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MORE ON GRAPHS WITH JUST THREE DISTINCT EIGENVALUES

Peter Rowlinson

In honour of Dragoš Cvetković on the occasion of his 75th birthday.

Let G be a connected non-regular non-bipartite graph whose adjacency matrix has spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$, where $k, l \in \mathbb{N}$ and $\rho > \mu > \lambda$. We show that if μ is non-main then $\delta(G) \geq 1 + \mu - \lambda\mu$, with equality if and only if G is of one of three types, derived from a strongly regular graph, a symmetric design or a quasi-symmetric design (with appropriate parameters in each case).

1. INTRODUCTION

Let G be a graph of order n with $(0, 1)$-adjacency matrix A. An eigenvalue σ of A is said to be an eigenvalue of G, and σ is a *main* eigenvalue if the eigenspace $\mathcal{E}_A(\sigma)$ is not orthogonal to the all-1 vector in $\mathbb{R}^n$. Always the largest eigenvalue, or *index*, of G is a main eigenvalue, and it is the only main eigenvalue if and only if G is regular. We say that G is an *integral* graph if every eigenvalue of G is an integer; and G is a *biregular* graph if it has just two different degrees. We use the notation of the monograph [**6**], where the basic properties of graph spectra can be found in Chapter 1.

Let $\mathcal{C}_1$ be the class of connected graphs with just three distinct eigenvalues, and let $\mathcal{C}_2$ be the class of connected graphs with exactly two main eigenvalues. It is an open problem to determine all the graphs in $\mathcal{C}_1$, and another open problem to determine all the graphs in $\mathcal{C}_2$. Here we continue the investigation of graphs in

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$\mathcal{C}_1 \cap \mathcal{C}_2$ begun in [**14**]. Independently the authors of [**3**] investigated the biregular graphs in $\mathcal{C}_1$, and it is not difficult to see that these are precisely the graphs in $\mathcal{C}_1 \cap \mathcal{C}_2$: this follows from [**3**, Theorem 4.3(i)] and [**14**, Lemma 2.2]. Some examples of biregular graphs in $\mathcal{C}_2 \setminus \mathcal{C}_1$ (of order 16) can be found in [**10**, Table 1].

In [**14**] it was noted that any graph G in $\mathcal{C}_1 \cap \mathcal{C}_2$ is either integral or complete bipartite. Examples include the following, where G has spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$ with $\rho > \mu > \lambda$.

(1) G is of *conical* type: here G is the cone over a strongly regular graph with parameters $\lambda^2\mu + \lambda^2 - \lambda\mu, \mu - \lambda\mu, 2\mu + \lambda, \mu$ (see [**12, 14**]);

(2) G is of *symmetric type*: here G is obtained from the incidence graph of a symmetric 2-$(q^3 - q + 1, q^2, q)$ design by adding all edges between blocks (see [**7**]);

(3) G is of *affine* type: here G is obtained from the incidence graph of an affine 2-$(q^3, q + 1, q^2, q)$ design by adding all edges between intersecting blocks (see [**8**]).

There are infinitely many graphs of each type. The graphs of conical type include the cones $K_1 \bigtriangledown H$, where H is the Petersen graph ($\mu = 1,\ \lambda = -2$), the Gewirtz graph ($\mu = 2,\ \lambda = -4$) or a Chang graph ($\mu = 4,\ \lambda = -2$). For graphs of symmetric type we have $\rho = q^3$, $\mu = q - 1$ and $\lambda = -q$, and for graphs of affine type we have $\rho = q^3 + q^2 + q$, $\mu = q$ and $\lambda = -q$. If G is a graph of any of these three types then μ is the non-main eigenvalue, and the minimum degree $\delta(G)$ is equal to $1 + \mu - \lambda\mu$.

To formulate a partial converse implicit in [**14**, Section 4] we extend the construction in (3) to any quasi-symmetric design which has 0 as one of its two intersection numbers. If $\mathcal{D}$ is such a design then we denote by $G_{\mathcal{D}}$ the graph obtained from the incidence graph of $\mathcal{D}$ by adding all edges between intersecting blocks: thus $G_{\mathcal{D}}$ is the the union of the block graph of $\mathcal{D}$ with the incidence graph of $\mathcal{D}$. We refer to the graphs $G_{\mathcal{D}}$ as graphs of *quasi-symmetric* type. It follows from [**14**, Theorem 4.2] that if μ is non-main and $\delta(G) = 1 + \mu - \lambda\mu$ then G is of conical, symmetric or quasi-symmetric type; moreover if G is of quasi-symmetric type, with $G = G_{\mathcal{D}}$, then the parameters of $\mathcal{D}$ are specified rational functions of λ and μ, and the intersection numbers are 0 and $-\lambda$. (We have strong integrality conditions as a consequence.) In Section 3 we show conversely that for any quasi-symmetric design $\mathcal{D}$ with these parameters and intersection numbers, the graph $G_{\mathcal{D}}$ lies in $\mathcal{C}_1 \cap \mathcal{C}_2$, with μ non-main and $\delta(G_{\mathcal{D}}) = 1 + \mu - \lambda\mu$. We shall see also that $\delta(G) \geq 1 + \mu - \lambda\mu$ whenever $G \in \mathcal{C}_1 \cap \mathcal{C}_2$ and μ is non-main. (For an example with strict inequality here we may take G to be the unique maximal exceptional graph of order 36 [**5**, Section 6.1]: for this graph we have $\delta(G) = 18$, $\mu = 5$ and $\lambda = -2$.)

To state one further observation, recall from [**1**] that a *multiplicative* graph is a connected graph whose distinct eigenvalues are $\rho, \mu, -\mu$, where $\rho > \mu > 0$. For such graphs, $\lambda = -\mu$, and in Section 4 we show that if $\delta(G) = 1 + \mu + \mu^2$ then $-\mu$ is a main eigenvalue. We deduce that the only multiplicative graphs in $\mathcal{C}_1 \cap \mathcal{C}_2$ with $\delta(G) = 1 + \mu + \mu^2$ are (a) the graphs of affine type, and (b) the cones over a strongly regular graph with parameters $\mu^3 + 2\mu^2, \mu^2 + \mu, \mu, \mu$. It was already shown in [**1**] that the graphs of type (b) lie in $\mathcal{C}_1$.

2. PRELIMINARIES

Here we note three results required in subsequent sections.

Lemma 2.1 [**14**, Lemma 2.2]. *A graph G in $\mathcal{C}_1 \cap \mathcal{C}_2$ has exactly two distinct degrees (say d_1, d_2), and these degrees determine an equitable bipartition of G. Moreover, if G has spectrum $\rho, \mu^{(k)}, \lambda^{(l)}$, where $\rho > \mu > \lambda$, then $d_h = \alpha_h^2 - \lambda\mu$, where $\alpha_h > 0$ $(h = 1, 2)$ and either*
(a) μ is non-main and $\alpha_1\alpha_2 = -\lambda(\mu + 1)$, or
(b) λ is non-main and $\alpha_1\alpha_2 = -\mu(\lambda + 1)$.

Next recall that a quasi-symmetric design is a 2-design $\mathcal{D}$ for which the number of points in the intersection of two blocks takes exactly two values. We say that $\mathcal{D}$ is a quasi-symmetric (a, b, c, d, e)-design with intersection numbers α and β $(\alpha < \beta)$ if it has a points, b blocks, each point lies in c blocks, each block contains d points, any two points lie in exactly e blocks, and two blocks intersect in α or β points. The *block graph* of $\mathcal{D}$ has the blocks of $\mathcal{D}$ as its vertices, with blocks adjacent if and only if they intersect in β points.

Theorem 2.2 [**11**, Theorem 8.3.14]. *The block graph of a quasi-symmetric (a, b, c, d, e)-design with intersection numbers α and β $(\alpha < \beta)$ is strongly regular with spectrum $\xi, \eta^{(a-1)}, \zeta^{(b-a)}$, where*

$$\xi = \frac{(c-1)d - \alpha(b-1)}{\beta - \alpha}, \quad \eta = \frac{c - d - e + \alpha}{\beta - \alpha}, \quad \zeta = \frac{\alpha - d}{\beta - \alpha}.$$

To formulate [**14**, Theorem 4.2(c)] in terms of a quasi-symmetric design, we define

$$(1) \quad f_1(\lambda, \mu) = \frac{\lambda(\lambda + \lambda\mu - \lambda^2\mu + \mu)}{\mu + 1}, \quad f_2(\lambda, \mu) = \frac{(1 + \mu - \lambda\mu)(\lambda + \lambda\mu - \lambda^2\mu + \mu)}{\lambda(\mu + 1)}.$$

Now we have

Theorem 2.3 [**14**, Theorem 4.2]. *Let G be a connected non-regular non-bipartite graph whose distinct eigenvalues are ρ, μ, λ where $\rho > \mu > \lambda$. If μ is non-main and $\delta(G) = 1 + \mu - \lambda\mu$ then G is of conical, symmetric or quasi-symmetric type. Moreover, if G is of quasi-symmetric type then $G = G_{\mathcal{D}}$ where $\mathcal{D}$ is a quasi-symmetric $(f_1(\lambda, \mu), f_2(\lambda, \mu), 1 + \mu - \lambda\mu,\ \lambda^2,\ 1 + \mu)$-design with intersection numbers 0 and $-\lambda$.*

3. GRAPHS OF QUASI-SYMMETRIC TYPE

Our main objective in this section is to prove the converse of Theorem 2.3. It has already been noted that the hypotheses are satisfied if G is of conical or symmetric type. Accordingly suppose that $G = G_{\mathcal{D}}$, where $\mathcal{D}$ is a quasi-symmetric

$(f_1(\lambda,\mu), f_2(\lambda,\mu), 1+\mu-\lambda\mu,\ \lambda^2,\ 1+\mu)$-design with intersection numbers 0 and $-\lambda$. Note that μ is a positive integer and λ is a negative integer. We show first that the distinct eigenvalues of G are ρ, μ, λ, where $\rho = -\lambda(1+\mu-\lambda\mu)$.

Let V_1 be the set of blocks of $\mathcal{D}$, let V_2 be the set of points of $\mathcal{D}$, and let G_1 be the block graph of $\mathcal{D}$, that is, the graph induced by V_1. From Theorem 2.2 we know that G_1 is strongly regular with eigenvalues ν, $\lambda+\mu$, λ, where $\nu = \lambda\mu(\lambda-1)$. Now let $m_1(x) = (x-\nu)\ (x-\lambda-\mu)(x-\lambda)$, $m(x) = (x-\rho)(x-\mu)(x-\lambda)$. Let G have adjacency matrix $A = \begin{pmatrix} A_1 & B^\top \\ B & O \end{pmatrix}$, and let $m(A) = \begin{pmatrix} X & Y^\top \\ Y & Z \end{pmatrix}$, both partitioned in accordance with $V_1 \dot\cup V_2$.

From the parameters and intersection numbers of $\mathcal{D}$ we see that $B^\top B = \lambda^2 I - \lambda A_1$ and $BB^\top = -\lambda\mu I + (\mu+1)J$, where each entry of J is 1. On expanding $m(A)$ and substituting for ρ, $B^\top B$ and $BB^\top$, we find that $X = m_1(A_1)$, and so $X = O$. Again, $Y^\top = (A_1^2 - (\lambda+\mu+\nu)A_1 + (\lambda+\mu)\nu I)B^\top$, from which it follows that $Y^\top B = -\lambda m_1(A_1) = O$. Since $BB^\top$ is invertible, we have $Y^\top = O$.

Further calculation shows that $Z = BA_1B^\top - \mu(1-\lambda+\lambda^2)(1+\mu)J + \lambda\mu(\lambda+\mu)I$. In what follows, we write $\mathbf{j}$ for an all-1 vector, its length determined by context. We have $BA_1B^\top\mathbf{j} = \lambda^2 BA_1\mathbf{j} = \lambda^2\nu B\mathbf{j} = \lambda^3\mu(\lambda-1)\ (1+\mu-\lambda\mu)\mathbf{j}$, and it follows that

$$Z\mathbf{j} = \lambda^3\mu(\lambda-1)(1+\mu-\lambda\mu)\mathbf{j} - \mu(1-\lambda+\lambda^2)(1+\mu)f_1(\lambda,\mu)\mathbf{j} + \lambda\mu(\lambda+\mu)\mathbf{j} = \mathbf{0}.$$

Now $BB^\top$ commutes with A_1 and hence with $BA_1B^\top$. The matrices $BB^\top$ and $BA_1B^\top$ are therefore simultaneously diagonalizable. Since $\mathbf{j}$ is an eigenvector of both of these matrices, there exist $f_1(\lambda,\mu)$ pairwise orthogonal eigenvectors, including $\mathbf{j}$, which are eigenvectors of both $BB^\top$ and $BA_1B^\top$. Let $\mathbf{x}$ be such a vector orthogonal to $\mathbf{j}$. We have $Z\mathbf{x} = BA_1B^\top\mathbf{x} + \lambda\mu(\lambda+\mu)\mathbf{x}$ and $BB^\top\mathbf{x} = -\lambda\mu\mathbf{x}$. Now $\lambda\mu B^\top\mathbf{x} = (B^\top B)B^\top\mathbf{x} = (\lambda^2-\lambda)A_1B^\top\mathbf{x}$, whence $A_1B^\top\mathbf{x} = (\lambda+\mu)B^\top\mathbf{x}$ and $BA_1B^\top\mathbf{x} = -\lambda\mu(\lambda+\mu)\mathbf{x}$. It follows that Z annihilates all $f_1(\lambda,\mu)$ pairwise orthogonal eigenvectors, and hence that $Z = O$. We deduce that $m(A) = 0$ and hence that the distinct eigenvalues of G are ρ, λ, μ as required. Now we can prove:

Theorem 3.1. *Let G be a connected non-regular non-bipartite graph whose distinct eigenvalues are ρ, μ, λ where $\rho > \mu > \lambda$. If μ is non-main then $\delta(G) \geq 1+\mu-\lambda\mu$ with equality if and only if G is of conical, symmetric or quasi-symmetric type $G_{\mathcal{D}}$, where $\mathcal{D}$ is a quasi-symmetric $(f_1(\lambda,\mu), f_2(\lambda,\mu), 1+\mu-\lambda\mu,\ \lambda^2,\ 1+\mu)$-design with intersection numbers 0 and $-\lambda$. (The functions f_1, f_2 are defined in Eq.(1).)*

Proof. As noted in [**7, 14**], the adjacency matrix A of G satisfies the equation

$$(A-\mu I)(A-\lambda I) = \mathbf{a}\mathbf{a}^\top, \tag{2}$$

where $A\mathbf{a} = \rho\mathbf{a}$ and each entry of $\mathbf{a}$ is positive. By Lemma 2.1, G has two degrees, say d_1 and d_2. Moreover $d_h = \alpha_h^2 - \lambda\mu$ $(h = 1,2)$ where, with a suitable ordering of vertices, $\mathbf{a} = \begin{pmatrix} \alpha_1\mathbf{j} \\ \alpha_2\mathbf{j} \end{pmatrix}$, $\alpha_1 > \alpha_2$ and $\alpha_1\alpha_2 = -\lambda(\mu+1)$. Let i, j be adjacent vertices with $\deg(i) = d_1$, $\deg(j) = d_2$. From Eq.(2), the number of i-j walks of length 2 is

$\alpha_1\alpha_2 + \lambda + \mu$, that is, $-\lambda\mu + \mu$. Now $-\lambda\mu + \mu \le \deg(j) - 1 = \delta(G) - 1$, and so $\delta(G) \ge 1+\mu-\lambda\mu$.

In the light of Theorem 2.3 and the argument above, it remains to show that when $G = G_{\mathcal{D}}$, the eigenvalue μ is non-main and $\delta(G) = 1+\mu-\lambda\mu$. In this case let $\begin{pmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{pmatrix}$ be the divisor matrix D determined by the equitable partition $V_1 \dot\cup V_2$, where V_1 is the set of blocks of $\mathcal{D}$ and V_2 is the set of points of $\mathcal{D}$. We have $r_{11} = \nu = \lambda\mu(\lambda - 1)$ and $r_{22} = 0$, and so D has trace $\rho + \lambda$. It follows from [**14**, Theorem 3.9.9] that G has exactly two main eigenvalues; moreover, since ρ is one main eigenvalue of G, the other is λ.

Finally, the vertices in V_1 have degree $\nu + \lambda^2 = -\lambda(-\lambda - \lambda\mu + \mu)$, while those in V_2 have the smaller degree $1 + \mu - \lambda\mu$. Thus $\delta(G) = 1 + \mu - \lambda\mu$ and the proof of the theorem is complete. □

4. FURTHER REMARKS

(1) A question which remains is whether there exist any graphs $G_{\mathcal{D}}$ of quasi-symmetric type that are not of affine type. As noted in [**14**], any candidate has $\lambda + \mu > 0$; moreover λ and μ are not coprime. The candidate with smallest $\mu - \lambda$ has order 625, with $\mu = 9$ and $\lambda = -6$. By Theorem 3.1 such a graph exists if and only if there exists a quasi-symmetric $(225, 400, 64, 36, 10)$-design $\mathcal{D}$. Then G_1, the block graph of $\mathcal{D}$, is strongly regular with parameters $400, 378, 357, 360$; according to [**2**], the existence of such a strongly regular graph is unknown. We note that the parameters of G_1 and $\mathcal{D}$ satisfy the four necessary conditions for the existence of $\mathcal{D}$ formulated in [**13**, Theorem 6].

(2) The authors of [**9**] investigate the strongly regular graphs G with a *strongly regular decomposition*. This means that $V(G) = V_1 \dot\cup V_2$, where (for $i = 1, 2$) the graph G_i induced by V_i is a strongly regular graph or a clique or a co-clique: then $V_1 \dot\cup V_2$ is an equitable partition. In the case that one of G_1, G_2 is a co-clique, the decomposition is said to be *improper*.

One can seek to generalize the results of [**9**] by requiring instead that G is a graph in $\mathcal{C}_1$ with an equitable bipartition and an associated strongly regular decomposition. If G is not strongly regular then $G \in \mathcal{C}_1 \cap \mathcal{C}_2$, and the graphs of Theorem 3.1 are examples of such a graph with an improper strongly regular decomposition.

Now suppose that G is a graph in $G \in \mathcal{C}_1 \cap \mathcal{C}_2$ having an improper strongly regular decomposition, with G_2 a non-trivial co-clique and G_1 a strongly regular graph with parameters n_1, r_1, e_1, f_1. We make two comments using the notation of Section 3. First, μ is the non-main eigenvalue of G for otherwise $-r_{12}r_{21} = \det(D) = \rho\mu > 0$, a contradiction. Secondly, if $i, j \in V_1$ $(i \ne j)$ then by Eq.(2) the (i, j)-entry of A^2 is given by

$$a_{ij}^{(2)} = \begin{cases} \alpha_1^2 & \text{if } i \not\sim j, \\ \alpha_1^2 + \lambda + \mu & \text{if } i \sim j. \end{cases}$$

Similarly, if $i, j \in V_2$ $(i \neq j)$ we have $a_{ij}^{(2)} = \alpha_2^2$. It follows that the neighbourhoods $\{h \in V_2 : h \sim i\}$ $(i \in V_1)$ form either a symmetric 2-design or a quasi-symmetric (a, b, c, d, e)-design with intersection numbers α, β, where $a = n_2$, $b = n_1$, $c = r_{21}$, $d = r_{12}$, $e = \alpha_2^2$, $\alpha = \alpha_1^2 - f_1$ and $\beta = \alpha_1^2 + \lambda + \mu - e_1$ (cf. [**9**, Theorem 2.9]).

(3) Our final remark concerns multiplicative graphs, for which $\lambda = -\mu$. Let G be a connected non-regular graph with spectrum $\rho, \mu^{(k)}, -\mu^{(l)}$, where $\rho > \mu > -\mu$ and one eigenvalue is non-main. Suppose that $\delta(G) = 1 + \mu + \mu^2$.

If $-\mu$ is non-main, then we obtain a contradiction as follows. In the notation of Section 3, we have $\alpha_2^2 = 1 + \mu$ as before, and $\alpha_1\alpha_2 = \mu(\mu - 1)$ by Lemma 2.1. Then $\alpha_1^2 = \mu^2(\mu - 1)^2/(\mu + 1)$. Since $\alpha_1 \neq 0$, we deduce that $\mu = 3$. Then $\alpha_1 = 3$, $\alpha_2 = 2$, $d_1 = 18$, $d_2 = 13$. Now $d_2 < \rho < d_1$, while $\rho + k\mu + l(-\mu) = 0$. Hence 3 divides ρ, and so $\rho = 15$.

From $A\mathbf{a} = \rho\mathbf{a}$ we have $\rho\alpha_1 = r_{11}\alpha_1 + r_{12}\alpha_2$ and $\rho\alpha_2 = r_{21}\alpha_1 + r_{22}\alpha_2$. Since also $r_{11} + r_{12} = d_1$ and $r_{21} + r_{22} = d_2$ we deduce that

$$r_{12} = \frac{\alpha_1(d_1 - \rho)}{\alpha_1 - \alpha_2} = 9, \quad r_{21} = \frac{\alpha_2(d_2 - \rho)}{\alpha_2 - \alpha_1} = 4.$$

Hence $9n_1 = 4n_2$, where $n_h = |V_h|$ $(h = 1, 2)$. Also $\|\mathbf{a}\|^2 = \rho^2 - \mu^2$, whence $9n_1 + 4n_2 = 216$. We deduce that $n_1 = 12$, $n_2 = 27$, $n = 39$. From the equations $15 + 3k - 3l = 0$, $1 + k + l = 39$, we obtain the contradiction $6 + 2k = 39$. This non-existence result can also be extracted from the results in [**3**].

We conclude that $-\mu$ is a main eigenvalue, and so μ is the non-main eigenvalue when $G \in \mathcal{C}_1 \cap \mathcal{C}_2$. We can then apply Theorem 3.1 with $\lambda = -\mu$. Then G is of conical or quasi-symmetric type. In the latter case, $f_1(-\mu, \mu) = \mu^3$ and $f_2(-\mu, \mu) = \mu^3 + \mu^2 + \mu$; moreover the block graph G_1 is strongly regular with parameters $\mu^3 + \mu^2 + \mu, \mu^3 + \mu^2, \mu^3 + \mu^2 - \mu, \mu^3 + \mu^2$. Thus $G_1 = \overline{(1 + \mu + \mu^2)K_\mu}$. It follows that G is of affine type because the blocks are partitioned into $1 + \mu + \mu^2$ parallel classes, with blocks taken as parallel when they are disjoint (see [**4**, Theorem 1.44]). Accordingly we have:

Proposition 4.1. *let G be a connected graph whose distinct eigenvalues are $\rho, \mu, -\mu$, where $\rho > \mu > -\mu$ and just one eigenvalue is non-main. Then $\delta(G) = 1 + \mu + \mu^2$ if and only if G is either (a) a graph of affine type, or (b) the cone over a strongly regular graph with parameters $\mu^3 + 2\mu^2, \mu^2 + \mu, \mu, \mu$.*

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