Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **x** (xxxx), xxx–xxx.
https://doi.org/10.2298/AADM161001004M

COMBINATORIAL IDENTITIES FOR APPELL POLYNOMIALS

Emanuele Munarini

Using the techniques of the modern umbral calculus, we derive several combinatorial identities involving s-Appell polynomials. In particular, we obtain identities for classical polynomials, such as the Hermite, Laguerre, Bernoulli, Euler, Nörlund, hypergeometric Bernoulli, and Legendre polynomials. Moreover, we obtain a generalization of the Carlitz identity for Bernoulli numbers and polynomials to arbitrary symmetric s-Appell polynomials.

1. INTRODUCTION

An *s-Appell polynomial sequence*, with $s \neq 0$, is a polynomial sequence $\{p_n(x)\}_{n\in\mathbb{N}}$ generated by the formal exponential series

$$p(x;t) = \sum_{n\geq 0} p_n(x)\, \frac{t^n}{n!} = g(t)\, \mathrm{e}^{sxt} \tag{1}$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ is an exponential series with $g_0 = 1$. So, $p_n(x)$ is a polynomial of degree n and

$$p_n(x) = \sum_{k=0}^{n} \binom{n}{k} g_{n-k} s^k x^k .$$

When $s = 1$, we have the usual *Appell polynomials* [**1**] [**27**, p. 86] [**28**]. In the literature, there are several classical sequences of this kind. Here below, we recall some of them. Other examples can be found in [**7**].

2010 Mathematics Subject Classification. 05A19, 05A40, 05A15.
Keywords and Phrases. Combinatorial sums, umbral calculus, orthogonal polynomials, formal series, generating functions.

1. The *ordinary powers* x^n , with exponential generating series e^{xt} .
2. The *generalized Hermite polynomials* $H_n^{(\nu)}(x)$, [**9**, Vol. 2, p. 192], with exponential generating series

$$\sum_{n\geq 0} H_n^{(\nu)}(x)\,\frac{t^n}{n!} = e^{2xt-\nu t^2} = e^{-\nu t^2}\,e^{2xt} \qquad (s=2)\,.$$

   For $\nu = 1$, we have the ordinary *Hermite polynomials* $H_n(x)$. Moreover, from [**9**, p. 250], it is easy to see that also the polynomials

$$K_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{n!}{k!} \frac{H_k(\sqrt{x})^2}{2^k} = \sum_{k=0}^{n} \binom{n}{k} \binom{2k}{k} k! 2^{n-2k} x^{n-k}$$

   form a 2-Appell sequence, having generating exponential series

$$\sum_{n\geq 0} K_n(x)\frac{t^n}{n!} = \frac{e^{2xt}}{\sqrt{1-2t}}\,.$$

3. The *Laguerre polynomials* $L_n^{(\alpha-n)}(x)$, [**9**, Vol. 2, p. 189, Formula (19)], with exponential generating series

$$\sum_{n\geq 0} L_n^{(\alpha-n)}(x)\,\frac{t^n}{n!} = (1+t)^{\alpha}\,e^{-xt} \qquad (s=-1)$$

   and the polynomials $\frac{x^n L_n^{(n)}(1/x)}{\Gamma(n+\alpha+1)}$, [**9**, Vol. 2, p. 189, Formula (18)], with exponential generating series

$$\sum_{n\geq 0} \frac{x^n L_n^{(\alpha)}(1/x)}{\Gamma(n+\alpha+1)}\,\frac{t^n}{n!} = \frac{J_\alpha(2\sqrt{t})}{t^{\alpha/2}}\,e^{xt}$$

   where $\Gamma(z)$ is the *Euler gamma function* and

$$J_\alpha(t) = \sum_{n\geq 0} \frac{(-1)^n}{n!\,\Gamma(n+\alpha+1)} \left(\frac{t}{2}\right)^{2n+\alpha}$$

   is a *Bessel function of the first kind* [**9**, Vol. 2, p. 4]. Notice that here the Laguerre polynomials are defined, as in [**27**, p. 108], by the exponential generating series

$$\sum_{n\geq 0} L_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \frac{e^{-\frac{xt}{1-t}}}{(1+t)^{\alpha}}\,.$$

4. The *generalized Bernoulli polynomials* $B_n^{(a)}(x)$ and the *generalized Euler polynomials* $E_n^{(a)}(x)$, [**27**, p. 93, p. 100] and [**9**, Vol. 3, p. 252], with exponential generating series

$$\sum_{n\geq 0} B_n^{(a)}(x)\,\frac{t^n}{n!} = \left(\frac{t}{e^t-1}\right)^{a} e^{xt} \quad \text{and} \quad \sum_{n\geq 0} E_n^{(a)}(x)\,\frac{t^n}{n!} = \left(\frac{2}{e^t+1}\right)^{a} e^{xt}\,.$$

We have the *generalized Bernoulli numbers* $B_n^{(a)} = B_n^{(a)}(0)$ and the *generalized Euler numbers* $E_n^{(a)} = 2^n E_n^{(a)}(1/2)$. For $a = 1$ we have the *ordinary Bernoulli polynomials* $B_n(x)$ and the *ordinary Euler polynomials* $E_n(x)$.

Bernoulli and Euler polynomials have been further generalized in several ways. For instance, we have the *Nörlund polynomials of order* k, [**21**] [**22**, Chapter 6] [**9**, Vol. 3, p. 253] [**4**] [**5**], with exponential generating series

$$\sum_{n\geq 0} B_n^{(k)}(x|\omega_1,\ldots,\omega_k)\,\frac{t^n}{n!} = \frac{\omega_1\cdots\omega_k t^k}{(\mathrm{e}^{\omega_1 t}-1)\cdots(\mathrm{e}^{\omega_k t}-1)}\,\mathrm{e}^{xt}$$

$$\sum_{n\geq 0} E_n^{(k)}(x|\omega_1,\ldots,\omega_k)\,\frac{t^n}{n!} = \frac{2^k}{(\mathrm{e}^{\omega_1 t}+1)\cdots(\mathrm{e}^{\omega_k t}+1)}\,\mathrm{e}^{xt}$$

with $\omega_1,\ldots,\omega_k \neq 0$, and the *hypergeometric Bernoulli polynomials* [**16**] [**14**], with exponential generating series

$$\sum_{n\geq 0} B_n^{[m]}(x)\,\frac{t^n}{n!} = \frac{1}{\mathrm{e}^t - 1 - t - \frac{t^2}{2!} - \cdots - \frac{t^{m-1}}{(m-1)!}}\,\frac{t^m}{m!}\,\mathrm{e}^{xt}\,.$$

In particular, the *Bernoulli* and *Euler numbers of order* k are defined by

$$B_n^{(k)}[\omega_1,\ldots,\omega_k] = B_n^{(k)}(0|\omega_1,\ldots,\omega_k)$$

$$E_n^{(k)}[\omega_1,\ldots,\omega_k] = 2^n E_n^{(k)}\Big(\frac{\omega_1+\cdots+\omega_k}{2}\Big|\omega_1,\ldots,\omega_k\Big)$$

and the *hypergeometric Bernoulli numbers* are defined by $B_n^{[m]} = B_n^{[m]}(0)$.

5. The polynomials $\widetilde{U}_n^{(\alpha)}(x)$ defined in [**17**], with exponential generating series

$$\sum_{n\geq 0} \widetilde{U}_n^{(\alpha)}(x)\,\frac{t^n}{n!} = \left(\frac{\mathrm{e}^t}{\mathrm{e}^t + \mathrm{e}^{-t} - 1}\right)^{\alpha}\mathrm{e}^{xt}\,.$$

6. The *modified Legendre polynomials*, defined by

$$P_n^*(x) = (x^2-1)^{n/2} P_n\Big(\frac{x}{\sqrt{x^2-1}}\Big) = \sum_{k=0}^{\lfloor n/2\rfloor}\binom{n}{k}\binom{n-k}{k}\frac{x^{n-2k}}{2^{2k}}\,,$$

where $P_n(x)$ is the *Legendre polynomial* of order n, [**9**, Vol. 3, p. 262], with exponential generating series

$$\sum_{n\geq 0} P_n^*(x)\,\frac{t^n}{n!} = I_0(t)\,\mathrm{e}^{xt}$$

where

$$I_0(t) = \sum_{n\geq 0}\frac{t^{2n}}{2^{2n}(n!)^2} = \sum_{n\geq 0}\binom{n}{n/2}\frac{\mathrm{even}(n)}{2^n}\,\frac{t^n}{n!}$$

is a *modified Bessel function of the first kind*. Also the polynomials

$$Q_n^*(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} \frac{1}{2^{2k}} P_{n-2k}^*(x) = \sum_{k=0}^{\lfloor n/2 \rfloor} \binom{n}{k} \binom{n-k}{k} \binom{2k}{k} \frac{x^{n-2k}}{2^{2k}} \,,$$

with exponential generating series

$$\sum_{n\geq 0} Q_n^*(x) \, \frac{t^n}{n!} = I_0(t)^2 \, \mathrm{e}^{xt} \,,$$

form an Appell sequence.

7. The *modified Charlier polynomials* [**10**] defined by

$$C_n^{(\alpha)}(x) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{\overline{n-k}} x^k \,,$$

where $\alpha^{\overline{m}} = \alpha(\alpha+1)(\alpha+2)\cdots(\alpha+m-1)$ are the rising factorials, with exponential generating series

$$\sum_{n\geq 0} C_n^{(\alpha)}(x) \, \frac{t^n}{n!} = \frac{1}{(1-t)^\alpha} \, \mathrm{e}^{xt} \,.$$

These polynomials are related to the Laguerre polynomials considered in the second example. Moreover, we also have the following interesting case [**10**]

$$\sum_{n\geq 0} C_n^{(\alpha)}(x-\alpha) \, \frac{t^n}{n!} = \left(\frac{\mathrm{e}^{-t}}{1-t} \right)^\alpha \mathrm{e}^{xt} \,.$$

For $\alpha = 1$, we have the *Riordan polynomials* (or *rencontres polynomials*)

$$D_n(x) = C_n^{(1)}(x-1) = \sum_{k=0}^{n} \binom{n}{k} (n-k)!(x-1)^k = \sum_{k=0}^{n} \binom{n}{k} d_{n-k} x^k \,, \tag{2}$$

with exponential generating series [**25**]

$$\sum_{n\geq 0} D_n(x) \, \frac{t^n}{n!} = \frac{\mathrm{e}^{(x-1)t}}{1-t} = \frac{\mathrm{e}^{-t}}{1-t} \, \mathrm{e}^{xt} \,.$$

Riordan polynomials have a simple combinatorial interpretation in terms of permutations (since d_n is the number of permutations with no fixed points of an n-set). This example can be generalized as follows. An $\mathcal{F}$*-enriched partition* of a finite set X is a partition $\pi = \{B_1, \ldots, B_k\}$ where each block B_i is endowed with a structure belonging to a combinatorial class $\mathcal{F}$. If $\widetilde{p}_n^{\mathcal{F}}$ denotes the number of the $\mathcal{F}$-enriched partitions with no singletons of an n-set, then the polynomials

$$p_n^{\mathcal{F}}(x) = \sum_{k=0}^{n} \binom{n}{k} \widetilde{p}_{n-k}^{\mathcal{F}} x^k$$

form an Appell sequence having exponential generating series

$$\sum_{n\geq 0} p_n^{\mathcal{F}}(x)\,\frac{t^n}{n!} = \mathrm{e}^{-t} f(t)\,\mathrm{e}^{xt}$$

where $f(t)$ is the exponential generating series of the class $\mathcal{F}$. In the case of Riordan polynomials, $\mathcal{F}$ is the class of permutations. See [**3**] for a different generalization of these polynomials.

s-Appell sequences are particular Sheffer sequences [**31**] [**27, 28**]. Indeed, a *Sheffer polynomial sequence* is a polynomial sequence $\{s_n(x)\}_{n\in\mathbb{N}}$ generated by the formal exponential series

$$\sum_{n\geq 0} s_n(x)\,\frac{t^n}{n!} = g(t)\,\mathrm{e}^{xf(t)} \tag{3}$$

where $g(t) = \sum_{n\geq 0} g_n \frac{t^n}{n!}$ is an exponential series with $g(0) = g_0 = 1$ and $f(t) = \sum_{n\geq 0} f_n \frac{t^n}{n!}$ is an exponential series with $f(0) = f_0 = 0$ and $f'(0) = f_1 \neq 0$. In particular, $s_n(0) = g_n$. A Sheffer sequence $\{s_n(x)\}_{n\in\mathbb{N}}$ can be represented by its *spectrum* $(g(t), f(t))$ and is equivalent to the infinite lower triangular matrix $S = [s_{n,k}]_{n,k\geq 0}$ generated by the coefficients of the polynomials. In particular, an s-Appell sequence has spectrum of the form $(g(t), st)$.

The set $\mathscr{S}$ of all Sheffer sequences (or Sheffer matrices) has a group structure with respect to the product defined by

$$(g_1(t), f_1(t))(g_2(t), f_2(t)) = (g_1(t)g_2(f_1(t)), f_2(f_1(t))) \tag{4}$$

and equivalent to the ordinary matrix product. The neutral element is $(1, t)$, corresponding to the identity matrix, and the inverses are given by

$$(g(t), f(t))^{-1} = \left(g(\widehat{f}(t))^{-1}, \widehat{f}(t)\right)$$

where $\widehat{f}(t)$ is the compositional inverse of $f(t)$.

The product in $\mathscr{S}$ defines the umbral composition of two Sheffer sequences. Specifically, the *umbral composition* of two Sheffer sequences $\{p_n(x)\}_{n\in\mathbb{N}}$ and $\{q_n(x)\}_{n\in\mathbb{N}}$ with spectrum $(g_1(t), f_1(t))$ and $(g_2(t), f_2(t))$, respectively, is the Sheffer sequence $\{p_n(x) \circledast q_n(x)\}_{n\in\mathbb{N}}$ with spectrum given by (4), i.e. with generating series

$$\sum_{n\geq 0} p_n(x) \circledast q_n(x)\,\frac{t^n}{n!} = g_1(t)g_2(f_1(t))\,\mathrm{e}^{xf_2(f_1(t))}\,.$$

The set $\mathscr{S}$ of Sheffer sequences can be endowed with another important (partial) operation. The *binomial convolution* of two Sheffer sequences $\{p_n(x)\}_{n\in\mathbb{N}}$ and $\{q_n(x)\}_{n\in\mathbb{N}}$ is defined as the polynomial sequence $\{p_n(x) * q_n(x)\}_{n\in\mathbb{N}}$, where

$$p_n(x) * q_n(x) = \sum_{k=0}^{n} \binom{n}{k} p_k(x)q_{n-k}(x)\,.$$

If $\sum_{n\geq 0} p_n(x)\,\frac{t^n}{n!} = g_1(t)\,\mathrm{e}^{xf_1(t)}$ and $\sum_{n\geq 0} q_n(x)\,\frac{t^n}{n!} = g_2(t)\,\mathrm{e}^{xf_2(t)}$, then the exponential generating series for the binomial convolution is

$$\sum_{n\geq 0} p_n(x) * q_n(x)\,\frac{t^n}{n!} = g_1(t)\,\mathrm{e}^{xf_1(t)} \cdot g_2(t)\,\mathrm{e}^{xf_2(t)} = g_1(t)g_2(t)\,\mathrm{e}^{x(f_1(t)+f_2(t))}\,.$$

So, the binomial convolution of two Sheffer sequences is a Sheffer sequence provided that $f_1'(0) + f_2'(0) \neq 0$. In this case, the binomial convolution, in terms of spectra, is given by

$$(g_1(t), f_1(t)) * (g_2(t), f_2(t)) = (g_1(t)g_2(t), f_1(t) + f_2(t))\,.$$

The set $\mathscr{A}^*$ of all s-Appell polynomial sequences, with $s \in \mathbb{R}$, $s \neq 0$, is a subgroup of the Sheffer group $\mathscr{S}$. Indeed, $(1,t) \in \mathscr{A}^*$, $(g_1(t), s_1t)(g_2(t), s_2t) = (g_1(t)g_2(s_1t), s_1s_2t) \in \mathscr{A}^*$ and $(g(t), st)^{-1} = (\frac{1}{g(t/s)}, t/s) \in \mathscr{A}^*$. In particular, the umbral composition of an s_1-Appell sequence with an s_2-Appell sequence is an s_1s_2-Appell sequence. Moreover, the binomial convolution of an s_1-Appell sequence with an s_2-Appell sequence is an (s_1+s_2)-Appell sequence, provided that $s_1 + s_2 \neq 0$.

The *cross sequence* of index λ (with $\lambda \in \mathbb{R}$) [**27**, p. 140] of an s-Appell sequence $\{p_n(x)\}_{n\in\mathbb{N}}$ with spectrum $(g(t), st)$ is the s-Appell sequence $\{p_n^{(\lambda)}(x)\}_{n\in\mathbb{N}}$ with spectrum $(g(t)^\lambda, st)$. For instance, the generalized Hermite, Bernoulli and Euler polynomials are cross sequences. In particular, the binomial convolution of two cross sequences of the same index λ relative to an r-Appell sequence and to an s-Appell sequence is a cross sequence of index λ of a $(r+s)$-Appell sequence, provided that $r + s \neq 0$.

In this paper, we will obtain several identities involving s-Appell polynomials by starting from combinatorial identities involving only ordinary powers. To obtain this result, we will use the linear algebra techniques of the modern umbral calculus due to G.-C. Rota [**27, 28, 29**].

Given an s-Appell polynomial sequence $\{p_n(x)\}_{n\in\mathbb{N}}$, with $s \neq 0$, we define a linear isomorphism $\varphi : \mathbb{R}[x] \to \mathbb{R}[x]$ by setting $\varphi(x^n) = p_n(x)$, for all $n \in \mathbb{N}$, and by extending it by linearity. This isomorphism has the following fundamental property.

Theorem 1. *Let $\{p_n(x)\}_{n\in\mathbb{N}}$ be an s-Appell polynomial sequence, with $s \neq 0$, and let α be an indeterminate. For every $n \in \mathbb{N}$, we have the identity*

$$\varphi((x+\alpha)^n) = p_n(x + \alpha/s)\,. \tag{5}$$

Proof. By generating series (1), we have the equations

$$\mathrm{e}^{\alpha t} p(x;t) = g(t)\mathrm{e}^{s(x+\alpha/s)t} = p(x + \alpha/s; t)$$

from which we obtain the identity

$$\sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} p_k(x) = p_n(x + \alpha/s)\,.$$

Then, by the linearity of φ, we have

$$\varphi((x+\alpha)^n) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} \varphi(x^k) = \sum_{k=0}^{n} \binom{n}{k} \alpha^{n-k} p_k(x) = p_n(x+\alpha/s)\,.$$

□

Then, we have

Theorem 2. *Let* $\{p_n(x)\}_{n\in\mathbb{N}}$ *be an s-Appell polynomial sequence, with* $s \neq 0$, *and let* α *and* β *be two indeterminates. If we have the identity*

$$\sum_{k=0}^{n} a_{n,k}(x+\alpha)^k = \sum_{k=0}^{n} b_{n,k}(x+\beta)^k\,, \tag{6}$$

then we also have the identity

$$\sum_{k=0}^{n} a_{n,k} p_k(x+\alpha/s) = \sum_{k=0}^{n} b_{n,k} p_k(x+\beta/s)\,. \tag{7}$$

Proof. Apply φ to the first identity, and then use Theorem 1. □

Theorem 2 permits to extend an elementary identity involving only ordinary powers to identities involving arbitrary Appell polynomials. In different words, we can say that (6) is an *umbral identity* from which we can derive all Appell identities (7), by means of the umbral map φ.

In next section, we obtain a symmetric binomial identity and then we use it to generalize the *Carlitz identity* for Bernoulli numbers and for Bernoulli polynomials to arbitrary symmetric s-Appell polynomials. In particular, we obtain a Carlitz-like identity for the generalized Hermite polynomials, for the modified Legendre polynomials, for the generalized Bernoulli and Euler polynomials and for the Nörlund polynomials.

Finally, in Section 2.2, we exemplify the application of Theorem 2 in the case of several umbral identities involving classical combinatorial numbers, such as the binomial coefficients, the Stirling numbers of the first and second kind, the Fibonacci and harmonic numbers.

2. CARLITZ-LIKE IDENTITIES

2.1 A symmetric binomial identity

Theorem 3. *Let* $\{p_n(x)\}_{n\in\mathbb{N}}$ *be an s-Appell polynomial sequence, with* $s \neq 0$, *and let* α *and* β *be two indeterminates. For every* $m, n \in \mathbb{N}$, *we have the symmetric identity*

$$\sum_{k=0}^{n} \binom{n}{k} (\beta-\alpha)^{n-k} p_{m+k}(x+\alpha/s) = \sum_{k=0}^{m} \binom{m}{k} (\alpha-\beta)^{m-k} p_{n+k}(x+\beta/s)\,. \tag{8}$$

Proof. From the two equivalent expansions

$$(x+\alpha)^m(x+\beta)^n = (x+\alpha)^m(x+\alpha+\beta-\alpha)^n = \sum_{k=0}^{n}\binom{n}{k}(\beta-\alpha)^{n-k}(x+\alpha)^{m+k}$$

$$(x+\alpha)^m(x+\beta)^n = (x+\beta+\alpha-\beta)^m(x+\beta)^n = \sum_{k=0}^{m}\binom{m}{k}(\alpha-\beta)^{m-k}(x+\beta)^{n+k},$$

we have the equation

$$\sum_{k=0}^{n}\binom{n}{k}(\beta-\alpha)^{n-k}(x+\alpha)^{m+k} = \sum_{k=0}^{m}\binom{m}{k}(\alpha-\beta)^{m-k}(x+\beta)^{n+k}.$$

So, by applying φ to this identity, and then by using the linearity of φ and Theorem 1, we obtain identity (8). □

Examples

1. For the generalized Hermite polynomials, with $s=2$, identity (8) becomes

$$\sum_{k=0}^{n}\binom{n}{k}(\beta-\alpha)^{n-k}H_{m+k}^{(\nu)}(x+\alpha/2) = \sum_{k=0}^{m}\binom{m}{k}(\alpha-\beta)^{m-k}H_{n+k}^{(\nu)}(x+\beta/2).$$

2. For the Laguerre polynomials, with $s=-1$, identity (8) becomes

$$\sum_{k=0}^{n}\binom{n}{k}(\beta-\alpha)^{n-k}L_{m+k}^{(\nu-m-k)}(x-\alpha) = \sum_{k=0}^{m}\binom{m}{k}(\alpha-\beta)^{m-k}L_{n+k}^{(\nu-n-k)}(x-\beta).$$

3. For the generalized Bernoulli and Euler polynomials, with $s=1$, identity (8) becomes

$$\sum_{k=0}^{n}\binom{n}{k}(\beta-\alpha)^{n-k}B_{m+k}^{(\nu)}(x+\alpha) = \sum_{k=0}^{m}\binom{m}{k}(\alpha-\beta)^{m-k}B_{n+k}^{(\nu)}(x+\beta).$$

$$\sum_{k=0}^{n}\binom{n}{k}(\beta-\alpha)^{n-k}E_{m+k}^{(\nu)}(x+\alpha) = \sum_{k=0}^{m}\binom{m}{k}(\alpha-\beta)^{m-k}E_{n+k}^{(\nu)}(x+\beta).$$

2.2 Symmetric Appell polynomials

We say that the s-Appell polynomials $p_n(x)$ are α-*symmetric* when

$$p_n(\alpha-x) = (-1)^n p_n(x).$$

This means that the polynomials $p_n(x)$ are symmetric with respect to the vertical line $x=\frac{\alpha}{2}$ when n is even, and are symmetric with respect to the point $(\frac{\alpha}{2},0)$ when n is odd. These polynomials are characterized as follows.

Theorem 4. *The s-Appell polynomials* $p_n(x)$ *, with spectrum* $(g(t), st)$ *, are* α-symmetric *if and only if*

$$\frac{g(-t)}{g(t)} = \mathrm{e}^{s\alpha t}\,. \tag{9}$$

Proof. The relations $p_n(\alpha - x) = (-1)^n p_n(x)$ are equivalent to the identity

$$g(t)\,\mathrm{e}^{s(\alpha-x)t} = g(-t)\,\mathrm{e}^{-sxt}$$

which, in turn, is equivalent to (9). □

The binomial convolution and the umbral composition of symmetric Appell sequences are still symmetric, as proved in next Theorem.

Theorem 5. *Let* $p_n(x)$ *be* α*-symmetric* r*-Appell polynomials, with* $r \neq 0$ *, and let* $q_n(x)$ *be* β*-symmetric* s*-Appell polynomials, with* $s \neq 0$ *. If* $\alpha = \beta$ *and* $r + s \neq 0$ *, then the binomial convolutions* $p_n(x) * q_n(x)$ *form an* α*-symmetric* $(r + s)$*-Appell sequence. If* $r = s$ *, then the binomial convolutions* $p_n(x) * q_n(x)$ *form an* $(\alpha + \beta)$*-symmetric s-Appell sequence. The umbral compositions* $p_n(x) \circledast q_n(x)$ *always form an* $(\alpha + s\beta)$*-symmetric* r*-Appell sequence. The cross sequence formed by the polynomials* $p_n^{(\lambda)}(x)$ *, with* $\lambda \neq 0$ *, is a* $(\lambda\alpha)$*-symmetric* r*-Appell sequence.*

Proof. Let $(g_1(t), rt)$ be the spectrum for the polynomials $p_n(x)$ and let $(g_2(t), st)$ be the spectrum for the polynomials $q_n(x)$. Then, by Theorem 4, we have $\frac{g_1(-t)}{g_1(t)} = \mathrm{e}^{r\alpha t}$ and $\frac{g_2(-t)}{g_2(t)} = \mathrm{e}^{s\beta t}$. So, the sequence of binomial convolutions $p_n(x) * q_n(x)$ has spectrum $(g_1(t)g_2(t), (r + s)t)$ and

$$\frac{g_1(-t)g_2(-t)}{g_1(t)g_2(t)} = \frac{g_1(-t)}{g_1(t)}\,\frac{g_2(-t)}{g_2(t)} = \mathrm{e}^{r\alpha t}\mathrm{e}^{s\beta t} = \mathrm{e}^{(r\alpha+s\beta)t}\,,$$

while the sequence of umbral convolutions $p_n(x) \circledast q_n(x)$ has spectrum $(g_1(t)g_2(rt), rst)$ and

$$\frac{g_1(-t)g_2(-rt)}{g_1(t)g_2(rt)} = \frac{g_1(-t)}{g_1(t)}\,\frac{g_2(-rt)}{g_2(rt)} = \mathrm{e}^{r\alpha t}\mathrm{e}^{rs\beta t} = \mathrm{e}^{r(\alpha+s\beta)t}\,.$$

Finally, the cross sequence formed by the polynomials $p_n^{(\lambda)}(x)$ has spectrum $(g_1(t)^\lambda, rt)$ and

$$\frac{g_1(-t)^\lambda}{g_1(t)^\lambda} = \left(\frac{g_1(-t)}{g_1(t)}\right)^\lambda = \left(\mathrm{e}^{r\alpha t}\right)^\lambda = \mathrm{e}^{r\lambda\alpha t}\,.$$

Now, the claims follows at once by Theorem 4. □

Examples

1. The generalized Hermite polynomials $H_n^{(a)}(x)$ are 0-symmetric.

2. The modified Legendre polynomials $P_n^*(x)$ and $Q_n^*(x)$ are 0-symmetric.
3. The generalized Bernoulli and Euler polynomials form two a-symmetric 1-Appell sequences, and the Nörlund polynomials of order k form two $(\omega_1 + \cdots + \omega_k)$-symmetric 1-Appell sequences.
4. The polynomials $\widetilde{U}_n^{(\alpha)}(x)$ are (-2α)-symmetric.

Bernoulli and Euler polynomials are related by the identity [**27**, p. 105]

$$E_n(x) = \frac{2^{n+1}}{n+1}\left[B_{n+1}\left(\frac{x+1}{2}\right) - B_{n+1}\left(\frac{x}{2}\right)\right] .$$

This suggests to consider the following transformation. Given an s-Appell sequence $\{p_n(x)\}_{n\in\mathbb{N}}$ and a real number $a \neq 0$, consider the polynomials

$$q_n(x) = \frac{a^{n+1}}{n+1}\left[p_{n+1}\left(\frac{x+1}{a}\right) - p_{n+1}\left(\frac{x}{a}\right)\right] . \tag{10}$$

These polynomials form an s-Appell sequence. Indeed, if the polynomials $p_n(x)$ have exponential generating series (1), then

$$\sum_{n\geq 0} q_n(x)\,\frac{t^n}{n!} = a\frac{p(\frac{x+1}{a};t) - p_0(\frac{x+1}{a})}{at} - a\frac{p(\frac{x}{a};t) - p_0(\frac{x}{a})}{at} = g(at)\frac{\mathrm{e}^t - 1}{t}\,\mathrm{e}^{sxt} .$$

By this remark and by Theorem 5, we have at once

Theorem 6. *If* $\{p_n(x)\}_{n\in\mathbb{N}}$ *is an* α*-symmetric* s*-Appell sequence, with* $s \neq 0$*, then also the sequence* $\{q_n(x)\}_{n\in\mathbb{N}}$ *defined by polynomials (10) is an* α*-symmetric* s*-Appell sequence.*

In 1971, Carlitz obtained [**6**] the following symmetric identity

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} B_{n+k}$$

for the Bernoulli numbers. This identity has been reproved and generalized in several ways by several authors [**10, 15, 20, 23, 24, 34**]. In next theorem, we generalized such an identity to symmetric Appell polynomials.

Theorem 7. *If* $\{p_n(x)\}_{n\in\mathbb{N}}$ *is an* α-symmetric s*-Appell sequence, with* $s \neq 0$*, then we have the Carlitz-like identities*

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} p_{m+k}(x)\,(sy)^{n-k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} p_{n+k}(\alpha - x - y)\,(sy)^{m-k} . \tag{11}$$

In particular, for $x = 0$ *and* $y = \alpha$*, we have the symmetric identity*

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} (s\alpha)^{n-k} g_{m+k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} (s\alpha)^{m-k} g_{n+k} \tag{12}$$

Proof. By Theorem 3, with $\alpha = 0$ and $\beta = sy$, we obtain the identity

$$\sum_{k=0}^{n} \binom{n}{k} (sy)^{n-k} p_{m+k}(x) = \sum_{k=0}^{m} \binom{m}{k} (-sy)^{m-k} p_{n+k}(x+y)\,.$$

Then, by using the α-symmetry on the right-hand side and by simplifying, we obtain identity (11). □

Examples

1. Since the generalized Hermite polynomials $H_n^{(a)}(x)$ form a 0-symmetric 2-Appell sequence, identity (11) becomes

$$(13) \quad (-1)^n \sum_{k=0}^{n} \binom{n}{k} H_{m+k}^{(a)}(x)(2y)^{n-k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} H_{n+k}^{(a)}(-x-y)(2y)^{m-k}\,.$$

   Let i_n be the number of involutions on an n-set [**33**, A000085]. Since $\sum_{n\geq 0} i_n \frac{t^n}{n!} = \mathrm{e}^{t+t^2/2}$, we have $i_n = H_n^{(-1/2)}(1/2)$. So, by setting $a = -1/2$, $x = 1/2$ and $y = -1$, identity (13) becomes

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 2^{n-k} i_{m+k} = \sum_{k=0}^{m} \binom{m}{k} (-1)^k 2^{m-k} i_{n+k}\,.$$

   Let now $i_n^* = \sum_{k=0}^{n} \binom{n}{k} i_k i_{n-k}$ be the binomial convolution of the numbers i_n [**33**, A000898]. Since $\sum_{n\geq 0} i_n^* \frac{t^n}{n!} = \mathrm{e}^{2t+t^2}$, we have $i_n = H_n^{(-1)}(1)$. So, by setting $a = -1$, $x = 1$ and $y = -2$, identity (13) becomes

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 4^{n-k} i_{m+k}^* = \sum_{k=0}^{m} \binom{m}{k} (-1)^k 4^{m-k} i_{n+k}^*\,.$$

   Finally, let $i_n^+ = \sum_{k=0}^{n} \binom{n}{k} i_k$ be the binomial cumulative sums of the numbers i_n [**33**, A005425]. Since $\sum_{n\geq 0} i_n^+ \frac{t^n}{n!} = \mathrm{e}^{2t+t^2/2}$, we have $i_n^+ = H_n^{(-1/2)}(1)$. So, by setting $a = -1/2$, $x = 1$ and $y = -2$, identity (13) becomes

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k 4^{n-k} i_{m+k}^+ = \sum_{k=0}^{m} \binom{m}{k} (-1)^k 4^{m-k} i_{n+k}^+\,.$$

   Notice that the numbers i_n^* and i_n^+ satisfy the same symmetric relation.

2. The modified Legendre polynomials $P_n^*(x)$ and $Q_n^*(x)$ form 0-symmetric 1-Appell sequences. So, for these polynomials, identity (11) becomes

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} P_{m+k}^*(x) y^{n-k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} P_{n+k}^*(-x-y) y^{m-k}$$

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} Q_{m+k}^*(x) y^{n-k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} Q_{n+k}^*(-x-y) y^{m-k}\,.$$

3. The generalized Bernoulli and Euler polynomials are a-symmetric 1-Appell sequences. So, identity (11) becomes

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} B_{m+k}^{(a)}(x)\, y^{n-k} = (-1)^m \sum_{k=0}^{m} \binom{n}{k} B_{n+k}^{(a)}(a-x-y)\, y^{m-k}$$
$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} E_{m+k}^{(a)}(x)\, y^{n-k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} E_{n+k}^{(a)}(a-x-y)\, y^{m-k}\,.$$

   In particular, identity (12) becomes

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} a^{n-k} B_{m+k}^{(a)} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} a^{m-k} B_{n+k}^{(a)}\,.$$

   Identities of this kind hold also for the Nörlund polynomials of order k.

4. Since the polynomials $\widetilde{U}_n^{(\alpha)}(x)$ form a (-2α)-symmetric 1-Appell sequence, identity (11) becomes

$$(-1)^n \sum_{k=0}^{n} \binom{n}{k} \widetilde{U}_{m+k}^{(\alpha)}(x)\, y^{n-k} = (-1)^m \sum_{k=0}^{m} \binom{m}{k} \widetilde{U}_{n+k}^{(\alpha)}(-2\alpha-x-y)\, y^{m-k}\,.$$

   In particular, for the numbers $\widetilde{U}_n^{(\alpha)} = \widetilde{U}_n^{(\alpha)}(0)$, we have the symmetric identity

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^k (2\alpha)^{n-k} \widetilde{U}_{m+k}^{(\alpha)} = \sum_{k=0}^{m} \binom{m}{k} (-1)^k (2\alpha)^{m-k} \widetilde{U}_{n+k}^{(\alpha)}\,.$$

3. SPECIAL IDENTITIES

In all Propositions of this section, we always consider an s-Appell sequence $\{p_n(x)\}_{n\in\mathbb{N}}$, with $s \neq 0$. In Lemmas we give umbral identities and in Propositions we give the corresponding identities for Appell polynomials (by applying Theorem 2). All these identities are exemplified only on generalized Hermite, Laguerre, and Bernoulli polynomials.

3.3 Identities involving binomial coefficients. I

Proposition 8. *For every $n \in \mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} (n-k)!\, p_k(x) = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}\, p_k(x+1/s)\,. \tag{14}$$

In particular, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}(n-k)!H_k^{(\nu)}(x)=\sum_{k=0}^{n}\binom{n}{k}d_{n-k}H_k^{(\nu)}(x+1/2)$$
$$\sum_{k=0}^{n}\binom{n}{k}(n-k)!L_k^{(\alpha-k)}(x)=\sum_{k=0}^{n}\binom{n}{k}d_{n-k}L_k^{(\alpha-k)}(x-1)$$
$$\sum_{k=0}^{n}\binom{n}{k}(n-k)!B_k^{(a)}(x)=\sum_{k=0}^{n}\binom{n}{k}d_{n-k}B_k^{(a)}(x+1)\,.$$

Proof. Apply Theorem 2 to identity (2), replacing x by $x+1$. □

Proposition 9. *For every* $n\in\mathbb{N}$, *we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(k+1)^{k-1}p_{n-k}(x+k/s)=p_n(x-1/s)\,. \tag{15}$$

In particular, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(k+1)^{k-1}H_{n-k}^{(a)}(x+k/2)=H_n^{(a)}(x-1/2)$$
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(k+1)^{k-1}L_{n-k}^{(\alpha-n+k)}(x-k)=L_n^{(\alpha-n)}(x+1)$$
$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(k+1)^{k-1}B_{n-k}^{(a)}(x+k)=B_n^{(a)}(x-1)\,.$$

Proof. Apply Theorem 2 to the Abel identity [**11**, Formula (1.117), p. 15]

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^k(k+1)^{k-1}(x+k)^{n-k}=(x-1)^n\,.$$

□

Lemma 10. *For every* $m,n\in\mathbb{N}$, *we have the identities*

$$\sum_{k=0}^{n}\binom{n}{k}F_{m+k}x^k=\sum_{k=0}^{n}\binom{n}{k}F_{m+2n-k}(x-1)^k \tag{16}$$
$$\sum_{k=0}^{n}\binom{n}{k}L_{m+k}x^k=\sum_{k=0}^{n}\binom{n}{k}L_{m+2n-k}(x-1)^k \tag{17}$$

where the F_n*'s are the* Fibonacci numbers *and the* L_n*'s are the* Lucas numbers*.*

Proof. From the equation $(1+\alpha x)^n = (1+\alpha+\alpha(x-1))^n$, we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} \alpha^k x^k = \sum_{k=0}^{n} \binom{n}{k} \alpha^k (1+\alpha)^{n-k} (x-1)^k .$$

Since $F_n = \frac{\varphi^n - \widehat{\varphi}^n}{\sqrt{5}}$ and $L_n = \varphi^n + \widehat{\varphi}^n$, where $\varphi = \frac{1+\sqrt{5}}{2}$ and $\widehat{\varphi} = \frac{1-\sqrt{5}}{2}$, by replacing α by φ (or by $\widehat{\varphi}$), we can obtain identity (16) (or identity (17)). □

Proposition 11. *For every* $m, n \in \mathbb{N}$, *we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k} F_{m+k} p_k(x) = \sum_{k=0}^{n} \binom{n}{k} F_{m+2n-k} p_k(x-1/s) \tag{18}$$

$$\sum_{k=0}^{n} \binom{n}{k} L_{m+k} p_k(x) = \sum_{k=0}^{n} \binom{n}{k} L_{m+2n-k} p_k(x-1/s) \tag{19}$$

In particular, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k} F_{m+k} H_k^{(\nu)}(x) = \sum_{k=0}^{n} \binom{n}{k} F_{m+2n-k} H_k^{(\nu)}(x-1/2)$$

$$\sum_{k=0}^{n} \binom{n}{k} F_{m+k} L_k^{(\alpha-k)}(x) = \sum_{k=0}^{n} \binom{n}{k} F_{m+2n-k} L_k^{(\alpha-k)}(x+1)$$

$$\sum_{k=0}^{n} \binom{n}{k} F_{m+k} B_k^{(a)}(x) = \sum_{k=0}^{n} \binom{n}{k} F_{m+2n-k} B_k^{(a)}(x-1) .$$

Proof. Apply Theorem 2 to identities (16) and (17). □

Lemma 12. *For every* $n \in \mathbb{N}$, *we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k x^k = \mathcal{H}_n (x+1)^n - \sum_{k=1}^{n} \frac{1}{k} (x+1)^{n-k} \tag{20}$$

where the $\mathcal{H}_n = 1 + \frac{1}{2} + \cdots + \frac{1}{n}$ *are the* harmonic numbers.

Proof. Consider the polynomials $\mathcal{H}_n(x) = \sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k x^k$ giving the left-hand side of identity (20). Since, the generating series of the harmonic numbers is

$$\mathcal{H}(t) = \sum_{n \geq 0} \mathcal{H}_n \, t^n = \frac{1}{1-t} \ln \frac{1}{1-t} ,$$

we have $\mathcal{H}(x;t) = \sum_{n \geq 0} \mathcal{H}_n(x) \, t^n = \frac{1}{1-t} \, \mathcal{H}\left(\frac{xt}{1-t}\right) = \frac{1}{1-t-xt} \ln \frac{1-t}{1-t-xt}$,
that is

$$\mathcal{H}(x;t) = \mathcal{H}((1+x)t) - \frac{1}{1-(1+x)t} \ln \frac{1}{1-t} .$$

This identity is equivalent to identity (20). □

Identity (20) also appears in [**19**, Formula (20)], where it has been proved in a slightly different way in the context of Riordan matrices.

Proposition 13. *For every* $n \in \mathbb{N}$, *we have the identity*

$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k p_k(x) = \mathcal{H}_n p_n(x+1/s) - \sum_{k=1}^{n} \frac{1}{k}\, p_{n-k}(x+1/s)\,. \tag{21}$$

In particular, we have the identities

$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k H_k^{(a)}(x) = \mathcal{H}_n H_n^{(a)}(x+1/2) - \sum_{k=1}^{n} \frac{1}{k}\, H_k(x+1/2)$$
$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k L_k^{(a-k)}(x) = \mathcal{H}_n L_n^{(a-k)}(x-1) - \sum_{k=1}^{n} \frac{1}{k}\, L_k^{(a-k)}(x-1)$$
$$\sum_{k=0}^{n} \binom{n}{k} \mathcal{H}_k B_k^{(a)}(x) = \mathcal{H}_n B_n^{(a)}(x+1) - \sum_{k=1}^{n} \frac{1}{k}\, B_k^{(a)}(x+1)\,.$$

Proof. Apply Theorem 2 to identity (20). □

3.4 Identities involving binomial coefficients. II

Lemma 14. *For every* $n \in \mathbb{N}$, *we have the identities*

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} x^k = \sum_{k=0}^{n} \binom{\alpha}{k}\binom{\alpha+\beta-k}{n-k} (x-1)^k \tag{22}$$

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta}{n-k} (-1)^k x^k = \sum_{k=0}^{n} \binom{\alpha}{k}\binom{\alpha+\beta-k}{n-k} (-1)^k (x+1)^k\,. \tag{23}$$

Equivalently, we have the identities

$$\sum_{k=0}^{n} \binom{\alpha+k}{k}\binom{\beta+n-k}{n-k} x^k = \sum_{k=0}^{n} \binom{\alpha+k}{k}\binom{\alpha+\beta+n+1}{n-k} (x-1)^k \tag{24}$$

$$\sum_{k=0}^{n} \binom{\alpha+k}{k}\binom{\beta+n-k}{n-k} (-1)^k x^k = \sum_{k=0}^{n} \binom{\alpha+k}{k}\binom{\alpha+\beta+n+1}{n-k} (-1)^k (x+1)^k \tag{25}$$

$$\sum_{k=0}^{n} \binom{\alpha+k}{k}\binom{\beta}{n-k} (-1)^{n-k} x^k = \sum_{k=0}^{n} \binom{\alpha+k}{k}\binom{\alpha-\beta+n}{n-k} (x-1)^k \tag{26}$$

$$\sum_{k=0}^{n} \binom{\alpha+k}{k}\binom{\beta}{n-k} x^k = \sum_{k=0}^{n} \binom{\alpha+k}{k}\binom{\alpha-\beta+n}{n-k} (-1)^{n-k} (x+1)^k \tag{27}$$

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta+n-k}{n-k} (-1)^{n-k} x^k = \sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta-\alpha+n}{n-k} (-1)^{n-k} (x-1)^k \tag{28}$$

$$\sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta+n-k}{n-k} x^k = \sum_{k=0}^{n} \binom{\alpha}{k}\binom{\beta-\alpha+n}{n-k} (x+1)^k\,. \tag{29}$$

Proof. The ordinary series

$$F(x,y;t) = (1+xt)^\alpha (1+yt)^\beta = \sum_{n\geq 0} \left[\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} x^x y^{n-k} \right] t^n$$

can also be expanded as

$$\begin{aligned} F(x,y;t) &= (1+(x-y)t+yt)^\alpha (1+yt)^\beta = \left(1+\frac{(x-y)t}{1+yt}\right)^\alpha (1+yt)^{\alpha+\beta} \\ &= \sum_{k\geq 0} (x-y)^k t^k (1+yt)^{\alpha+\beta-k} = \sum_{n\geq 0} \left[\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\alpha+\beta-k}{n-k} (x-y)^k y^{n-k} \right] t^n . \end{aligned}$$

Comparing the coefficient of t^n in both the expansions of $F(x,y;t)$ and setting $y=1$, we obtain identity (22). Replacing x by $-x$, we obtain identity (23).

All other identities are equivalent to (22) and (23), due to the fact that $\binom{-\alpha-1}{k} = (-1)^k \binom{\alpha+k}{k}$. So, replacing α by $-\alpha-1$ and β by $-\beta-1$, we have (24) and (25). Replacing only α by $-\alpha-1$, we have (26) and (27). Replacing only β by $-\beta-1$, we have (28) and (29). □

Remark 15. Identity (22), with x replaced by $x+1$, admits a simple bijective proof (extending the bijective proof for the Vandermonde identity). Given an a-set A and a b-set B, consider the family

$$\mathcal{V}[A,B] = \{(S,X,Y) \,:\, S\subseteq X\,,\ X\subseteq A\,,\ Y\subseteq B\,,\ |X|+|Y|=n\}\,.$$

By defining the weight of an element (S,X,Y) of $\mathcal{V}[A,B]$ as $w(S,X,Y) = x^{|S|}$, the weight of the entire family $\mathcal{V}[A,B]$ is

$$w(\mathcal{V}[A,B]) = \sum_{(S,X,Y)\in\mathcal{V}[A,B]} w(S,X,Y) = \sum_{X\subseteq A} \sum_{Y\subseteq B} \sum_{S\subseteq X} x^{|S|} = \sum_{k=0}^{n} \binom{a}{k} \binom{b}{n-k} (x+1)^k .$$

Now, assuming A and B disjoint, consider the family

$$\mathcal{W}[A,B] = \{(T,Z) \,:\, T\subseteq A\,,\ Z\subseteq A\cup B\,,\ T\cap Z=\varnothing\,,\ |T|+|Z|=n\}\,.$$

By defining the weight of an element (T,Z) of $\mathcal{W}[A,B]$ as $w(T,Z) = x^{|T|}$, the weight of the entire family $\mathcal{W}[A,B]$ is

$$w(\mathcal{W}[A,B]) = \sum_{(T,Z)\in\mathcal{W}[A,B]} w(T,Z) = \sum_{T\subseteq A} \sum_{Z\subseteq (A\cup B)\setminus T} x^{|T|} = \sum_{k=0}^{n} \binom{a}{k} \binom{a+b-k}{n-k} x^k .$$

The map $\psi : \mathcal{V}[A,B] \to \mathcal{W}[A,B]$, defined by $\psi(S,X,Y) = (S,(X\cup Y)\setminus S)$, is a bijection preserving the weight. So $w(\mathcal{V}[A,B]) = w(\mathcal{W}[A,B])$, which is essentially identity (22).

Remark 16. We can obtain several other interesting identities by specializing (22), ..., (29). For instance, for $\alpha = \beta = n$, we have

$$\sum_{k=0}^{n} \binom{n}{k}^2 x^k = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n-k} (x-1)^k \tag{30}$$

$$\sum_{k=0}^{n} \binom{n}{k}^2 (-1)^k x^k = \sum_{k=0}^{n} \binom{n}{k} \binom{2n-k}{n-k} (-1)^k (x+1)^k \tag{31}$$

$$\sum_{k=0}^{n} \binom{n+k}{k} \binom{2n-k}{n-k} x^k = \sum_{k=0}^{n} \binom{n+k}{k} \binom{3n+1}{n-k} (x-1)^k \tag{32}$$

$$\sum_{k=0}^{n} \binom{n+k}{k} \binom{2n-k}{n-k} (-1)^k x^k = \sum_{k=0}^{n} \binom{n+k}{k} \binom{3n+1}{n-k} (-1)^k (x+1)^k \tag{33}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} x^k = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (x-1)^k \tag{34}$$

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} x^k = \sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} (-1)^{n-k} (x+1)^k . \tag{35}$$

These identities are not new. For instance, identity (30) appears in [**11**, Formula (3.65), p. 30], and identity (35) is Simons's identity [**32**] (see also [**18**]).

Proposition 17. *For every $n \in \mathbb{N}$, we have the identities*

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} p_k(x) = \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\alpha+\beta-k}{n-k} p_k(x-1/s) \tag{36}$$

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} (-1)^k p_k(x) = \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\alpha+\beta-k}{n-k} (-1)^k p_k(x+1/s) . \tag{37}$$

In particular, we have the identities

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} H_k^{(\nu)}(x) = \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\alpha+\beta-k}{n-k} H_k^{(\nu)}(x-1/2)$$

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} L_k^{(\alpha-k)}(x) = \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\alpha+\beta-k}{n-k} L_k^{(\alpha-k)}(x+1)$$

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} B_k^{(a)}(x) = \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\alpha+\beta-k}{n-k} B_k^{(a)}(x-1)$$

and

$$\sum_{k=0}^{n} \binom{\alpha}{k} \binom{\beta}{n-k} (-1)^k H_k^{(\nu)}(x) = \sum_{k=0}^{n} \binom{\alpha}{k} \binom{\alpha+\beta-k}{n-k} (-1)^k H_k^{(\nu)}(x+1/2)$$

$$\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}(-1)^k L_k^{(\alpha-k)}(x)=\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\alpha+\beta-k}{n-k}(-1)^k L_k^{(\alpha-k)}(x-1)$$
$$\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\beta}{n-k}(-1)^k B_k^{(a)}(x)=\sum_{k=0}^{n}\binom{\alpha}{k}\binom{\alpha+\beta-k}{n-k}(-1)^k B_k^{(a)}(x+1)\,.$$

Proof. Apply φ to identities (22) and (23). □

These identities can be specialized as in the previous Remark. We consider just a couple of them in the next two propositions.

Proposition 18. *For every* $n\in\mathbb{N}$, *we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}^2 p_k(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n-k}p_k(x-1/s)\,. \tag{38}$$

In particular, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}^2 H_k^{(a)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n-k}H_k^{(a)}(x-1/2)$$
$$\sum_{k=0}^{n}\binom{n}{k}^2 L_k^{(\alpha-k)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{n-k}L_k^{(\alpha-k)}(x+1)$$
$$\sum_{k=0}^{n}\binom{n}{k}^2 B_k^{(a)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{2n-k}{k}B_k^{(a)}(x-1)\,.$$

Proof. Apply Theorem 2 to identities (30). □

Proposition 19. *For every* $n\in\mathbb{N}$, *we have the identity*

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}p_k(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}p_k(x+1/s)\,. \tag{39}$$

In particular, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}H_k^{(a)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}H_k^{(a)}(x+1/2)$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}L_k^{(\alpha-k)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}L_k^{(\alpha-k)}(x-1)$$
$$\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}B_k^{(a)}(x)=\sum_{k=0}^{n}\binom{n}{k}\binom{n+k}{k}(-1)^{n-k}B_k^{(a)}(x+1)\,.$$

Proof. Apply Theorem 2 to Simons's identity (35). □

The proof of identity (22) given in Remark 15 can be generalized as follows.

Lemma 20. *For every* $n, k \in \mathbb{N}$, *we have the identity*

$$\sum_{i_1,\ldots,i_k=0}^{n} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_k}{i_k} \binom{\beta}{n-i_1-\cdots-i_k} (x+1)^k = \tag{40}$$
$$= \sum_{i_1,\ldots,i_k=0}^{n} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_k}{i_k} \binom{\alpha_1+\cdots+\alpha_k+\beta-i_1-\cdots-i_k}{n-i_1-\cdots-i_k} x^k .$$

Proof. Let A_j be a set of size a_j, for $j = 1, \ldots, k$, and let B be a set of size b. Let $\mathcal{V}[A_1, \ldots, A_k, B]$ be the family consisting of the elements $(S_1, X_1, \ldots, S_k, X_k, Y)$, where $S_j \subseteq X_j \subseteq A_j$ for $j = 1, \ldots, k$, $Y \subseteq B$ and $|X_1|+\cdots+|X_k|+|Y| = n$. Then define the weight of an element of $\mathcal{V}[A_1, \ldots, A_k, B]$ as $w(S_1, X_1, \ldots, S_k, X_k, Y) = x^{|S_1|+\cdots+|S_k|}$.

Now, assuming $A_1, \ldots, A_k$ and B disjoint, let $\mathcal{W}[A_1, \ldots, A_k, B]$ be the family consisting of the elements $(T_1, \ldots, T_k, Z)$, where $T_j \subseteq A_j$ for $j = 1, \ldots, k$, $Z \subseteq A_1 \cup \cdots \cup A_k \cup B$ and $|T_1| + \cdots + |T_k| + |Z| = n$. Then, define the weight of an element of $\mathcal{W}[A_1, \ldots, A_k, B]$ as $w(T_1, \ldots, T_k, Z) = x^{|T_1|+\cdots+|T_k|}$.

Finally, let $\psi : \mathcal{V}[A_1, \ldots, A_k, B] \to \mathcal{W}[A_1, \ldots, A_k, B]$ be the map defined by

$$\psi(S_1, X_1, \ldots, S_k, X_k, Y) = (S_1, \ldots, S_k, (X_1 \cup \cdots \cup X_k \cup Y) \setminus (S_1 \cup \cdots \cup S_k)) .$$

This map is a bijection preserving the weight. So, we have $w(\mathcal{V}[A_1, \ldots, A_k, B]) = w(\mathcal{W}[A_1, \ldots, A_k, B])$. This is identity (40) with $\alpha_1, \ldots, \alpha_k$ and β non-negative integers. Since it holds for infinite values, it can be extended to an identity where $\alpha_1, \ldots, \alpha_k$ and β are abstract symbols. ☐

Proposition 21. *For every* $n, k \in \mathbb{N}$, *we have the identity*

$$\sum_{i_1,\ldots,i_k=0}^{n} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_k}{i_k} \binom{\beta}{n-i_1-\cdots-i_k} p_k(x+1/s) = \tag{41}$$
$$= \sum_{i_1,\ldots,i_k=0}^{n} \binom{\alpha_1}{i_1} \cdots \binom{\alpha_k}{i_k} \binom{\alpha_1+\cdots+\alpha_k+\beta-i_1-\cdots-i_k}{n-i_1-\cdots-i_k} p_k(x) .$$

Proof. Apply Theorem 2 to identity (40). ☐

3.5 Identities involving Stirling numbers

Lemma 22. *For every* $n \in \mathbb{N}$, *we have the identity*

$$\sum_{k=0}^{n} \left\{ {n \atop k} \right\} k! x^k = \sum_{k=0}^{n} \left\{ {n+1 \atop k+1} \right\} (-1)^{n-k} k! (x+1)^k . \tag{42}$$

Proof. Recalling that

$$\sum_{n\geq k}\left\{{n \atop k}\right\}\frac{t^n}{n!}=\frac{1}{k!}\,(\mathrm{e}^t-1)^k \quad\text{and}\quad \sum_{n\geq k}\left\{{n+1 \atop k+1}\right\}\frac{t^n}{n!}=\frac{1}{k!}\,\mathrm{e}^t(\mathrm{e}^t-1)^k\,, \tag{43}$$

we have

$$\begin{aligned}\sum_{n\geq 0}\left[\sum_{k=0}^{n}\left\{{n \atop k}\right\}k!x^k\right]\frac{t^n}{n!}&=\frac{1}{1-x(\mathrm{e}^t-1)}=\\ &=\frac{\mathrm{e}^{-t}}{1+(x+1)(\mathrm{e}^{-t}-1)}=\sum_{n\geq 0}\left[\sum_{k=0}^{n}\left\{{n+1 \atop k+1}\right\}(-1)^{n-k}k!(x+1)^k\right]\frac{t^n}{n!}\,.\end{aligned}$$

Taking the coefficient of $\frac{t^n}{n!}$ in the first and in the last expression, we obtain identity (42). □

Proposition 23. *For every $n\in\mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n}\left\{{n \atop k}\right\}k!p_k(x)=\sum_{k=0}^{n}\left\{{n+1 \atop k+1}\right\}(-1)^{n-k}k!p_k(x+1/s)\,.$$

In particular, we have the identities

$$\begin{aligned}\sum_{k=0}^{n}\left\{{n \atop k}\right\}k!H_k^{(a)}(x)&=\sum_{k=0}^{n}\left\{{n+1 \atop k+1}\right\}(-1)^{n-k}k!H_k^{(a)}(x+1/2)\\ \sum_{k=0}^{n}\left\{{n \atop k}\right\}k!L_k^{(a-k)}(x)&=\sum_{k=0}^{n}\left\{{n+1 \atop k+1}\right\}(-1)^{n-k}k!L_k^{(a-k)}(x-1)\\ \sum_{k=0}^{n}\left\{{n \atop k}\right\}k!B_k^{(a)}(x)&=\sum_{k=0}^{n}\left\{{n+1 \atop k+1}\right\}(-1)^{n-k}k!B_k^{(a)}(x+1)\,.\end{aligned}$$

Proof. Apply Theorem 2 to identity (42). □

Lemma 24. *For every $n\in\mathbb{N}$, we have the identity*

$$\sum_{k=0}^{n}\left[{n \atop k}\right](\alpha-\beta)^{n-k}(x+\alpha)^k=\sum_{k=0}^{n}\left[{n+1 \atop k+1}\right](\alpha-\beta)^{n-k}(x+\beta)^k\,. \tag{44}$$

Proof. If $\alpha=\beta$, identity (44) is trivially true. Suppose $\alpha\neq\beta$. Recalling that

$$x^{\overline{n}}=\sum_{k=0}^{n}\left[{n \atop k}\right]x^k \qquad\text{and}\qquad (x+1)^{\overline{n}}=\sum_{k=0}^{n}\left[{n+1 \atop k+1}\right]x^k$$

where $x^{\overline{n}}=x(x+1)\cdots(x+n-1)$ is a rising factorial power, then we have

$$\sum_{k=0}^{n}\left[{n \atop k}\right](\alpha-\beta)^{n-k}(x+\alpha)^k=(\alpha-\beta)^n\left(\frac{x+\alpha}{\alpha-\beta}\right)^{\overline{n}}=$$

$$= (\alpha-\beta)^n \left(\frac{x+\beta}{\alpha-\beta}+1\right)^{\overline{n}} = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (\alpha-\beta)^{n-k}(x+\beta)^k .$$

□

Proposition 25. *For every* $n \in \mathbb{N}$ *, we have the identity*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\alpha-\beta)^{n-k} p_k(x+\alpha/s) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (\alpha-\beta)^{n-k} p_k(x+\beta/s) . \tag{45}$$

For $(\alpha,\beta)=(1,0)$ *and for* $(\alpha,\beta)=(0,1)$ *, we have the identities*

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} p_k(x+1/s) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} p_k(x) \tag{46}$$

and

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^{n-k} p_k(x) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (-1)^{n-k} p_k(x+1/s) . \tag{47}$$

In particular, we have the identities

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\alpha-\beta)^{n-k} H_k^{(a)}(x+\alpha/2) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (\alpha-\beta)^{n-k} H_k^{(a)}(x+\beta/2)$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\alpha-\beta)^{n-k} L_k^{(a-k)}(x-\alpha) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (\alpha-\beta)^{n-k} L_k^{(a-k)}(x-\beta)$$

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\alpha-\beta)^{n-k} B_k^{(a)}(x+\alpha) = \sum_{k=0}^{n} \begin{bmatrix} n+1 \\ k+1 \end{bmatrix} (\alpha-\beta)^{n-k} B_k^{(a)}(x+\beta) .$$

Proof. Apply Theorem 2 to identity (44). □

Lemma 26. *For every* $n \in \mathbb{N}$ *, we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k} (n-k)^n x^k = \sum_{k=0}^{n} \binom{n}{k} \left\{ {n \atop n-k} \right\} (n-k)!(x+1)^k \tag{48}$$

$$\sum_{k=0}^{n} \binom{n}{k} (n-k+1)^n x^k = \sum_{k=0}^{n} \binom{n}{k} \left\{ {n+1 \atop n-k+1} \right\} (n-k)!(x+1)^k . \tag{49}$$

Proof. To obtain the first identity, replace y by $\mathrm{e}^t - 1$ in the obvious identity

$$\sum_{k=0}^{n} \binom{n}{k} x^k (y+1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} (x+1)^k y^{n-k}$$

and then extract the coefficient of $\frac{t^n}{n!}$. To obtain the second identity, replace y by $\mathrm{e}^t - 1$ in the previous identity, then multiply both members by e^t and finally extract the coefficient of $\frac{t^n}{n!}$. □

Proposition 27. *For every* $n \in \mathbb{N}$, *we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k} (n-k)^n p_k(x) = \sum_{k=0}^{n} \binom{n}{k} \left\{ {n \atop n-k} \right\} (n-k)! p_k(x+1/s) \tag{50}$$

$$\sum_{k=0}^{n} \binom{n}{k} (n-k+1)^n p_k(x) = \sum_{k=0}^{n} \binom{n}{k} \left\{ {n+1 \atop n-k+1} \right\} (n-k)! p_k(x+1/s) \,. \tag{51}$$

In particular, we have the identities

$$\begin{aligned}
\sum_{k=0}^{n} \binom{n}{k} (n-k)^n H_k(x) &= \sum_{k=0}^{n} \binom{n}{k} \left\{ {n \atop n-k} \right\} (n-k)! H_k(x+1/2) \\
\sum_{k=0}^{n} \binom{n}{k} (n-k+1)^n H_k(x) &= \sum_{k=0}^{n} \binom{n}{k} \left\{ {n+1 \atop n-k+1} \right\} (n-k)! H_k(x+1/2) \\
\sum_{k=0}^{n} \binom{n}{k} (n-k)^n L_k^{(\alpha-k)}(x) &= \sum_{k=0}^{n} \binom{n}{k} \left\{ {n \atop n-k} \right\} (n-k)! L_k^{(\alpha-k)}(x-1) \\
\sum_{k=0}^{n} \binom{n}{k} (n-k+1)^n L_k^{(\alpha-k)}(x) &= \sum_{k=0}^{n} \binom{n}{k} \left\{ {n+1 \atop n-k+1} \right\} (n-k)! L_k^{(\alpha-k)}(x-1) \\
\sum_{k=0}^{n} \binom{n}{k} (n-k)^n B_k^{(a)}(x) &= \sum_{k=0}^{n} \binom{n}{k} \left\{ {n \atop n-k} \right\} (n-k)! B_k^{(a)}(x+1) \\
\sum_{k=0}^{n} \binom{n}{k} (n-k+1)^n B_k^{(a)}(x) &= \sum_{k=0}^{n} \binom{n}{k} \left\{ {n+1 \atop n-k+1} \right\} (n-k)! B_k^{(a)}(x+1) \,.
\end{aligned}$$

Proof. Apply Theorem 2 to identities (48) and (49). □

Lemma 28. *For every* $n \in \mathbb{N}$, *we have the identity*

$$\sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k \atop n} \right\} (x+1)^{n-k} = \sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k+1 \atop n+1} \right\} x^{n-k} \,. \tag{52}$$

Proof. Taking the coefficient of $\frac{t^n}{n!}$ from the series on both sides of the obvious equation

$$\frac{(\mathrm{e}^t-1)^m}{m!}\, \mathrm{e}^{(x+1)t} = \frac{\mathrm{e}^t\,(\mathrm{e}^t-1)^m}{m!}\, \mathrm{e}^{xt} \,,$$

we have the identity

$$\sum_{k=0}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} (x+1)^{n-k} = \sum_{k=0}^{n} \binom{n}{k} \left\{ {k+1 \atop m+1} \right\} x^{n-k} \,.$$

Then, replacing n by $2n$ and m by n, we obtain identity (52). □

Proposition 29. *For every* $n \in \mathbb{N}$, *we have the identity*

$$\sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k \atop n} \right\} p_{n-k}(x+1) = \sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k+1 \atop n+1} \right\} p_{n-k}(x) . \tag{53}$$

In particular, we have the identities

$$\sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k \atop n} \right\} H_{n-k}^{(\nu)}(x+1/2) = \sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k+1 \atop n+1} \right\} H_{n-k}^{(\nu)}(x)$$

$$\sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k \atop n} \right\} L_{n-k}^{(\alpha-n+k)}(x-1) = \sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k+1 \atop n+1} \right\} L_{n-k}^{(\alpha-n+k)}(x)$$

$$\sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k \atop n} \right\} B_{n-k}^{(a)}(x+1) = \sum_{k=0}^{n} \binom{2n}{n+k} \left\{ {n+k+1 \atop n+1} \right\} B_{n-k}^{(a)}(x) .$$

Lemma 30. *For every* $n, r \in \mathbb{N}$, *we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x+k\alpha)^{n+r} = n!\, \alpha^n \sum_{k=0}^{r} \binom{n+r}{n+k} \left\{ {n+k \atop n} \right\} \alpha^k x^{r-k} \tag{54}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} (x-k\alpha)^{n+r} = n!\, \alpha^n \sum_{k=0}^{r} \binom{n+r}{n+k} \left\{ {n+k \atop n} \right\} (-1)^k \alpha^k x^{r-k} . \tag{55}$$

In particular, for $r = 0$, *we have the identities*

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{n-k} (x+k\alpha)^{n} = n!\, \alpha^n \tag{56}$$

$$\sum_{k=0}^{n} \binom{n}{k} (-1)^{k} (x-k\alpha)^{n} = n!\, \alpha^n . \tag{57}$$

Proof. Let $\Delta = E - I$ be the finite difference operator, i.e. the linear operator defined by $\Delta f(x) = f(x+1) - f(x)$. Then, for every $m \in \mathbb{N}$, we have

$$\Delta^m \mathrm{e}^{xt} = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \mathrm{e}^{(x+k)t} = \mathrm{e}^{xt} \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} \mathrm{e}^{kt} = m!\, \mathrm{e}^{xt}\, \frac{(\mathrm{e}^t - 1)^m}{m!} .$$

So, by comparing the coefficient of $\frac{t^n}{n!}$ in the first and in the last member, we have

$$\Delta^m x^n = m! \sum_{k=m}^{n} \binom{n}{k} \left\{ {k \atop m} \right\} x^{n-k} .$$

Replacing n by $n+r$ and m by n, we obtain

$$\Delta^n x^{n+r} = n! \sum_{k=n}^{n+r} \binom{n+r}{n+k} \left\{ {n+k \atop n} \right\} x^{n+r-k} ,$$

that is

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}(x+k)^{n+r}=n!\sum_{k=0}^{r}\binom{n+r}{k}\left\{{k \atop n}\right\}x^{r-k}\,.$$

Finally, replacing x by x/α, we obtain identity (54). □

Notice that identity (57) also appears in [**30**].

Proposition 31. *For every $n,r\in\mathbb{N}$, we have the identities*

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}p_{n+r}(x+k\alpha/s)=n!\,\alpha^{n}\sum_{k=0}^{r}\binom{n+r}{n+k}\left\{{n+k \atop n}\right\}\alpha^{k}p_{r-k}(x) \tag{58}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}p_{n+r}(x-k\alpha/s)=n!\,\alpha^{n}\sum_{k=0}^{r}\binom{n+r}{n+k}\left\{{n+k \atop n}\right\}(-1)^{k}\alpha^{k}p_{r-k}(x)\,. \tag{59}$$

For $r=0$, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}p_{n}(x+k\alpha/s)=n!\,\alpha^{n} \tag{60}$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{k}p_{n}(x-k\alpha/s)=n!\,\alpha^{n}\,. \tag{61}$$

In particular, we have the identities

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}H_{n+r}^{(\nu)}(x+k\alpha/2)=n!\,\alpha^{n}\sum_{k=0}^{r}\binom{n+r}{n+k}\left\{{n+k \atop n}\right\}\alpha^{k}H_{r-k}^{(\nu)}(x)$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}L_{n+r}^{(\nu-n-r)}(x-k\alpha)=n!\,\alpha^{n}\sum_{k=0}^{r}\binom{n+r}{n+k}\left\{{n+k \atop n}\right\}\alpha^{k}L_{r-k}^{(\nu-r+k)}(x)$$

$$\sum_{k=0}^{n}\binom{n}{k}(-1)^{n-k}B_{n+r}^{(\nu)}(x+k\alpha)=n!\,\alpha^{n}\sum_{k=0}^{r}\binom{n+r}{n+k}\left\{{n+k \atop n}\right\}\alpha^{k}B_{r-k}^{(\nu)}(x)\,.$$

Proof. Apply Theorem 2 to identity (54). □

Lemma 32. *For every $m,n\in\mathbb{N}$, we have the identity*

$$\sum_{k=0}^{m}\binom{m}{k}\left\{{n \atop k}\right\}k!\,x^{k}=\sum_{k=0}^{m}\binom{m}{k}\left\{{n \atop k}\right\}_{m-k}k!\,(x-1)^{k} \tag{62}$$

where the coefficients $\left\{{n \atop k}\right\}_{m}$ are the m-Stirling numbers of the second kind.

Proof. By the first of identities (43), we have

$$(1+x(\mathrm{e}^{t}-1))^{m}=\sum_{k=0}^{n}\binom{n}{k}k!\frac{(\mathrm{e}^{t}-1)^{k}}{k!}=\sum_{n\geq 0}\left[\sum_{k=0}^{m}\binom{m}{k}\left\{{n \atop k}\right\}k!\,x^{k}\right]\frac{t^{n}}{n!}\,.$$

On the other hand, recalling [**2, 8**] that the exponential generating series for the m-Stirling numbers of the second kind is

$$\sum_{n\geq k} \left\{ n \atop k \right\}_m \frac{t^n}{n!} = \mathrm{e}^{mt}\, \frac{1}{k!}\, (\mathrm{e}^t - 1)^k \,,$$

we also have

$$\begin{aligned}(1 + x(\mathrm{e}^t - 1))^m &= \mathrm{e}^{mt}(1 + (x-1)\mathrm{e}^{-t}(\mathrm{e}^t - 1))^m \\ &= \sum_{k=0}^{m} \binom{m}{k} k!\, \mathrm{e}^{(m-k)t}\, \frac{(\mathrm{e}^t-1)^k}{k!} = \sum_{n\geq 0} \left[\sum_{k=0}^{m} \binom{m}{k} \left\{ n \atop k \right\}_{m-k} k!\, (x-1)^k \right] \frac{t^n}{n!} \,.\end{aligned}$$

By comparing the coefficients of $\frac{t^n}{n!}$ in both the expressions, we obtain (62). □

Proposition 33. *For every* $m, n \in \mathbb{N}$*, we have the identity*

$$\sum_{k=0}^{m} \binom{m}{k} \left\{ n \atop k \right\} k!\, p_k(x) = \sum_{k=0}^{m} \binom{m}{k} \left\{ n \atop k \right\}_{m-k} k!\, p_k(x - 1/s) \,. \tag{63}$$

In particular, we have the identities

$$\sum_{k=0}^{m} \binom{m}{k} \left\{ n \atop k \right\} k!\, H_k^{(\nu)}(x) = \sum_{k=0}^{m} \binom{m}{k} \left\{ n \atop k \right\}_{m-k} k!\, H_k^{(\nu)}(x - 1/2) \tag{64}$$

$$\sum_{k=0}^{m} \binom{m}{k} \left\{ n \atop k \right\} k!\, L_k^{(\alpha-k)}(x) = \sum_{k=0}^{m} \binom{m}{k} \left\{ n \atop k \right\}_{m-k} k!\, L_k^{(\alpha-k)}(x + 1) \tag{65}$$

$$\sum_{k=0}^{m} \binom{m}{k} \left\{ n \atop k \right\} k!\, B_k^{(a)}(x) = \sum_{k=0}^{m} \binom{m}{k} \left\{ n \atop k \right\}_{m-k} k!\, B_k^{(a)}(x - 1) \,. \tag{66}$$

Proof. Apply Theorem 2 to identity (62). □

REFERENCES

1. P. Appell: *Sur une classe de polynomes*, Ann. Sci. École Norm. Sup. (2) **9** (1880), 119–144.
2. A. Z. Broder: *The r-Stirling numbers*, Discrete Math. **49** (1984), 241–259.
3. S. Capparelli, M. M. Ferrari, E. Munarini, N. Zagaglia Salvi: *A Generalization of the "*Problème des Rencontres*"*, J. Integer Seq. **21** (2018), Article 18.2.8 (26 pages).
4. L. Carlitz: *Eulerian numbers and polynomials of higher order*, Duke Math. J. **27** (1960), 401–423.
5. L. Carlitz: *Products of Appell polynomials*, Collect. Math. **15** (1963), 245–258.
6. L. Carlitz: *Problem 795*, Math. Mag. **44** (1971), 107.
7. B. C. Carlson: *Polynomials satisfying a binomial theorem*, J. Math. Anal. Appl. **32** (1970), 543–558.

8. M. d'Ocagne: *Sur une classe de nombres remarquables*, Amer. J. Math. **9** (1887), 353–380.

9. A. Erdélyi (ed.): "*Higher Trascendental Functions*", The Bateman Manuscript project, Vols. I-III, McGraw-Hill, New York 1953.

10. I. M. Gessel: *Applications of the classical umbral calculus*, Algebra Universalis **49** (2003), 397–434.

11. H. W. Gould: "*Combinatorial Identities, A Standardized Set of Tables Listing 500 Binomial Coefficient Summations*", second edition, Morgantown, W. Va. 1972.

12. H. W. Gould, J. Quaintance: *Bernoulli numbers and a new binomial transform identity*, J. Integer Seq. **17** (2014), Article 14.2.2.

13. R. L. Graham, D. E. Knuth, O. Patashnik: "*Concrete Mathematics*". Addison-Wesley, Reading, MA, 1989.

14. A. Hassen, H. D. Nguyen: *Hypergeometric Bernoulli polynomials and Appell sequences*, Int. J. Number Theory **4** (2008), 767–774.

15. Y. He: *Some results for Carlitz's q-Bernoulli numbers and polynomials*, Appl. Anal. Discrete Math. **8** (2014), 304–319.

16. F. T. Howard: *Numbers generated by the reciprocal of* $e^x - 1 - x$, Math. Comput. **31** (1977), no. 138, 581–598.

17. J. Y. Kang, C. S. Ryoo: *A research on the new polynomials involved with the central factorial numbers, Stirling numbers and others polynomials*, J. Inequal. Appl. **26** (2014), 10 pp.

18. E. Munarini: *Generalization of a binomial identity*, Integers **5** (2005), #A15.

19. E. Munarini: *Riordan matrices and sums of harmonic numbers*, Appl. Anal. Discrete Math. **5** (2011), 176–200.

20. A. F. Neto: *Carlitzs identity for the Bernoulli numbers and Zeon algebra*, J. Integer Seq. **18** (2015), Article 15.5.6.

21. N. E. Nörlund: *Mémoire sur les polynomes de Bernoulli*, Acta Math. **43** (1920), 121–196.

22. N. E. Nörlund: "*Vorlesungen über Differenzenrechnung*", Springer-Verlag, Berlin, (1924). Reprinted by Chelsea Publishing Company, Bronx, New York, (1954).

23. C. Pita Ruiz: *Carlitz–type and other Bernoulli identities*, J. Integer Seq. **19** (2016), Article 16.1.8.

24. H. Prodinger: *A Short proof of Carlitz's Bernoulli number identity*, J. Integer Seq. **17** (2014), Article 14.4.1.

25. J. Riordan: "*An introduction to combinatorial analysis*", John Wiley & Sons, New York 1958.

26. J. Riordan: "*Combinatorial identities*", John Wiley & Sons, New York-London-Sydney, 1968.

27. S. Roman: "*The Umbral Calculus*", Pure and Applied Mathematics, **111**, Academic Press, New York, 1984.

28. S. M. Roman, G.-C. Rota: *The umbral calculus*, Advances in Math. **27** (1978), 95–188.

29. G.-C. Rota: *The number of partitions of a set*, Amer. Math. Monthly **71** (1964), 498–504.

30. S. Ruiz: *An algebraic identity leading to Wilson's theorem*, Math. Gaz. **80** (1996), 579–582.

31. I. M. Sheffer: *Some properties of polynomial sets of type zero*, Duke Math. J. **5** (1939), 590–622.

32. S. Simons: *A curious identity*, Math. Gaz. **85** (2001), 296–298.

33. N. J. A. Sloane: "*On-Line Encyclopedia of Integer Sequences*", http://oeis.org/.

34. P. Vassilev, M. V. Missana: *On one remarkable identity involving Bernoulli numbers*, Notes on Number Theory and Discrete Mathematics **11** (2005), 22–24.

Emanuele Munarini
Politecnico di Milano
Dipartimento di Matematica
Piazza Leonardo da Vinci 32
20133 Milano, Italy
E–mail: emanuele.munarini@polimi.it

(Received 01.10.2016)
(Revised 04.04.2018)