Think of traversing along the path $[v_0, v_1, v_2, v_3]$. Notice that the entries of the Fiedler vector increase along this path.

In general, a Fiedler vector of $T \square H$ may not have such a property, where T is a tree and H is any graph. The following result gives us a sufficient condition for the Fiedler vector of $T \square H$ to have the monotonicity property. We require a notation. Let G be a graph and S be a subset of vertices and edges of G. Let $V(S)$ be the set of vertices involved in S (including the endvertices of edges). Let $u \in V(G)$. Then by the distance $dist(u, S)$ of u from S we mean $\min_{v \in V(S)} dist(u, v)$, where $dist(u, v)$ means the usual distance between u and v.

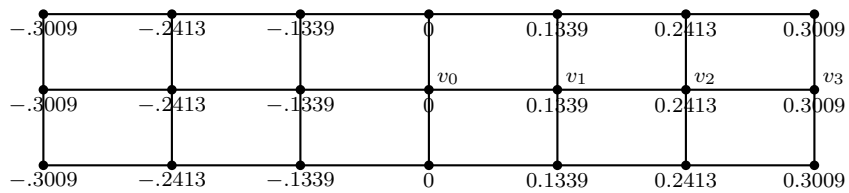


Figure 4: Monotonicity of Fiedler vector

Proposition 8. *Let T be a tree with $a(T)$ simple and H be a graph such that $a(T) < a(H)$. Let Z be a Fiedler vector of $G = T\Box H$. Let $P = [v_0, v_1, \ldots, v_k]$ be a path such that $dist(\mathcal{C}(G,Z), v_{i+1}) > dist(\mathcal{C}(G,Z), v_i)$, $i = 0, 1, \ldots, k-1$ with $dist(\mathcal{C}(G,Z), v_0) = 0$. (In other words, P starts from the characteristic set and goes strictly away from the characteristic set.) Then the entries of Z monotonically increase (decrease, or are identically zero) along P if $Z(v_0)$ is positive (negative, or zero).*

Proof. Let $V(T) = \{u_1, \ldots, u_m\}$ and $V(H) = \{x_1, \ldots, x_n\}$. Let X be a Fiedler vector of T. Recall that $|\mathcal{C}(T,X)| = 1$ and $\mathcal{C}(T,X)$ is either an edge or a singleton vertex. Assume first that $\mathcal{C}(T,X) = [u_1, u_2]$ and $X(u_1) > 0$. By Theorem 3, we may assume that the Fiedler vector $Z = X \otimes \mathbb{1}$. By Theorem 5,

$$\mathcal{C}(G,Z) \text{ is the set of edges } \big\{[(u_1,x_i),(u_2,x_i)] \,:\, i = 1, \ldots, n\big\}.$$

Assume, $v_0 = (u_1, x_1)$. As $dist(\mathcal{C}(G,Z), v_{i+1}) > dist(\mathcal{C}(G,Z), v_i)$, it follows that $v_i = (u_{r_i}, x_1)$, where $P' = [u_1, u_{r_1}, u_{r_2}, \ldots, u_{r_k}]$ is a path in T which starts at u_1 and does not pass through u_2. By Fiedler's monotonicity theorem [**13**, Item (3.14)], the entries of Y increase along P'. As $Z(v_0) = X(u_1)$ and $Z(v_i) = X(u_{r_i})$, for $i = 1, \ldots, k$. The proof is complete in this case. The proof is similar in the case when $\mathcal{C}(T,X)$ is a single vertex. ∎

As another application of our observations, we construct an infinite class of graphs with nonisomorphic Perron branches, whose characteristic set consists of vertices only.

Proposition 9. *Let T be a Type I tree with a characteristic vertex v with two nonisomorphic Perron branches at v. Let H be any graph such that $a(T) < a(H)$. Let $G_1 = T\Box H$ and $G_k = G_{k-1}\Box H$ for $k = 2, 3, \ldots$. Then the class consisting of G_k, $k = 1, 2, \ldots$ is an infinite class of graphs with nonisomorphic Perron branches, whose characteristic set consists of vertices only.*

Proof. Let X be a Fiedler vector of T and $C(T,X) = \{v\}$. Since $a(T) < a(H)$, using Theorem 3, $a(G_1) = a(T)$ and $X_1 = X \otimes \mathbb{1}$ is a Fiedler vector of G_1. By Theorem 5, $\mathcal{C}(G_1, X_1) = \{v\}\Box H^* = \{v\} \times V(H)$ consists of the vertices (v, w_i), where $w_i \in H$.

It will help to imagine the tree T (on the plane) first, and then put a (vertical) copy of H at each vertex of T, and then put edges between two adjacent copies of H, that is, a vertex $h \in H$ at u goes to the same vertex at w when $[u,w] \in T$. If we delete the whole copy at the vertex v from G_1, we will have a disconnected graph. In fact, if B is any branch of T at v, then $B_1 = B\Box H$ is a branch of

G_1 at the characteristic set $\mathcal{C}(G_1, X_1) = \{v\} \times V(H)$. Thus if P_1 and P_2 are two nonisomorphic branches at $\{v\}$, then $P_1 \Box H$ and $P_2 \Box H$ are two nonisomorphic branches at $\mathcal{C}(G_1, X_1)$. The fact that these two branches $P_i \Box H$, $i = 1, 2$, are Perron branches, follows from the fact that the valuation given to a vertex $(u, w) \in P_1 \Box H$ by the Fiedler vector $X_1 = X \otimes \mathbb{1}$ is precisely the valuation given to $u \in P_1$ by X, that is, $X_1((u, w)) = X(u)$. Hence, if P_1 was a branch with its vertices negatively valuated by X, then $P_1 \Box H$ is a branch at $\mathcal{C}(G_1, X_1) = \{v\} \times V(H)$ with its vertices negatively valuated by X_1.

Put $G_{i+1} = G_i \Box H$, $i = 1, 2, \ldots$. Repeating our arguments, we see that G_{i+1} has a Fiedler vector X_{i+1} for which $\mathcal{C}(G_{i+1}, X_{i+1})$ consists of vertices only and there are two nonisomorphic Perron branches at $\mathcal{C}(G_{i+1}, X_{i+1})$. ∎

3. Graph direct product

Other names for direct product are *categorical product, Kronecker product, tensor product*. Recall that the adjacency in the direct product $F \times H$ of the graphs F and H is determined by the following rule: $(u_i, v_j) \sim (u_r, v_s)$ in $F \times H$ if $u_i \sim u_r$ and $v_j \sim v_s$. Note that, if $u_i \sim u_r$ in F and $v_j \sim v_s$ in H, then $(u_i, v_j) \sim (u_r, v_s)$ and $(u_i, v_s) \sim (u_r, v_j)$ in $F \times H$. Thus $|E(F \times H)| = 2|E(F)||E(H)|$.

The following description of $L(F \times H)$ in terms of $L(F)$ and $L(H)$ can be found in Kaveh and Alinejad in [**20**].

Proposition 10. *Let F and H be two graphs on m and n vertices, respectively. Then*

$$L(F \times H) = D(F) \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H),$$

where $D(G)$ is the diagonal degree matrix of a graph G.

Recall that the Cartesian product of two connected graphs is always a connected graph. But this is not true for the direct product, as one can see in $P_2 \times P_4$. Let F and H be two connected graphs. A necessary and sufficient condition for $F \times H$ to be connected is known.

Theorem 11. (Imrich and Klavžar [**19**]) *Let F and H be graphs with at least one edge. Then $F \times H$ is connected if and only if both F and H are connected and at least one of them is non-bipartite. Furthermore, if both F and H are connected and bipartite, then $F \times H$ has exactly two connected components.*

In terms of algebraic connectivity, it now follows that for graphs F and H with $a(F) > 0$ and $a(H) > 0$, we may have $a(F \times H) = 0$. This happens if and only if both F and H are bipartite. Hence we cannot, in general, predict the Laplacian eigenvalues of $F \times H$, only from the Laplacian eigenvalues of F and H. Indeed, given $a(F) > 0$ and $a(H) > 0$, we require another additional information 'whether one of F and H is non-bipartite' in order to predict that $a(F \times H) > 0$. Bipartiteness cannot be predicted from the Laplacian spectrum as there are Laplacian cospectral graphs F and H (see Figure 5) in which only one is bipartite. These are taken from van Dam and Haemers [**29**]. We shall need them later.

In the case both F and H are regular, the graph $F \times H$ is regular and it is known (see for example Cvetković[**10**, Section 2.5]) that if λ_i, $i = 1, \ldots, m$ and μ_j, $j = 1, \ldots, n$ are ordinary (adjacency) eigenvalues of F and H, respectively, then

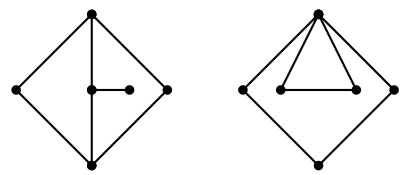


Figure 5: Nonisomorphic Laplacian cospectral graphs F and H

$\lambda_i\mu_j$, $i = 1, \ldots, m$ and μ_j, $j = 1, \ldots, n$ are the adjacency eigenvalues of $F \times H$. Hence, when F and H are regular, the spectrum of $L(F\times H)$ can easily be computed from the spectrum of $L(F)$ and $L(H)$.

Theorem 12. *Let F and H be connected regular graphs on m and n vertices and of regularities r and s, respectively. Suppose that $S(F) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $S(H) = (\mu_1, \mu_2, \ldots, \mu_n)$. Then*

$$r\mu_j + \lambda_i s - \lambda_i\mu_j \in S(F \times H)$$

each for $i = 1, \ldots, m$ and $j = 1, \ldots, n$. Indeed, if X_i is an eigenvector of $L(F)$ corresponding to λ_i and Y_j is an eigenvector of $L(H)$ corresponding to μ_j, then $X_i \otimes Y_j$ is an eigenvector of $L(F \times H)$ corresponding to $r\mu_j + \lambda_i s - \lambda_i\mu_j$.

In the case when one of the two graphs F and H is regular one can describe $S(F \times H)$ partially using that of F and H.

Proposition 13. *Let F be a regular graph on m vertices with regularity r and H be any graph on n vertices. Suppose that $S(H) = (\mu_1, \ldots, \mu_n)$. Then*

$$r\mu_j \in S(F \times H)$$

for each $j = 1, 2 \ldots, n$.

Proof. Let Y_j be an eigenvector of $L(H)$ for the eigenvalue μ_j. By Proposition 10, we have

$$\begin{aligned} L(F \times H) &= D(F) \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H) \\ &= rI_m \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H). \end{aligned}$$

Thus $r\mu_j$ is an eigenvalue of $L(F \times H)$ afforded by the eigenvector $\mathbb{1} \otimes Y_j$ for $j = 1, \ldots, n$. ∎

The following is an immediate corollary.

Corollary 14. *Let F be a regular graph and H be any graph. If $F \times H$ is Laplacian integral, then H is Laplacian integral.*

Proof. Suppose that F is with regularity r and $S(H) = (\mu_1, \ldots, \mu_n)$. It follows from Proposition 13 that $r\mu_j \in S(F \times H)$ for each $j = 1, 2, \ldots, n$. Since $F \times H$ is Laplacian integral, we have $r\mu_j$ is an integer for each $j = 1, 2, \ldots, n$. Hence μ_j are rational numbers and hence they are integers. ∎

In general, the direct product of two Laplacian integral graphs is not necessarily Laplacian integral. For example, $S(K_3) = (0, 3, 3)$ and $S(P_3) = (0, 1, 3)$ but $S(K_3 \times P_3) = (0, 1.2679, 1.2679, 2, 2, 2, 4.7321, 4.7321, 6)$. In fact $K_n \times P_3$, $n \geq 3$, is not Laplacian integral though K_n and P_3 are Laplacian integral graphs. This follows from the next proposition.

Proposition 15. *Let $n \geq 3$. Then*

(i) $0 \in S(K_n \times P_3)$ *with multiplicity* 1,

(ii) $3n - 3 \in S(K_n \times P_3)$ *with multiplicity* 1,

(iii) $\dfrac{3n - 3 + \sqrt{n^2 - 2n + 9}}{2} \in S(K_n \times P_3)$ *with multiplicity* $n - 1$,

(iv) $\dfrac{3n - 3 - \sqrt{n^2 - 2n + 9}}{2} \in S(K_n \times P_3)$ *with multiplicity* $n - 1$, *and*

(v) $n - 1 \in S(K_n \times P_3)$ *with multiplicity* n.

In particular, $K_n \times P_3$ is not Laplacian integral.

Proof. Let $G = K_n \times P_3$, $n \geq 3$. Note that $S(K_n) = (0, n, \ldots, n)$, $S(P_3) = (0, 1, 3)$ and G is of regularity $n - 1$. Thus, applying Proposition 13, 0, $n - 1$ and $3n - 3 \in S(G)$. Let $\hat{L}_1$ be the diagonal matrix $diag(n - 1, 2n - 2, n - 1)$. By Proposition 10, we have

$$L(G) = \begin{bmatrix} \hat{L}_1 & \hat{L}_2 & \cdots & \hat{L}_2 \\ \hat{L}_2 & \hat{L}_1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \hat{L}_2 \\ \hat{L}_2 & \cdots & \hat{L}_2 & \hat{L}_1 \end{bmatrix}, \text{ where } \hat{L}_2 = \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix}.$$

Observe that if λ is an eigenvalue of $\hat{L}_1 - \hat{L}_2$ afforded by the eigenvector X, then

$$\begin{bmatrix} X \\ -X \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} X \\ \mathbf{0} \\ -X \\ \vdots \\ \mathbf{0} \end{bmatrix}, \ldots, \begin{bmatrix} X \\ \mathbf{0} \\ \mathbf{0} \\ \vdots \\ -X \end{bmatrix}$$

are $n - 1$ linearly independent eigenvectors corresponding to the eigenvalue λ of $L(G)$. Furthermore, each of these eigenvectors is orthogonal to $\mathbf{1} \otimes \begin{bmatrix} 1 & 0 & -1 \end{bmatrix}^T$, an eigenvector of $L(G)$ corresponding to the eigenvalue $n - 1$.

It can be verified that the eigenvalues of $\hat{L}_1 - \hat{L}_2$ are $n-1$, and $\frac{3n-3\pm\sqrt{n^2-2n+9}}{2}$ and that $\sqrt{n^2 - 2n + 9}$ is not an integer for $n \geq 3$. ∎

It is natural to wonder whether we can have classes of graphs on which Laplacian integrality of F and H is equivalent to Laplacian integrality of $F \times H$. The following result supplies such a class.

Corollary 16. *Let F and H be two regular graphs of regularity r and s, respectively. Then $F \times H$ is Laplacian integral if and only if both F and H are Laplacian integral.*

Proof. Follows easily from Theorem 12 and Corollary 14. ∎

Another interesting problem is the following. Suppose that the graphs F and H are Laplacian integral. Can we give a necessary and sufficient condition on F and H so that $F \times H$ is Laplacian integral? By Corollary 16, the condition 'F and H are regular' is clearly sufficient. However, it is not necessary. Indeed, $P_3 \times P_3$ being the disjoint union of the cycle C_4 and the star $K_{1,4}$ has Laplacian eigenvalues $S(P_3 \times P_3) = (0, 0, 1, 1, 1, 2, 2, 4, 5)$.

Given a graph G, let $\Delta(G)$ denote the maximum degree of vertices in G and $\rho(L(G))$ denote the spectral radius of $L(G)$. Next we provide an upper bound for the algebraic connectivity of the direct product. This is our main result of the section.

Theorem 17. *Let F and H be two graphs and $G = F \times H$. Then*

$$a(G) \leq \Delta(F)\Delta(H) - \Big(\Delta(F) - \rho(L(F))\Big)\Big(\Delta(H) - \rho(L(H))\Big). \tag{2}$$

The bound is sharp.

Proof. By Proposition 10, we know that $L(G) = D(F) \otimes L(H) + L(F) \otimes D(H) - L(F) \otimes L(H)$. Let X and Y be unit eigenvectors of $L(F)$ and $L(H)$ corresponding to the eigenvalues $a(F)$ and $a(H)$, respectively. Let $Z = X \otimes Y$. Note that $Z \perp \mathbf{1}$. Thus

$$\begin{aligned} a(G) &\leq Z^T L(G) Z \\ &= Z^T(D(F) \otimes L(H))Z + Z^T(L(F) \otimes D(H))Z - Z^T(L(F) \otimes L(H))Z \\ &\leq \Delta(F)\rho(L(H)) + \rho(L(F))\Delta(H) - \rho(L(F))\rho(L(H)) \\ &= \Delta(F)\Delta(H) - (\Delta(F) - \rho(L(F))(\Delta(H) - \rho(L(H))). \end{aligned}$$

The bound given is sharp as the equality holds in the case where $F = C_{2m}$ and $H = C_{2n}$, $m, n \geq 2$. Indeed, as F and H are bipartite, we have by Theorem 11, $a(F \times H) = 0$. On the other hand it is well known that $\rho(L(C_n)) = 4$, for even $n \geq 4$; see [**11**]. Thus

$$\Delta(F)\Delta(H) - \Big(\Delta(F) - \rho(L(F))\Big)\Big(\Delta(H) - \rho(L(H))\Big) = 4 - (-2)(-2) = 0.$$

∎

A natural problem is to characterize the class of graphs attaining the bound in Theorem 17. The following result supplies an answer in a special case.

Proposition 18. *Let $F = C_{2n}$, $n \geq 2$, and H a bipartite connected graph. Then the equality holds in Equation* (2) *if and only if H is regular.*

Proof. As F and H are bipartite, we have $a(F \times H) = 0$. Hence, the equality holds in (2) if and only if $\rho(L(H)) = 2\Delta(H)$. Suppose first that $\rho(L(H)) = 2\Delta(H)$. Anderson and Morley [**1**] have proved that

$$\rho(L(G')) \leq \max\Big\{d(u) + d(v) \,:\, [u, v] \in E(G')\Big\},$$

for any graph G' and the equality holds if and only if G' is a bipartite semiregular graph. As $\rho(L(H)) \leq \max\Big\{d(u) + d(v) \,:\, [u, v] \in E(H)\Big\} \leq 2\Delta(H) = \rho(L(H))$,

we see that H is semiregular. Further, as $d(u) + d(v) = 2\Delta(H)$ for some edge $[u, v] \in E(H)$, we see that H is regular. Conversely, if H is regular, then again by the result of Anderson and Morley $\rho(L(H)) = 2\Delta(H)$ and the equality in (2) holds. ∎

4. Graph strong product

The strong product $F \boxtimes H$ of two graphs F and H is a graph where the adjacency is determined by the following rule: (u_i, v_j) and (u_r, v_s) are adjacent in $F \boxtimes H$ if either ($u_i = u_r$ and $v_j \sim v_s$ in H) or ($u_i \sim u_r$ in F and $v_j = v_s$) or ($u_i \sim u_r$ in F and $v_j \sim v_s$ in H). See Figure 6.

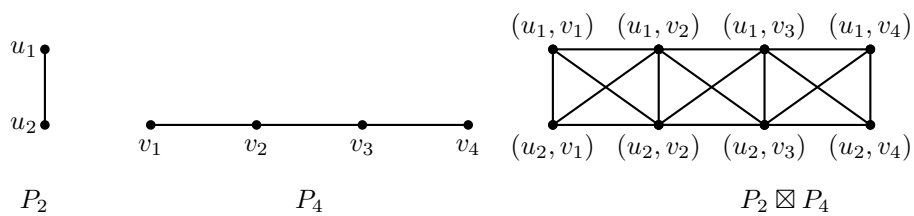


Figure 6: Graph strong product

Remark 19. Let F and H be two graphs. As $E(F \boxtimes H) = E(F \square H) \cup E(F \times H)$ and $E(F \square H) \cap E(F \times H) = \emptyset$ one immediately sees that

$$A(F \boxtimes H) = A(F \square H) + A(F \times H) \quad \text{and} \quad L(F \boxtimes H) = L(F \square H) + L(F \times H).$$

Thus, if $|V(F)| = m$, $|V(H)| = n$, then $|E(F \boxtimes H)| = m|E(H)| + n|E(F)| + 2|E(F)||E(H)|$.

In general, the complete description of the Laplacian spectrum of $F \boxtimes H$ cannot be obtained only from the Laplacian spectra of F and H. This is clear from the following example.

Example 20. Consider the nonisomorphic Laplacian cospectral graphs F and H shown in Figure 5. One can verify that

$$S(F \boxtimes P_2) = (0, 1.5279, 4, 6, 6, 6, 6, 6, 6, 6, 10, 10.4721)$$

and

$$S(H \boxtimes P_2) = (0, 1.5279, 4, 4, 6, 6, 6, 6, 8, 8, 8, 10.4721).$$

Though $S(F) = S(H)$, the graphs $F \boxtimes P_2$ and $H \boxtimes P_2$ have different Laplacian spectra.

In the case both F and G are regular the following description of the Laplacian spectrum of $F \boxtimes H$ can easily be obtained from Theorems 3 and 12.

Theorem 21. *Let F and H be connected regular graphs on m and n vertices and of regularities r and s, respectively. Suppose that $S(F) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $S(H) = (\mu_1, \mu_2, \ldots, \mu_n)$. Then $(1+s)\lambda_i + (1+r)\mu_j - \lambda_i\mu_j \in S(F \boxtimes H)$, for $i = 1, \ldots, m$, and $j = 1, \ldots, n$. Indeed, if X_i is an eigenvector of $L(F)$ corresponding to λ_i and Y_j is an eigenvector of $L(H)$ corresponding to μ_j, then $X_i \otimes Y_j$ is an eigenvector of $L(F \times H)$ corresponding to $(1+s)\lambda_i + (1+r)\mu_j - \lambda_i\mu_j$.*

In the case when one of the two graphs F and H is regular one can give a partial description of $S(F \times H)$, similar to that of Proposition 13.

Proposition 22. *Let F be a regular graph on m vertices with regularity r and H be any graph on n vertices. Suppose that $S(H) = (\mu_1, \ldots, \mu_n)$. Then*

$$(1+r)\mu_j \in S(F \boxtimes H)$$

for each $j = 1, 2, \ldots, n$. Indeed if Y_j is an eigenvector of $L(H)$ for μ_j, then $\mathbb{1} \otimes Y_j$ is an eigenvector of $L(F \boxtimes H)$ for $(1+r)\mu_j$.

We have two corollaries similar to those of Corollaries 14 and 16.

Corollary 23. *Let F be a regular graph and H be any graph. If $F \boxtimes H$ is Laplacian integral, then H is Laplacian integral.*

Corollary 24. *Let F and H be two regular graphs of regularity r and s, respectively. Then $F \boxtimes H$ is Laplacian integral if and only if both F and H are Laplacian integral.*

In general the strong product of two Laplacian integral graphs is not a Laplacian integral graph. For example, $P_3 \boxtimes P_3$ is not Laplacian integral.

Example 25. We show that $P_3 \boxtimes P_3$ is not Laplacian integral. Observe that $L(G) = \begin{bmatrix} \hat{L}_1 & \hat{L}_2 & \mathbf{0} \\ \hat{L}_2 & \hat{L}_3 & \hat{L}_2 \\ \mathbf{0} & \hat{L}_2 & \hat{L}_1 \end{bmatrix}$, where $\hat{L}_1 = \begin{bmatrix} 3 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 3 \end{bmatrix}$, $\hat{L}_2 = \begin{bmatrix} -1 & -1 & 0 \\ -1 & -1 & -1 \\ 0 & -1 & -1 \end{bmatrix}$ and $\hat{L}_3 = \begin{bmatrix} 5 & -1 & 0 \\ -1 & 8 & -1 \\ 0 & -1 & 5 \end{bmatrix}$. Thus, if λ is an eigenvalue of $\hat{L}_1$ with a corresponding eigenvector X, then λ is an eigenvalue of $L(G)$ with a corresponding eigenvector $\begin{bmatrix} X \\ \mathbf{0} \\ -X \end{bmatrix}$. It is easy to see that the eigenvalues $\hat{L}_1$ are $3, 4-\sqrt{3}, 4+\sqrt{3}$. Hence G is not Laplacian integral.

5. Graph lexicographic product

Let F and H be two graphs on vertex sets $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively. The lexicographic product $F[H]$ of F and H is the product of F and H where the adjacency is determined by the following rule: (u_i, v_j) and (u_r, v_s) are adjacent in $F[H]$ if either ($u_i \sim u_j$ in F) or ($u_i = u_r$ and $v_j \sim v_s$ in H). Note that,

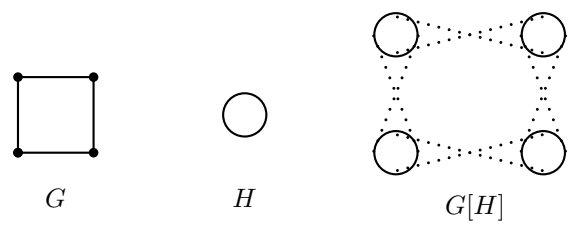


Figure 7: Graph lexicographic product

the lexicographic product of F and H can be obtained from F by substituting a copy H_{u_i} (say), of H for every vertex u_i of F and by joining all vertices of H_{u_i} with all vertices of H_{u_r} if $u_i \sim u_r$ in F. Thus it is also known as composition or substitution; see Harary[**18**].

If $|V(F)| = m$, $|V(H)| = n$, then $|V(F[H])| = mn = |V(H[F])|$, $|E(F[H])| = m|E(H)| + n^2|E(F)|$ and $|E(H[F])| = n|E(F)| + m^2|E(H)|$.

From the definition of lexicographic product it follows that if F and H are two nontrivial graphs with at least two vertices, then $F[H]$ is connected if and only if F is connected. Thus $F[H] \ncong H[F]$ whenever one of F or H is disconnected. Even in the case both the factors are connected, it need not commute. For example, taking $F = K_2$ and $H = P_3$, we see that $F[H]$ has 13 edges but $H[F]$ has only 11 edges. In [**19**], Imrich and Klavžar have characterized the cases when the lexicographic product commutes.

The following result describes the Laplacian matrix of $F[H]$ using the Laplacian matrices of F and H.

Proposition 26. *Let F and H be graphs on m and n vertices, respectively. Then*

$$L(F[H]) = L(F\Box H) + L(F) \otimes (J_n - I_n) + D(F) \otimes L(K_n). \tag{3}$$

Proof. Let F be on vertices $\{u_1, u_2, \ldots, u_m\}$ and H be on vertices $\{v_1, v_2, \ldots, v_n\}$. It follows from the construction that the adjacency matrix of $F[H]$ is $A(F[H]) = I_m \otimes A(H) + A(F) \otimes J_n$ and the degree matrix of $F[H]$ is $D(F[H]) = I_m \otimes D(H) + D(F) \otimes nI_n$. Thus,

$$\begin{aligned}
L(F[H]) &= I_m \otimes D(H) + D(F) \otimes nI_n - I_m \otimes A(H) - A(F) \otimes J_n \\
&= I_m \otimes D(H) + D(F) \otimes nI_n + I_m \otimes \Big(L(H) - D(H)\Big) + \Big(L(F) - D(F)\Big) \otimes J_n \\
&= I_m \otimes L(H) + L(F) \otimes J_n + D(F) \otimes (nI_n - J_n) \\
&= L(F\Box H) + L(F) \otimes (J_n - I_n) + D(F) \otimes L(K_n).
\end{aligned}$$

■

Our next result describes the complete Laplacian spectrum of $F[H]$ using the Laplacian spectra of F and H.

Theorem 27. *Let F be a connected graph with vertex set $\{u_1, \ldots, u_m\}$ and H be any graph of order n. Suppose that $S(F) = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ and $S(H) = (\mu_1, \mu_2, \ldots, \mu_n)$. Then*

(i) $\lambda_i n \in S(F[H])$ *for* $i = 1, \ldots, m$,

(ii) $\mu_j + d(u_i)n \in S(F[H])$ *for* $i = 1, 2, \ldots, m$ *and* $j = 2, \ldots, n$.

Thus

$$a(F[H]) = min\Big\{a(H) + \delta(F)n,\ a(F)n\Big\},$$

where $\delta(G)$ denotes the minimum degree vertices in a graph G.

Proof. Using Proposition 26

$$\begin{aligned} L(F[H]) &= L(F\Box H) + L(F) \otimes (J_n - I_n) + D(F) \otimes L(K_n) \\ &= L(F) \otimes I_n + I_m \otimes L(H) + L(F) \otimes J_n - L(F) \otimes I_n + D(F) \otimes L(K_n) \\ &= I_m \otimes L(H) + L(F) \otimes J_n + D(F) \otimes L(K_n). \end{aligned}$$

Let X_i and Y_j be eigenvectors of $L(F)$ and $L(H)$ corresponding to the eigenvalues λ_i and μ_j, respectively. Observe that for $i = 1, 2, \ldots, m$,

$$\begin{aligned} L(F[H])(X_i \otimes \mathbb{1}) &= \Big(I_m \otimes L(H) + L(F) \otimes J_n + D(F) \otimes L(K_n)\Big)(X_i \otimes \mathbb{1}) \\ &= \lambda_i n(X_i \otimes \mathbb{1}). \end{aligned}$$

Thus $\lambda_i n \in S(F[H])$ for $i = 1, \ldots, m$. Now for $i = 1, 2, \ldots, m$ and $j = 2, \ldots, n$,

$$\begin{aligned} L(F[H])(\mathbf{e}_i \otimes Y_j) &= \Big(I_m \otimes L(H) + L(F) \otimes J_n + D(F) \otimes L(K_n)\Big)(\mathbf{e}_i \otimes Y_j) \\ &= \Big(\mu_j + d(u_i)n\Big)(\mathbf{e}_i \otimes Y_j). \end{aligned}$$

Thus $\mu_j + d(u_i)n \in S(F[H])$, for $i = 1, \ldots, m$ and $j = 2, \ldots, n$. ∎

Example 28. We give an illustration of Theorem 27. Take $F = K_2$, $H = P_3$, and thus $m = 2, n = 3$ here. The graph $F[H]$ is shown in Figure 8.

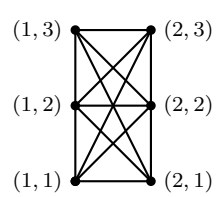


Figure 8: $K_2[P_3]$

Notice that

$$L(F) = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix};\ S(F) = \{0, 2\};\ v_{1,F} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, v_{2,F} = \begin{bmatrix} 1 \\ -1 \end{bmatrix};$$

$$L(H) = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix};\ S(H) = \{0,1,3\};\ v_{1,H} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, v_{2,H} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, v_{3,H} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix};$$

$$L(K_n) = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix};$$

$$L(F[H]) = \begin{array}{c} (1,1) \\ (1,2) \\ (1,3) \\ (2,1) \\ (2,2) \\ (2,3) \end{array} \begin{bmatrix} 4 & -1 & 0 & -1 & -1 & -1 \\ -1 & 5 & -1 & -1 & -1 & -1 \\ 0 & -1 & 4 & -1 & -1 & -1 \\ -1 & -1 & -1 & 4 & -1 & 0 \\ -1 & -1 & -1 & -1 & 5 & -1 \\ -1 & -1 & -1 & 0 & -1 & 4 \end{bmatrix} = \left[\begin{array}{ccc|ccc} 1 & -1 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & 0 & -1 & 1 \end{array}\right]$$

$$+ \left[\begin{array}{ccc|ccc} 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ 1 & 1 & 1 & -1 & -1 & -1 \\ \hline -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \\ -1 & -1 & -1 & 1 & 1 & 1 \end{array}\right] + \left[\begin{array}{ccc|ccc} 2 & -1 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & -1 & -1 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & -1 & 2 \end{array}\right]$$

$$= I_m \otimes L(H) + L(F) \otimes J_n + D(F) \otimes L(K_n)$$

Now, take $Y_2 = v_{2,H}$, the eigenvector corresponding to eigenvalue $\mu_2 = 1$ for H, and $i = 1$ so that $d(u_1) = 1$. Then repeating the computation in the third equation block in the proof, we have

$$\begin{aligned} L(F[H])(\mathbf{e}_1 \otimes Y_2) &= (I_2 \otimes L(H) + L(F) \otimes J_3 + D(F) \otimes L(K_3))(\mathbf{e}_1 \otimes v_{2,H}) \\ &= \mathbf{e}_1 \otimes 1 v_{2,H} + L(F)\mathbf{e}_1 \otimes J_3 v_{2,H} + D(F)\mathbf{e}_1 \otimes L(K_3) v_{2,H} \\ &= \mathbf{e}_1 \otimes v_{2,H} + \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 0 \\ -3 \end{bmatrix} \\ &= \begin{bmatrix} 4 \\ 0 \\ -4 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 4\mathbf{e}_1 \otimes Y_2 = \Big(\mu_2 + d(u_1)n\Big)(\mathbf{e}_1 \otimes Y_2). \end{aligned}$$

The following construction is immediate.

Corollary 29. *Let F and H be two graphs of order m and n, respectively. Then $F[H]$ is Laplacian integral if and only if both F and H are Laplacian integral.*

By Theorem 27, $a(F[H]) = min\Big\{a(H)+\delta(F)n,\ a(F)n\Big\}$. The following result tells when each of the cases occur.

Corollary 30. *Let F be a connected graph of order m and H be any graph of order n. If $F \neq K_m$, then $a(F[H]) = a(F)n$. Furthermore, if $F = K_m$, then $a(F[H]) = a(H) + (m-1)n$.*

Proof. It is known (Fiedler [**11**]) that if $F \neq K_m$, then $a(F) < \delta(F)$. Thus if F is not a complete graph, then $a(F[H]) = a(F)n$. If $F = K_m$, then $a(H)+\delta(F)n = a(H) + (m-1)n = \Big(a(H) - n\Big) + mn = \Big(a(H) - n\Big) + a(F)n \leq a(F)n$. Thus by Theorem 27, $a(F[H]) = a(H) + \delta(F)n = a(H) + (m-1)n$. ∎

The following is an immediate corollary.

Corollary 31. *Let T be a tree of order m and H be any graph of order n. Then $a(T[H]) = a(T)n$.*

The next result follows from Theorem 27 and describes the characteristic set of the lexicographic product. This is our main result of this section.

Theorem 32. *Let F and H be connected graphs on vertices $\{u_1, u_2, \ldots, u_m\}$ and $\{v_1, v_2, \ldots, v_n\}$, respectively. Let H^* be the empty graph on $V(H)$.*

1. *If $F \neq K_m$, then the following statements hold.*
   (a) $a(F[H]) = a(F)n$.
   (b) *If X is a Fiedler vector of F, then $Z = X \otimes \mathbb{1}$ is a Fiedler vector of $F[H]$.*
   (c) $\tilde{C}(F[H], X \otimes \mathbb{1}) = \tilde{C}(F, X)[H^*]$.
2. *If $F = K_m$, then the following statements hold.*
   (a) $a(F[H]) = a(H) + (m-1)n$.
   (b) *If Y is a Fiedler vector of H, then $Z = \mathbf{e}_i \otimes Y$ are Fiedler vectors of $F[H]$ for each $i = 1, \ldots, m$. Hence $\mathbb{1} \otimes Y$ is also a Fiedler vector.*
   (c) *$\tilde{C}(F[H], \mathbf{e}_i \otimes Y)$ is isomorphic to the disjoint union of $\tilde{C}(H, Y)$ with $m-1$ copies of H^*.*

Proof. Part (a) of both the items follow from Corollary 30. Part (b) of item 1 follows from the proof of Theorem 27. Following the same proof for part (b) of item 2, we see that $\mathbf{e}_i \otimes Y$ is a Fielder vector for $F[H]$ as $d(u_i) = m-1$ for each $i = 1, \ldots, m$. Hence $\mathbb{1} \otimes Y$ is also a Fiedler vector.

To prove part (c) of item 1, note first that the (u_1, v_j) entry of $X \otimes \mathbb{1}$ is $X(u_1)$ for each $j = 1, \ldots, n$. Hence if $u_1 \in \mathcal{C}(F, X)$ is a characteristic vertex, then each of the vertices (u_1, v_j), $j = 1, \ldots, n$, are characteristic vertices of $F[H]$ with respect to $X \otimes \mathbb{1}$. If $[u_1, u_2] \in \mathcal{C}(F, X)$ is a characteristic edge, then each of the edges $[(u_1, v_j), (u_2, v_k)]$, $1 \leq j, k \leq n$ is a characteristic edge of $F[H]$ with respect to $X \otimes \mathbb{1}$. Hence the characteristic graph $\tilde{C}(F[H], X \otimes \mathbb{1})$ is obtained by taking the composition of the characteristic graph $\tilde{C}(F, X)$ with H^*.

To prove part (c) of item 2, assume that we are considering $Z = \mathbf{e}_1 \otimes Y$. Notice that the value given by this vector to the vertex (u_i, v_j) is 0 for each $i = 2, \ldots, m$, $j = 1, \ldots, n$. As Y has a negative entry, say $Y(v_1) < 0$ and a positive entry, say $Y(v_2) > 0$, we see that $Z(u_1, v_1) < 0$ and $Z(u_1, v_2) > 0$. Furthermore, the vertices

(u_1, v_1) and (u_1, v_2) are adjacent to each (u_i, v_j) for $i = 2, \ldots, m$, $j = 1, \ldots, n$. Hence, each (u_i, v_j) for $i = 2, \ldots, m$, $j = 1, \ldots, n$, is a characteristic vertex of $F[H]$ with respect to Z. These vertices are the disjoint union of $m - 1$ copies of H^*. Also notice that, $Z(u_1, v_j) = Y(v_j)$ for each $j = 1, \ldots, n$. As the subgraph of $F[H]$ induced by the vertices $\{(u_1, v_1), \ldots, (u_1, v_n)\}$ is isomorphic to H, its contribution to the characteristic graph of $F[H]$ with respect to Z is isomorphic to $\tilde{C}(H, Y)$. ∎

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