

## ON HILBERT'S INEQUALITY ON TIME SCALES

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In this paper, we will prove some new dynamic inequalities of Hilbert's type on time scales. Our results as special cases extend some obtained dynamic inequalities on time scales and also contain some integral and discrete inequalities as special cases. We prove our main results by using some algebraic inequalities, Hölder's inequality, Jensen's inequality and a simple consequence of Keller's chain rule on time scales.

### 1. Introduction

The original integral Hilbert's inequality is given by

$$(1) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \pi \left( \int_0^\infty f^2(x) dx \right)^{1/2} \left( \int_0^\infty g^2(x) dx \right)^{1/2},$$

where  $f(x)$ ,  $g(x)$  are nonnegative functions and satisfy

$$\int_0^\infty f^2(x) dx < \infty, \quad \text{and} \quad \int_0^\infty g^2(x) dx < \infty.$$

The constant  $\pi$  is the best possible (see [8]). This inequality has been extended by Hardy and Riesz by introducing a pair of conjugate exponents  $p$  and  $q$  with  $1/p + 1/q = 1$ , and proved that (see [8])

$$(2) \quad \int_0^\infty \int_0^\infty \frac{f(x)g(y)}{x+y} dx dy \leq \frac{\pi}{\sin \frac{\pi}{p}} \left( \int_0^\infty f^p(x) dx \right)^{1/p} \left( \int_0^\infty g^q(x) dx \right)^{1/q},$$

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where the constant  $\pi/\sin(\pi/p)$  is the best possible. In [7] Hardy proved the discrete version of (2) which is given by

$$(3) \quad \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} \leq \frac{\pi}{\sin \frac{\pi}{p}} \left( \sum_{m=1}^{\infty} a_m^p \right)^{1/p} \left( \sum_{n=1}^{\infty} b_n^q \right)^{1/q},$$

where  $p > 1$ ,  $1/p + 1/q = 1$ ,  $\{a_m\}_{m=1}^{\infty}$  and  $\{b_n\}_{n=1}^{\infty}$  are nonnegative sequences such that

$$\sum_{m=1}^{\infty} a_m^p < \infty, \quad \text{and} \quad \sum_{n=1}^{\infty} b_n^q < \infty.$$

In the last decades a lot of results which generalize and extend (2) and (3) has been obtained by several authors, we refer to the paper [11] and the papers they are cited. For more details we refer the reader to the papers [9, 10, 11] and the paper [6] which discuss the development of the discrete and continuous Hilbert-type inequalities. Pachpatte in [11] established several new inequalities similar to Hilbert's inequality. One of them is given by

$$(4) \quad \int_0^a \int_0^b \frac{F^p(s)G^q(t)}{s+t} ds dt \leq D(p, q, a, b) \left( \int_0^a (a-s)(F^{p-1}(s)f(s))^2 ds \right)^{\frac{1}{2}} \\ \times \left( \int_0^b (b-t)(G^{q-1}(t)g(t))^2 dt \right)^{\frac{1}{2}},$$

where  $p, q \geq 1$ ,  $F(s) = \int_0^s f(\tau)d\tau \geq 0$  and  $G(t) = \int_0^t g(\theta)d\theta \geq 0$ , for  $s, \tau \in (0, a)$  and  $t, \theta \in (0, b)$ , and

$$D(p, q, a, b) = \frac{1}{2} pq \sqrt{ab}.$$

The discrete version of (4), which has been obtained by Pachpatte [11] is given by

$$(5) \quad \sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{m+n} \leq C(p, q, k, r) \left( \sum_{m=1}^k (k+1-m)(A_m^{p-1} a_m)^2 \right)^{\frac{1}{2}} \\ \times \left( \sum_{n=1}^r (r+1-n)(B_n^{q-1} b_n)^2 \right)^{\frac{1}{2}},$$

where  $p, q \geq 1$ ,  $A_m = \sum_{s=1}^m a_s \geq 0$  and  $B_n = \sum_{t=1}^n b_t \geq 0$ , for  $m = 1, 2, \dots, k$  and  $n = 1, 2, \dots, r$  where  $k$  and  $r$  are natural numbers, and

$$C(p, q, k, r) = \frac{1}{2} pq \sqrt{kr}.$$

In [10] Young-Ho Kim extended the inequality (4) and proved that

$$(6) \quad \int_0^a \int_0^b \frac{F^p(s)G^q(t)}{(s^\alpha + t^\alpha)^{\frac{1}{\alpha}}} ds dt \leq D(p, q, a, b) \left( \int_0^a (a-s)(F^{p-1}(s)f(s))^2 ds \right)^{\frac{1}{2}} \\ \times \left( \int_0^b (b-t)(G^{q-1}(t)g(t))^2 dt \right)^{\frac{1}{2}},$$

where  $p, q \geq 1$ ,  $\alpha > 0$ ,  $F(s) = \int_0^s f(\tau)d\tau \geq 0$  and  $G(t) = \int_0^t g(\theta)d\theta \geq 0$ , for  $s, \tau \in (0, a)$ , and  $t, \theta \in (0, b)$ , and

$$D(p, q, a, b) = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} pq\sqrt{ab}.$$

The discrete version of (6) which has been obtained by Young-Ho Kim [10] and can be considered as the extension of (5) due to Pachpatte [11] is given by

$$(7) \quad \sum_{m=1}^k \sum_{n=1}^r \frac{A_m^p B_n^q}{(m^\alpha + n^\alpha)^{\frac{1}{\alpha}}} \leq C(p, q, k, r, \alpha) \left( \sum_{m=1}^k (k+1-m)(A_m^{p-1} a_m)^2 \right)^{\frac{1}{2}} \\ \times \left( \sum_{n=1}^r (r+1-n)(B_n^{q-1} b_n)^2 \right)^{\frac{1}{2}},$$

where  $p, q \geq 1$ ,  $\alpha > 0$ ,  $A_m = \sum_{s=1}^m a_s \geq 0$  and  $B_n = \sum_{t=1}^n b_t \geq 0$ , for  $m = 1, 2, \dots, k$  and  $n = 1, 2, \dots, r$  where  $k$  and  $r$  are natural numbers, and

$$C(p, q, k, r, \alpha) = \left(\frac{1}{2}\right)^{\frac{1}{\alpha}} pq\sqrt{kr}.$$

In recent years the study of dynamic inequalities on time scales has received a lot of attention (see the book [1]). The general idea is to prove a result for a dynamic inequality where the domain of the unknown function is a so-called time scale  $\mathbb{T}$ , which may be an arbitrary closed subset of the real numbers  $\mathbb{R}$ . The cases when the time scale is equal to the reals or to the integers represent the classical theories of integral and of discrete inequalities. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus, i.e., when  $\mathbb{T} = \mathbb{R}$ ,  $\mathbb{T} = \mathbb{N}$  and  $\mathbb{T} = q^{\mathbb{N}_0} = \{q^t : t \in \mathbb{N}_0\}$  where  $q > 1$ . For recent results of Hilbert's type inequalities on time scales, we refer the reader to the recent book [2].

Following this trend and to develop the study of dynamic inequalities on time scales we will prove some new inequalities of Hilbert's type on time scales. The results as special cases, when  $\mathbb{T} = \mathbb{R}$  and  $\mathbb{T} = \mathbb{N}$  contain the inequalities (6) and (7) due to Young-Ho Kim and also generalize the results by Pachpatte to time scales. The technique in this paper depends on the application of the chain rule, Hölder's inequality, Jensen's inequality on time scales and some algebraic inequalities.

Before we present our main result, let us recall essentials about time scales. For more details of time scale analysis, we refer the reader to the two books by Bohner and Peterson [3], [4] which summarize and organize much of the time scale calculus.

A time scale  $\mathbb{T}$  is an arbitrary nonempty closed subset of the real numbers  $\mathbb{R}$ . We assume throughout that  $\mathbb{T}$  has the topology that it inherits from the standard topology on the real numbers  $\mathbb{R}$ . Let  $a, b \in \mathbb{T}$ , the interval  $[a, b]$  in time scale  $\mathbb{T}$  is defined by  $[a, b] := \{t \in \mathbb{T} : a \leq t \leq b\}$ . The forward jump operator and the backward jump operator are defined by:

$$\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}, \quad \text{and} \quad \rho(t) := \sup\{s \in \mathbb{T} : s > t\}.$$

A point  $t \in \mathbb{T}$  is said to be right-dense if  $\sigma(t) = t$ , right-scattered if  $\sigma(t) > t$ , left-dense if  $\rho(t) = t$  and is left-scattered if  $\rho(t) < t$ . A function  $f : \mathbb{T} \rightarrow \mathbb{R}$  is said to be right-dense continuous (*rd*-continuous) provided it is continuous at all right-dense points in

$\mathbb{T}$  and its left-sided limits exist (finite) at all left-dense points in  $\mathbb{T}$ . The set of all such rd-continuous functions is denoted by  $C_{rd}(\mathbb{T}, \mathbb{R})$ . The graininess  $\mu$  for a time scale  $\mathbb{T}$  is define by  $\mu(t) := \sigma(t) - t$ , and for a function  $f : \mathbb{T} \rightarrow \mathbb{R}$  the notation  $f^\sigma(t)$  denotes  $f(\sigma(t))$ . For a function  $f : \mathbb{T} \rightarrow \mathbb{R}$  the delta derivative is defined by

$$(8) \quad f^\Delta(t) := \lim_{s \rightarrow t, \sigma(t) \neq t} \frac{f^\sigma(t) - f(t)}{\sigma(t) - t}.$$

Here are some basic formulas involving delta derivatives:  $f^\sigma = f + \mu f^\Delta$ ,  $(fg)^\Delta = f^\Delta g + f^\sigma g^\Delta = fg^\Delta + f^\Delta g^\sigma$ ,  $(\frac{f}{g})^\Delta = \frac{f^\Delta g - fg^\Delta}{gg^\sigma}$ , where  $f, g$  are delta differentiable and  $gg^\sigma \neq 0$  in the last formula. For  $a, b \in \mathbb{T}$ , and a delta differentiable function  $f$  the Cauchy integral of  $f^\Delta$  is defined by  $\int_a^b f^\Delta(t) \Delta t = f(b) - f(a)$ . The integration by parts formula on time scales is given by

$$(9) \quad \int_a^b f(t)g^\Delta(t) \Delta t = f(t)g(t)|_a^b - \int_a^b f^\Delta(t)g^\sigma(t) \Delta t.$$

The chain rule formula, (see [3, Theorem 1.90]) that we will use in this paper is

$$(10) \quad (u^\gamma(t))^\Delta = \gamma \left( \int_0^1 [hu^\sigma(t) + (1-h)u(t)]^{\gamma-1} dh \right) u^\Delta(t).$$

where  $\gamma > 1$  and  $u : \mathbb{T} \rightarrow \mathbb{R}$  is delta differentiable function. The Hölder's inequality, (see [3, Theorem 6.13]) on time scales is given by

$$(11) \quad \int_a^b |f(t)g(t)| \Delta t \leq \left[ \int_a^b |f(t)|^\gamma \Delta t \right]^{\frac{1}{\gamma}} \left[ \int_a^b |g(t)|^\nu \Delta t \right]^{\frac{1}{\nu}},$$

where  $a, b \in \mathbb{T}$  and  $f, g \in C_{rd}(\mathbb{T}, \mathbb{R})$ ,  $\gamma > 1$  and  $1/\gamma + 1/\nu = 1$ .

**Theorem 1.** [Fubini's Theorem [5, Theorem 6.13]] Let  $f$  be bounded and delta integrable over a rectangle  $R = [a, b) \times [c, d)$  and suppose that the single integral

$$I(t) = \int_c^d f(t, s) \Delta_2 s,$$

exists for each  $t \in [a, b)$ . Then the iterated integral

$$\int_a^b I(t) \Delta_1 t = \int_a^b \Delta_1 t \int_c^d f(t, s) \Delta_2 s,$$

exists and the equality

$$(12) \quad \iint_R f(t, s) \Delta_1 t \Delta_2 s = \int_a^b \Delta_1 t \int_c^d f(t, s) \Delta_2 s,$$

holds.

It is evident from Theorem 1 that we can interchange the roles  $t$  and  $s$ , that is, we may assume the existence of the double integral and existence of the single integral

$$k(s) = \int_a^b f(t, s) \Delta_1 t,$$

for each  $s \in [c, d)$ . Then Theorem 1 will state the existence of the iterated integral

$$\int_c^d k(s) \Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t, s) \Delta_1 t,$$

and the equality

$$(13) \quad \iint_{\mathbb{R}} f(t, s) \Delta_1 t \Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t, s) \Delta_1 t.$$

If together with the double integral  $\iint_{\mathbb{R}} f(t, s) \Delta_1 t \Delta_2 s$  there exist both single integrals  $I$  and  $K$ , then the formulas (12) and (13) will hold simultaneously, i.e.,

$$(14) \quad \int_a^b \Delta_1 t \int_c^d f(t, s) \Delta_2 s = \int_c^d \Delta_2 s \int_a^b f(t, s) \Delta_1 t.$$

**Theorem 2 (Jensen's Inequality [1]).** *Let  $a, b \in \mathbb{T}$  and  $c, d \in \mathbb{R}$ . Suppose that  $g \in C_{rd}([a, b]_{\mathbb{T}}, (c, d))$  and  $h \in C_{rd}([a, b]_{\mathbb{T}}, \mathbb{R})$  are nonnegative with*

$$\int_a^b h(s) \Delta s > 0.$$

*If  $\Phi \in C((c, d), \mathbb{R})$  is convex, then*

$$(15) \quad \Phi \left( \frac{\int_a^b h(s) g(s) \Delta s}{\int_a^b h(s) \Delta s} \right) \leq \frac{\int_a^b h(s) \Phi(g(s)) \Delta s}{\int_a^b h(s) \Delta s}.$$

## 1. MAIN RESULTS

In this section, we will prove the main results. Throughout this paper, we will assume (usually without mentioning) that the functions in the statements of the theorems are right-dense continuous nonnegative functions and the integrals considered exist and finite. We also assume that all the constants and the boundaries of the integrals that appear in the inequalities are real numbers greater than or equal to zero. In particular, we will assume that  $\gamma > 0$  and  $h, l \geq 1$  be real numbers, and  $p > 1, q > 1$  with  $1/p + 1/q = 1$ .

First, we prove the basic lemma that will be needed in the proofs of the main results and can be considered as the extension of power rules for integrals. The proof depends on the application of the time scales chain rule.

**Lemma 1.** *Let  $\mathbb{T}$  be a time scale with  $x, a \in \mathbb{T}$  such that  $x \geq a$ . If  $\alpha \geq 1$ , then*

$$(16) \quad \left( \int_a^{\sigma(x)} f(\tau) \Delta \tau \right)^\alpha \leq \alpha \int_a^{\sigma(x)} f(\eta) \left( \int_a^{\sigma(\eta)} f(\tau) \Delta \tau \right)^{\alpha-1} \Delta \eta.$$

*Proof.* Define

$$(17) \quad F(x) := \int_a^x f(\tau) \Delta \tau.$$

Applying the chain rule (10), we see that

$$(18) \quad (F^\alpha(x))^\Delta = \alpha \int_0^1 [hF^\sigma(x) + (1-h)F(x)]^{\alpha-1} dh F^\Delta(x).$$

Since  $F(x)$  is nondecreasing and  $\sigma(x) \geq x$ , we have

$$(19) \quad \begin{aligned} [F^\alpha(x)]^\Delta &\leq \alpha \int_0^1 [hF^\sigma(x) + (1-h)F^\sigma(x)]^{\alpha-1} dh f(x) \\ &= \alpha \int_0^1 [F^\sigma(x)]^{\alpha-1} dh f(x) = \alpha [F^\sigma(x)]^{\alpha-1} f(x). \end{aligned}$$

Integrating both sides of (19) from  $a$  to  $\sigma(x)$ , we have

$$(20) \quad \int_a^{\sigma(x)} [F^\alpha(\eta)]^\Delta \Delta \eta \leq \alpha \int_a^{\sigma(x)} f(\eta) [F^\sigma(\eta)]^{\alpha-1} \Delta \eta.$$

Since  $F(a) = 0$ , we get

$$(21) \quad \int_a^{\sigma(x)} [F^\alpha(\eta)]^\Delta \Delta \eta = (F^\sigma(x))^\alpha.$$

Substituting (21) into (20), we have

$$\left( \int_a^{\sigma(x)} f(\tau) \Delta \tau \right)^\alpha \leq \alpha \int_a^{\sigma(x)} f(\eta) \left( \int_a^{\sigma(\eta)} f(\tau) \Delta \tau \right)^{\alpha-1} \Delta \eta,$$

which is the desired inequality (16). The proof is complete. ■

Now, we are ready to state and prove the main results in this paper.

**Theorem 3.** Let  $\mathbb{T}$  be a time scale with  $s, t, t_0, x, y \in \mathbb{T}$ , and define

$$(22) \quad A(s) := \int_{t_0}^s a(\tau) \Delta \tau, \quad \text{and} \quad B(t) := \int_{t_0}^t b(\tau) \Delta \tau.$$

Then for  $\sigma(s) \in [t_0, x]$  and  $\sigma(t) \in [t_0, y]$ , we have

$$(23) \quad \begin{aligned} &\int_{t_0}^x \int_{t_0}^y \frac{A^h(\sigma(s)) B^l(\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\ &\leq C(h, l, p, \gamma) \left[ \int_{t_0}^x (\sigma(x) - \sigma(s)) (a(s) A^{h-1}(\sigma(s)))^q \Delta s \right]^{\frac{1}{q}} \\ &\quad \times \left[ \int_{t_0}^y (\sigma(y) - \sigma(t)) (b(t) B^{l-1}(\sigma(t)))^q \Delta t \right]^{\frac{1}{q}}. \end{aligned}$$

where

$$(24) \quad C(h, l, p, \gamma) := hl \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} (x - t_0)^{\frac{1}{p}} (y - t_0)^{\frac{1}{p}}.$$

*Proof.* By using the inequality (16), we obtain

$$(25) \quad A^h(\sigma(s)) \leq h \int_{t_0}^{\sigma(s)} a(\eta) A^{h-1}(\sigma(\eta)) \Delta\eta,$$

and

$$(26) \quad B^l(\sigma(t)) \leq l \int_{t_0}^{\sigma(t)} b(\eta) B^{l-1}(\sigma(\eta)) \Delta\eta.$$

Applying Hölder's inequality (11) on the right hand side of (25) with indices  $p$  and  $q$ , we have

$$(27) \quad \int_{t_0}^{\sigma(s)} a(\eta) A^{h-1}(\sigma(\eta)) \Delta\eta \leq (\sigma(s) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(s)} (a(\eta) A^{h-1}(\sigma(\eta)))^q \Delta\eta \right)^{\frac{1}{q}}.$$

Applying Hölder's inequality (11) on the right hand side of (26) with indices  $p$  and  $q$ , we have also that

$$(28) \quad \int_{t_0}^{\sigma(t)} b(\eta) B^{l-1}(\sigma(\eta)) \Delta\eta \leq (\sigma(t) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(t)} (b(\eta) B^{l-1}(\sigma(\eta)))^q \Delta\eta \right)^{\frac{1}{q}}.$$

From (25)-(28), we get

$$(29) \quad \begin{aligned} A^h(\sigma(s)) B^l(\sigma(t)) &\leq hl (\sigma(s) - t_0)^{\frac{1}{p}} (\sigma(t) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(s)} (a(\eta) A^{h-1}(\sigma(\eta)))^q \Delta\eta \right)^{\frac{1}{q}} \\ &\quad \times \left( \int_{t_0}^{\sigma(t)} (b(\eta) B^{l-1}(\sigma(\eta)))^q \Delta\eta \right)^{\frac{1}{q}}. \end{aligned}$$

Using the elementary inequality (see [9])

$$(30) \quad \left\{ \prod_{i=1}^n a_i \right\}^{\frac{1}{n}} \leq \left\{ \sum_{i=1}^n \frac{a_i^\gamma}{n} \right\}^{\frac{1}{\gamma}}, \quad 0 < \gamma,$$

for nonnegative real numbers  $a_i$ , for  $i = 1, 2, \dots, n$ , we observe that

$$(31) \quad (a_1 a_2)^{\frac{1}{2}} \leq \left( \frac{a_1^\gamma + a_2^\gamma}{2} \right)^{\frac{1}{\gamma}}.$$

Setting  $a_1 = (\sigma(s) - t_0)$  and  $a_2 = (\sigma(t) - t_0)$  in (31), we get that

$$(32) \quad [(\sigma(s) - t_0)(\sigma(t) - t_0)]^{\frac{1}{2}} \leq \left[ \frac{(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma}{2} \right]^{\frac{1}{\gamma}}.$$

Therefore,

$$(33) \quad [(\sigma(s) - t_0)(\sigma(t) - t_0)]^{\frac{1}{p}} \leq \left[ \frac{(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma}{2} \right]^{\frac{2}{p\gamma}}.$$

From (29) and (33), we have

$$\begin{aligned}
 A^h(\sigma(s))B^l(\sigma(t)) &\leq hl \left( \frac{(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma}{2} \right)^{\frac{2}{p\gamma}} \\
 &\quad \times \left( \int_{t_0}^{\sigma(s)} (a(\eta)A^{h-1}(\sigma(\eta)))^q \Delta\eta \right)^{\frac{1}{q}} \\
 &\quad \times \left( \int_{t_0}^{\sigma(t)} (b(\eta)B^{l-1}(\sigma(\eta)))^q \Delta\eta \right)^{\frac{1}{q}}.
 \end{aligned}
 \tag{34}$$

Dividing both sides of (34) by  $[(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma]^{\frac{2}{p\gamma}}$ , we get

$$\begin{aligned}
 \frac{A^h(\sigma(s))B^l(\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} &\leq hl \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} \left( \int_{t_0}^{\sigma(s)} (a(\eta)A^{h-1}(\sigma(\eta)))^q \Delta\eta \right)^{\frac{1}{q}} \\
 &\quad \times \left( \int_{t_0}^{\sigma(t)} (b(\eta)B^{l-1}(\sigma(\eta)))^q \Delta\eta \right)^{\frac{1}{q}}.
 \end{aligned}
 \tag{35}$$

Integrating both sides of (35) from  $t_0$  to  $y$  and from  $t_0$  to  $x$  and applying Hölder's inequality with indices  $p$  and  $q$ , we obtain

$$\begin{aligned}
 &\int_{t_0}^x \int_{t_0}^y \frac{A^h(\sigma(s))B^l(\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
 &\leq hl \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} (x - t_0)^{\frac{1}{p}} (y - t_0)^{\frac{1}{p}} \left[ \int_{t_0}^x \left( \int_{t_0}^{\sigma(s)} (a(\eta)A^{h-1}(\sigma(\eta)))^q \Delta\eta \right) \Delta s \right]^{\frac{1}{q}} \\
 &\quad \times \left[ \int_{t_0}^y \left( \int_{t_0}^{\sigma(t)} (b(\eta)B^{l-1}(\sigma(\eta)))^q \Delta\eta \right) \Delta t \right]^{\frac{1}{q}}.
 \end{aligned}
 \tag{36}$$

Applying Fubini's Theorem 1 on the right hand side of (36), we have

$$\begin{aligned}
 &\int_{t_0}^x \int_{t_0}^y \frac{A^h(\sigma(s))B^l(\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
 &\leq hl \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} (x - t_0)^{\frac{1}{p}} (y - t_0)^{\frac{1}{p}} \left[ \int_{t_0}^x (x - \sigma(s))(a(s)A^{h-1}(\sigma(s)))^q \Delta s \right]^{\frac{1}{q}} \\
 &\quad \times \left[ \int_{t_0}^y (y - \sigma(t))(b(t)B^{l-1}(\sigma(t)))^q \Delta t \right]^{\frac{1}{q}}. \\
 &= C(h, l, p, \gamma) \left[ \int_{t_0}^x (x - \sigma(s))(a(s)A^{h-1}(\sigma(s)))^q \Delta s \right]^{\frac{1}{q}} \\
 &\quad \times \left[ \int_{t_0}^y (y - \sigma(t))(b(t)B^{l-1}(\sigma(t)))^q \Delta t \right]^{\frac{1}{q}}.
 \end{aligned}$$



By using the facts that  $\sigma(x) \geq x$  and  $\sigma(y) \geq y$ , we obtain

$$\begin{aligned} & \int_{t_0}^x \int_{t_0}^y \frac{A^h(\sigma(s))B^l(\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\ & \leq C(h, l, p, \gamma) \left[ \int_{t_0}^x (\sigma(x) - \sigma(s))(a(s)A^{h-1}(\sigma(s)))^q \Delta s \right]^{\frac{1}{q}} \\ & \quad \times \left[ \int_{t_0}^y (\sigma(y) - \sigma(t))(b(t)B^{l-1}(\sigma(t)))^q \Delta t \right]^{\frac{1}{q}}, \end{aligned}$$

which is the desired inequality (23). The proof is complete. ■

**Remark 1.** If we apply the inequality (30) on the right-hand side of inequality (23), and proceeding as the proof of Theorem 3, we get the following inequality

$$\begin{aligned} & \int_{t_0}^x \int_{t_0}^y \frac{A^h(\sigma(s))B^l(\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\ & \leq C_0(h, l, p, \gamma) \left\{ \left( \int_{t_0}^x (\sigma(x) - \sigma(s))(a(s)A^{h-1}(\sigma(s)))^q \Delta s \right)^\gamma \right. \\ (37) \quad & \left. + \left( \int_{t_0}^y (\sigma(y) - \sigma(t))(b(t)B^{l-1}(\sigma(t)))^q \Delta t \right)^\gamma \right\}^{\frac{2}{q\gamma}}, \end{aligned}$$

where

$$C_0(h, l, p, \gamma) := hl \left( \frac{1}{2} \right)^{\frac{2}{\gamma}} (x - t_0)^{\frac{1}{p}} (y - t_0)^{\frac{1}{p}}.$$

As a special case of Theorem 3 when  $\mathbb{T} = \mathbb{R}$  we have  $\sigma(x) = x$ ,  $\sigma(y) = y$ ,  $\sigma(s) = s$  and  $\sigma(t) = t$  and we get the following result.

**Corollary 1.** Assume that  $a(s)$  and  $b(t)$  are nonnegative functions and define

$$A(s) := \int_0^s a(\tau) d\tau, \quad \text{and} \quad B(t) := \int_0^t b(\tau) d\tau.$$

Then

$$\begin{aligned} & \int_0^x \int_0^y \frac{A^h(s)B^l(t)}{(s^\gamma + t^\gamma)^{\frac{2}{p\gamma}}} ds dt \\ & \leq C_1(h, l, p, \gamma) \left[ \int_0^x (x - s)(a(s)A^{h-1}(s))^q ds \right]^{\frac{1}{q}} \\ (38) \quad & \times \left[ \int_0^y (y - t)(b(t)B^{l-1}(t))^q dt \right]^{\frac{1}{q}}, \end{aligned}$$

where

$$C_1(h, l, p, \gamma) := \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} hl(xy)^{\frac{1}{p}}.$$

**Remark 2.** If we put  $p = q = 2$  in the inequality (38), then we get the result due to Young-Ho Kim [10, Theorem 3.1].

As a special case of Theorem 3 when  $\mathbb{T} = \mathbb{Z}$  we have  $\sigma(x) = x + 1$ ,  $\sigma(y) = y + 1$ ,  $\sigma(s) = s + 1$  and  $\sigma(t) = t + 1$  and we get the following result.

**Corollary 2.** *Assume that  $a(n)$  and  $b(m)$  are nonnegative sequences and define*

$$A(n) = \sum_{s=0}^n a(s), \quad \text{and} \quad B(m) = \sum_{k=0}^m b(k).$$

Then

$$(39) \quad \begin{aligned} & \sum_{n=1}^N \sum_{m=1}^M \frac{A^h(n)B^l(m)}{((n+1)^\gamma + (m+1)^\gamma)^{\frac{2}{p\gamma}}} \\ & \leq C_2(h, l, p, \gamma) \left( \sum_{n=1}^N (N+1 - (n+1))(a(n)A^{h-1}(n))^q \right)^{\frac{1}{q}} \\ & \quad \times \left( \sum_{m=1}^M (M+1 - (m+1))(b(m)B^{l-1}(m))^q \right)^{\frac{1}{q}}, \end{aligned}$$

where

$$C_2(h, l, p, \gamma) := \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} hl(NM)^{\frac{1}{p}}.$$

**Remark 3.** *If we put  $p = q = 2$  in the inequality (39), then we get the result due to Young-Ho Kim [10, Theorem 2.1].*

**Remark 4.** *If we take  $h = l = 1$ , then the inequality (23) becomes*

$$(40) \quad \begin{aligned} & \int_{t_0}^x \int_{t_0}^y \frac{A^\sigma(s)B^\sigma(t)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\ & \leq C_3(p, \gamma) \left[ \int_{t_0}^x (\sigma(x) - \sigma(s))(a(s))^q \Delta s \right]^{\frac{1}{q}} \left[ \int_{t_0}^y (\sigma(y) - \sigma(t))(b(t))^q \Delta t \right]^{\frac{1}{q}}, \end{aligned}$$

where

$$C_3(p, \gamma) := \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} (x - t_0)^{\frac{1}{p}} (y - t_0)^{\frac{1}{p}}.$$

In the next theorems, we assume that there exist two functions  $\Phi$  and  $\Psi$  which are real-valued, nonnegative, convex, and submultiplicative functions defined on  $[0, \infty)$ . The function  $\Phi$  is said to be a submultiplicative on  $[0, \infty)$  if  $\Phi(xy) \leq \Phi(x)\Phi(y)$ , for  $x, y \geq 0$ .

**Theorem 4.** *Let  $\mathbb{T}$  be a time scale with  $s, t, t_0, x, y \in \mathbb{T}$ ,  $A(s)$  and  $B(t)$  be as defined in Theorem 3. Furthermore assume that*

$$(41) \quad F(s) := \int_{t_0}^s f(\tau) \Delta \tau, \quad \text{and} \quad G(t) := \int_{t_0}^t g(\eta) \Delta \eta.$$

Then for  $\sigma(s) \in [t_0, x]$  and  $\sigma(t) \in [t_0, y]$ , we have that

$$\begin{aligned}
 & \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
 & \leq D(p, \gamma) \left\{ \int_{t_0}^x (\sigma(x) - \sigma(s)) \left( f(s) \Phi \left[ \frac{a(s)}{f(s)} \right] \right)^q \Delta s \right\}^{\frac{1}{q}} \\
 (42) \quad & \times \left\{ \int_{t_0}^y (\sigma(y) - \sigma(t)) \left( g(t) \Psi \left[ \frac{b(t)}{g(t)} \right] \right)^q \Delta t \right\}^{\frac{1}{q}},
 \end{aligned}$$

where

$$(43) \quad D(p, \gamma) := \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} \left\{ \int_{t_0}^x \left( \frac{\Phi(F^\sigma(s))}{F^\sigma(s)} \right)^p \Delta s \right\}^{\frac{1}{p}} \left\{ \int_{t_0}^y \left( \frac{\Psi(G^\sigma(t))}{G^\sigma(t)} \right)^p \Delta t \right\}^{\frac{1}{p}}.$$

*Proof.* Since  $\Phi$  is convex and submultiplicative function, we get by applying Jensen's inequality that

$$\begin{aligned}
 \Phi(A^\sigma(s)) &= \Phi \left( \frac{F(\sigma(s)) \int_{t_0}^{\sigma(s)} f(\tau) \frac{a(\tau)}{f(\tau)} \Delta \tau}{\int_{t_0}^{\sigma(s)} f(\tau) \Delta \tau} \right) \\
 &\leq \Phi(F(\sigma(s))) \Phi \left( \frac{\int_{t_0}^{\sigma(s)} f(\tau) \frac{a(\tau)}{f(\tau)} \Delta \tau}{\int_{t_0}^{\sigma(s)} f(\tau) \Delta \tau} \right) \\
 (44) \quad &\leq \frac{\Phi(F^\sigma(s))}{F^\sigma(s)} \int_{t_0}^{\sigma(s)} f(\tau) \Phi \left[ \frac{a(\tau)}{f(\tau)} \right] \Delta \tau.
 \end{aligned}$$

Applying Hölder's inequality with indices  $p$  and  $q$  on the right hand side of (44), we see

$$(45) \quad \Phi(A^\sigma(s)) \leq \frac{\Phi(F^\sigma(s))}{F^\sigma(s)} (\sigma(s) - t_0)^{\frac{1}{p}} \left\{ \int_{t_0}^{\sigma(s)} \left( f(\tau) \Phi \left[ \frac{a(\tau)}{f(\tau)} \right] \right)^q \Delta \tau \right\}^{\frac{1}{q}}.$$

Also, since  $\Psi$  is convex and submultiplicative function, we get by applying Jensen's inequality and Hölder's inequality with indices  $p$  and  $q$  that

$$(46) \quad \Psi(B^\sigma(t)) \leq \frac{\Psi(G^\sigma(t))}{G^\sigma(t)} (\sigma(t) - t_0)^{\frac{1}{p}} \left\{ \int_{t_0}^{\sigma(t)} \left( g(\eta) \Psi \left[ \frac{b(\eta)}{g(\eta)} \right] \right)^q \Delta \eta \right\}^{\frac{1}{q}}.$$

Form (45) and (46), we have

$$\begin{aligned}
 & \Phi(A^\sigma(s))\Psi(B^\sigma(t)) \\
 & \leq (\sigma(s) - t_0)^{\frac{1}{p}} (\sigma(t) - t_0)^{\frac{1}{p}} \left( \frac{\Phi(F^\sigma(s))}{F^\sigma(s)} \left\{ \int_{t_0}^{\sigma(s)} \left( f(\tau) \Phi \left[ \frac{a(\tau)}{f(\tau)} \right] \right)^q \Delta \tau \right\}^{\frac{1}{q}} \right) \\
 (47) \quad & \times \left( \frac{\Psi(G^\sigma(t))}{G^\sigma(t)} \left\{ \int_{t_0}^{\sigma(t)} \left( g(\eta) \Psi \left[ \frac{b(\eta)}{g(\eta)} \right] \right)^q \Delta \eta \right\}^{\frac{1}{q}} \right).
 \end{aligned}$$

Applying the inequality (30) on the term  $(\sigma(s) - t_0)^{\frac{1}{p}}(\sigma(t) - t_0)^{\frac{1}{p}}$ , we get the following inequality

$$\begin{aligned}
& \Phi(A^\sigma(s))\Psi(B^\sigma(t)) \\
& \leq \left( \frac{(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma}{2} \right)^{\frac{2}{p\gamma}} \\
& \quad \times \left( \frac{\Phi(F^\sigma(s))}{F^\sigma(s)} \left\{ \int_{t_0}^{\sigma(s)} \left( f(\tau) \Phi \left[ \frac{a(\tau)}{f(\tau)} \right] \right)^q \Delta\tau \right\}^{\frac{1}{q}} \right) \\
(48) \quad & \quad \times \left( \frac{\Psi(G^\sigma(t))}{G^\sigma(t)} \left\{ \int_{t_0}^{\sigma(t)} \left( g(\eta) \Psi \left[ \frac{b(\eta)}{g(\eta)} \right] \right)^q \Delta\eta \right\}^{\frac{1}{q}} \right).
\end{aligned}$$

From (48), we have

$$\begin{aligned}
& \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \\
& \leq \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} \left( \frac{\Phi(F^\sigma(s))}{F^\sigma(s)} \left\{ \int_{t_0}^{\sigma(s)} \left( f(\tau) \Phi \left[ \frac{a(\tau)}{f(\tau)} \right] \right)^q \Delta\tau \right\}^{\frac{1}{q}} \right) \\
(49) \quad & \quad \times \left( \frac{\Psi(G^\sigma(t))}{G^\sigma(t)} \left\{ \int_{t_0}^{\sigma(t)} \left( g(\eta) \Psi \left[ \frac{b(\eta)}{g(\eta)} \right] \right)^q \Delta\eta \right\}^{\frac{1}{q}} \right).
\end{aligned}$$

Integrating both sides of (49) from  $t_0$  to  $y$  and from  $t_0$  to  $x$ , we obtain

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} \int_{t_0}^x \left( \frac{\Phi(F^\sigma(s))}{F^\sigma(s)} \left\{ \int_{t_0}^{\sigma(s)} \left( f(\tau) \Phi \left[ \frac{a(\tau)}{f(\tau)} \right] \right)^q \Delta\tau \right\}^{\frac{1}{q}} \right) \Delta s \\
(50) \quad & \quad \times \int_{t_0}^y \left( \frac{\Psi(G^\sigma(t))}{G^\sigma(t)} \left\{ \int_{t_0}^{\sigma(t)} \left( g(\eta) \Psi \left[ \frac{b(\eta)}{g(\eta)} \right] \right)^q \Delta\eta \right\}^{\frac{1}{q}} \right) \Delta t.
\end{aligned}$$

Applying Hölder's inequality with indices  $p$  and  $q$  on the right hand side of (50), we have

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} \left\{ \int_{t_0}^x \left( \frac{\Phi(F^\sigma(s))}{F^\sigma(s)} \right)^p \Delta s \right\}^{\frac{1}{p}} \left\{ \int_{t_0}^x \int_{t_0}^{\sigma(s)} \left( f(\tau) \Phi \left[ \frac{a(\tau)}{f(\tau)} \right] \right)^q \Delta \tau \Delta s \right\}^{\frac{1}{q}} \\
& \quad \times \left\{ \int_{t_0}^y \left( \frac{\Psi(G^\sigma(t))}{G^\sigma(t)} \right)^p \Delta t \right\}^{\frac{1}{p}} \left\{ \int_{t_0}^y \int_{t_0}^{\sigma(t)} \left( g(\eta) \Psi \left[ \frac{b(\eta)}{g(\eta)} \right] \right)^q \Delta \eta \Delta t \right\}^{\frac{1}{q}} . \\
& = D(p, \gamma) \left\{ \int_{t_0}^x \int_{t_0}^{\sigma(s)} \left( f(\tau) \Phi \left[ \frac{a(\tau)}{f(\tau)} \right] \right)^q \Delta \tau \Delta s \right\}^{\frac{1}{q}} \\
(51) \quad & \times \left\{ \int_{t_0}^y \int_{t_0}^{\sigma(t)} \left( g(\eta) \Psi \left[ \frac{b(\eta)}{g(\eta)} \right] \right)^q \Delta \eta \Delta t \right\}^{\frac{1}{q}} .
\end{aligned}$$

Applying Fubini's Theorem 1 on the right hand side of (51), we obtain

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B(\sigma(t)))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq D(p, \gamma) \left\{ \int_{t_0}^x (x - \sigma(s)) \left( f(s) \Phi \left[ \frac{a(s)}{f(s)} \right] \right)^q \Delta s \right\}^{\frac{1}{q}} \\
(52) \quad & \times \left\{ \int_{t_0}^y (y - \sigma(t)) \left( g(t) \Psi \left[ \frac{b(t)}{g(t)} \right] \right)^q \Delta t \right\}^{\frac{1}{q}} .
\end{aligned}$$

Since  $\sigma(x) \geq x$  and  $\sigma(y) \geq y$ , we have

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq D(p, \gamma) \left\{ \int_{t_0}^x (\sigma(x) - \sigma(s)) \left( f(s) \Phi \left[ \frac{a(s)}{f(s)} \right] \right)^q \Delta s \right\}^{\frac{1}{q}} \\
& \quad \times \left\{ \int_{t_0}^y (\sigma(y) - \sigma(t)) \left( g(t) \Psi \left[ \frac{b(t)}{g(t)} \right] \right)^q \Delta t \right\}^{\frac{1}{q}} .
\end{aligned}$$

The proof is complete. ■

**Remark 5.** If we apply the inequality (30) on the right-hand side of inequality (42), and proceeding as the proof of Theorem 4, we get the following inequality

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))}{(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq D_0(p, \gamma) \left\{ \left[ \int_{t_0}^x (\sigma(x) - \sigma(s)) \left( f(s) \Phi \left[ \frac{a(s)}{f(s)} \right] \right)^q \Delta s \right]^\gamma \right. \\
(53) \quad & \left. + \left[ \int_{t_0}^y (\sigma(y) - \sigma(t)) \left( g(t) \Psi \left[ \frac{b(t)}{g(t)} \right] \right)^q \Delta t \right]^\gamma \right\}^{\frac{2}{q\gamma}} ,
\end{aligned}$$

where

$$D_0(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{\gamma}} \left\{ \int_{t_0}^x \left( \frac{\Phi(F^\sigma(s))}{F^\sigma(s)} \right)^p \Delta s \right\}^{\frac{1}{p}} \left\{ \int_{t_0}^y \left( \frac{\Psi(G^\sigma(t))}{G^\sigma(t)} \right)^p \Delta t \right\}^{\frac{1}{p}}.$$

As a special case of Theorem 4 when  $\mathbb{T} = \mathbb{R}$ , we have  $\sigma(x) = x$ ,  $\sigma(y) = y$ ,  $\sigma(s) = s$  and  $\sigma(t) = t$ . Then we get the following result.

**Corollary 3.** Assume that  $a(s)$ ,  $b(t)$ ,  $f(s)$  and  $g(t)$  are nonnegative functions and define

$$A(s) := \int_0^s a(\tau) d\tau, \quad B(t) := \int_0^t b(\tau) d\tau, \quad F(s) := \int_0^s f(\tau) d\tau, \quad G(t) := \int_0^t g(\tau) d\tau.$$

Then

$$(54) \quad \int_0^x \int_0^y \frac{\Phi(A(s))\Psi(B(t))}{(s^\gamma + t^\gamma)^{\frac{2}{p\gamma}}} ds dt \leq D_1(p, \gamma) \left\{ \int_0^x (x-s) \left( f(s) \Phi \left[ \frac{a(s)}{f(s)} \right] \right)^q ds \right\}^{\frac{1}{q}} \\ \times \left\{ \int_0^y (y-t) \left( g(t) \Psi \left[ \frac{b(t)}{g(t)} \right] \right)^q dt \right\}^{\frac{1}{q}},$$

where

$$D_1(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} \left\{ \int_0^x \left( \frac{\Phi(F(s))}{F(s)} \right)^p ds \right\}^{\frac{1}{p}} \left\{ \int_0^y \left( \frac{\Psi(G(t))}{G(t)} \right)^p dt \right\}^{\frac{1}{p}}.$$

**Remark 6.** If we put  $p = q = 2$  in the inequality (54), then we get the result due to Young-Ho Kim [10, Theorem 3.3].

As a special case of Theorem 4 when  $\mathbb{T} = \mathbb{Z}$  we have  $\sigma(x) = x + 1$ ,  $\sigma(y) = y + 1$ ,  $\sigma(s) = s + 1$  and  $\sigma(t) = t + 1$  and we get the following result.

**Corollary 4.** Assume that  $a(n)$ ,  $b(m)$ ,  $f(n)$  and  $g(m)$  are nonnegative sequences and define

$$A(n) := \sum_{s=0}^n a(s), \quad B(m) := \sum_{k=0}^m b(k), \quad F(n) := \sum_{s=0}^n f(s), \quad G(m) := \sum_{k=0}^m g(k).$$

Then

$$(55) \quad \sum_{n=1}^N \sum_{m=1}^M \frac{\Phi(A(n))\Psi(B(m))}{((n+1)^\gamma + (m+1)^\gamma)^{\frac{2}{p\gamma}}} \\ \leq D_2(p, \gamma) \left\{ \sum_{n=1}^N (N+1 - (n+1)) \left( f(n) \Phi \left[ \frac{a(n)}{f(n)} \right] \right)^q \right\}^{\frac{1}{q}} \\ \times \left\{ \sum_{m=1}^M (M+1 - (m+1)) \left( g(m) \Psi \left[ \frac{b(m)}{g(m)} \right] \right)^q \right\}^{\frac{1}{q}}.$$

where

$$D_2(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} \left\{ \sum_{n=1}^N \left( \frac{\Phi(F(n))}{F(n)} \right)^p \right\}^{\frac{1}{p}} \left\{ \sum_{m=1}^M \left( \frac{\Psi(G(m))}{G(m)} \right)^p \right\}^{\frac{1}{p}}.$$

**Remark 7.** If we put  $p = q = 2$  in the inequality (55), then we get the result due to Young-Ho Kim [10, Theorem 2.3].

**Theorem 5.** Let  $\mathbb{T}$  be a time scale with  $s, t, t_0, x, y \in \mathbb{T}$ . Furthermore assume that

$$(56) \quad A(s) := \frac{1}{s - t_0} \int_{t_0}^s a(\tau) \Delta\tau, \quad \text{and} \quad B(t) := \frac{1}{t - t_0} \int_{t_0}^t b(\eta) \Delta\eta,$$

then for  $\sigma(s) \in [t_0, x]$  and  $\sigma(t) \in [t_0, y]$ , we have

$$(57) \quad \begin{aligned} & \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s)) \Psi(B^\sigma(t)) (\sigma(s) - t_0) (\sigma(t) - t_0)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\ & \leq H(p, \gamma) \left\{ \int_{t_0}^x (\sigma(x) - \sigma(s)) (\Phi[a(s)])^q \Delta s \right\}^{\frac{1}{q}} \\ & \quad \times \left\{ \int_{t_0}^y (\sigma(y) - \sigma(t)) (\Psi[b(t)])^q \Delta t \right\}^{\frac{1}{q}}, \end{aligned}$$

where

$$H(p, \gamma) := \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} (x - t_0)^{\frac{1}{p}} (y - t_0)^{\frac{1}{p}}.$$

*Proof.* From (56), we see

$$(58) \quad \Phi(A^\sigma(s)) = \Phi \left( \frac{1}{\sigma(s) - t_0} \int_{t_0}^{\sigma(s)} a(\tau) \Delta\tau \right).$$

Applying Jensen's inequality on the right hand side of (58), we get

$$(59) \quad \Phi(A^\sigma(s)) \leq \frac{1}{\sigma(s) - t_0} \int_{t_0}^{\sigma(s)} \Phi[a(\tau)] \Delta\tau.$$

Applying Hölder's inequality with indices  $p$  and  $q$  on the right hand side of (59), we have

$$(60) \quad \Phi(A^\sigma(s)) \leq \frac{1}{\sigma(s) - t_0} (\sigma(s) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(s)} (\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}}.$$

This implies that

$$(61) \quad \Phi(A^\sigma(s)) (\sigma(s) - t_0) \leq (\sigma(s) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(s)} (\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}}.$$

Similarly, we obtain

$$(62) \quad \Psi(B^\sigma(t)) (\sigma(t) - t_0) \leq (\sigma(t) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(t)} (\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}}.$$

From (61) and (62), we get

$$\begin{aligned}
& \Phi(A^\sigma(s))\Psi(B^\sigma(t))(\sigma(s) - t_0)(\sigma(t) - t_0) \\
& \leq (\sigma(s) - t_0)^{\frac{1}{p}} (\sigma(t) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(s)} (\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}} \\
(63) \quad & \times \left( \int_{t_0}^{\sigma(t)} (\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}}.
\end{aligned}$$

Applying the inequality (30) on the term  $(\sigma(s) - t_0)^{\frac{1}{p}}(\sigma(t) - t_0)^{\frac{1}{p}}$ , we get the following inequality

$$\begin{aligned}
& \Phi(A^\sigma(s))\Psi(B^\sigma(t))(\sigma(s) - t_0)(\sigma(t) - t_0) \\
& \leq \left( \frac{(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma}{2} \right)^{\frac{2}{p\gamma}} \left( \int_{t_0}^{\sigma(s)} (\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}} \\
(64) \quad & \times \left( \int_{t_0}^{\sigma(t)} (\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}}.
\end{aligned}$$

Dividing both sides of (64) by  $[(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma]^{\frac{2}{p\gamma}}$ , we get that

$$\begin{aligned}
& \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))(\sigma(s) - t_0)(\sigma(t) - t_0)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \\
(65) \quad & \leq \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} \left( \int_{t_0}^{\sigma(s)} (\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}} \left( \int_{t_0}^{\sigma(t)} (\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}}.
\end{aligned}$$

Integrating both sides of (65) from  $t_0$  to  $y$  and from  $t_0$  to  $x$ , we obtain

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))(\sigma(s) - t_0)(\sigma(t) - t_0)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} \left[ \int_{t_0}^x \left( \int_{t_0}^{\sigma(s)} (\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}} \Delta s \right] \\
(66) \quad & \times \left[ \int_{t_0}^y \left( \int_{t_0}^{\sigma(t)} (\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}} \Delta t \right].
\end{aligned}$$

Applying Hölder's inequality again with indices  $p$  and  $q$  on the right hand side of (66), we



obtain

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))(\sigma(s)-t_0)(\sigma(t)-t_0)}{((\sigma(s)-t_0)^\gamma + (\sigma(t)-t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} (x-t_0)^{\frac{1}{p}} (y-t_0)^{\frac{1}{p}} \left[ \int_{t_0}^x \left( \int_{t_0}^{\sigma(s)} (\Phi[a(\tau)])^q \Delta \tau \right) \Delta s \right]^{\frac{1}{q}} \\
& \quad \times \left[ \int_{t_0}^y \left( \int_{t_0}^{\sigma(t)} (\Psi[b(\eta)])^q \Delta \eta \right) \Delta t \right]^{\frac{1}{q}} . \\
& = H(p, \gamma) \left[ \int_{t_0}^x \left( \int_{t_0}^{\sigma(s)} (\Phi[a(\tau)])^q \Delta \tau \right) \Delta s \right]^{\frac{1}{q}} \\
(67) \quad & \times \left[ \int_{t_0}^y \left( \int_{t_0}^{\sigma(t)} (\Psi[b(\eta)])^q \Delta \eta \right) \Delta t \right]^{\frac{1}{q}} .
\end{aligned}$$

Applying Fubini's Theorem 1 on the right-hand side of (67), we get

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))(\sigma(s)-t_0)(\sigma(t)-t_0)}{((\sigma(s)-t_0)^\gamma + (\sigma(t)-t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq H(p, \gamma) \left\{ \int_{t_0}^x (x-\sigma(s))(\Phi[a(s)])^q \Delta s \right\}^{\frac{1}{q}} \\
(68) \quad & \times \left\{ \int_{t_0}^y (y-\sigma(t))(\Psi[b(t)])^q \Delta t \right\}^{\frac{1}{q}} .
\end{aligned}$$

By using the fact  $\sigma(x) \geq x$  and  $\sigma(y) \geq y$ , we obtain

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))(\sigma(s)-t_0)(\sigma(t)-t_0)}{((\sigma(s)-t_0)^\gamma + (\sigma(t)-t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq H(p, \gamma) \left\{ \int_{t_0}^x (\sigma(x)-\sigma(s))(\Phi[a(s)])^q \Delta s \right\}^{\frac{1}{q}} \\
& \quad \times \left\{ \int_{t_0}^y (\sigma(y)-\sigma(t))(\Psi[b(t)])^q \Delta t \right\}^{\frac{1}{q}} .
\end{aligned}$$

which is the desired inequality (57). The proof is complete. ■

**Remark 8.** If we apply the inequality (30) on the right-hand side of inequality (57), and proceeding as the proof of Theorem 5, we get the following inequality

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{(\sigma(s)-t_0)(\sigma(t)-t_0)}{((\sigma(s)-t_0)^\gamma + (\sigma(t)-t_0)^\gamma)^{\frac{2}{p\gamma}}} \Phi(A^\sigma(s))\Psi(B^\sigma(t)) \Delta s \Delta t \\
& \leq H_0(p, \gamma) \left\{ \left[ \int_{t_0}^x (\sigma(x)-\sigma(s))(\Phi[a(s)])^q \Delta s \right]^\gamma \right. \\
(69) \quad & \left. + \left[ \int_{t_0}^y (\sigma(y)-\sigma(t))(\Psi[b(t)])^q \Delta t \right]^\gamma \right\}^{\frac{2}{q\gamma}} ,
\end{aligned}$$

where

$$H_0(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{\gamma}} [(x-t_0)(y-t_0)]^{\frac{1}{p}}.$$

As a special case of Theorem 5 when  $\mathbb{T} = \mathbb{R}$ , we have  $\sigma(x) = x$ ,  $\sigma(y) = y$ ,  $\sigma(s) = s$  and  $\sigma(t) = t$  and we get the following result.

**Corollary 5.** *Assume that  $a(s)$  and  $b(t)$  are nonnegative functions and define*

$$A(s) := \frac{1}{s} \int_0^s a(\tau) d\tau \quad \text{and} \quad B(t) := \frac{1}{t} \int_0^t b(\tau) d\tau.$$

Then

$$(70) \quad \int_0^x \int_0^y \frac{st\Phi(A(s))\Psi(B(t))}{(s^\gamma + t^\gamma)^{\frac{2}{p\gamma}}} ds dt \leq H_1(p, \gamma) \left\{ \int_0^x (x-s)[\Phi(a(s))]^q ds \right\}^{\frac{1}{q}} \\ \times \left\{ \int_0^y (y-t)[\Psi(b(t))]^q dt \right\}^{\frac{1}{q}},$$

where

$$H_1(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} (xy)^{\frac{1}{p}}.$$

**Remark 9.** *If we put  $p = q = 2$  in the inequality (70), then we get the result due to Young-Ho Kim [10, Theorem 3.4].*

As a special case of Theorem 5 when  $\mathbb{T} = \mathbb{Z}$  we have  $\sigma(x) = x + 1$ ,  $\sigma(y) = y + 1$ ,  $\sigma(s) = s + 1$  and  $\sigma(t) = t + 1$ . Then we get the following result.

**Corollary 6.** *Assume that  $a(n)$  and  $b(m)$  are nonnegative sequences and define*

$$A(n) := \frac{1}{n} \sum_{s=0}^n a(s), \quad \text{and} \quad B(m) := \frac{1}{m} \sum_{k=0}^m b(k).$$

Then

$$(71) \quad \sum_{n=1}^N \sum_{m=1}^M \frac{(n+1)(m+1)\Phi(A(n))\Psi(B(m))}{((n+1)^\gamma + (m+1)^\gamma)^{\frac{2}{p\gamma}}} \\ \leq H_2(p, \gamma) \left\{ \sum_{n=1}^N (N+1 - (n+1))[\Phi(a(n))]^q \right\}^{\frac{1}{q}} \\ \times \left\{ \sum_{m=1}^M (M+1 - (m+1))[\Psi(b(m))]^q \right\}^{\frac{1}{q}},$$

where

$$H_2(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} (NM)^{\frac{1}{p}}.$$

**Remark 10.** *If we put  $p = q = 2$  in the inequality (71), then we get the result due to Young-Ho Kim [10, Theorem 2.5].*

In the following theorem, we prove a new dynamic inequality with two different weighted functions.

**Theorem 6.** *Let  $\mathbb{T}$  be a time scale with  $s, t, t_0, x, y \in \mathbb{T}$ ,  $F$  and  $G$  be as defined in Theorem 4. Furthermore assume that*

$$(72) \quad A(s) := \frac{1}{F(s)} \int_{t_0}^s f(\tau) a(\tau) \Delta\tau, \quad \text{and} \quad B(t) := \frac{1}{G(t)} \int_{t_0}^t g(\eta) b(\eta) \Delta\eta.$$

Then for  $\sigma(s) \in [t_0, x]$  and  $\sigma(t) \in [t_0, y]$ , we have

$$(73) \quad \begin{aligned} & \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s)) \Psi(B^\sigma(t)) F^\sigma(s) G^\sigma(t)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\ & \leq K(p, \gamma) \left[ \int_{t_0}^x (\sigma(x) - \sigma(s)) (f(s) \Phi[a(s)])^q \Delta s \right]^{\frac{1}{q}} \\ & \quad \times \left[ \int_{t_0}^y (\sigma(y) - \sigma(t)) (g(t) \Psi[b(t)])^q \Delta t \right]^{\frac{1}{q}}, \end{aligned}$$

where

$$K(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} (x - t_0)^{\frac{1}{p}} (y - t_0)^{\frac{1}{p}}.$$

*Proof.* From (72), we see

$$(74) \quad \Phi(A^\sigma(s)) = \Phi\left(\frac{1}{F^\sigma(s)} \int_{t_0}^{\sigma(s)} f(\tau) a(\tau) \Delta\tau\right).$$

Applying Jensen's inequality on the right hand side of (74), we observe that

$$\Phi(A^\sigma(s)) \leq \frac{1}{F^\sigma(s)} \int_{t_0}^{\sigma(s)} f(\tau) \Phi[a(\tau)] \Delta\tau.$$

Applying Hölder's inequality with indices  $p$  and  $q$  on the right hand side of (74), we obtain

$$(75) \quad \Phi(A^\sigma(s)) \leq \frac{(\sigma(s) - t_0)^{\frac{1}{p}}}{F^\sigma(s)} \left( \int_{t_0}^{\sigma(s)} (f(\tau) \Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}}.$$

From (75), we get

$$(76) \quad \Phi(A^\sigma(s)) F^\sigma(s) \leq (\sigma(s) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(s)} (f(\tau) \Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}}.$$

Similarly, we obtain

$$(77) \quad \Psi(B^\sigma(t)) G^\sigma(t) \leq (\sigma(t) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(t)} (g(\eta) \Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}}.$$

From (76) and (77), we observe that

$$\begin{aligned}
& \Phi(A^\sigma(s))\Psi(B^\sigma(t))F^\sigma(s)G^\sigma(t) \\
& \leq (\sigma(s) - t_0)^{\frac{1}{p}}(\sigma(t) - t_0)^{\frac{1}{p}} \left( \int_{t_0}^{\sigma(s)} (f(\tau)\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}} \\
(78) \quad & \times \left( \int_{t_0}^{\sigma(t)} (g(\eta)\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}}.
\end{aligned}$$

Applying the inequality (30) on the term  $(\sigma(s) - t_0)^{\frac{1}{p}}(\sigma(t) - t_0)^{\frac{1}{p}}$ , we get the following inequality

$$\begin{aligned}
& \Phi(A^\sigma(s))\Psi(B^\sigma(t))F^\sigma(s)G^\sigma(t) \\
& \leq \left( \frac{(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma}{2} \right)^{\frac{2}{p\gamma}} \left( \int_{t_0}^{\sigma(s)} (f(\tau)\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}} \\
(79) \quad & \times \left( \int_{t_0}^{\sigma(t)} (g(\eta)\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}}.
\end{aligned}$$

Dividing both sides of (79) by  $[(\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma]^{\frac{2}{p\gamma}}$ , we get that

$$\begin{aligned}
& \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))F^\sigma(s)G^\sigma(t)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \\
(80) \quad & \leq \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} \left( \int_{t_0}^{\sigma(s)} (f(\tau)\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}} \left( \int_{t_0}^{\sigma(t)} (g(\eta)\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}}.
\end{aligned}$$

Integrating both sides of (80) from  $t_0$  to  $y$  and from  $t_0$  to  $x$ , we obtain

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))F^\sigma(s)G^\sigma(t)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq \left( \frac{1}{2} \right)^{\frac{2}{p\gamma}} \left( \int_{t_0}^x \left( \int_{t_0}^{\sigma(s)} (f(\tau)\Phi[a(\tau)])^q \Delta\tau \right)^{\frac{1}{q}} \Delta s \right) \\
(81) \quad & \times \left( \int_{t_0}^y \left( \int_{t_0}^{\sigma(t)} (g(\eta)\Psi[b(\eta)])^q \Delta\eta \right)^{\frac{1}{q}} \Delta t \right).
\end{aligned}$$

Applying Hölder's inequality again with indices  $p$  and  $q$  on the right hand side of (81), we

have

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))F^\sigma(s)G^\sigma(t)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} (x - t_0)^{\frac{1}{p}} (y - t_0)^{\frac{1}{p}} \left[ \int_{t_0}^x \left( \int_{t_0}^{\sigma(s)} (f(\tau)\Phi[a(\tau)])^q \Delta \tau \right) \Delta s \right]^{\frac{1}{q}} \\
& \quad \times \left[ \int_{t_0}^y \left( \int_{t_0}^{\sigma(t)} (g(\eta)\Psi[b(\eta)])^q \Delta \eta \right) \Delta t \right]^{\frac{1}{q}}. \\
& = K(p, \gamma) \left[ \int_{t_0}^x \left( \int_{t_0}^{\sigma(s)} (f(\tau)\Phi[a(\tau)])^q \Delta \tau \right) \Delta s \right]^{\frac{1}{q}} \\
(82) \quad & \times \left[ \int_{t_0}^y \left( \int_{t_0}^{\sigma(t)} (g(\eta)\Psi[b(\eta)])^q \Delta \eta \right) \Delta t \right]^{\frac{1}{q}}.
\end{aligned}$$

Applying Fubini's Theorem 1 on the right-hand side of (82), we get

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))F^\sigma(s)G^\sigma(t)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq K(p, \gamma) \left[ \int_{t_0}^x (x - \sigma(s))(f(s)\Phi[a(s)])^q \Delta s \right]^{\frac{1}{q}} \\
(83) \quad & \times \left[ \int_{t_0}^y (y - \sigma(t))(g(t)\Psi[b(t)])^q \Delta t \right]^{\frac{1}{q}}.
\end{aligned}$$

By using the fact  $\sigma(x) \geq x$  and  $\sigma(y) \geq y$ , we obtain

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))F^\sigma(s)G^\sigma(t)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq K(p, \gamma) \left[ \int_{t_0}^x (\sigma(x) - \sigma(s))(f(s)\Phi[a(s)])^q \Delta s \right]^{\frac{1}{q}} \\
& \quad \times \left[ \int_{t_0}^y (\sigma(y) - \sigma(t))(g(t)\Psi[b(t)])^q \Delta t \right]^{\frac{1}{q}},
\end{aligned}$$

which is the desired inequality (73). The proof is complete. ■

**Remark 11.** If we apply the inequality (30) on the right-hand side of inequality (73), and proceeding as the proof of Theorem 6, we get the following inequality

$$\begin{aligned}
& \int_{t_0}^x \int_{t_0}^y \frac{\Phi(A^\sigma(s))\Psi(B^\sigma(t))F^\sigma(s)G^\sigma(t)}{((\sigma(s) - t_0)^\gamma + (\sigma(t) - t_0)^\gamma)^{\frac{2}{p\gamma}}} \Delta s \Delta t \\
& \leq K_0(p, \gamma) \left\{ \left[ \int_{t_0}^x (\sigma(x) - \sigma(s))(f(s)\Phi[a(s)])^q \Delta s \right]^\gamma \right. \\
(84) \quad & \left. + \left[ \int_{t_0}^y (\sigma(y) - \sigma(t))(g(t)\Psi[b(t)])^q \Delta t \right]^\gamma \right\}^{\frac{2}{q\gamma}},
\end{aligned}$$

where

$$K_0(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{\gamma}} (x-t)^{\frac{1}{p}} (y-t)^{\frac{1}{p}}.$$

As a special case of Theorem 6 when  $\mathbb{T} = \mathbb{R}$ , we have  $\sigma(x) = x$ ,  $\sigma(y) = y$ ,  $\sigma(s) = s$  and  $\sigma(t) = t$  and we get the following result.

**Corollary 7.** *Assume that  $a(s)$ ,  $b(t)$ ,  $f(s)$  and  $g(t)$  are nonnegative functions and define*

$$\begin{aligned} A(s) &: = \frac{1}{F(s)} \int_0^s f(\tau)a(\tau)d\tau, \quad B(t) := \frac{1}{G(t)} \int_0^t g(\tau)b(\tau)d\tau, \\ F(s) &: = \int_0^s f(\tau)d\tau, \quad \text{and } G(t) := \int_0^t g(\tau)d\tau. \end{aligned}$$

Then

$$\begin{aligned} \int_0^x \int_0^y \frac{\Phi(A(s))\Psi(B(t))F(s)G(t)}{(s^\gamma + t^\gamma)^{\frac{2}{p\gamma}}} ds dt &\leq K_1(p, \gamma) \left[ \int_0^x (x-s)(f(s)\Phi[a(s)])^q ds \right]^{\frac{1}{q}} \\ (85) \quad &\times \left\{ \int_0^y (y-t)(g(t)\Psi[b(t)])^q dt \right\}^{\frac{1}{q}}, \end{aligned}$$

where

$$K_1(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} (xy)^{\frac{1}{p}}.$$

**Remark 12.** *If we put  $p = q = 2$  in the inequality (85), then we get the result due to Young-Ho Kim [10, Theorem 3.5].*

As a special case of Theorem 6 when  $\mathbb{T} = \mathbb{Z}$  we have  $\sigma(x) = x + 1$ ,  $\sigma(y) = y + 1$ ,  $\sigma(s) = s + 1$  and  $\sigma(t) = t + 1$  and we get the following result.

**Corollary 8.** *Assume that  $a(n)$ ,  $b(m)$ ,  $f(n)$  and  $g(m)$  are nonnegative sequences and define*

$$\begin{aligned} A(n) &: = \frac{1}{F(n)} \sum_{s=0}^n f(s)a(s), \quad B(m) := \frac{1}{G(m)} \sum_{k=0}^m g(k)b(k), \\ F(n) &: = \sum_{s=0}^n f(s), \quad \text{and } G(m) := \sum_{k=0}^m g(k). \end{aligned}$$

Then

$$\begin{aligned} \sum_{n=1}^N \sum_{m=1}^M \frac{F(n)G(m)\Phi(A(n))\Psi(B(m))}{((n+1)^\gamma + (m+1)^\gamma)^{\frac{2}{p\gamma}}} \\ (86) \quad &\leq K_2(p, \gamma) \left\{ \sum_{n=1}^N (N+1 - (n+1)) (f(n)\Phi[a(n)])^q \right\}^{\frac{1}{q}} \\ &\times \left\{ \sum_{m=1}^M (M+1 - (m+1)) (g(m)\Psi[b(m)])^q \right\}^{\frac{1}{q}}, \end{aligned}$$

where

$$K_2(p, \gamma) := \left(\frac{1}{2}\right)^{\frac{2}{p\gamma}} (NM)^{\frac{1}{p}}.$$

**Remark 13.** If we put  $p = q = 2$  in the inequality (86), then we get the result due to Young-Ho Kim [10, Theorem 2.6].

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