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CONNECTIVITY AND SOME OTHER PROPERTIES OF GENERALIZED SIERPIŃSKI GRAPHS

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If G is a graph and n a positive integer, then the generalized Sierpiński graph S_G^n is a fractal-like graph that uses G as a building block. The construction of S_G^n generalizes the classical Sierpiński graphs S_p^n, where the role of G is played by the complete graph K_p. An explicit formula for the number of connected components in S_G^n is given and it is proved that the (edge-)connectivity of S_G^n equals the (edge-)connectivity of G. It is demonstrated that S_G^n contains a 1-factor if and only if G contains a 1-factor. Hamiltonicity of generalized Sierpiński graphs is also discussed.

1. INTRODUCTION

Sierpiński graphs of base 3 were often studied because of their relation to the famous Sierpiński triangle fractal(s), see [**27**] for the latter. Sierpiński graphs of an arbitrary base still posses the same fractal-like structure. The generalized Sierpiński graphs considered in this paper form even more general structures in which a base graph G can be arbitrary and the resulting graphs are self-similar fractal-like graphs with a structure originating from G.

Sierpiński graphs S_p^n equipped with the Sierpiński labeling were introduced in [**15**]. Earlier, unlabelled graphs almost (but not completely) the same as the unlabelled Sierpiński graphs were studied by Della Vecchia and Sanges in [**5**] as a model for interconnection networks under the name WK-recursive networks. The Sierpiński labeling which was, by the way, motivated by investigations of a type of universal topological spaces, turned out to be the key tool in investigations

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of several aspects of Sierpiński graphs and their applications. In particular, this labeling enabled Romik [**22**] to solve the so-called P2-decision problem for the Tower of Hanoi puzzle by solving it on S_3^n by means of an automaton which is now known as Romik's automaton. Almost a decade later Hinz and Holz auf der Heide [**12**] generalized Romik's automaton to an arbitrary p. For more information on the applicability of Sierpiński graphs in the theory of the Tower of Hanoi see the book [**13**]. For additional investigations and applications of Sierpiński graphs see the recent papers [**1, 16**] and the survey [**14**] which contains no less than 121 references.

In 2011, Gravier, Kovše, and Parreau [**11**] proposed an extension of Sierpiński graphs in which an arbitrary graph can serve as a building block in the construction, while in the case of the classical Sierpiński graphs this role is reserved for complete graphs. The authors of [**11**] announced several results about generalized Sierpiński graphs concerning their automorphism groups, distinguishing number, and perfect codes. Unfortunately, the corresponding paper has not (yet) been written up. The first published papers on the generalized Sierpiński graphs are thus due to Geetha and Somasundaram [**10**] and to Rodríguez-Velázquez and Tomás-Andreu [**20**], dealing with their total chromatic number and Randić index, respectively. Both papers appeared in 2015 and were submitted at almost the same time in the fall of 2014.

Afterwards—very recently—several papers followed that examine the generalized Sierpiński graphs. First, the work from [**20**] on the Randić index was extended in [**8**] to the generalized Randić index. In [**21**] the chromatic number, vertex cover number, clique number, and domination number of generalized Sierpiński graphs are investigated, while in [**19**] the focus is on their Roman domination number. An extensive study of metric aspects of generalized Sierpiński graphs has just been posted in [**9**]. For another metric aspect of generalized Sierpiśki graphs, the strong metric dimension, see [**7**].

In this paper we consider a couple of fundamental properties of generalized Sierpiński graphs. In the next section we formally introduce these fractal-like graphs, recall or deduce some of their properties, and define additional concepts needed. In Section 3 we first give an explicit formula for the number of connected components of generalized Sierpiński graphs. We follow with a proof that their connectivity and edge-connectivity are the same as those of the base graph. Then, in Section 4, we prove that a generalized Sierpiński graph admits a perfect matching if and only if the base graph admits it. We then discuss hamiltonicity and observe that it is not preserved by the construction, that is, even if a base graph G is Hamiltonian, its generalized Sierpiński graphs S_G^n need not be. We close the paper with a connection between the generalized Sierpiński graphs and the so-called clone graphs that were recently introduced in [**18**].

To conclude the introduction we point out that the notation $S(G, n)$ is used in the first papers on the generalized Sierpiński graphs. To emphasize that these graphs extend Sierpiński graphs S_p^n, and to be in line with the notation for Sierpiński-type graphs as proposed in [**14**], we have decided to denote the generalized Sierpiński graphs with S_G^n.

2. PRELIMINARIES

If $n \in \mathbb{N}$, then set $[n] = \{1, \ldots, n\}$ and $[n]_0 = \{0, \ldots, n-1\}$. The minimum degree and the maximum degree of a graph G are denoted with $\delta(G)$ and $\Delta(G)$, respectively. A *perfect matching* of G is a set of independent edges F such that every vertex of G is an endvertex of some edge of F. The subgraph of G induced by the edges of F is then a 1-*factor*. The connectivity and edge-connectivity of G are denoted with $\kappa(G)$ and $\kappa'(G)$, respectively. For other graph theory concepts not defined here we refer to [**26**].

Now we (finally) formally introduce the generalized Sierpiński graphs. Let G be a graph of order p with $V(G) = [p]_0$. Then the *generalized Sierpiński graph* S_G^n is the graph with the vertex set $[p]_0^n$, where the vertex $s = s_n \ldots s_1$ is adjacent to the vertex $t = t_n \ldots t_1$ if and only if there exists a $d \in [n]$ such that

- $s_\delta = t_\delta$ for $\delta > d$,
- $s_d \neq t_d$ and $\{s_d, t_d\} \in E(G)$, and
- $s_\delta = t_d$ and $t_\delta = s_d$ for $\delta < d$.

The edge set of S_G^n can be written in a more compact form as follows:

$$E(S_G^n) = \left\{ \{\underline{s}ij^{d-1}, \underline{s}ji^{d-1}\} : \ d \in [n], \underline{s} \in [p]_0^{n-d}, \{i,j\} \in E(G) \right\} .$$

In Fig. 1 a graph G of order 14 and the graph S_G^2 are shown.

Clearly, $|V(S_G^n)| = p^n$. It is also not difficult to observe that $\Delta(S_G^n) = \Delta(G) + 1$ and $\delta(S_G^n) = \delta(G)$, cf. [**21**]. That the minimum degree of G is preserved in S_G^n follows from the fact that $\deg_{S_G^n}(i^n) = \deg_G(i)$ for any vertex $i \in V(G)$.

If $\underline{s} \in [p]_0^{n-1}$ and H is a subgraph of G, then let $\underline{s}H$ denote the subgraph of S_G^n isomorphic to H with the vertex set $\{\underline{s}i : \ i \in V(H)\}$. In particular, $\underline{s}G = \underline{s}S_G^1$ is isomorphic to G for any $\underline{s} \in [p]_0^{n-1}$. Hence S_G^n contains p^{n-1} disjoint subgraphs isomorphic to G, and the vertex sets of these subgraphs partition $V(S_G^n)$. If $X \subseteq V(G)$ and $F \subseteq E(G)$, then we will also use the notation $\underline{s}X$ and $\underline{s}F$ to denote the set of vertices and edges in $\underline{s}S_G^1$ corresponding to X and F, respectively. See Fig. 2 for an illustration of this notation on $S_{C_4}^3$.

To conclude the preliminaries we state the following fact that will be useful later and follows directly from the definition of the generalized Sierpiński graphs.

Remark 1. *Let G be a graph, $i \in V(G)$, and $n \geq 2$. Then a vertex $i^{n-1}j$ of $i^{n-1}S_G^1$ has a neighbor $s \notin i^{n-1}S_G^1$ if and only if s is of the form $i^{n-2}ji$, where $\{i,j\} \in E(G)$.*

For an example consider the subgraph $11C_4$ of $S_{C_4}^3$ in Fig. 2. The vertices 110 and 112 have respective neighbors 101 and 121 outside $11C_4$ because $\{0,1\} \in E(C_4)$ and $\{1,2\} \in E(C_4)$. On the other hand, the vertex 113 has no neighbor outside $11C_4$.

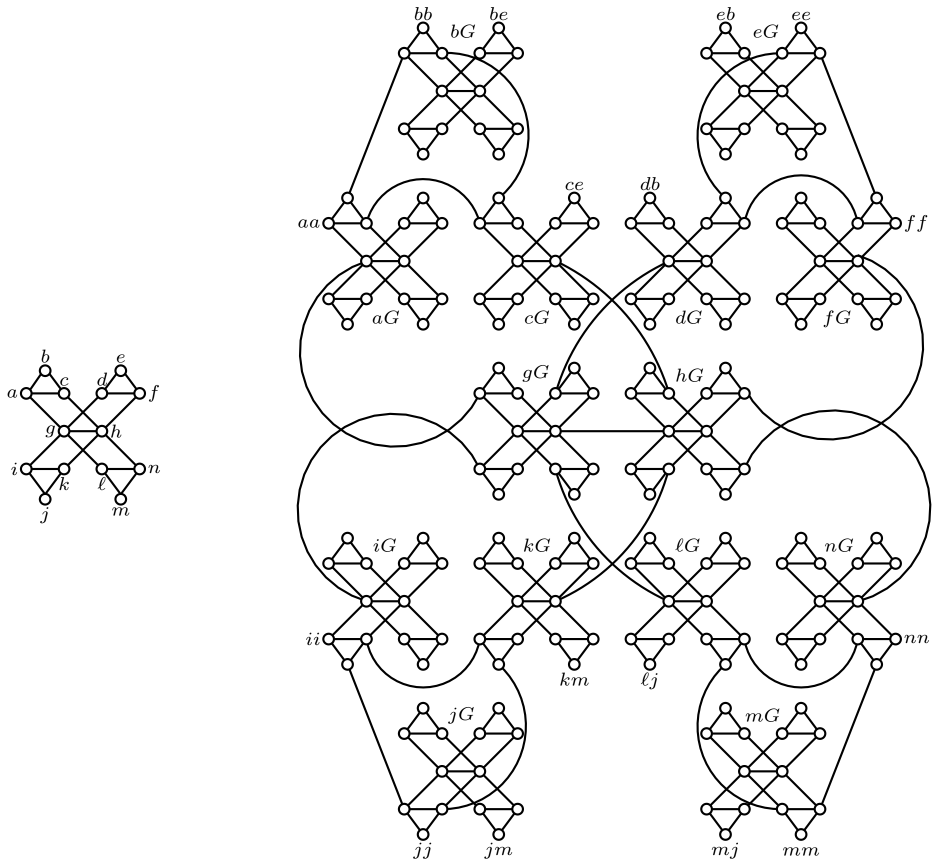

Figure 1: Graphs G (left) and S_G^2 (right)

3. CONNECTIVITY

In this section we give an explicit formula for the number of connected components of S_G^n and determine its connectivity and edge-connectivity.

Let $\#\mathrm{cc}(G)$ denote the number of connected components of a graph G and let $\mathcal{C}(G)$ be the set of its connected components.

Lemma 2. *If G is a graph and $n \geq 2$, then*

$$\#\mathrm{cc}(S_G^n) = \#\mathrm{cc}(S_G^{n-1}) \cdot |V(G)| + \#\mathrm{cc}(G) - |V(G)|\,.$$

Proof. To prove the lemma we will prove the formula

$$\#\mathrm{cc}(S_G^n) = \sum_{C \in \mathcal{C}(G)} \left[|V(C)| \cdot \left(\#\mathrm{cc}(S_G^{n-1}) - 1\right) + 1\right]\,. \tag{1}$$

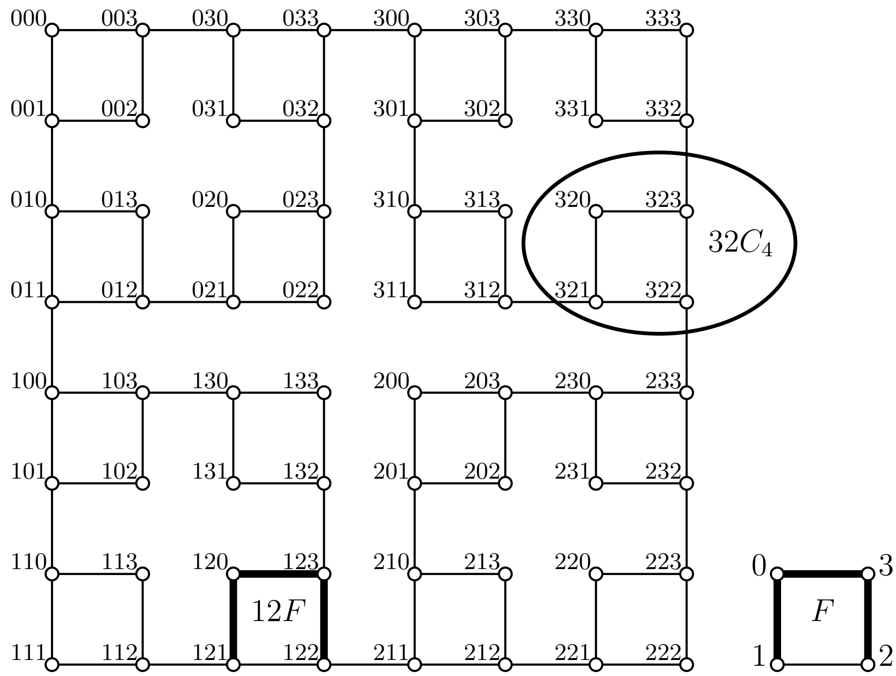

Figure 2: $S^3_{C_4}$ together with its labeling, subgraph $32C_4$, and edge subset $12F$

Let C be an arbitrary connected component of G and $x \in V(C)$. We consider the following two cases.

Suppose first that C consists of a single vertex x. Then (by Remark 1), if $y \in V(G)$, $y \neq x$, no edge of the graph S^n_G connects a vertex of the subgraph xS^{n-1}_G with a vertex of the subgraph yS^{n-1}_G of S^n_G. Hence xS^{n-1}_G induces in S^n_G as many connected components as there are components in S^{n-1}_G, that is, $\#\mathrm{cc}(S^{n-1}_G)$, cf. Fig. 3. Since $|V(C)| = 1$, this number can in turn be written as $|V(C)| \cdot \big(\#\mathrm{cc}(S^{n-1}_G) - 1\big) + 1$.

Assume now that C has at least two vertices and let $x \in C$. By Remark 1, only the edges from C can lead to edges between the subgraph xS^{n-1}_G and a subgraph yS^{n-1}_G, where $y \neq x$. Hence in the subgraph xS^{n-1}_G (being isomorphic to S^{n-1}_G) we find $\#\mathrm{cc}(S^{n-1}_G) - 1$ connected components that have no neighbor outside xS^{n-1}_G. Summing over all vertices of C this gives $|V(C)| \big(\#\mathrm{cc}(S^{n-1}_G) - 1\big)$ connected components. It remains to show that there is an edge between the subgraphs of the form yS^{n-1}_G, $y \in V(C)$, because this will give us precisely one additional component. This is indeed true, since C is connected and for an arbitrary edge $\{y, z\}$ in C there is an edge $\{yz^{n-1}, zy^{n-1}\}$ in S^n_G connecting yS^{n-1}_G with zS^{n-1}_G (in particular, connecting the components $yz^{n-2}C$ and $zy^{n-2}C$). (We again refer to Fig. 3.) Therefore, when $|V(C)| \geq 2$, the component C yields exactly

$|V(C)| \cdot \big(\#\mathrm{cc}(S_G^{n-1}) - 1\big) + 1$ connected components in S_G^n.

We have thus proved (1) from where the assertion of the lemma follows directly. □

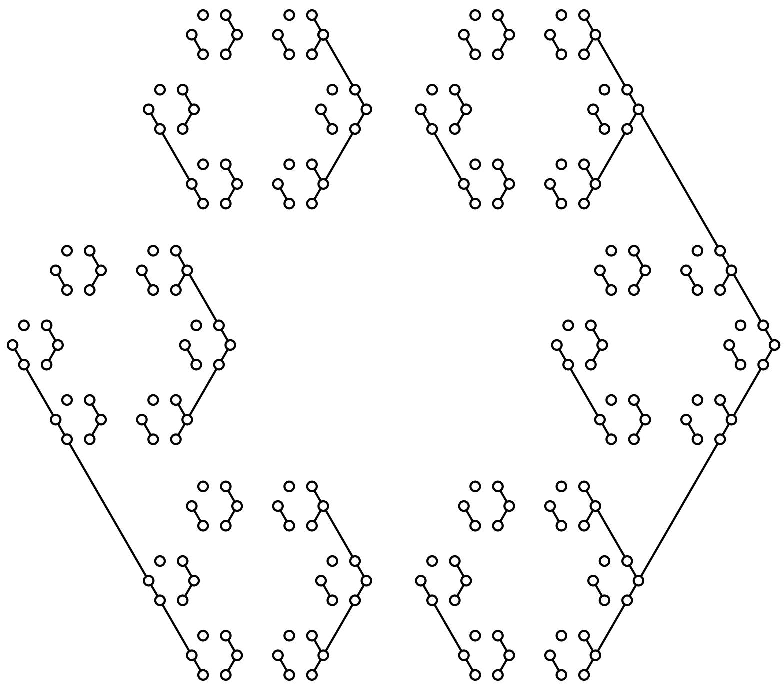

Figure 3: S_G^3, where G is the disjoint union of K_1, K_2, and P_3

Solving the recursion of Lemma 2 yields the following result.

Theorem 3. *If G is a graph and $n \geq 2$, then*

$$\#\mathrm{cc}(S_G^n) = \frac{1}{|V(G)| - 1}\Big[|V(G)|^n(\#\mathrm{cc}(G) - 1) + |V(G)| - \#\mathrm{cc}(G)\Big]\,.$$

Note that Theorem 3 in particular asserts that if $\#\mathrm{cc}(G) = 1$, then $\#\mathrm{cc}(S_G^n) = 1$, that is, if G is connected, then S_G^n is connected. We now continue with the connectivity of generalized Sierpiński graphs.

If x is a vertex of a graph G and U a set of its vertices, then an x, U*-fan* is a set of paths from x to U such that any two of them share only the vertex x. We will use the following result proved by Dirac [**6**], cf. [**26**, Theorem 4.2.23].

Theorem 4 (Fan Lemma)**.** *A graph G is k-connected if and only if $|V(G)| \geq k+1$ and, for any choice of x, U with $|U| \geq k$, there exists an x, U-fan of size k.*

We also need the following edge version of the Fan Lemma. Since we were not able to find it explicitly (with a proof) in the literature, we prove it here.

Lemma 5. *Let G be a graph with $\kappa'(G) = k$ ($k \geq 1$). If $x \in V(G)$ and U is a k-set of vertices of G, where $x \notin U$, then there exist k edge-disjoint paths between x and the respective vertices of U.*

Proof. Let G, x, and U be as stated. Let G' be the graph obtained from G by adding a new vertex z and connecting z by an edge with each vertex from U. Then it is straightforward to see that $\kappa'(G') = k$. Hence, by the edge version of Menger's theorem, in G' there exist k edge-disjoint x,z-paths. Omitting the edges of these paths that are incident with z, the asserted paths between x and the vertices of U in G are obtained. □

Now all is ready for our first main result.

Theorem 6. *If $n \in \mathbb{N}$ and G is a graph, then $\kappa(S_G^n) = \kappa(G)$ and $\kappa'(S_G^n) = \kappa'(G)$.*

Proof. Since $S_G^1 = G$, there is nothing to be proved for $n = 1$. In the rest of the proof we thus assume that $n > 1$. Furthermore, since it is known that $\kappa(S_{K_p}^n) = \kappa'(S_{K_p}^n) = \kappa(K_p) = \kappa'(K_p) = p-1$ ([**3**, Theorem 2.3], see also [**13**, Exercise 4.7]), we may also assume in the rest that G is not a complete graph. In addition, the result is also clear if G is not connected. So we are left with a connected, non-complete graph G and $n > 1$.

Let X be a separating set of G with $|X| = \kappa(G)$ and let H and H' be arbitrary connected components of $G - X$. If $i \in V(H)$, then we claim that $i^{n-1}X = \{i^{n-1}j : \ j \in X\}$ is a separating set of S_G^n. For this sake consider an arbitrary vertex $t = i^{n-1}j \in V(i^{n-1}H')$. By Remark 1, any possible neighbor of t that does not lie in $i^{n-1}S_G^1$ is of the form $i^{n-2}ji$, where $\{j,i\} \in E(G)$. But X is a separating set of G hence no such edge $\{j,i\}$ exists. Moreover, the fact that X is a separating set of G (and hence a separating set of $i^{n-1}S_G^1$) also implies that $i^{n-1}H'$ and $i^{n-1}H$ lie in different connected components of $S_G^n \setminus i^{n-1}X$. This proves the claim from which we conclude that $\kappa(S_G^n) \leq |i^{n-1}X| = |X| = \kappa(G)$.

To prove that $\kappa(S_G^n) \geq \kappa(G)$, let $s = \underline{s}i\overline{s}$ and $t = \underline{s}j\overline{t}$ be arbitrary vertices of S_G^n, where $\underline{s} \in [p]_0^{n-d}$ ($d \in [n]$), $i,j \in [p]_0$ ($i \neq j$), and $\overline{s}, \overline{t} \in [p]_0^{d-1}$. By the global version of Menger's theorem it suffices to show that there exist $\kappa(G)$ pairwise internally disjoint paths between s and t. This reduces to showing that there are $\kappa(G)$ pairwise internally disjoint paths between $s' = i\overline{s}$ and $t' = j\overline{t}$ in S_G^d. (Indeed, once we have these paths we just add the prefix $\underline{s}$ to all the vertices of the paths to obtain $\kappa(G)$ pairwise internally disjoint paths between s and t in S_G^n.)

We proceed by induction on d, where the base case $d = 1$ is trivial since in that case $s', t' \in V(S_G^1)$, and S_G^1 is isomorphic to G. Let now $d \geq 2$, so that $s' \in iS_G^{d-1}$ and $t' \in jS_G^{d-1}$. To simplify the notation set $\kappa = \kappa(G)$. Let $P_1, \ldots, P_\kappa$ be pairwise internally disjoint i,j-paths in G and let the consecutive vertices of P_ℓ, $\ell \in [\kappa]$, be $i = k_{\ell,1}, k_{\ell,2}, \ldots, k_{\ell,r_\ell} = j$. By the induction hypothesis, $\kappa(iS_G^{d-1}) \geq \kappa$, and hence Theorem 4 (having in mind that iS_G^{d-1} is not complete, so that $|iS_G^{d-1}| > \kappa$), implies that there exists an $s', \{ik_{1,2}^{d-1}, \ldots, ik_{\kappa,2}^{d-1}\}$-fan in iS_G^{d-1}. Let $Q_1, \ldots, Q_\kappa$ be the paths of this fan and note that if $s' \in \{ik_{1,2}^{d-1}, \ldots, ik_{\kappa,2}^{d-1}\}$, then one of these paths is the path of length 0. Similarly, since $\kappa(jS_G^{d-1}) \geq \kappa$, there exists a $t', \{jk_{1,r_1-1}^{d-1}, \ldots, jk_{\kappa,r_\kappa-1}^{d-1}\}$-fan in jS_G^{d-1}; let $Q'_1, \ldots, Q'_\kappa$ be the paths of this fan. We now construct internally disjoint s', t'-paths $R_1, \ldots, R_\kappa$ as follows. If $\lambda \in [\kappa]$, then let R_λ be a path of the form

$$\begin{aligned} i\overline{s} \;&\to\; Q_\lambda \to ik_{\lambda,2}^{d-1} \to k_{\lambda,2}i^{d-1} \to P[k_{\lambda,2}i^{d-1}, k_{\lambda,2}k_{\lambda,3}^{d-1}] \to \cdots \\ &\to\; P[k_{\lambda,r_\lambda-1}k_{\lambda,r_\lambda-2}^{d-1}, k_{\lambda,r_\lambda-1}j^{d-1}] \to k_{\lambda,r_\lambda-1}j^{d-1} \to jk_{\lambda,r_\lambda-1}^{d-1} \to Q'_\lambda \to j\bar{t}\,, \end{aligned}$$

where $P[k_{\lambda,\ell}\overline{x}, k_{\lambda,\ell}\overline{y}]$ denotes an arbitrary fixed $k_{\lambda,\ell}\overline{x}, k_{\lambda,\ell}\overline{y}$-path in $k_{\lambda,\ell}S_G^{d-1}$. It is clear that $R_1, \ldots, R_\kappa$ are internally disjoint s', t'-paths which proves that $\kappa(S_G^n) \geq \kappa(G)$ and consequently $\kappa(S_G^n) = \kappa(G)$.

The proof for the edge-connectivity proceeds along similar lines as the above proof for the vertex-connectivity. In short, given an edge-cut F of a graph G, one can prove that if H is an arbitrary connected component of $G - F$ and $i \in V(H)$, then the set of edges $i^{n-1}F$ forms an edge-cut of S_G^n. Consequently, $\kappa'(S_G^n) \leq \kappa'(G)$. To prove the other inequality, $\kappa'(G)$ internally edge-disjoint paths are constructed between arbitrary two vertices of S_G^n. For this sake we apply Lemma 5 and combine the edge-fans with the lifts of the $\kappa'(G)$ edge-disjoint paths from G to S_G^d in an analogous way as we have lifted the paths for vertex-connectivity. □

4. ADDITIONAL ASPECTS OF S_G^n

In this section additional properties of generalized Sierpiński graphs S_G^n are considered. We first characterize the graphs S_G^n that admit perfect matchings and discuss the number of their perfect matchings. We follow with the hamiltonicity and show that if G is a graph with $\delta(G) \leq 2$, then S_G^n is not Hamiltonian. At the end we point to the connection between the graphs S_G^2 and the so-called clone covers.

4.1. Perfect matchings

If G is a graph, then $o(G)$ denotes the number of odd components of G. To characterize the generalized Sierpiński graphs that admit 1-factors, we will apply the following celebrated result.

Theorem 7 (Tutte's 1-factor theorem)**.** *A graph G has a perfect matching if and only if $o(\langle V(G) - X\rangle) \leq |X|$ holds for any $X \subseteq V(G)$.*

In addition, we will also need:

Lemma 8. [**26**, Remark 3.3.5] *Let G be a graph with an even number of vertices. If G has no 1-factor, then there exists $X \subseteq V(G)$ such that $o(G - X) \geq |X| + 2$.*

Our next main result reads as follows.

Theorem 9. *Let $n \in \mathbb{N}$ and let G be a connected graph. Then S_G^n has a 1-factor if and only if G has a 1-factor.*

Proof. If G has a 1-factor then it is easy to see that S_G^n also has a 1-factor. Indeed, if F is a 1-factor of G, then

$$\bigcup_{\underline{s}\in[p]_0^{n-1}} \underline{s}F$$

forms a 1-factor of S_G^n.

Conversely, assume that G has no 1-factor. If G is of odd order, then S_G^n also is of odd order because $|V(S_G^n)| = |V(G)|^n$. So neither S_G^n has a 1-factor. Hence assume in the rest that $|V(G)|$ is even. By Theorem 7 there exists a set $X \subseteq V(G)$, $X \neq \emptyset$, such that $o(G - X) > |X|$. Let $H_1, \dots, H_k$ be the odd components of $G - X$.

Let i be an arbitrary vertex of H_1 and consider the subgraph $i^{n-1}S_G^1$ of S_G^n. Applying Remark 1 again we infer that only the vertices from $i^{n-1}H_1 \cup i^{n-1}X$ (the latter being removed) can be adjacent to vertices outside of $i^{n-1}S_G^1$. Hence since X separates $H_2, \dots, H_k$ in G (and hence $i^{n-1}X$ separates $i^{n-1}H_2, \dots, i^{n-1}H_k$ in $i^{n-1}S_G^1$), the subgraphs $i^{n-1}H_2, \dots, i^{n-1}H_k$ form odd components of $S_G^n - i^{n-1}X$. Since Lemma 8 implies that $k \geq |X|+2$, we conclude that $o(S_G^n - i^{n-1}X) \geq k-1 \geq |i^{n-1}X| + 1$. Hence, by Theorem 7, S_G^n has no 1-factor. □

Let $\#\mathrm{pm}(G)$ denote the number of perfect matchings in G. To determine this number is a computationally hard problem, cf. [**25**]. Since S_G^n contains $|V(G)|^{n-1}$ disjoint subgraphs isomorphic to G, we find that

$$\#\mathrm{pm}(S_G^n) \geq (\#\mathrm{pm}(G))^{|V(G)|^{n-1}}\,.$$

Hence it seems an interesting problem to determine $\#\mathrm{pm}(S_G^n)$, at least asymptotically. A related problem is to determine the number of matchings of S_G^n. In this direction Teufl and Wagner [**24**, Example 6.3] determined the asymptotic behavior of the number of matchings of S_3^n, while Chen et al. in [**4**, Proposition 5] obtained a parallel result by computing the average entropy of S_3^n. We also refer to [**2, 23**] additional investigations of perfect matchings in related graphs.

4.2. Hamiltonicity

Another fundamental property of generalized Sierpiński graphs that would be interesting to look at is their hamiltonicity. It is well-known that S_p^n is Hamiltonian for $p \geq 3$ and $n \geq 1$, see [**5**, Theorem 2.2] and [**15**, Proposition 3]. In the general case the situation seems to be more involved, which we illustrate with the following observation.

Proposition 10. *If G is a graph of order at least 4 with $\delta(G) \leq 2$ and if $n \geq 2$, then S_G^n is not Hamiltonian.*

Proof. Let G be a graph of order at least 4 with $\delta(G) = 2$, and let $n \geq 2$. Let i be a vertex of G of degree 2 and let $N(i) = \{j, k\}$. Clearly, $\{i^{n-1}j, i^{n-1}k\}$ is a separating set of $i^{n-1}S_G^1$. Since G has at least 4 vertices, $i^{n-1}S_G^1 - \{i^{n-1}j, i^{n-1}k\}$ has at least two components, one of them being the isolated vertex i^n. Applying Remark 1 again, these components are also components considered in $S_G^n - \{i^{n-1}j, i^{n-1}k\}$. Since (by Remark 1) the only vertices of $i^{n-1}S_G^1$ that connect to other subgraphs $\underline{s}S_G^1$ are of the form $i^{n-1}\ell$, where $\{i, \ell\} \in E(G)$, we infer that $S_G^n - \{i^{n-1}j, i^{n-1}k\}$ consists of at least three components, hence S_G^n is not Hamiltonian.

Suppose next that G has a vertex i of degree 1, where j is its neighbor. Then $i^{n-1}j$ is a cut vertex of S_G^n and hence S_G^n is again not Hamiltonian. □

Proposition 10 thus implies that the hamiltonicity of G does not guarantee the hamiltonicity of S_G^n. For instance, this happens if G is a cycle or a graph obtained from a cycle by adding arbitrary edges to it just by keeping one vertex of degree 2. On the other hand, if $\kappa(G) = 2$ and G is Hamiltonian-connected (recall that G is *Hamiltonian-connected* if there is a Hamiltonian path connecting any two fixed vertices of G, cf. [**17**]), then it is not difficult to observe that S_G^2 is Hamiltonian. For instance, such a graph is obtained from the disjoint union of two complete graphs on four vertices by identifying an edge of one of them with an edge of the other.

Based on the above discussion we propose:

Problem 11. *Characterize Hamiltonian generalized Sierpiński graphs. In particular, characterize Hamiltonian generalized Sierpiński graphs among the graphs G with $\kappa(G) = 2$.*

4.3. Clone covers

We conclude by pointing to a construction from [**18**] that is similar to the second power of generalized Sierpiński graphs. If G is a graph with $V(G) = [p]_0$, then let $\mathrm{Clone}_g(G)$ be the graph obtained from the collection of $|V(G)|$ vertex-deleted subgraph $G_i = G - i$ by adding, for each edge i, j of G an edge joining the vertex i of G_j with the vertex j of G_i. Then

$$\mathrm{Clone}_g(G) = S_G^2 - \{ii : i \in [p]_0\}.$$

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