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FAULT-TOLERANT DESIGNS IN TRIANGULAR LATTICE NETWORKS

Ayesha Shabbir, Tudor Zamfirescu

We present fault-tolerant designs in the form of $\mathbf{P}_k^j$-*graphs* and $\mathbf{C}_k^j$-*graphs*, in which n processing units are interlinked as parts of a triangular lattice network, and l of these n units, forming a chain or cycle of maximal length, are used to solve some task. These graphs can tolerate the failure of up to two components or communication links, keeping constant performance. We extend the results to triangular lattices on the torus and Möbius strip.

1. INTRODUCTION

1.1 Motivation and history

Suppose n processing units are interlinked, and l of these n units, forming a chain of maximal length, are used to solve some task. To have a fault-tolerant self-stable system, it is necessary that in case of failure of any single unit, another chain of l units not containing the faulty unit can replace the originally used chain. One uses graphs to represent the units and the links, see [**1**], [**5**]. This is the kind of application addressed in the present paper. Fault-tolerant designs are widely used in engineering and computer sciences.

A finite graph is called a $\mathbf{P}_k^j$-*graph* ($\mathbf{C}_k^j$-*graph*), if it is k-connected and any set of j vertices is missed by some longest path (cycle). One can interpret any such graph as a j-fault-tolerant design, which is perfectly reliable, because its performance remains of constant quality, whether or not any j of its components fail.

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A related situation is that of j-fault hamiltonicity (see e.g. [**8**], [**9**]). However, in that case the (maximal) length of the used circuit changes when failure occurs, that is, the system is not perfectly stable.

Research on $\mathbf{P}_k^j$- and $\mathbf{C}_k^j$-graphs started in 1966, when Gallai asked the question "do we have graphs in which every vertex is missed by some longest path?" [**4**]. In 1972, Zamfirescu asked for $\mathbf{P}_k^j$- and $\mathbf{C}_k^j$-graphs of small (if possible minimal) order [**14**]. The question was also asked for the special case of planar graphs. In 2001, with lattice networks in mind, he asked about the existence of such graphs in geometrical lattices [**16**]. Any answer to this question is a j-fault-tolerant design in lattice networks, also known as grid or mesh networks and widely used in distributed parallel computation, distributed control and wired circuits.

In [**7**] Nadeem, Shabbir and Zamfirescu proved the existence of $\mathbf{P}_1^1$-, $\mathbf{P}_2^1$- and $\mathbf{C}_2^1$-graphs in the infinite square lattice and hexagonal lattice of the Euclidean plane. In [**3**] Dino and Zamfirescu presented $\mathbf{P}_1^1$- and $\mathbf{P}_2^1$-graphs in the infinite equilateral triangular lattice $\mathcal{T}$. Moreover, Shabbir and Zamfirescu proved the existence of $\mathbf{P}_k^1$- and $\mathbf{C}_k^1$-graphs (for $k = 1, 2$) in finite square and hexagonal lattices on tori and Möbius strips, see [**11**].

In this note, first we prove the existence of $\mathbf{C}_2^j$-graphs (for $j = 1, 2$) in $\mathcal{T}$, and then by considering (finite) triangular lattices on the torus and on the Möbius strip we construct $\mathbf{P}_k^1$- and $\mathbf{C}_2^j$-graphs ($k = 1, 2; j = 1, 2$) there. Any finite connected graph in $\mathcal{T}$ has connectivity at most 3. For $j \geq 3$, no $\mathbf{P}_1^j$- or $\mathbf{C}_2^j$-graph (whether planar or not) is known (see [**12**], p. 79).

1.2 Auxiliary results

Let G, H and K be graphs homeomorphic to the graphs G', H' and K' in Fig. 1(a), (b) and 3(a), respectively. The graph K contains ten subgraphs isomorphic to the graph of Fig. 3(b). The variables v, x, y, z, t and w denote the number of vertices of degree 2 on paths corresponding to edges shown on the respective figures as well.

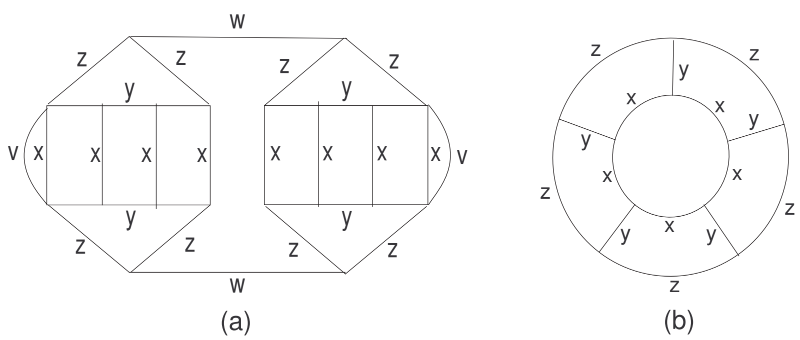

Figure 1:

We need the following lemmas.

Lemma 1. *Let $v \leq x \leq w$. The longest paths of G have empty intersection if $x + v = y + 2z + w + 1$.*

Proof. The possible longest paths of G are shown in Fig. 2, where the paths in b), c), d) and e) have equal lengths. The longest paths of G have empty intersection if the paths in Fig. 2, a) and b) are among the longest ones and consequently the paths in f), g), h), i), j), k), l), m) and n) are not longer. The lengths of the paths shown in Fig. 2 a), b), f), g), h), i), j), k), l), m) and n) are respectively

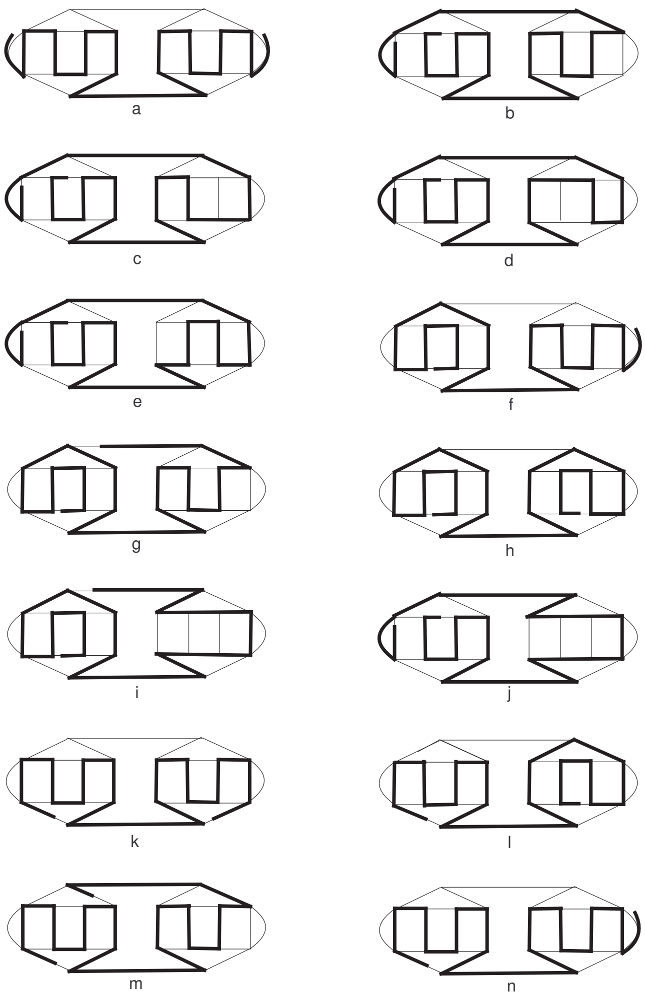

Figure 2:

$$
\begin{array}{ll}
a = 8x + 2v + 2y + 2z + w + 18, & j = 5x + v + 4y + 4z + 2w + 20, \\
b = 7x + v + 3y + 4z + 2w + 19, , & k = 8x + 2y + 4z + w + 18, \\
f = 8x + v + 3y + 4z + w + 19, & l = 8x + 3y + 5z + w + 19, \\
g = 7x + 3y + 5z + 2w + 19, & m = 7x + 2y + 5z + 2w + 19, \\
h = 8x + 4y + 6z + w + 20, & n = 8x + v + 2y + 3z + w + 18. \\
i = 5x + 4y + 5z + 2w + 20, &
\end{array}
$$

The equality $a = b$ is equivalent to $x+v = y+2z+w+1$. Moreover, $v \leq x \leq w$ implies that a is not smaller than f, g, h, i, j, k, l, m or n. $\square$

Lemma 1 also appears – without proof – in [**7**].

Lemma 2. [**9**] *The longest cycles of H have empty intersection if $2y \geq 3x + 1$.*

Lemma 3. *Any two vertices of K are missed by some longest cycle of K if $2x \geq y + 2z + 1$ and $2t = x + y + 3z + 3w + 8$.*

The proof of Lemma 3 is similar to that of Lemma 1 and will therefore be omitted.

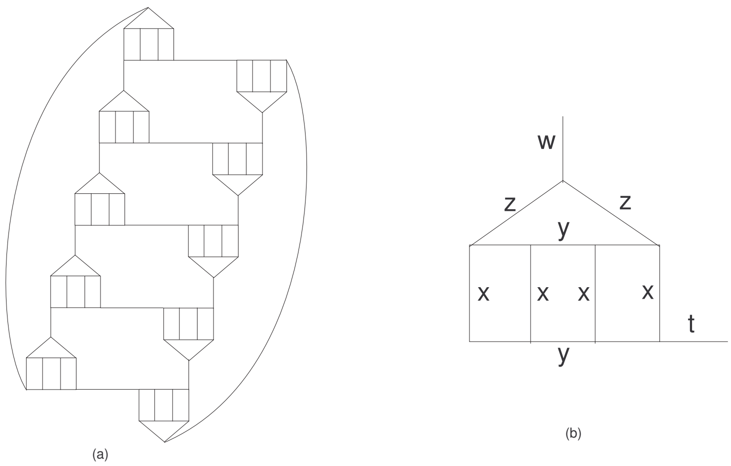

Figure 3:

2. MAIN RESULTS

Now, we present our main results.

2.1 Triangular lattice in the plane

In this section we prove the existence of $\mathbf{C}_2^j$-graphs (for $j = 1, 2$) in the infinite triangular lattice $\mathcal{T}$.

Theorem 1. *In $\mathcal{T}$ we have a $\mathbf{C}_2^1$-graph of order* 60.

Proof. The conditions of Lemma 2 are satisfied for $x = 0$ and $y = 10$. We are led to a graph of order 60 which is embeddable in $\mathcal{T}$, see Fig. 4. ☐

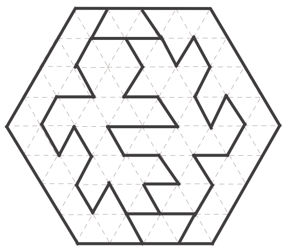

Figure 4: A $\mathbf{C}_2^1$-graph in $\mathcal{T}$.

Theorem 2. *There exists a $\mathbf{C}_2^2$-subgraph of $\mathcal{T}$ of order* 375.

Proof. For $y = 1, z = 0, x = 2, t = 19$ and $w = 9$, the conditions of Lemma 3 are verified and the corresponding graph K is of order 375 and has the desired property. Fig. 5 reveals an embedding of K in $\mathcal{T}$. ☐

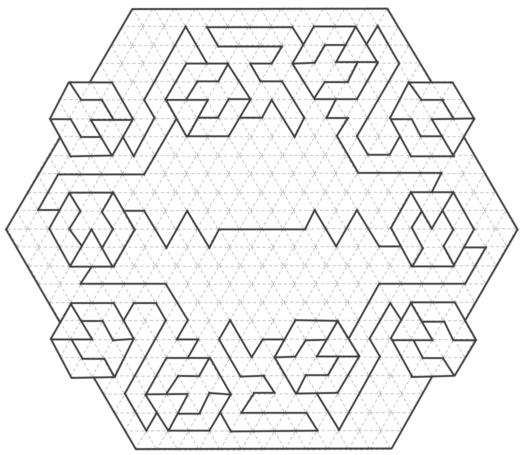

Figure 5: A $\mathbf{C}_2^2$-graph in $\mathcal{T}$.

Once the existence of a certain $\mathbf{P}_k^j$- or $\mathbf{C}_k^j$-graph in the plane was established, and small examples have been found, one can be interested in embeddings on surfaces of higher genus, provided the graphs have smaller order. This is what we do in the next two sections.

2.2 Toroidal triangular lattices

We start with the definition of a toroidal triangular lattice.

We consider an $(m+1)\times(n+1)$ parallelogram (with $(m+1)(n+1)$ vertices) in $\mathcal{T}$. By identifying opposite vertices on the boundary as indicated on Fig. 6, we obtain the *toroidal triangular lattice* $\mathcal{T}^T_{m,n}$. It has mn vertices.

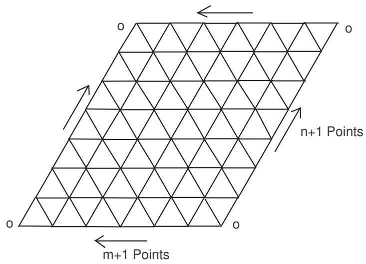

Figure 6:

In the rest of the section we present $\mathbf{P}^1_k$-subgraphs (for $k = 1,2$) and $\mathbf{C}^j_2$-subgraphs (for $j = 1,2$) of triangular toroidal lattices.

From Theorem 1 in [**11**] we deduce the existence of a spanning $\mathbf{P}^1_1$-graph in $\mathcal{T}^T_{5,4}$ of order 20. But we can do better.

Walther and Voss [**13**] and Zamfirescu [**15**] independently exhibited a $\mathbf{P}^1_1$-graph of order 12, see Fig. 7(a), which remained to date the smallest known $\mathbf{P}^1_1$-graph.

Theorem 3. *There exists a spanning* $\mathbf{P}^1_1$*-subgraph of* $\mathcal{T}^T_{4,3}$.

Proof. An embedding of the graph of Fig. 7(a) in $\mathcal{T}^T_{4,3}$ is shown in Fig. 7(b). $\square$

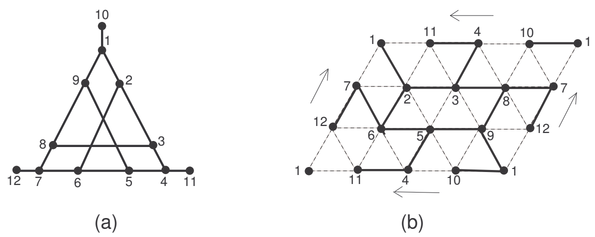

Figure 7: A spanning $\mathbf{P}^1_1$-graph in $\mathcal{T}^T_{4,3}$.

The existence of a $\mathbf{P}^1_2$-graph in $\mathcal{T}^T_{10,10}$ follows from Theorem 2 in [**11**]. That graph has 80 vertices.

Theorem 4. *There exists a planar $\mathbf{P}_2^1$-graph in $\mathcal{T}_{7,8}^T$ of order* 48.

Proof. It can easily be checked that the values $y = 1, z = 0$, and $x = v = w = 2$ satisfy the conditions of Lemma 1. The resulting graph G is a planar $\mathbf{P}_2^1$-graph of order 48. Fig. 8 presents an embedding of G in the toroidal lattice $\mathcal{T}_{7,8}^T$. ☐

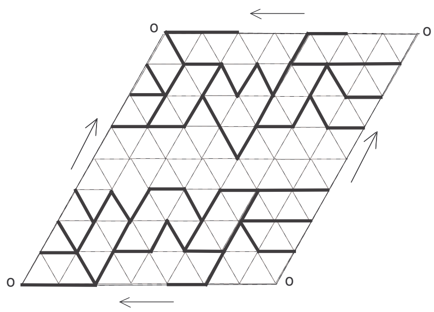

Figure 8: A $\mathbf{P}_2^1$-subgraph of $\mathcal{T}_{7,8}^T$.

Theorem 5. *In $\mathcal{T}_{5,3}^T$ we have a spanning planar $\mathbf{C}_2^1$-graph.*

Proof. Since the toroidal square lattice $\mathcal{L}_{5,3}^T$ (for a definition, see [**11**]) is a spanning subgraph of $\mathcal{T}_{5,3}^T$, and in $\mathcal{L}_{5,3}^T$ we have a spanning planar $\mathbf{C}_2^1$-graph (see [**11**]), the theorem follows, see Fig. 9. ☐

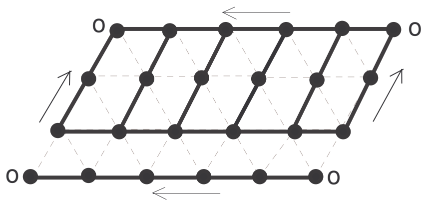

Figure 9: A spanning $\mathbf{C}_2^1$-subgraph of $\mathcal{L}_{5,3}^T$.

Regarding $\mathbf{C}_k^2$-graphs, we present the following existence result.

Theorem 6. *In $\mathcal{T}_{25,11}^T$ there exists a planar $\mathbf{C}_2^2$-graph of order* 235.

Proof. To obtain a graph as required we will use Lemma 3 once again. Now we take $y = 1$, $z = 0$, $x = 2$, $t = 7$ and $w = 1$, which satisfy the conditions of the lemma, and the resulting graph K is a planar $\mathbf{C}_2^2$-graph of order 235. Fig. 10 presents an embedding of K in $\mathcal{T}_{25,11}^T$. ☐

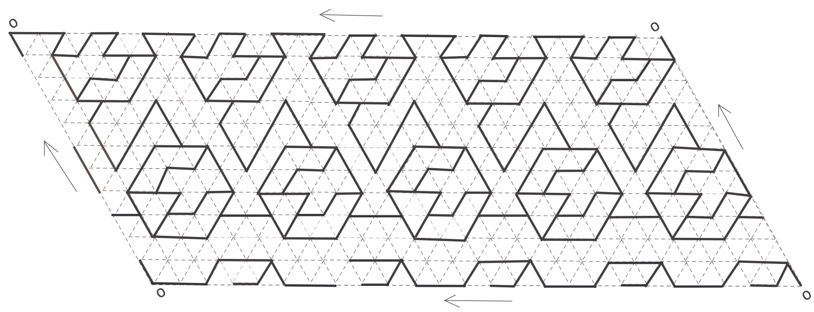

Figure 10: A $\mathbf{C}_2^2$-subgraph of $\mathcal{T}_{25,11}^T$.

2.3 Triangular lattices on Möbius strips

We define the lattice $\mathcal{T}_{m,n}^M$ on the Möbius strip, according to Fig. 11(a) and (b). When n is odd it has mn vertices and for an even value of n its order is $\frac{n}{2}(2m+1)$.

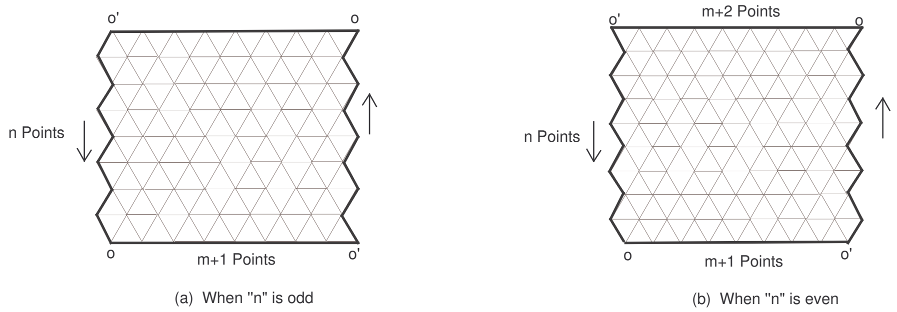

Figure 11:

Theorem 7. *There exists a planar* $\mathbf{P}_1^1$*-subgraph of* $\mathcal{T}_{4,4}^M$ *of order* 17.

Proof. This follows from Theorem 6 in [**11**], and uses the smallest known planar $\mathbf{P}_1^1$-graph (see Fig. 12(a)), which was found by Schmitz [**10**]. See Fig. 12(b). ☐

From Theorem 7 in [**11**] we know that there exists a planar $\mathbf{P}_2^1$-graph of order 112 in $\mathcal{T}_{10,14}^M$. The following result presents a smaller example.

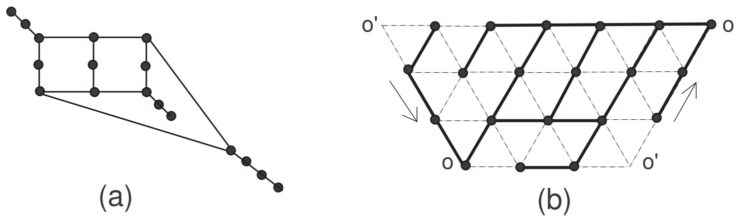

Figure 12: A $\mathbf{P}_1^1$-graph and an embedding of it in $\mathcal{T}_{4,4}^M$.

Theorem 8. *In $\mathcal{T}_{8,11}^M$ we have a planar $\mathbf{P}_2^1$-graph of order* 64.

Proof. Let us take $y = 0, z = 1$ and $x = v = w = 3$ in Lemma 1. The conditions of the lemma are verified for these values and the resulting graph G is a planar $\mathbf{P}_2^1$-graph of order 64. Fig. 13 illustrates an embedding of G in $\mathcal{T}_{8,11}^M$. $\square$

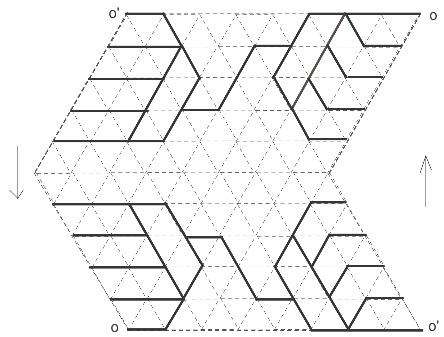

Figure 13: A $\mathbf{P}_2^1$-subgraph of $\mathcal{T}_{8,11}^M$.

Theorem 9. *The lattice $\mathcal{T}_{4,3}^M$ contains a spanning $\mathbf{C}_2^1$-graph.*

Proof. This follows from Theorem 8 in [**9**]. The graph used is presented by Zamfirescu, see Fig. 14(a)[**6**]. For the embedding, see Fig. 14(b). $\square$

The existence of a $\mathbf{C}_2^2$-graph in a triangular lattice on the Möbius strip is established in the next theorem.

Theorem 10. *In $\mathcal{T}_{12,33}^M$ we have a planar $\mathbf{C}_2^2$-graph of order* 235.

Proof. The graph K of Theorem 6 is also embeddable in $\mathcal{T}_{12,33}^M$, which is shown in Fig. 15. $\square$

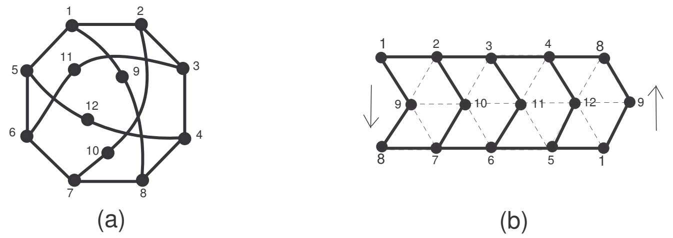

Figure 14: A $\mathbf{C}_2^1$-graph and an embedding of it in $\mathcal{T}_{4,3}^M$.

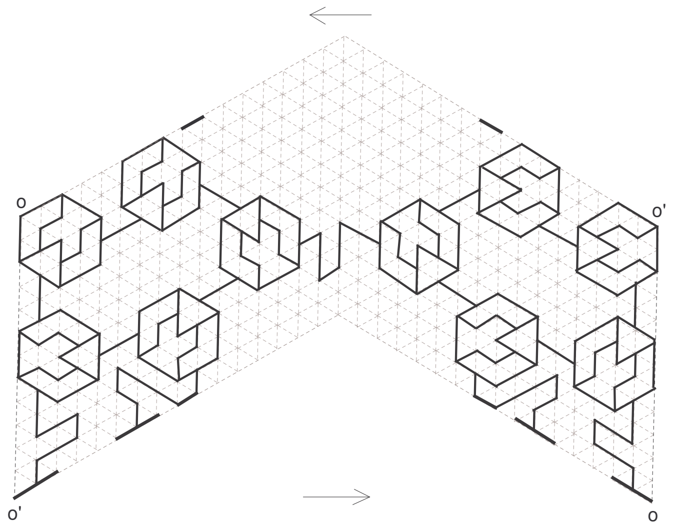

Figure 15: A $\mathbf{C}_2^2$-subgraph of $\mathcal{T}_{12,33}^M$.

We conclude this paper with the following open problem.
Problem: Do there exist $\mathbf{P}_2^2$-graphs in $\mathcal{T}$?

For $\mathbf{P}_2^1$-graphs in various triangular lattice networks, see [**2**].

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