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COUNTING WATER CELLS IN BARGRAPHS OF COMPOSITIONS AND SET PARTITIONS

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In this paper, we consider statistics on compositions and set partitions represented geometrically as bargraphs. By a *water cell*, we mean a unit square exterior to a bargraph that lies along a horizontal line between any two squares contained within the area subtended by the bargraph. That is, if a large amount of a liquid were poured onto the bargraph from above and allowed to drain freely, then the water cells are precisely those cells where the liquid would collect. In this paper, we count both compositions and set partitions according to the number of descents and water cells in their bargraph representations and determine generating function formulas for the joint distributions on the respective structures. Comparable generating functions that count non-crossing and non-nesting partitions are also found. Finally, we determine explicit formulas for the sign balance and for the first moment of the water cell statistic on set partitions, providing both algebraic and combinatorial proofs.

1. INTRODUCTION

Bargraphs have recently been studied from several perspectives and have found various refined enumerations. Recall that a bargraph is a first quadrant lattice path starting at the origin and ending upon its first return to the x-axis in which there are three types of steps–an up step $u = (0, 1)$, a down step $d = (0, -1)$ and a horizontal step $h = (1, 0)$–such that the first step is u with no d directly following a u and vice versa. Bargraphs go by different names, such as

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wall polyominoes [**11**] and skylines [**13**], and have proven to be an effective way of visualizing compositions [**5**] and other discrete structures [**18**]. Prellberg and Brak [**23**] and Feretíc [**12**] were among the first to study statistics on bargraphs and found a generating function with two variables x and y keeping track of the number of horizontal and up steps, respectively. Blecher et al. later considered levels [**3**], peaks [**4**], descents [**5**], area [**5**] and other statistics [**1, 6**], and further refined counts were given by Deutsch and Elizalde in [**10**], where bargraphs were viewed as Motzkin paths without peaks or valleys. There is also some interest in generating function formulas related to bargraphs in statistical physics [**21, 22**], where they are used to model certain kinds of polymers.

Given a bargraph B, four points (x, y), $(x+1, y)$, $(x+1, y+1)$, $(x, y+1)$ that lie along B or within the area it subtends in the first quadrant determine what is called a *cell* of B. If B contains m horizontal steps, then it can be identified as a sequence of columns $t = t_1t_2\cdots t_m$ such that the j-th column from the left contains exactly t_j cells. Here, we consider a new statistic and study its distribution on several classes of bargraphs. Let us call a unit square s in the first quadrant a *water cell* of the bargraph B if s lies outside of the area subtended by B but along a horizontal line connecting two cells of B. That is, imagine one were to pour a large amount of a liquid on (a three-dimensional representation of) the impermeable boundary B from above, where liquid in which there is no basin for it to collect is understood to flow off of B altogether (with no restriction at the left or right endpoints of B to stop the flow). Then a water cell is a square s that would be immersed in the liquid by virtue of their being columns of strictly greater height both to the left and to the right of the column of B above which s lies, with the height of s (above the x-axis) less than or equal the minimum of the two greater heights. Thus, in addition to addressing a new class of restrictions on compositions and related structures, our results below could be of potential physical interest if bargraphs were used to model some kind of surface.

Figure 1: The bargraph $B = (1, 3, 6, 2, 4, 6, 2, 5, 2, 1, 3, 2)$ with $wc(B) = 12$.

Let $wc(B)$ denote the number of water cells of B. For example, if

$$B = (1, 3, 6, 2, 4, 6, 2, 5, 2, 1, 3, 2),$$

then columns 4, 5, 7, 9 and 10 contain 4, 2, 3, 1 and 2 water cells, respectively, which implies $wc(B) = 12$, see Figure 1. Here, we will be interested in computing distribution polynomials of the form

$$A_{\mathcal{T}_n}(q)=\sum_{\lambda\in\mathcal{T}_n} q^{wc(\lambda)}, \qquad n\geq 1,$$

for various restricted subsets $\mathcal{T}_n$ of the set of bargraphs having n horizontal steps.

We remark that the $wc(\lambda)$ statistic value for a bargraph $\lambda=\lambda_1\lambda_2\cdots\lambda_m$ has an alternative description as follows. Given $\ell\geq 1$, let us say that λ has an ℓ-impulse of type 1 at index $i\in[m]=\{1,2,\ldots,m\}$ if $\lambda_{i-1}<\ell\leq\lambda_i$ and an ℓ-impulse of type 2 if $\lambda_i\geq\ell>\lambda_{i+1}$, where $\lambda_0=\lambda_{m+1}=0$. That is, there exists an up step at height ℓ forming part of the left side of the i-th column of λ in the former case and a down step at height ℓ forming part of the right side of the i-th column in the latter. Suppose λ has ℓ-impulses of type 1 at indices $i_1<i_2<\cdots<i_{t_\ell}$ and ℓ-impulses of type 2 at $j_1<j_2\cdots<j_{t_\ell}$. Observe that $i_r\leq j_r<i_{r+1}-1$ for all r and consider the sum $s_\ell=\sum_{r=1}^{t_\ell-1}(i_{r+1}-j_r-1)$. Note that s_ℓ equals the sum total of the number of positions separating adjacent type 2 and type 1 ℓ-impulses, which is seen to be the same as the number of water cells of λ of height ℓ above the x-axis. Considering all possible ℓ implies $wc(\lambda)=\sum_{\ell\geq 1}s_\ell$. For example, if B is as above, then the $(s_1,\ldots,s_6)$ vector is given by $(s_1,\ldots,s_6)=(0,1,4,2,3,2)$, with $wc(B)=12=\sum_{\ell=1}^{6}s_\ell$.

Recall that a *composition* of a positive integer n is any sequence $\sigma=\sigma_1\cdots\sigma_m$ of positive integers, called *parts*, whose sum is n. We denote the set of all compositions of n having m parts by $\mathcal{C}_{n,m}$. Given $\sigma\in\mathcal{C}_{n,m}$, let bar_σ denote the bargraph having m horizontal steps such that the height of the i-th column is σ_i for $1\leq i\leq m$. Note that n corresponds to the area in the first quadrant subtended by bar_σ. For example, the bargraph B represented in Figure 1 above corresponds to some $\sigma\in\mathcal{C}_{37,12}$. We will subsequently represent compositions of an integer as bargraphs in this manner when considering various statistics. For example, when speaking of the number of water cells of a composition σ, we will be referring to the application of this statistic to bar_σ. See the text [**14**] and references contained therein for previously considered statistics on compositions.

A *partition* of a set is a collection of nonempty, mutually disjoint subsets, called *blocks*, whose union is the set. Let $\mathcal{P}_{n,k}$ denote the set of partitions of $[n]$ having k blocks. Recall that $|\mathcal{P}_{n,k}|=S(n,k)$, the classical Stirling number of the second kind (see [**25**, A008277]). We represent partitions having k blocks as a subset of the bargraphs whose members satisfy a restricted growth condition with respect to their column heights. Considering the wc statistic on this subset of the bargraphs leads to a new polynomial generalization of $S(n,k)$. We refer the reader to the text [**17**] and references contained therein for other kinds of statistics on set partitions.

The organization of this paper is as follows. In the next section, we find the generating function of the joint distribution for the statistics on bargraphs of compositions recording the number of descents and water cells. In the third section, the comparable distribution is found for partitions of $[n]$ having k blocks. We remark that the generating function for the distribution of wc by itself on $\mathcal{P}_{n,k}$

has a particularly elegant form expressible as a finite product. From this, simple explicit formulas, which can also be explained bijectively, may be deduced for the total number of water cells within members of $\mathcal{P}_{n,k}$ and for the sign balance of this statistic. In the final two sections, we compute the generating function for the distribution of wc on non-crossing and non-nesting partitions of $[n]$. As a consequence, one obtains new polynomial generalizations of the Catalan sequence– one in terms of continued fractions and another in terms of a solution to a certain functional equation. Our derivation in the latter case leads to a new proof of the fact that there are the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$ non-nesting partitions of size n.

2. COUNTING WATER CELLS IN COMPOSITIONS

Recall that if $\pi = \pi_1\pi_2\cdots\pi_m$ is a composition, then a *descent* refers to an index $i \in [m-1]$ such that $\pi_i > \pi_{i+1}$. To count compositions according to the number of water cells, we first consider the restricted version of the problem where part sizes are required to belong to $[d]$ for some d. Let $\mathcal{C}_n^{(d)}$ be the set of compositions of n whose parts belong to $[d]$ for $d \geq 1$. Define the distribution polynomial $a_n^{(d)} = a_n^{(d)}(y, p, q)$ by

$$a_n^{(d)} = \sum_{\pi \in \mathcal{C}_n^{(d)}} y^{\nu(\pi)} p^{des(\pi)} q^{wc(\pi)}, \qquad n \geq 1,$$

where $\nu(\pi)$ and $des(\pi)$ denote the number of parts and descents, respectively, in the composition π. Define $f_d = f_d(x, y) = f_d(x, y; p, q)$ for $d \geq 1$ by $f_d = 1 + \sum_{n \geq 1} a_n^{(d)} x^n$. That is, $f_d(x, y)$ is the generating function that counts the members π of $\mathcal{C}_{n,m}$ whose parts all belong to $[d]$ according to the number of descents and number of water cells (marked by p and q, respectively). Similarly, let g_d, h_d and ℓ_d count the same compositions π in $\mathcal{C}_{n,m}$, but now according to the number of descents and water cells in πd, $d\pi$ and $d\pi d$, respectively.

We have the following system of recurrences, where arguments are suppressed unless otherwise needed.

Lemma 1. *If $d \geq 2$, then*

$$f_d = f_{d-1} + x^d y g_{d-1} h_d, \tag{1}$$

$$g_d = (1 + x^d y \ell_d) g_{d-1}, \tag{2}$$

$$h_d = (1 - x^{d-1} y(1-p))(1 + x^d y \ell_d) h_{d-1}, \tag{3}$$

and

$$\ell_d(x, y) = (1 - x^{d-1} y(1-p)q)(1 + x^d y \ell_d(x, y)) \ell_{d-1}(x, yq), \tag{4}$$

with $f_1 = g_1 = h_1 = \ell_1 = \frac{1}{1-xy}$.

Proof. The initial conditions when $d = 1$ are clear, so assume $d \geq 2$. Suppose $\pi \in \mathcal{C}_n^{(d)}$ for some n. To show (1), consider whether or not π contains a part of size d. If not, then the contribution towards the generating function (gf) is f_{d-1}. If so, then consider the leftmost occurrence of the letter d and write $\pi = \pi' d\pi''$, where π' and π'' are possibly empty and π' contains no parts of size d. Note that the distribution of water cells in π' and π'' are independent of one another since all parts belong to $[d]$ with a d separating the two sections. Then π' and π'' have gf's of g_{d-1} and h_d, respectively, by definition, with the letter d contributing an additional factor of $x^d y$. Thus, the gf for all $\pi \in \mathcal{C}_n^{(d)}$ containing at least one d is given by $x^d y g_{d-1} h_d$. Combining this with the prior case implies (1). A similar argument applies to (2) where here the sections π' and π'' contribute g_{d-1} and ℓ_d, respectively.

To show (3), first suppose π contains no parts of size d. If π does not start with $d-1$, then the contribution towards the gf is $h_{d-1} - x^{d-1} y h_{d-1}$, by subtraction, whereas compositions π starting with $d-1$ give $x^{d-1} yp h_{d-1}$ since a descent at the very beginning is introduced. If π contains d, consider the rightmost occurrence of d and write $\pi = \pi' d\pi''$, where π'' contains no parts of size d. Note that π'' contributes $(1 - x^{d-1} y(1-p))h_{d-1}$, by the previous case, while $\pi' d$ is enumerated by $x^d y \ell_d$. Combining the previous cases yields a gf of

$$(1 - x^{d-1} y(1-p))h_{d-1} + x^d y(1 - x^{d-1} y(1-p))h_{d-1}\ell_d,$$

which gives (3). Similar reasoning applies to (4), except now $(d-1)$-ary π have weight $(1 - x^{d-1} y(1-p)q)\ell_{d-1}(x, yq)$. Note that the replacement of y with yq in the second argument of ℓ_{d-1} accounts for the extra row of water cells (at height d) that arises in going from $(d-1)\pi(d-1)$ to $d\pi d$. If $\pi = \pi' d\pi''$ where π'' is $(d-1)$-ary, then π'' has the same weight as $(d-1)$-ary π in the prior case, while $\pi' d$ is enumerated by $x^d y \ell_d(x, y)$. Combining cases implies (4) and completes the proof. □

We have the following explicit formula for $f_d(x, y)$.

Theorem 2. *For all $d \geq 1$,*

$$\begin{aligned} f_d(x,y) &= \frac{1}{1-xy} \\ &\quad + \sum_{k=2}^{d} x^k y \frac{a_{k-1}(x, yq; p, q)}{a_k(x, y; p, q)} \prod_{m=1}^{k-1} \frac{(1 - x^m y(1-p))(a_{m-1}(x, yq; p, q))^2}{(a_m(x, y; p, q))^2}, \end{aligned} \tag{5}$$

where $a_m(x, y; p, q) = 1 - xy \sum_{j=0}^{m-1} x^j q^{m-1-j} \prod_{i=1}^{j} (1 - x^i y(1-p)q^{m-i})$.

Proof. Let $\ell_d^*(x, y) = \frac{1}{\ell_d(x,y)}$ and $v_d(y) = \frac{1}{1 - x^{d-1} y(1-p)q}$. Then

$$\ell_d^*(x, y) = v_d(y)\ell_{d-1}^*(x, yq) - x^d y$$

for $d\geq 2$, with $\ell_1^*(x,y)=1-xy$. By induction on d, we have

$$\ell_d^*(x,y)=\prod_{j=2}^{d}v_j(yq^{d-j})-y\sum_{j=1}^{d}x^jq^{d-j}\prod_{i=j+1}^{d}v_i(yq^{d-i}).$$

Hence,

$$\ell_d(x,y)=\frac{\prod_{j=1}^{d-1}(1-x^jy(1-p)q^{d-j})}{1-xy\sum_{j=0}^{d-1}x^jq^{d-1-j}\prod_{i=1}^{j}(1-x^iy(1-p)q^{d-i})}.$$

By (2) and (3) and induction on d, we have for all $d\geq 1$,

$$g_d(x,y)=\prod_{m=1}^{d}\frac{1-xy\sum_{j=0}^{m-2}x^jq^{m-1-j}\prod_{i=1}^{j}(1-x^iy(1-p)q^{m-i})}{1-xy\sum_{j=0}^{m-1}x^jq^{m-1-j}\prod_{i=1}^{j}(1-x^iy(1-p)q^{m-i})}$$

and

$$h_d(x,y)=\prod_{m=1}^{d-1}(1-x^my(1-p))\prod_{m=1}^{d}\frac{1-xy\sum_{j=0}^{m-2}x^jq^{m-1-j}\prod_{i=1}^{j}(1-x^iy(1-p)q^{m-i})}{1-xy\sum_{j=0}^{m-1}x^jq^{m-1-j}\prod_{i=1}^{j}(1-x^iy(1-p)q^{m-i})}.$$

Thus, by (1), we have

$$f_d(x,y)=\frac{1}{1-xy}+\sum_{k=2}^{d}x^kyg_{k-1}(x,y)h_k(x,y),$$

which leads to the desired result. □

Remark: Letting $d=2$ and $y=1$ in (5) yields

$$f_2(x,1)=\frac{1}{1-x}+\frac{x^2(1-xq)(1-x(1-p))}{(1-x)^2(1-xq-x^2+x^3(1-p)q)},$$

which reduces to $\frac{1}{1-x-x^2}=\sum_{n\geq 0}F_nx^n$ when $p=q=1$, where F_n is the Fibonacci number defined by $F_n=F_{n-1}+F_{n-2}$ if $n\geq 2$, with $F_0=F_1=1$. See, e.g., [**8, 9**] for other previously considered generalizations of F_n.

Let $[m]_q=1+q+\cdots+q^{m-1}$ for a positive integer m, with $[0]_q=0$. Taking $p=1$ in (5) gives

$$f_d(x,y;1,q)=\frac{1}{1-xy}+\sum_{k=2}^{d}x^ky\frac{b_{k-1}(x,yq;q)}{b_k(x,y;q)}\prod_{m=1}^{k-1}\frac{(b_{m-1}(x,yq;q))^2}{(b_m(x,y;q))^2},$$

where $b_m(x,y;q)=1-xyq^{m-1}\left(\frac{1-(x/q)^m}{1-x/q}\right)=1-xyq^{m-1}[m]_{x/q}$. This gives the following explicit formula.

Corollary 3. *If $d \geq 1$, then*

$$f_d(x,y;1,q) = \frac{1}{1-xy} + \sum_{k=2}^{d} x^k y \frac{1-xyq^{k-1}[k-1]_{x/q}}{1-xyq^{k-1}[k]_{x/q}} \prod_{m=1}^{k-1} \frac{(1-xyq^{m-1}[m-1]_{x/q})^2}{(1-xyq^{m-1}[m]_{x/q})^2}. \tag{6}$$

Letting $q=1$ in (6) gives

$$\begin{aligned} f_d(x,y;1,1) &= \frac{1}{1-xy} + \sum_{k=2}^{d} \frac{x^k y}{(1-xy[k]_x)(1-xy[k-1]_x)} \\ &= \frac{1}{1-xy} + \sum_{k=2}^{d} \left(\frac{1}{1-(x+\cdots+x^k)y} - \frac{1}{1-(x+\cdots+x^{k-1})y} \right) \\ &= \frac{1}{1-(x+\cdots+x^d)y}, \end{aligned}$$

as required.

Remark: Note that $\mathcal{C}_n^{(d)}$ when $d \geq n$ coincides with the set $\mathcal{C}_n = \cup_{m=1}^{n} \mathcal{C}_{n,m}$ of all compositions of n. Letting $d \to \infty$ in (6) thus gives the generating function counting members of $\mathcal{C}_{n,m}$ according to the distribution of water cells. Below, we have computed the distribution of wc on $\mathcal{C}_n$ for $1 \leq n \leq 11$ by extracting the coefficient of x^n in $f_{11}(x,1;1,q)$:

$$\begin{aligned} &1, \quad 2, \quad 4, \quad 8, \quad q+15, \quad q^2+4q+27, \quad q^3+5q^2+11q+47, \\ &2q^4+4q^3+16q^2+27q+79, \quad q^6+q^5+8q^4+16q^3+42q^2+58q+130, \\ &q^8+6q^6+7q^5+25q^4+50q^3+97q^2+117q+209, \\ &q^{10}+q^9+4q^8+5q^7+19q^6+31q^5+74q^4+126q^3+212q^2+221q+330. \end{aligned}$$

3. COUNTING WATER CELLS IN SET PARTITIONS

If the blocks B_i of $\Pi = B_1/B_2/\cdots/B_k \in \mathcal{P}_{n,k}$ are arranged in such a way that $\min(B_1) < \min(B_2) < \cdots < \min(B_k)$, then π is said to be in *standard form*. Equivalently, one may express Π arranged in standard form sequentially as $\pi = \pi_1\pi_2\cdots\pi_n$ where $i \in B_{\pi_i}$ for all i (see, e.g., [**26**]). Then π is known as the *canonical sequential form* and satisfies the restricted growth condition (see, e.g., [**20, 27**]), meaning that $\pi_{i+1} \leq \max(\pi_1\pi_2\cdots\pi_i)+1$ for all $i \in [n-1]$. Note that this implies that the first occurrence of the letter ℓ in π precedes the first occurrence of $\ell+1$ for all $\ell \in [k-1]$.

For example, if

$$\Pi = \{1,3,11\},\{2,6\},\{4,5,8,10\},\{7,12\},\{9\} \in \mathcal{P}_{12,5},$$

then $\pi = 121332435314$. Here, we represent the restricted growth sequences associated with members of $\mathcal{P}_{n,k}$ as bargraphs and consider the descent and water cell statistics on these bargraphs. The bargraph for $\pi \in \mathcal{P}_{12,5}$ above is illustrated in Figure 2 below and has $des(\pi) = 5$ and $wc(\pi) = 7$. By a slight abuse of language, we will at times refer to the associated sequences π and also the corresponding bargraphs as belonging to $\mathcal{P}_{n,k}$. Furthermore, when we speak of a combinatorial statistic defined on a set of partitions (here, des or wc), we are referring to its application to the representative set of bargraphs.

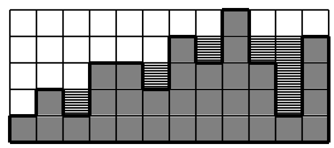

Figure 2: Bargraph representation of $\pi = 121332435314 \in \mathcal{P}_{12,5}$ with $wc(\pi) = 7$.

In this section, we enumerate members of $\mathcal{P}_{n,k}$, expressed in terms of bargraphs, according to the joint distribution for the number of descents and water cells (marked by p and q, respectively). Explicit formulas are given in the case $q = -1$ and also for the q-derivative at $q = 1$. Both algebraic and combinatorial proofs of these formulas are provided.

3.1 Generating function formulas

To determine the distribution of wc on partitions, we first need to determine the distribution on k-ary words. Recall that a *k-ary word* $w = w_1w_2\cdots$ is a sequence in which all of the letters belong to $[k]$. Given $k \geq 1$, let $a_k(x) = a_k(x;p,q)$ and $b_k(x) = b_k(x;p,q)$ count k-ary words w according to the number of descents and water cells in $kw(k+1)$ and in kw, respectively. Then we have the following recurrence relations.

Lemma 4. *We have*

$$a_k(x) = (1 - x(1-p)q)(1 + xa_k(x))a_{k-1}(xq), \qquad k \geq 2, \tag{7}$$

and

$$b_k(x) = (1 - x(1-p))(1 + xa_k(x))b_{k-1}(x), \qquad k \geq 2, \tag{8}$$

with $a_1(x) = b_1(x) = \frac{1}{1-x}$.

Proof. The initial conditions are clear from the definitions. Let w be a k-ary word. To show (7), we first count w which do not contain the letter k. The generating function for all such w not starting with $k-1$ according to the wc-value of $kw(k+1)$ is given by $a_{k-1}(xq) - xqa_{k-1}(xq)$ (by subtraction, since words enumerated by

$a_{k-1}(xq)$ that start with $k-1$ have gf $xqa_{k-1}(xq)$). Note that $(k-1)$-ary words are counted in the same way by the preceding gf as they are by $a_k(x)$, since the substitution of x by xq has the effect of adding a row of water cells (at height k) to the bargraph representation of $kw(k+1)$ which is not present in $(k-1)wk$. On the other hand, if w is a $(k-1)$-ary word starting with $k-1$, then an extra descent is added in going from $(k-1)wk$ to $kw(k+1)$ (and hence a factor of p is introduced). Thus, the gf for such words is given by $xpqa_{k-1}(xq)$. Combining the two previous cases gives a gf of $(1-x(1-p)q)a_{k-1}(xq)$ for all w not containing k.

On the other hand, if w contains a letter k, then write $w = w'kw''$, where w' is k-ary and w'' is $(k-1)$-ary. One can then count the water cells in $kw(k+1)$ by counting separately those in $kw'k$ and in $kw''(k+1)$. By similar reasoning as before, the section $kw''(k+1)$ has gf $x(1-x(1-p)q)a_{k-1}(xq)$, where here the extra factor of x accounts for the extra letter k. The section $kw'k$ has gf $a_k(x)$, as we have already considered the terminal k. Thus, the gf counting k-ary words containing at least one letter k is given by $x(1-x(1-p)q)a_k(x)a_{k-1}(xq)$. Combining this case with the previous implies (7). A proof along similar lines may also be given for (8); note that the replacement of x with xq is not needed here since there is no longer an understood letter to the right that is larger. Thus, k-ary words not containing k contribute $(1-x(1-p))b_{k-1}(x)$, whereas those containing at least one k (and hence are of the form $w = w'kw''$ above) contribute $x(1-x(1-p))a_k(x)b_{k-1}(x)$. Combining these two cases yields (8) and completes the proof. □

Let $Q_k(x) = Q_k(x;p,q) = x^k b_k(x;p,q)\prod_{j=1}^{k-1} a_j(x;p,q)$ for $k \geq 1$ be the generating function for the joint distribution of the descent and water cell statistics on partitions of $[n]$ having k blocks.

Theorem 5. *For all $k \geq 1$, we have*

$$Q_k(x;p,q) = x^k \frac{\prod_{j=0}^{k-1}(1-x(1-p)q^j)^{k-1-j}\prod_{m=1}^{k-1} f_m(xq;p,q)}{f_k(x;p,q)\prod_{m=1}^{k-1}(f_m(x;p,q))^2},$$

where $f_m(x;p,q) = 1 - xq^{m-1} - \sum_{j=0}^{m-2} xq^j \prod_{s=1}^{m-1-j}(1-x(1-p)q^{s+j})$. In particular, for $p=1$, we have

$$Q_k(x;1,q) = x^k\left(\frac{1-([k]_q-1)x}{1-[k]_q x}\right)\prod_{j=1}^{k-1}\frac{1-([j]_q-1)x}{(1-[j]_q x)^2}, \qquad k \geq 1. \tag{9}$$

Proof. By (7), we have

$$a_k(x) = \frac{(1-x(1-p)q)a_{k-1}(xq)}{1-x(1-x(1-p)q)a_{k-1}(xq)},$$

with $a_1(x) = \frac{1}{1-x}$. In order to solve this recurrence relation, define $a_k(x) = v_k(x)/w_k(x)$ for all k. By the recurrence for $a_k(x)$, we have

$$v_k(x) = (1-x(1-p)q)v_{k-1}(xq) \text{ and } w_k(x) = w_{k-1}(xq) - x(1-x(1-p)q)v_{k-1}(xq),$$

with $v_1(x)=1$ and $w_1(x)=1-x$. Hence, by induction on k, we obtain

$$v_k(x)=\prod_{j=1}^{k-1}(1-x(1-p)q^j)$$

and then

$$w_k(x)=1-xq^{k-1}-\sum_{j=0}^{k-2}xq^j\prod_{s=1}^{k-1-j}(1-x(1-p)q^{s+j}).$$

Thus, for all $k\geq 1$,

$$a_k(x)=\frac{\prod_{j=1}^{k-1}(1-x(1-p)q^j)}{1-xq^{k-1}-\sum_{j=0}^{k-2}xq^j\prod_{s=1}^{k-1-j}(1-x(1-p)q^{s+j})}.$$

From (8), we obtain

$$b_k(x)=\frac{(1-x(1-p))^{k-1}}{1-x}\prod_{m=2}^{k}(1+xa_m(x)),$$

which implies

$$\begin{aligned}
&b_k(x)\\
&=\frac{(1-x(1-p))^{k-1}}{1-x}\prod_{m=2}^{k}\left(1+\frac{x\prod_{j=1}^{m-1}(1-x(1-p)q^j)}{1-xq^{m-1}-\sum_{j=0}^{m-2}xq^j\prod_{s=1}^{m-1-j}(1-x(1-p)q^{s+j})}\right)\\
&=\frac{(1-x(1-p))^{k-1}}{1-x}\prod_{m=2}^{k}\frac{1-xq^{m-1}-\sum_{j=0}^{m-3}xq^{j+1}\prod_{s=1}^{m-2-j}(1-x(1-p)q^{s+1+j})}{1-xq^{m-1}-\sum_{j=0}^{m-2}xq^j\prod_{s=1}^{m-1-j}(1-x(1-p)q^{s+j})}.
\end{aligned}$$

Thus by the fact that $Q_k(x)=x^kb_k(x)\prod_{j=1}^{k-1}a_j(x)$, we complete the proof. □

Note that $Q_k(x;1,1)=\frac{x^k}{\prod_{i=1}^{k}(1-ix)}=\sum_{n\geq k}S(n,k)x^n$ for $k\geq 1$, as expected. By (9), we have

$$\frac{Q_k(x;1,q)}{Q_{k-1}(x;1,q)}=\frac{x(1-([k]_q-1)x)}{(1-[k-1]_qx)(1-[k]_qx)},\qquad k\geq 1.$$

Cross-multiplying, and comparing the coefficients of x^n on both sides of the resulting equation, yields the following recurrence relation for $a_{n,k}=a_{n,k}(q)=\sum_{\pi\in\mathcal{P}_{n,k}}q^{wc(\pi)}$.

Corollary 6. *If $n\geq k\geq 2$, then*

$$\begin{aligned}
a_{n,k}&=(2[k-1]_q+q^{k-1})a_{n-1,k}\\
&\quad+a_{n-1,k-1}-[k]_q[k-1]_qa_{n-2,k}-([k]_q-1)a_{n-2,k-1}.
\end{aligned}\tag{10}$$

Remark: Taking $q = 1$ in (10) gives the four-term Stirling number recurrence for $n \geq k \geq 2$:

$$S(n,k) = (2k-1)S(n-1,k)+S(n-1,k-1)-k(k-1)S(n-2,k)-(k-1)S(n-2,k-1),$$

which can be shown to be equivalent to the defining relation $S(n,k) = S(n-1,k-1) + kS(n-1,k)$.

Extracting the coefficient of x^n in $\sum_{k\geq 1} Q_k(x;1,q)$ for $1 \leq n \leq 9$ gives the following values for $a_n = \sum_{k=1}^{n} a_{n,k}$:

$$\begin{aligned}
&a_1 = 1, \quad a_2 = 2, \quad a_3 = 5, \quad a_4 = 2q + 13, \quad a_5 = 4q^2 + 14q + 34,\\
&a_6 = 2q^4 + 10q^3 + 37q^2 + 65q + 89,\\
&a_7 = 4q^6 + 14q^5 + 47q^4 + 114q^3 + 213q^2 + 252q + 233,\\
&a_8 = 2q^9 + 12q^8 + 38q^7 + 113q^6 + 245q^5 + 483q^4 + 772q^3 + 981q^2 + 884q + 610,\\
&a_9 = 4q^{12} + 18q^{11} + 58q^{10} + 160q^9 + 387q^8 + 790q^7 + 1471q^6 + 2353q^5 + 3373q^4\\
&\qquad + 4054q^3 + 3968q^2 + 2914q + 1596.
\end{aligned}$$

Note that a_n when $q = 1$ reduces to the n-th Bell number $B_n = \sum_{k=1}^{n} S(n,k)$ (see [**25**, A000110]), as required.

3.2 First moment and sign balance formulas

We first determine a formula for the total number of water cells within all members of $\mathcal{P}_{n,k}$. Differentiating formula (9) with respect to q, and setting $q = 1$, gives

$$\begin{aligned}
&\frac{\partial}{\partial q} Q_k(x;1,q)\mid_{q=1}\\
&\quad = \frac{x^{k+1}}{\prod_{j=1}^{k}(1-jx)}\left(\frac{\binom{k}{2}}{1-kx} - \frac{\binom{k}{2}}{1-(k-1)x} + \sum_{j=1}^{k-1}\left(\frac{j(j-1)}{1-jx} - \frac{\binom{j}{2}}{1-(j-1)x}\right)\right)\\
&\quad = \frac{x^{k+1}}{\prod_{j=1}^{k}(1-jx)}\left(\frac{\binom{k}{2}}{1-kx} + \frac{\binom{k-2}{2}-1}{1-(k-1)x} + \sum_{j=1}^{k-2}\left(\frac{j(j-1)}{1-jx} - \frac{\binom{j+1}{2}}{1-jx}\right)\right)\\
&\quad = \frac{x^{k+1}}{\prod_{j=1}^{k}(1-jx)}\left(\frac{\binom{k}{2}}{1-kx} + \sum_{j=1}^{k-1}\frac{(\binom{j-1}{2}-1)}{1-jx}\right).
\end{aligned}$$

Extracting the coefficient of x^n in the last formula yields the following result.

Corollary 7. *If $n \geq k \geq 2$, then the total number of water cells within the bargraphs of all members of $\mathcal{P}_{n,k}$ is given by*

$$\binom{k}{2}\sum_{i=k}^{n-1} k^{n-i-1}S(i,k) + \sum_{j=1}^{k-1}\sum_{i=k}^{n-1}\left(\binom{j-1}{2}-1\right)j^{n-i-1}S(i,k).$$

Combinatorial proof of Corollary 7.

To determine the total number of water cells, suppose $\pi = 1\pi^{(1)} \cdots k\pi^{(k)} \in \mathcal{P}_{n,k}$, where each $\pi^{(\ell)}$ is ℓ-ary for $1 \leq \ell \leq k$. We first count cells lying above a column corresponding to a letter r in $\pi^{(j)}$ for some $2 \leq j \leq k-1$. Then $\pi^{(j)} = \alpha r \beta$, where α and β are j-ary and possibly empty. Suppose that the length of β is $i-1$ for some $1 \leq i \leq n-k$. Then $\pi - \{r\beta\}$ corresponds to a partition of size $n-i$ having k blocks and hence there are $S(n-i,k)$ possibilities for these letters, with j^{i-1} possibilities for β. For each choice of the other letters, considering all possible values of r yields a contribution of $\sum_{r=1}^{j}(j-r) = \binom{j}{2}$ towards the overall number of water cells, since there exists a letter j to the left of r and a letter $j+1$ to the right where $j \geq r$. Considering all possible i and j, the total number of cells occurring to the left of the leftmost column of height k within members of $\mathcal{P}_{n,k}$ is given by

$$\sum_{j=2}^{k-1}\sum_{i=1}^{n-k}\binom{j}{2}j^{i-1}S(n-i,k).$$

We now count water cells occurring to the right of the leftmost k column. Equivalently, we fix a particular square c within the $n\times k$ rectangular grid of squares in the first quadrant bounding the x- and y-axes, and count all members of $\mathcal{P}_{n,k}$ having c as a water cell where the first column of height k in the bargraph lies to the left of c. Suppose c corresponds to the $(n-i+1)$-st position (from the left) and has height j. Note that we then must have $2 \leq i \leq n-k$ and $2 \leq j \leq k$. Furthermore, the following two conditions are necessary and sufficient in order for c to correspond to a water cell in this case: (i) the letter corresponding to position $n-i+1$ must be $< j$ and belong to $\pi^{(k)}$, and (ii) at least one letter $\geq j$ occurs within the final $i-1$ positions. We count the members $\pi \in \mathcal{P}_{n,k}$ satisfying (i) and (ii). By (i), there are $S(n-i,k)$ possibilities for the first $n-i$ positions of π and $j-1$ possibilities for the $(n-i+1)$-st letter. By condition (ii) and subtraction, there are $k^{i-1}-(j-1)^{i-1}$ possibilities for the remaining letters of π. Considering all possible i and j, we get

$$\sum_{j=2}^{k}\sum_{i=2}^{n-k}(j-1)\left(k^{i-1}-(j-1)^{i-1}\right)S(n-i,k)$$

water cells within the members of $\mathcal{P}_{n,k}$ that are to the right of the first column of height k.

Combining this case with the previous implies that the total number of water cells is given by

$$\sum_{j=2}^{k-1}\sum_{i=1}^{n-k}\binom{j}{2}j^{i-1}S(n-i,k)+\sum_{j=2}^{k}\sum_{i=2}^{n-k}(j-1)\left(k^{i-1}-(j-1)^{i-1}\right)S(n-i,k)$$
$$=\sum_{j=1}^{k-1}\sum_{i=1}^{n-k}\binom{j}{2}j^{i-1}S(n-i,k)+\sum_{j=1}^{k-1}\sum_{i=1}^{n-k}j\left(k^{i-1}-j^{i-1}\right)S(n-i,k)$$
$$=\sum_{j=1}^{k-1}\sum_{i=1}^{n-k}\left(\binom{j-1}{2}-1\right)j^{i-1}S(n-i,k)+\binom{k}{2}\sum_{i=1}^{n-k}k^{i-1}S(n-i,k),$$

which matches the formula in Corollary 7 above, upon replacing i by $n-i$ in both sums. □

We now determine the extent to which members of $\mathcal{P}_{n,k}$ having an even *wc* value outnumber (or are outnumbered by) those having an odd value. Taking $q=-1$ in (9), and noting $[k]_q\mid_{q=-1}=\frac{1-(-1)^k}{2}$, implies

$$Q_k(x;1,-1)=\frac{x^k(1+x)^{\lfloor k/2\rfloor}}{(1-x)^k},\qquad k\geq 1.$$

Extracting the coefficient of x^n yields the following result.

Corollary 8. *If $n\geq k\geq 1$, then*

$$a_{n,k}(-1)=\sum_{j=0}^{\lfloor k/2\rfloor}\binom{\lfloor k/2\rfloor}{j}\binom{n-j-1}{k-1}. \tag{11}$$

It is instructive to provide a bijective proof of the prior formula.

Combinatorial proof of Corollary 8.

We treat the case when k is even, the odd case being similar which we leave to the reader. Let $\mathcal{P}^*_{n,k}$ denote the subset of $\mathcal{P}_{n,k}$ consisting of $\pi=1\pi^{(1)}2\pi^{(2)}\cdots k\pi^{(k)}$ satisfying the following conditions: (i) $\pi^{(2)}=\pi^{(4)}=\cdots=\pi^{(k-2)}=\varnothing$, (ii) $\pi^{(i)}$ consists of a (possibly empty) string of 1's if $i\in[k-1]$ is odd, and (iii) $\pi^{(k)}$ can be expressed as

$$\pi^{(k)}=\alpha_k\alpha_{k-1}\cdots\alpha_1,$$

wherein if $j\in[k]$ is even, then α_j is empty or consists of a single letter j, and if j is odd, then α_j is empty or consists of a single letter j preceded by a (possibly empty) sequence of 1's. Note that the bargraphs of members of $\mathcal{P}^*_{n,k}$ contain an even number (possibly zero) of water cells since the number in each column is seen to be even. Furthermore, we have $|\mathcal{P}^*_{n,k}|=\sum_{j=0}^{k/2}\binom{k/2}{j}\binom{n-j-1}{k-1}$. To realize this, suppose that exactly j of the α_i of even index are nonempty, whence there are $\binom{k/2}{j}$ choices for such α_i. If $1\leq i\leq\frac{k}{2}$, let $x_i=|\pi^{(2i-1)}|$ and for $\frac{k}{2}+1\leq i\leq k$, let

$x_i = |\alpha_{2i-k-1}|$. Note that $x_1 + \cdots + x_k = n - k - j$, with $x_i \geq 0$ for all i. Thus, there are $\binom{n-j-1}{k-1}$ choices for the x_i, which taken together with the choice of the j α_i of even index above, uniquely determines a member of $\mathcal{P}^*_{n,k}$. Considering all possible j then implies the stated formula for $|\mathcal{P}^*_{n,k}|$.

To complete the proof of (11), it suffices to identify a wc-parity reversing involution of $\mathcal{P}_{n,k} - \mathcal{P}^*_{n,k}$. Suppose $\pi = 1\pi^{(1)} \cdots k\pi^{(k)} \in \mathcal{P}_{n,k} - \mathcal{P}^*_{n,k}$. Let i_o be the largest index $i \in [k-1]$ (if it exists) such that either (a) i is odd and $\pi^{(i)}$ contains some letter $s > 1$, or (b) i is even with $\pi^{(i)} \neq \varnothing$. If i_o satisfies (a) and $s > 1$ is the leftmost such letter of $\pi^{(i_o)}$, then replace s by $s+1$ if s is even and by $s-1$ if s is odd to obtain π'. On the other hand, if i_o satisfies (b), and t is the first letter of $\pi^{(i_o)}$, then replace t by $t+1$ or $t-1$ depending on if t is odd or even to obtain π'. One may verify that the mapping $\pi \mapsto \pi'$ where defined is an involution of $\mathcal{P}_{n,k} - \mathcal{P}^*_{n,k}$ that reverses the parity of $wc(\pi)$ for all π. It is not defined for those π satisfying conditions (i) and (ii) in the previous paragraph.

For such π, we extend the mapping as follows. Let j_o be the largest $j \in [k]$ such that the letter j fails to occur in a way prescribed for the α_j in condition (iii) above. That is, the sections α_i for $j_o + 1 \leq i \leq k$ follow the rules defined above, whereas α_{j_o} does not. If j_o is even, then there is a letter $\ell \leq j_o$ directly preceding the rightmost occurrence of j_o, which we replace by $\ell + 1$ if ℓ is odd and by $\ell - 1$ if ℓ is even. If j_o is odd, then there is a letter $\ell \in [2, j_o]$ somewhere to the left of the rightmost j_o in π. Replace the leftmost such ℓ with $\ell + 1$ or $\ell - 1$ depending on if ℓ is even or odd. Let $\widetilde{\pi}$ denote the resulting partition in all cases above. Then the mapping $\pi \mapsto \widetilde{\pi}$ is a parity-changing involution defined for the remaining members of $\mathcal{P}_{n,k} - \mathcal{P}^*_{n,k}$. Combining the mappings $\pi \mapsto \pi'$ and $\pi \mapsto \widetilde{\pi}$ yields the desired involution. $\square$

It is also possible to determine explicitly the total number of water cells within all members of $\mathcal{P}_n = \cup_{k=1}^n \mathcal{P}_{n,k}$. To do so, we first define

$$A_k(x) = \sum_{n \geq k} \sum_{i=k}^{n-1} k^{n-i-1} S(i,k) \frac{x^n}{n!}$$

and

$$B_k(x,w) = \sum_{n \geq k} \left(\sum_{j=1}^{k-1} \sum_{i=k}^{n-1} \left(\binom{j-1}{2} - 1 \right) j^{n-i-1} w^j S(i,k) \right) \frac{x^n}{n!}.$$

By the definitions, we have $\frac{d}{dx} A_k(x) = kA_k(x) + (e^x - 1)^k / k!$ with $A_k(0) = 0$. Thus,

$$A_k(x) = \int_0^x \frac{(e^t - 1)^k e^{k(x-t)}}{k!} dt,$$

which implies

$$A(x) = \sum_{k \geq 2} \binom{k}{2} A_k(x) = \frac{1}{2}(e^x - 1) - e^{e^x - 1}(e^x - 1) + \frac{1}{2} e^{e^x + 2x} \int_0^x e^{-e^t} dt.$$

On the other hand, the function $B(x,w)=\sum_{k\geq 2}B_k(x,w)$ satisfies $B(0,w)=0$ and

$$\frac{\partial}{\partial x}B(x,w)=w\frac{\partial}{\partial w}B(x,w)+\sum_{k\geq 2}\left(\frac{(e^x-1)^k}{2k!}\sum_{j=1}^{k-1}j(j-3)w^j\right).$$

Solving the preceding first order linear partial differential equation implies

$$\begin{aligned}B(x,w)&=\frac{1}{2}we^{we^x+x-1}(we^x-2)\int_0^x e^{(1-w)e^t}dt\\&-\frac{1}{2}we^{we^x+x}(we^x-2)\int_0^x e^{-we^t}dt\\&+\frac{e^{(e^x-1)w}(2w(1-w)e^x+w^2+3w-3)+we^{e^x-1}((w-1)e^x+2-3w)}{2(w-1)^2}\\&-\frac{1}{2}(we^x-3).\end{aligned}$$

By Corollary 7, the exponential generating function for the total number of water cells within the bargraphs of all members of $\mathcal{P}_n$ is given by

$$\begin{aligned}&A(x)+B(x,1)\\&=\sum_{k\geq 2}\sum_{n\geq k}\left(\binom{k}{2}\sum_{i=k}^{n-1}k^{n-i-1}S(i,k)+\sum_{j=1}^{k-1}\sum_{i=k}^{n-1}\left(\binom{j-1}{2}-1\right)j^{n-i-1}S(i,k)\right)\frac{x^n}{n!}.\end{aligned}$$

From the preceding, we have

$$\begin{aligned}&A(x)+B(x,1)\\&=1+\frac{2x-3}{4}e^{e^x+2x-1}+(2-x)e^{e^x+x-1}-\frac{9}{4}e^{e^x-1}+e^{e^x+x}\int_0^x e^{-e^t}dt\\&=\frac{2x-3}{4}\frac{d^2}{dx^2}(e^{e^x-1})+\frac{11-6x}{4}\frac{d}{dx}(e^{e^x-1})-\frac{9}{4}e^{e^x-1}+\frac{d}{dx}\left(e^{e^x}\int_0^x e^{-e^t}dt\right).\end{aligned}$$

Hence, the coefficient of $\frac{x^n}{n!}$ in $A(x)+B(x,1)$ is given by

$$-\frac{3}{4}B_{n+2}+\frac{2n+11}{4}B_{n+1}-\frac{6n+9}{4}B_n+\sum_{j=0}^{n}\binom{n+1}{j}B_j\widetilde{B}_{n-j},$$

where $\widetilde{B}_n$ is the n-th complementary Bell number (see [**25**, A000587]) defined by

$$e^{1-e^x}=\sum_{n\geq 0}\widetilde{B}_n\frac{x^n}{n!}.$$

By [**2**], the coefficient of $\frac{x^n}{n!}$ in $e^{e^x}\int_0^x e^{-e^t}dt$ is given by $B_n(C+O(e^{-\kappa n/\log n}))$, where $C=0.5963473622\cdots$ and κ is a positive constant. Moreover, $\frac{B_{n+h}}{B_n}=$

$\frac{(n+h)!}{n!r^h}(1+O(\log n/n))$ uniformly for $h=O(\log n)$, where r is the positive root of $re^r=n+1$, see [**7**]. Hence, we have the following result.

Theorem 9. *The total number of water cells within the bargraphs of all members of $\mathcal{P}_n$ for $n\geq 1$ is given by*

$$-\frac{3}{4}B_{n+2}+\frac{2n+11}{4}B_{n+1}-\frac{6n+9}{4}B_n+\sum_{j=0}^{n}\binom{n+1}{j}B_j\widetilde{B}_{n-j}$$
$$=\frac{n+1}{4r^2}\left((2n+11)r-3n-6+4rC-6r^2\right)B_n(1+O(\log n/n)),$$

where $C=0.5963473622\cdots$ and r is the positive root of $re^r=n+1$.

4. WATER CELLS IN NON-CROSSING PARTITIONS

A sequence $\beta=\beta_1\beta_2\cdots\beta_m$ of positive integers having largest letter ℓ and that is onto $[\ell]$ for some ℓ is called a *pattern*. Then a sequence $\alpha=\alpha_1\alpha_2\cdots\alpha_n$ is said to *avoid* β if α contains no subsequence that is order-isomorphic to β. Recall that *non-crossing* partitions are those whose canonical sequential form avoids the pattern 1212 (see, e.g., [**16**]) and are enumerated by the Catalan number C_n. In this section, we restrict the wc statistic to bargraphs corresponding to non-crossing partitions and consider the resulting generalization of C_n. For examples of other types of parameters considered on non-crossing partitions, see, e.g., [**24, 28**].

We first consider the distribution of wc where there is an understood copy of the largest letter following the sequential form of a partition. Given $i\geq 1$, let $g_i=g_i(x,q)$ denote the generating function that counts non-crossing partitions π of $[n]$ having i blocks according to the number of water cells in πi. The g_i are determined recursively as follows.

Lemma 10. *We have*

$$g_i(x,q)=\frac{xg_{i-1}(x,q)}{(1-x)(1-xq^{i-1})}+\frac{x}{1-xq^{i-1}}\sum_{\ell=2}^{i-1}q^{\ell-1}g_\ell(x,q)g_{i-\ell}(x,q),\quad i\geq 2, \tag{12}$$

with $g_1(x,q)=\frac{x}{1-x}$.

Proof. Let $\pi\in\mathcal{P}_{n,i}$ avoid 1212. Then π may be decomposed as

$$\pi=1^{a_1}\pi^{(1)}1^{a_2}\cdots\pi^{(r)}1^{a_{r+1}},$$

where $a_1,\ldots,a_r>0$, $a_{r+1}\geq 0$ and $\pi^{(j)}$ is nonempty for $1\leq j\leq r$ and contains no 1's. Note that π avoiding 1212 implies $\max(\pi^{(j)})<\min(\pi^{(j+1)})$ for all j. We will refer to an individual $\pi^{(j)}$ as a *unit*. We first count all π containing one unit, i.e., $r=1$. Then within $\pi=1^{a_1}\pi^{(1)}1^{a_2}$ the section $\pi^{(1)}$ contributes $g_{i-1}(x,q)$ towards the gf, while the 1^{a_1} and 1^{a_2} sequences contribute $\frac{x}{1-x}$ and $\frac{1}{1-xq^{i-1}}$, respectively.

(Note that each 1 within the 1^{a_2} sequence contributes xq^{i-1} due to the understood letter i that follows π.) This accounts for the first term on the right side of (12).

We now count π where $r > 1$. Suppose that the initial section

$$\pi' = 1^{a_1}\pi^{(1)} \cdots 1^{a_{r-1}}\pi^{(r-1)}1^{a_r}$$

contains exactly ℓ blocks. Note that $2 \le \ell \le i-1$ since each $\pi^{(i)}$ is nonempty. Then π' contributes $xq^{\ell-1}g_\ell(x,q)$ towards the gf as $a_r > 0$, upon appending a single 1 to an arbitrary partition enumerated by $g_\ell(x,q)$ (which adds $\ell-1$ water cells). The section $\pi^{(r)}$ is accounted for by $g_{i-\ell}(x,q)$ since it contains $i-\ell$ blocks with each letter belonging to $[\ell+1,i]$ (note that adding the same amount to each column of a bargraph preserves the number of water cells). The $1^{a_{r+1}}$ sequence is then seen to contribute $\frac{1}{1-xq^{i-1}}$. Altogether, the gf for $\pi \in \mathcal{P}_{n,i}$ containing more than one unit and $i-\ell$ blocks in the final unit is given by $\frac{xq^{\ell-1}}{1-xq^{i-1}}g_\ell(x,q)g_{i-\ell}(x,q)$. Summing over $2 \le \ell \le i-1$ gives the gf for all π containing more than one unit and completes the proof. □

Given $k \ge 1$, let $f_k(x,q)$ denote the generating function that counts non-crossing partitions of $[n]$ having k blocks according to the number of water cells. The $f_k(x,q)$ satisfy the following recurrence relation.

Lemma 11. *We have*

$$f_k(x,q) = \frac{x}{(1-x)^2}f_{k-1}(x,q) + \frac{x}{1-x}\sum_{\ell=2}^{k-1} q^{\ell-1}g_\ell(x,q)f_{k-\ell}(x,q), \quad k \ge 2, \tag{13}$$

with $f_1(x,q) = \frac{x}{1-x}$.

Proof. We consider cases based on the decomposition of non-crossing $\pi \in \mathcal{P}_{n,k}$ used in the proof of the previous lemma. Then $\frac{x}{(1-x)^2}f_{k-1}(x,q)$ accounts for all π such that $r = 1$ in this decomposition. Note that here neither initial nor terminal sequences of 1's contribute to the statistic value $wc(\pi)$. On the other hand, if π contains more than one unit and π' is as defined previously and contains exactly ℓ blocks, then the contribution of π' towards the gf is $xq^{\ell-1}g_\ell(x,q)$ since the first letter of $\pi^{(r)}$ acts as the $\ell+1$ (when considering $\pi'(\ell+1)$). As each letter in $\pi^{(r)}$ is strictly greater than the letters in all other units, it follows that $\pi^{(r)}$ contributes $f_{k-\ell}(x,q)$, with the terminal (possibly empty) sequence of 1's giving an additional factor of $\frac{1}{1-x}$. Altogether, the gf for $\pi \in \mathcal{P}_{n,k}$ containing more than one unit and exactly $k-\ell$ blocks in the final unit equals $\frac{xq^{\ell-1}}{1-x}g_\ell(x,q)f_{k-\ell}(x,q)$. Summing over $2 \le \ell \le k-1$ yields the gf for all π containing more than one unit and combining with the previous case gives (13). □

We wish to solve the recurrence relations in Lemmas 10 and 11. In order to do so, we define $G(u) = G(x,q;u) = \sum_{k\ge1} g_k(x,q)u^k$ and $F(u) = F(x,q;u) =$

$\sum_{k\geq 1} f_k(x,q)u^k$. Lemma 10 gives

$$g_i(x,q) - xq^{i-1}g_i(x,q) = \frac{x}{1-x}g_{i-1}(x,q) + x\sum_{\ell=2}^{i-1} q^{\ell-1}g_\ell(x,q)g_{i-\ell}(x,q), \quad i \geq 2,$$

with $g_1(x,q) = \frac{x}{1-x}$. Multiplying both sides of the last recurrence by u^i, and summing over $i \geq 2$, we obtain

$$G(u) - \frac{x}{q}G(qu) - xu = \frac{xu}{1-x}G(u) + \frac{x}{q}\left(G(qu) - \frac{xqu}{1-x}\right)G(u),$$

which implies

$$G(u) = \frac{xu + \frac{x}{q}G(qu)}{1 - xu - \frac{x}{q}G(qu)}.$$

Thus,

$$G(u) = -1 + \cfrac{1}{1 - x(u-1/q) - \cfrac{x/q}{1-x(qu-1/q) - \cfrac{x/q}{1-x(q^2u-1/q) - \cfrac{x/q}{\ddots}}}}.$$

By Lemma 11, we have

$$F(u) - \frac{xu}{1-x} = \frac{xu}{(1-x)^2}F(u) + \frac{x/q}{1-x}(G(qu) - xqu/(1-x))F(u),$$

which implies

$$F(u) = \frac{xu}{1 - x(1+u) - \frac{x}{q}G(qu)}.$$

Hence, we can state the following result.

Theorem 12. *We have*

$$F(x,q;u) = \cfrac{x}{1 - x(1+u-1/q) - \cfrac{x/q}{1-x(qu-1/q) - \cfrac{x/q}{1-x(q^2u-1/q) - \cfrac{x/q}{1-x(q^3u-1/q) - \cfrac{x/q}{\ddots}}}}}. \tag{14}$$

Letting $q = u = 1$ in (14) gives

$$F(x,1;1) = \cfrac{x}{1 - x - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{\ddots}}}}.$$

Let $C(x)$ denote the Catalan number generating function, i.e.,

$$C(x) = \frac{1-\sqrt{1-4x}}{2x} = \sum_{n\geq 0} C_n x^n.$$

By a well-known continued fraction expansion of $C(x)$ (see, e.g., [**19**]), we have

$$C(x) = \cfrac{1}{1 - \cfrac{x}{1 - \cfrac{x}{1 - \cfrac{x}{\ddots}}}}.$$

This implies

$$F(x,1;1) = \frac{x}{1 - x - xC(x)} = C(x) - 1,$$

where the second equality may be verified by using $xC(x)^2 = C(x) - 1$. This agrees with the well-known fact that the gf for the number of non-crossing partitions is $C(x)$ (note that the sum for $F(x,1;1)$ starts at $n = 1$ instead of $n = 0$, and so we must subtract 1 from $C(x)$).

5. NON-NESTING PARTITIONS

Non-nesting partitions of $[n]$ are those whose sequential form avoid the pattern 1221 (see, e.g., [**16**]) and are also enumerated by C_n. In this section, we consider the restriction of wc to the set of non-nesting partitions. The distribution which arises will be seen to differ markedly from the one obtained above in the non-crossing case.

Our approach in this section will be to work with related arrays of polynomials first instead of arguing directly using generating functions as before, since the non-nesting structure seems to preclude the latter. Given $1 \le i \le k \le n$, let $\mathcal{A}_{n,k,i}$ denote the set of non-nesting partitions of $[n]$ having k blocks ending in the letter i and let $\mathcal{A}_{n,k} = \cup_{i=1}^{k} \mathcal{A}_{n,k,i}$. We represent $\pi \in \mathcal{A}_{n,k}$ as $\pi = 1\pi^{(1)}2\pi^{(2)}\cdots k\pi^{(k)}$, where each $\pi^{(\ell)}$ is ℓ-ary. Let $a(n,k,i) = a_{p,q}(n,k,i)$ denote the joint distribution on $\mathcal{A}_{n,k,i}$ for the statistics recording the length of $\pi^{(k)}$ and the wc-value (marked by p and q, respectively). That is,

$$a_{p,q}(n,k,i) = \sum_{\pi \in \mathcal{A}_{n,k,i}} p^{|\pi^{(k)}|} q^{wc(\pi)}.$$

Let $d(n,k,i) = d_{p,q}(n,k,i)$ denote the same distribution except when computing wc, there is an understood k following π; that is,

$$d_{p,q}(n,k,i) = \sum_{\pi \in \mathcal{A}_{n,k,i}} p^{|\pi^{(k)}|} q^{wc(\pi k)}.$$

It suffices to study the latter distribution.

Proposition 13. *For all n, k, i, we have*

$$a_{p,q}(n,k,i) = d_{pq^{i-k},q}(n,k,i). \tag{15}$$

Proof. First note that $\pi \in \mathcal{A}_{n,k,i}$ implies $\pi^{(k)}$ is increasing if nonempty, for otherwise π would contain an occurrence of 1221, with the second "2" and "1" belonging to $\pi^{(k)}$. Thus $\pi^{(k)}$ increasing implies that when the letter k is appended to π, there are $k-i$ water cells created above each column corresponding to a letter in $\pi^{(k)}$ within the bargraph representation of π. Since the water cells of π are otherwise unchanged, this implies that the number of water cells is increased by $(k-i)|\pi^{(k)}|$. As the variable p marks the length of $\pi^{(k)}$ in $a_{p,q}(n,k,i)$, we have $d_{p,q}(n,k,i) = a_{pq^{k-i},q}(n,k,i)$, which implies (15). □

To determine $d_{p,q}(n,k,i)$, we first study the special case when $p=1$. Let $b(n,k,i) = b_q(n,k,i)$ represent $d_{1,q}(n,k,i)$ and let $b(n,k) = \sum_{i=1}^{k} b(n,k,i)$ for $n \geq k \geq 1$. The sequence $b(n,k,i)$ is determined recursively as follows.

Lemma 14. *We have*

$$(16)\quad b(n,k,i) = qb(n-1,k-1,i) + q^{k-i}\sum_{j=1}^{i} b(n-1,k,j), \quad n>k \text{ and } 1 \leq i \leq k-2,$$

(17)

$$b(n,k,k-1) = q(b(n-1,k-1,k-1) - b(n-2,k-2)) + q\sum_{j=1}^{k-1} b(n-1,k,j), \quad n>k,$$

and

$$(18)\qquad b(n,k,k) = b(n-1,k-1) + b(n-1,k), \quad n \geq k \geq 2,$$

with $b(n,1,1) = 1$ *for all* $n \geq 1$.

Proof. We first show (16). To do so, consider cases based on the length of $\pi^{(k)}$ within $\pi \in \mathcal{A}_{n,k,i}$ where $i \in [k-2]$. If $\pi^{(k)}$ is of length 1, then we remove the single letter k of π (in the penultimate position), which results in a partition π' enumerated by $b(n-1,k-1,i)$. Note that the column in the bargraph corresponding to the rightmost letter i of the partition πk has one more water cell above it than the comparable column in $\pi'(k-1)$. Since all other water cells of π remain unchanged, it follows that the weight of π for which $|\pi^{(k)}| = 1$ is $qb(n-1,k-1,i)$. On the other hand, if $|\pi^{(k)}| > 1$, then $\pi^{(k)}$ is increasing so that the removal of the final i results in a member of $\mathcal{A}_{n-1,k,j}$ for some $j \in [i]$. As there are $k-i$ water cells above the last column of π (due to the understood k that follows), considering all possible j gives a contribution of $q^{k-i}\sum_{j=1}^{i} b(n-1,k,j)$. Combining this case with the prior yields (16). A similar proof applies to (17) except that now in the first case above where $|\pi^{(k)}| = 1$, the resulting partition $\pi' \in \mathcal{A}_{n-1,k-1,k-1}$ must contain at least two letters $k-1$. We correct for this by subtracting the weight of all members of $\mathcal{A}_{n-1,k-1,k-1}$ containing only a single $k-1$, which is $b(n-2,k-2)$, from $b(n-1,k-1,k-1)$. Finally, considering whether or not the final k within a member of $\mathcal{A}_{n,k,k}$ is the only k yields (18) and completes the proof. □

Let $b_{n,k}(v)=\sum_{i=1}^{k} b(n,k,i)v^i$ for $n\geq k\geq 1$. Note that $b_{n,k}(1)=b(n,k)$, by the definitions. Then (16)-(18) together imply

$$\begin{aligned} b_{n,k}(v) &= v^k(b_{n-1,k-1}(1)+b_{n-1,k}(1)) - v^{k-1}qb_{n-2,k-2}(1)+qb_{n-1,k-1}(v)\\ &\quad + q^k\sum_{j=1}^{k} b(n-1,k,j)\sum_{i=j}^{k-1}(v/q)^i\\ &= v^k b_{n-1,k-1}(1) - v^{k-1}qb_{n-2,k-2}(1)+qb_{n-1,k-1}(v)\\ &\quad + \frac{q^{k+1}}{q-v}b_{n-1,k}(v/q) - \frac{v^{k+1}}{q-v}b_{n-1,k}(1), \qquad 2\leq k\leq n, \end{aligned} \tag{19}$$

with $b_{n,0}(v)=\delta_{n,0}$. Let $b_n(u,v)=\sum_{k=1}^{n} b_{n,k}(v)u^k$ for $n\geq 1$. Multiplying both sides of (19) by u^k, and summing over $2\leq k\leq n$, yields

$$\begin{aligned} b_n(u,v) &= uvb_{n-1}(uv,1) - u^2vqb_{n-2}(uv,1) + uqb_{n-1}(u,v) + \frac{q}{q-v}b_{n-1}(uq,v/q)\\ &\quad - \frac{v}{q-v}b_{n-1}(uv,1), \qquad n\geq 2, \end{aligned} \tag{20}$$

with $b_0(u,v)=1$, $b_1(u,v)=uv$. Let $B(x;u,v)=B_q(x;u,v)$ be given by

$$B(x;u,v)=\sum_{n\geq 0} b_n(u,v)x^n.$$

Multiplying both sides of (20) by x^n, and summing over $n\geq 2$, yields the following functional equation satisfied by $B(x;u,v)$.

Theorem 15. *The generating function $B(x;u,v)$ is determined by the functional equation*

$$\begin{aligned} &(1-xuq)B(x;u,v) - \frac{xq}{q-v}B(x;uq,v/q)\\ &\qquad = 1-x-xuq+xv\left(u-xu^2q-\frac{1}{q-v}\right)B(x;uv,1). \end{aligned} \tag{21}$$

Corollary 16. *We have $B(x;1,1)=C(x)$ when $q=1$.*

Proof. Letting $q=1$ in (21), one obtains the following equation for $B(x;u,v)=B_1(x;u,v)$:

$$\left(1-xu-\frac{x}{1-v}\right)B(x;u,v) = 1-x-xu+xv\left(u-xu^2-\frac{1}{1-v}\right)B(x;uv,1). \tag{22}$$

Taking $u=\frac{1}{v}$ in (22) gives

$$\left(1-\frac{x}{v}-\frac{x}{1-v}\right)B(x;1/v,v) = 1-x-\frac{x}{v}+x\left(1-\frac{x}{v}-\frac{v}{1-v}\right)B(x;1,1). \tag{23}$$

Applying the *kernel method* (see, e.g., [**15**]), and letting $v = \frac{1+\sqrt{1-4x}}{2}$ in (23), implies

$$B(x;1,1) = -\frac{1-x-\frac{x}{v}}{x-\frac{x^2}{v}-\frac{xv}{1-v}} = \frac{x-\frac{x}{1-v}}{\frac{x^2}{1-v}-\frac{xv}{1-v}} = \frac{v}{v-x} = \frac{1}{v} = C(x),$$

as required. □

Remark: The derivation in Corollary 16 provides an apparently new proof of the fact that there are C_n non-nesting partitions of $[n]$.

In order to determine $B(x;1,1)$ for q in general, we will need the following limiting value.

Lemma 17. *If $|q| < 1$ and $|u|, |uv| \le 1$, then*

$$\lim_{n\to\infty}(B(x;uq^n,v/q^n)) = 1-x+x(1+uv)B(x;uv,1).$$

Proof. Let $n,m \ge 1$ and $R_m = \sum_{k=1}^{m}\sum_{i=1}^{k-1} b(m,k,i)(uq^n)^k(v/q^n)^i$. Then we have

$$|R_m| \le |q^n|\sum_{k=1}^{m}\sum_{i=1}^{k-1}|b(m,k,i)u^kv^i| \le |q^n|\sum_{k=1}^{m}\sum_{i=1}^{k-1}b(m,k,i) \le |q^n|C_m.$$

Thus, for $|x| \le 1/4$,

$$|\sum_{m\ge2}R_mx^m| \le |q|^n\sum_{m\ge2}C_m|x|^m \to 0,$$

as $n \to \infty$ since $|q| < 1$. Let $T_m = \sum_{k=1}^{m} b(m,k,k)(uv)^k$ for $m \ge 1$, with $T_0 = 1$. Then we have

$$\lim_{n\to\infty}(B(x;uq^n,v/q^n)) = \sum_{m\ge0}T_mx^m.$$

To compute $\sum_{m\ge0}T_mx^m$, first note that (18) implies

$$T_m - uv = \sum_{k=2}^{m} b_{m-1,k-1}(1)(uv)^k + \sum_{k=2}^{m} b_{m-1,k}(1)(uv)^k,$$

or equivalently $T_m = (1+uv)b_{m-1}(uv,1)$ for $m \ge 2$, with $T_1 = uv$, $T_0 = 1$. Multiplying both sides of this last equality by x^m, and summing over $m \ge 2$, yields the desired result. □

We have the following explicit formula for $B(x;1,1)$, where we make use of the standard notation $(x;q)_n = \prod_{j=0}^{n-1}(1-xq^j)$ for $n \ge 1$, with $(x;q)_0 = 1$.

Theorem 18. *If $|q| < 1$, then*

$$B(x;1,1) = \frac{\sum_{j\ge1}\frac{(-x)^{j-1}q^{\binom{j}{2}}}{(q;q)_{j-1}(xq;q)_j}(1-x-xq^j)}{1+\sum_{j\ge1}\frac{(-x)^jq^{\binom{j}{2}}}{(q;q)_j(xq;q)_j}(2-q^j-xq^j(1-q^j))}.$$

Proof. Let $\alpha(u,v) = 1 - \frac{x}{1-xuq} + x\left(uv - \frac{v}{(q-v)(1-xuq)}\right)B(x;uv,1)$ and $\beta(u,v) = \frac{xq}{(q-v)(1-xuq)}$. By (21), we have

$$B(x;u,v) = \alpha(u,v) + \beta(u,v)B(x;uq,v/q).$$

Iteration of this equation where $|q| < 1$ leads to

$$\begin{aligned} B(x;u,v) &= \sum_{j\geq 0}\alpha(uq^j,v/q^j)\prod_{i=0}^{j-1}\beta(uq^i,v/q^i) \\ &\quad + \prod_{i\geq 0}\beta(uq^i,v/q^i)\lim_{n\to\infty}(B(x;uq^n,v/q^n)). \end{aligned}$$

Thus, by Lemma 17, we have for all $\ell \geq 0$,

$$\begin{aligned} B(x;q^\ell,1/q^\ell) &= \sum_{j\geq 1}\frac{(-x)^{j-1}q^{\binom{j}{2}}}{(q^{\ell+1};q)_{j-1}(xq^{\ell+1};q)_j}(1-x-xq^{j+\ell}) \\ &\quad - B(x;1,1)\sum_{j\geq 1}\frac{(-x)^j q^{\binom{j}{2}}}{(q^{\ell+1};q)_j(xq^{\ell+1};q)_j}(2-q^{j+\ell}-xq^{j+\ell}(1-q^{j+\ell})). \end{aligned}$$

Setting $\ell = 0$ in the last equation, and solving for $B(x;1,1)$, yields the desired result. □

Remark: Generalizing slightly the preceding proof yields a comparable formula for $B(x;u,1)$.

We now consider the full array $d_{p,q}(n,k,i)$. Let $d(n,k) = \sum_{i=1}^{k} d(n,k,i)$. Extending the reasoning used in the proof of Lemma 14 above gives the following system of recurrences.

Lemma 19. *We have*

$$d(n,k,i) = pqb(n-1,k-1,i) + pq^{k-i}\sum_{j=1}^{i} d(n-1,k,j), \quad n > k \text{ and } 1 \leq i \leq k-2,$$

$$d(n,k,k-1) = pq(b(n-1,k-1,k-1) - b(n-2,k-2)) + pq\sum_{j=1}^{k-1} d(n-1,k,j), \quad n > k,$$

and

$$d(n,k,k) = b(n-1,k-1) + pd(n-1,k), \quad n \geq k \geq 2,$$

with $d(n,1,1) = p^{n-1}$ for all $n \geq 1$.

Define

$$d_n(u,v) = \sum_{k=1}^{n}\sum_{i=1}^{k} d(n,k,i)u^k v^i, \qquad n \geq 1,$$

with $d_0(u,v) = 1$, and let $D(x;u,v) = D_{p,q}(x;u,v)$ be given by $D(x;u,v) = \sum_{n\geq 0} d_n(u,v)x^n$. We wish to find $D(x;1,1)$. Using Lemma 19, one obtains the following functional equation:

$$\begin{aligned} D(x;u,v) &= (1-xp-xupq) + xupqB(x;u,v) + xuv(1-xupq)B(x;uv,1) \\ &\quad + \frac{xpq}{q-v}D(x;uq,v/q) - \frac{xvp}{q-v}D(x;uv,1). \end{aligned} \tag{24}$$

We first consider the $q = 1$ case of (24). Taking $q = 1$ in (24), and then $v = \frac{1}{u} = 1 - xp$, implies that $D_p(x) = D_{p,1}(x;1,1)$ satisfies

$$D_p(x) = 1 - \frac{x}{(1-xp)^2}\left(p - (1-2xp)C(x) - pB(x;(1-xp)^{-1}, 1-xp)\right),$$

where $B(x;(1-xp)^{-1}, 1-xp)$ is given by (23) with $v = 1 - xp$. Simplifying the last equation gives the unexpectedly simple formula

$$D_p(x) = 1 + \frac{x(C(x)-p)}{1-p+xp^2}. \tag{25}$$

Observe that $D_p(x)$ is the generating function that counts non-nesting partitions of size n according to the statistic for the number of letters to the right of the first occurrence of the maximal letter (i.e., the length of $\pi^{(k)}$ in the notation above). Taking $p = 1$ in (25) implies $D_1(x) = C(x)$, as required.

Proceeding as in the proof of Theorem 18 to solve (24) yields the following formula for $D(x;1,1)$ for general p and q.

Theorem 20. *If $|p| \leq 1$ and $|q| < 1$, then*

$$D(x;1,1) = \frac{\sum_{j\geq 0} \frac{(-xp)^j q^{\binom{j+1}{2}}}{(q;q)_j}\alpha_j(x,p,q)}{1 + \sum_{j\geq 1} \frac{(-xp)^j q^{\binom{j}{2}}}{(q;q)_j}},$$

where $\alpha_j(x,p,q) = 1 - xp + xB(x;1,1) + xpq^{j+1}(B(x;q^j,1/q^j) - xB(x;1,1) - 1)$ and $B(x;1,1)$ and $B(x;q^j,1/q^j)$ are as before.

REFERENCES

1. M. Archibald, A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, *Shedding light on words*, Appl. Anal. Discrete Math., **11:1** (2017), 216–231.
2. W. Asakly, A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, S. Wagner, *Set partition asymptotics and a conjecture of Gould and Quaintance*, J. Math. Anal. Appl., **416** (2014), 672–682.
3. A. Blecher, C. Brennan, A. Knopfmacher, *Levels in bargraphs*, Ars Math. Contemp., **9** (2015), 287–300.

4. A. Blecher, C. Brennan, A. Knopfmacher, *Peaks in bargraphs*, Trans. Royal Soc. S. Afr., **71** (2016), 97–103.
5. A. Blecher, C. Brennan, A. Knopfmacher, *Combinatorial parameters in bargraphs*, Quaest. Math., **39** (2016), 619–635.
6. A. Blecher, C. Brennan, A. Knopfmacher, T. Mansour, *The perimeter of words*, Discrete Math., **340** (2017), 2456–2465.
7. E. R. Canfield, *Engel's inequality for Bell numbers*, J. Combin. Theory Ser. A, **72** (1995), 184–187.
8. L. Carlitz, *Fibonacci notes 3: q-Fibonacci numbers*, Fibonacci Quart., **12** (1974), 317–322.
9. J. Cigler, *Some algebraic aspects of Morse code sequences*, Discrete Math. Theoret. Comput. Sci., **6(1)** (2003), 55–68.
10. E. Deutsch, S. Elizalde, *Statistics on bargraphs viewed as cornerless Motzkin paths*, Discrete Appl. Math., **221** (2017), 54–66.
11. P. Duchon, *q-Grammars and wall polyominoes*, Ann. Comb., **3** (1999), 311–321.
12. S. Feretić, *A perimeter enumeration of column-convex polyominoes*, Discrete Math. Theor. Comput. Sci., **9** (2007), 57–84.
13. A. Geraschenko, *An investigation of skyline polynomials*, pre-print (2016), available at http://people.brandeis.edu/ gessel/47a/geraschenko.pdf.
14. S. Heubach, T. Mansour, *Combinatorics of Compositions and Words* (Discrete Mathematics and its Applications), Chapman & Hall/CRC, Boca Raton, 2009.
15. Q. H. Hou, T. Mansour, *Kernel method and systems of functional equations with several conditions*, J. Comput. Appl. Math., **235:5** (2011), 1205–1212.
16. M. Klazar, *On abab-free and abba-free set partitions*, European J. Combin., **17** (1996), 53–68.
17. T. Mansour, *Combinatorics of Set Partitions* (Discrete Mathematics and its Applications), Chapman & Hall/CRC, Boca Raton, 2012.
18. T. Mansour, M. Shattuck, *Bargraph statistics on words and set partitions*, J. Difference Equ. Appl., **23:6** (2017), 1025–1046.
19. T. Mansour, A. Vainshtein, *Restricted permutations, continued fractions, and Chebyshev polynomials*, Electron. J. Combin., **7** (2000), #R17.
20. S. Milne, *A q-analog of restricted growth functions, Dobinski's equality, and Charlier polynomials*, Trans. Amer. Math. Soc., **245** (1978), 89–118.
21. J. Osborn, T. Prellberg, *Forcing adsorption of a tethered polymer by pulling*, J. Stat. Mech., (2010), P09018.
22. A. Owczarek, T. Prellberg, *Exact solution of the discrete* (1+1)*-dimensional SOS model with field and surface interactions*, J. Stat. Phys., **70:5/6** (1993), 1175–1194.
23. T. Prellberg, R. Brak, *Critical exponents from nonlinear functional equations for partially directed cluster models*, J. Stat. Phys., **78** (1995), 701–730.
24. R. Simion, *Combinatorial statistics on non-crossing partitions*, J. Combin. Theory Ser. A, **66** (1994), 270–301.
25. N. J. Sloane, *The On-Line Encyclopedia of Integer Sequences*, http://oeis.org, 2010.

26. D. Stanton, D. White, *Constructive Combinatorics*, Springer, New York, 1986.

27. C. G. Wagner, *Generalized Stirling and Lah numbers*, Discrete Math., **160** (1996), 199–218.

28. F. Yano, H. Yoshida, *Some set partition statistics in non-crossing partitions and generating functions*, Discrete Math., **307** (2007), 3147–3160.

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