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JOINT APPROXIMATION OF ANALYTIC FUNCTIONS BY SHIFTS OF THE RIEMANN AND PERIODIC HURWITZ ZETA-FUNCTIONS

Antanas Laurinčikas, Renata Macaitienė *

We present some new results on the simultaneous approximation with given accuracy, uniformly on compact subsets of the critical strip, of a collection of analytic functions by discrete shifts of the Riemann and periodic Hurwitz zeta-functions. We prove that the set of such shifts has a positive lower density. For this, we apply the linear independence over the field of rational numbers of certain sets related to the zeta-functions.

1. INTRODUCTION

Since Bohr and Courant works of the second decade of the later century, it is well known that the set of values of the Riemann zeta-function $\zeta(s)$, $s = \sigma + it$, is dense in $\mathbb{C}$. A qualitative leap in the Bohr–Courant theory was made by Voronin [**28**], who obtained an infinite-dimensional generalization of the Bohr–Courant theorem [**3**]. More precisely, Voronin discovered the famous universality property of the function $\zeta(s)$, which, roughly speaking, means that a wide class of analytic functions can be approximated by shifts $\zeta(s + i\tau)$, $\tau \in \mathbb{R}$. Let $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, $\mathcal{K}$ be the class of compact subsets of the strip D with connected complements, and let $H_0(K)$ for $K \in \mathcal{K}$ denote the class of continuous nonvanishing functions on K that are analytic in the interior of K. Then the Voronin theorem [**28**] states that, for all $K \in \mathcal{K}$, $f \in H_0(K)$, and $\varepsilon > 0$, the set of shifts $\zeta(s + i\tau)$,

* Corresponding author. Renata Macaitienė
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$\tau \in \mathbb{R}$, satisfying

$$\sup_{s \in K} |\zeta(s+i\tau) - f(s)| < \varepsilon$$

has a positive lower density. A similar statement is also true for discrete shifts $\zeta(s+ikh)$, $k \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, with every fixed $h > 0$.

After Voronin's work, it turned out that the majority of classical zeta- and L-functions are universal in the Voronin sense. An example is the Hurwitz zeta-function $\zeta(s,\alpha)$ with parameter $0 < \alpha \leq 1$ defined, for $\sigma > 1$, by the series

$$\zeta(s,\alpha) = \sum_{m=0}^{\infty} \frac{1}{(m+\alpha)^s}$$

and analytically continued to the whole complex plane, except for a simple pole at the point $s = 1$ with residue 1. The function $\zeta(s,\alpha)$, differently from $\zeta(s)$, except for the values $\alpha = 1$ ($\zeta(s,1) = \zeta(s)$) and $\alpha = \frac{1}{2}$ ($\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s)$), has no Euler's product over primes, and this is reflected in the universality of $\zeta(s,\alpha)$.

Denote by $H(K)$ with $K \in \mathcal{K}$ the class of continuous functions on K that are analytic in the interior of K. Thus, $H_0(K) \subset H(K)$, $K \in \mathcal{K}$. If α is a transcendental or rational number $\neq 1, \frac{1}{2}$, then, for all $K \in \mathcal{K}$, $f \in H(K)$, and $\varepsilon > 0$, the set of shifts $\zeta(s+i\tau,\alpha)$, $\tau \in \mathbb{R}$, satisfying

$$\sup_{s \in K} |\zeta(s+i\tau,\alpha) - f(s)| < \varepsilon$$

has a positive lower density.

The universality of $\zeta(s,\alpha)$ with algebraic irrational α remains an open problem. The universality of $\zeta(s,\alpha)$ was considered in [**9, 1, 17**] and [**12**]. Discrete versions of universality for $\zeta(s,\alpha)$ on the approximation of analytic functions by shifts $\zeta(s+ikh,\alpha)$ were obtained by Bagchi [**1**] (the case of rational α), Laurinčikas and Macaitienė [**18**], Buivydas, Laurinčikas, Macaitienė, and Rašytė [**6**] (the case of transcendental α and rational $\exp\{\frac{2\pi}{h}\}$), and Laurinčikas [**13**] (the case of linear independence over $\mathbb{Q}$ of the set $\left\{(\log(m+\alpha) : m \in \mathbb{N}_0), \frac{2\pi}{h}\right\}$).

A more interesting but more complicated is the joint universality of zeta- and L-functions. In this case, the simultaneous approximation of a collection of analytic functions by a collection of shifts of zeta- or L-functions is considered. It is clear that, in this case, a certain kind of independence for a collection of zeta-functions is necessary. In the case of Hurwitz zeta-functions, this can be ensured by algebraic independence of the parameters $\alpha_1, \dots, \alpha_r$. Then [**11, 19, 24**], for all $K_j \in \mathcal{K}$, $f_j \in H(K_j)$, $j = 1, \dots, r$, and $\varepsilon > 0$, the set of shifts $\zeta(s+i\tau,\alpha_1), \dots, \zeta(s+i\tau,\alpha_r)$, $\tau \in \mathbb{R}$, satisfying the inequality

$$\sup_{1 \leq j \leq r} \sup_{s \in K_j} |\zeta(s+i\tau,\alpha_j) - f_j(s)| < \varepsilon$$

has a positive lower density. In [**12**], the algebraic independence of the numbers $\alpha_1, \dots, \alpha_r$ was replaced by the weaker hypothesis that the set $\{\log(m+\alpha_j) : m \in$

$\mathbb{N}_0,\ j=1,\dots,r\}$ is linearly independent over $\mathbb{Q}$. Mishou [**22**] ($r=2$) and Dubickas [**7**] proved joint universality theorems for Hurwitz zeta-functions with dependent transcendental parameters $\alpha_1,\dots,\alpha_r$.

We have already mentioned the thesis [**1**], where the first discrete universality theorem for Hurwitz zeta-functions with rational parameter was obtained. The paper [**26**] is devoted to the joint discrete universality of a collection $\zeta(s,\alpha_1),\dots,\zeta(s,\alpha_r)$ with rational $\alpha_1,\dots,\alpha_r$ and $h>0$, whereas, in [**15**] the case of $\alpha_1,\dots,\alpha_r$ for which the set

$$\left\{(\log(m+\alpha_j) : m\in\mathbb{N}_0,\ j=1,\dots,r),\ \frac{2\pi}{h}\right\}$$

is linearly independent over $\mathbb{Q}$ was considered. In [**14**] the latter set was replaced by

$$\{(h_j\log(m+\alpha_j) : m\in\mathbb{N}_0,\ j=1,\dots,r),2\pi\}$$

with positive $h_1,\dots,h_r$.

Several joint universality theorems for zeta-functions are of mixed character: the joint universality of collections consisting of zeta-functions with Euler's product and zeta-functions without such a product is investigated. The first result in this direction belongs to a Japanese mathematician Mishou [**21**]. He proved a mixed joint universality theorem for the functions $\zeta(s)$ and $\zeta(s,\alpha)$ with transcendental α. Discrete versions of the Mishou theorem were obtained in [**4, 5, 16**].

The Mishou theorem was generalized for various zeta-functions. Much attention was devoted to the so-called periodic zeta-functions. Let $\mathfrak{a}=\{a_m : m\in\mathbb{N}\}$ be a periodic sequence of complex numbers. Then the periodic zeta-function $\zeta(s;\mathfrak{a})$, a generalization of $\zeta(s)$, is defined, for $\sigma>1$, by the Dirichlet series

$$\zeta(s;\mathfrak{a})=\sum_{m=1}^{\infty}\frac{a_m}{m^s}$$

and can be meromorphically continued to the whole complex plane. Let $\mathfrak{b}=\{b_m : m\in\mathbb{N}_0\}$ be another periodic sequence of complex numbers. Then the periodic Hurwitz zeta-function $\zeta(s,\alpha;\mathfrak{b})$, a generalization of $\zeta(s,\alpha)$, is given, for $\sigma>1$, by the series

$$\zeta(s,\alpha;\mathfrak{b})=\sum_{m=0}^{\infty}\frac{b_m}{(m+\alpha)^s}$$

and also has a meromorphic continuation to the whole complex plane. In [**10**], the Mishou theorem was generalized for $\zeta(s;\mathfrak{a})$ and $\zeta(s,\alpha;\mathfrak{b})$ with multiplicative sequence $\mathfrak{a}$. In [**8**], a mixed joint universality theorem for $\zeta(s)$ and a collection of periodic Hurwitz zeta-functions was proved. The aim of this paper is mixed joint discrete theorems for the latter functions.

For $j=1,\dots,r$, let α_j, $0<\alpha_j\le 1$, be fixed parameters, let $\mathfrak{a}_j=\{a_{mj} : m\in\mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_j\in\mathbb{N}$, and let $\zeta(s,\alpha_j;\mathfrak{a}_j)$ denote the corresponding periodic Hurwitz zeta-function. Moreover, let

$$L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, \pi) = \Big\{ \big(\log p : p \in \mathbb{P} \big), \\ \big(\log(m+\alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r \big), \frac{2\pi}{h} \Big\}, \ h > 0,$$

where $\mathbb{P}$ denotes the set of all prime numbers.

Theorem 1. *Suppose that the set* $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, \pi)$ *is linearly independent over* $\mathbb{Q}$. *Let* $K, K_1, \dots, K_r \in \mathcal{K}$, $f \in H_0(K)$, *and* $f_1 \in H(K_1), \dots, f_r \in H(K_r)$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \Big\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon, \\ \sup_{1\le j\le r} \sup_{s\in K_j} |\zeta(s+ikh, \alpha_j; \mathfrak{a}_j) - f_j(s)| < \varepsilon \Big\} \quad > \quad 0.$$

Here $\#A$ denotes the cardinality of a set A.

Theorem 1 can be generalized as follows. Let $l_j \in \mathbb{N}$, let $\mathfrak{a}_{jl} = \{a_{mjl} : m \in \mathbb{N}_0\}$ be a periodic sequence of complex numbers with minimal period $k_{jl} \in \mathbb{N}$ for $l = 1, \dots, l_j$, $j = 1, \dots, r$, and let $\zeta(s, \alpha_j; \mathfrak{a}_{jl})$ denote the corresponding periodic Hurwitz zeta-function. Moreover, let k_j be the least common multiple of the periods $k_{j1}, \dots, k_{jl_j}$, $j = 1, \dots, r$, and

$$B_j = \begin{pmatrix} a_{0j1} & a_{0j2} & \dots & a_{0jl_j} \\ a_{1j1} & a_{1j2} & \dots & a_{2jl_j} \\ \dots & \dots & \dots & \dots \\ a_{k_j-1,j1} & a_{k_j-1,j2} & \dots & a_{k_j-1,jl_j} \end{pmatrix}, \quad j = 1, \dots, r.$$

Theorem 2. *Suppose that the set* $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, \pi)$ *is linearly independent over* $\mathbb{Q}$ *and that* $\operatorname{rank}(B_j) = l_j$, $j = 1, ..., r$. *Let* $K, K_{jl} \in \mathcal{K}$, *and let* $f \in H_0(K)$, $f_{jl} \in H(K_{jl})$, $j = 1, ..., r$, $l = 1, \dots, l_j$. *Then, for every* $\varepsilon > 0$,

$$\liminf_{N\to\infty} \frac{1}{N+1} \# \Big\{ 0 \le k \le N : \sup_{s\in K} |\zeta(s+ikh) - f(s)| < \varepsilon, \\ \sup_{1\le j\le r} \sup_{1\le l\le l_j} \sup_{s\in K_{jl}} |\zeta(s+ikh, \alpha_j; \mathfrak{a}_{jl}) - f_{jl}(s)| < \varepsilon \Big\} \quad > \quad 0.$$

For $h > 0$, $h_1 > 0, \dots, h_r > 0$, define

$$L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi) \quad = \quad \Big\{ \big(h \log p : p \in \mathbb{P} \big), \\ \big(h_j \log(m+\alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r \big), 2\pi \Big\}.$$

Theorem 3. *Suppose that the set* $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi)$ *is linearly independent over* $\mathbb{Q}$ *and that* $\operatorname{rank}(B_j) = l_j$, $j = 1, \dots, r$. *Let* K, K_{jl}, *and* f, f_{jl},

$j=1,\dots,r$, $l=1,\dots,l_j$, be the same as in Theorem 2. Then, for every $\varepsilon>0$,

$$\liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{0\le k\le N:\ \sup_{s\in K}|\zeta(s+ikh)-f(s)|<\varepsilon,$$
$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|\zeta(s+ikh_j,\alpha_j;\mathfrak{a}_{jl})-f_{jl}(s)|<\varepsilon\Big\}\quad>\quad 0.$$

Clearly, Theorems 1 and 2 are corollaries of Theorem 3. Therefore, it suffices to prove Theorem 3. We will also give some remarks on the proof of Theorem 2 (in fact, of Theorem 4, which is the main tool for the proof of Theorem 2), which is shorter than that of Theorem 3.

It is not difficult to see that the set $L(\mathbb{P},\alpha_1,\dots,\alpha_r,h,h_1,\dots,h_r,\pi)$ is linearly independent over $\mathbb{Q}$ if the numbers $\alpha_1,\dots,\alpha_r$ and e^π are algebraically independent over $\mathbb{Q}$ and the numbers $h,h_1,\dots,h_r$ are rational. By the Nesterenko theorem [**25**] the numbers π and e^π are algebraically independent over $\mathbb{Q}$. Therefore, in Theorem 2, we can take, for example, $\alpha_j=q_j\pi$ with rational q_j such that $q_j\pi<1$, $j=1,\dots,r$, and rational h. Obviously, it is always possible to choose the numbers a_{mjl} such that $\operatorname{rank}(B_j)=l_j$, $j=1,\dots,r$.

2. MAIN LEMMAS

We start with the definition of a certain topological group. Denote by γ the unit circle on the complex plane and define two infinite-dimensional tori

$$\hat{\Omega}=\prod_p\gamma_p\quad\text{and}\quad\Omega=\prod_{m=0}^{\infty}\gamma_m,$$

where $\gamma_p=\gamma$ for all primes $p\in\mathbb{P}$, and $\gamma_m=\gamma$ for all $m\in\mathbb{N}_0$. By the Tikhonov theorem, $\hat{\Omega}$ and Ω with the product topology and pointwise multiplication are compact topological Abelian groups. Moreover, let

$$\underline{\Omega}=\hat{\Omega}\times\Omega_1\times\dots\times\Omega_r,$$

where $\Omega_j=\Omega$ for $j=1,\dots,r$. Then, by the Tikhonov theorem again, $\underline{\Omega}$ is a compact topological Abelian group. Denote by $\mathcal{B}(\mathbb{X})$ the Borel σ-field of the space $\mathbb{X}$. Then, on $(\underline{\Omega},\mathcal{B}(\underline{\Omega}))$, there exists a probability Haar measure m_H, and we have the probability space $(\underline{\Omega},\mathcal{B}(\underline{\Omega}),m_H)$. Denote by $\hat{\omega}(p)$ the projection of $\hat{\omega}\in\hat{\Omega}$ to γ_p, $p\in\mathbb{P}$ and by $\omega_j(m)$ the projection of $\omega_j\in\Omega_j$ to γ_m, $m\in\mathbb{N}_0$. Let $\underline{\omega}=(\hat{\omega},\omega_1,\dots,\omega_r)$ be elements of $\underline{\Omega}$.

For $A\in\mathcal{B}(\underline{\Omega})$, define

$$Q_{N,\underline{h}}(A)=\frac{1}{N+1}\#\Big\{0\le k\le N:\ \Big(\big(p^{-ikh}:p\in\mathbb{P}\big),$$
$$\big((m+\alpha_1)^{-ikh_1}:m\in\mathbb{N}_0\big),\dots,\big((m+\alpha_r)^{-ikh_r}:m\in\mathbb{N}_0\big)\Big)\in A\Big\}.$$

Lemma 1. *Suppose that the set* $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi)$ *is linearly independent over* $\mathbb{Q}$. *Then* $Q_{N,\underline{h}}$ *converges weakly to the Haar measure* m_H *as* $N \to \infty$.

Proof. For $\underline{k} = (k_p, k_{jm} : k_p,\ k_{jm} \in \mathbb{Z},\ p \in \mathbb{P},\ m \in \mathbb{N}_0,\ j = 1, \dots, r)$, denote by $g_{N,\underline{h}}(\underline{k})$ the Fourier transform of $Q_{N,\underline{h}}$. Then, we have that (see [**8**])

$$g_{N,\underline{h}}(\underline{k}) = \int\limits_{\Omega} \prod_p{}' \hat{\omega}^{k_p} \prod_{j=1}^{r} \prod_{m=0}^{\infty}{}' \omega_j^{k_{jm}}(m) \mathrm{d}Q_{N,\underline{h}},$$

where the sign $'$ means that only a finite number of integers k_p and k_{jm} are distinct from zero. Thus, by the definition of $Q_{N,\underline{h}}$,

$$\begin{aligned} g_{N,\underline{h}}(\underline{k}) &= \frac{1}{N+1} \sum_{k=0}^{N} \prod_p{}' p^{-ik_p kh} \prod_{j=1}^{r} \prod_{m=0}^{\infty}{}' (m+\alpha_j)^{-ik_{jm}kh_j} \\ &= \frac{1}{N+1} \sum_{k=0}^{N} \exp\Big\{ -ik\Big(\sum_p{}' hk_p \log p \\ &\quad + \sum_{j=1}^{r} \sum_{m=0}^{\infty}{}' h_j k_{jm} \log(m+\alpha_j) \Big) \Big\}. \end{aligned} \tag{1}$$

Obviously,

$$g_{N,\underline{h}}(\underline{0}) = 1. \tag{2}$$

Now suppose that $\underline{k} \neq \underline{0}$. Then we observe that

$$A_{\underline{h}}(\underline{k}) \overset{def}{=} \exp\left\{ -i \left(\sum_p{}' hk_p \log p + \sum_{j=1}^{r} \sum_{m=0}^{\infty}{}' h_j k_{jm} \log(m+\alpha_j) \right) \right\} \neq 1. \tag{3}$$

Indeed, if the latter inequality were not true, then we would have

$$A_{\underline{h}}(\underline{k}) = \mathrm{e}^{2\pi i v}$$

for some $v \in \mathbb{Z}$. Hence,

$$\sum_p{}' hk_p \log p + \sum_{j=1}^{r} \sum_{m=0}^{\infty}{}' h_j k_{jm} \log(m+\alpha_j) - 2\pi v_1 = 0$$

for some $v_1 \in \mathbb{Z}$, and this contradicts the linear independence of the set $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi)$. Thus, (3) is valid. Now, using (1) we find that, in the case $\underline{k} \neq \underline{0}$,

$$g_{N,\underline{h}}(\underline{k}) = \frac{1 - A_{\underline{h}}^{N+1}(\underline{k})}{(N+1)(1 - A_{\underline{h}}(\underline{k}))}.$$

Therefore, by (2) we have

$$\lim_{N\to\infty} g_{N,\underline{h}}(\underline{k}) = \begin{cases} 1 & \text{if} \quad \underline{k} = \underline{0}, \\ 0 & \text{if} \quad \underline{k} \neq \underline{0}. \end{cases}$$

This proves the lemma because the right-hand side of the last equality is the Fourier transform of the Haar measure m_H. □

We continue with a result from ergodic theory. Let

$$\underline{a}_{\underline{h}} = \left(\left(p^{-ih} : p \in \mathbb{P}\right), \left((m+\alpha_1)^{-ih_1} : m \in \mathbb{N}_0\right), \dots, \left((m+\alpha_r)^{-ih_r} : m \in \mathbb{N}_0\right)\right),$$

and, for $\underline{\omega} \in \underline{\Omega}$, define $\underline{\Phi}_{\underline{h}}(\underline{\omega}) = \underline{a}_{\underline{h}}\underline{\omega}$. Since $\underline{a}_{\underline{h}} \in \underline{\Omega}$, we have that $\underline{\Phi}_{\underline{h}}$ is a measurable measure-preserving transformation of $\underline{\Omega}$. We recall that a set $A \in \mathcal{B}(\underline{\Omega})$ is called invariant with respect to $\underline{\Phi}_{\underline{h}}$ if the sets A and $\underline{\Phi}_{\underline{h}}(A)$ coincide m_H-almost surely. The transformation $\underline{\Phi}_{\underline{h}}$ is ergodic if the σ-field of invariant sets consists only of the sets of m_H-measure 1 or 0.

Lemma 2. *Suppose that the set $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi)$ is linearly independent over $\mathbb{Q}$. Then the transformation $\underline{\Phi}_{\underline{h}}$ is ergodic.*

Proof. For proving Lemma 1, we have already used the fact that a character χ of $\underline{\Omega}$ is of the form

$$\chi(\underline{\omega}) = {\prod_p}' \hat{\omega}^{k_p}(p) \prod_{j=1}^{r} {\prod_{m=0}^{\infty}}' \omega_j^{k_{jm}}(m), \quad \underline{\omega} \in \underline{\Omega}.$$

Therefore, in view of (3), we have that, for $\underline{k} \neq \underline{0}$,

$$\chi(\underline{a}_{\underline{h}}) = A_{\underline{h}}(\underline{k}) \neq 1. \tag{4}$$

Let A be an invariant set of $\underline{\Phi}_{\underline{h}}$, and let $I_A(\underline{\omega})$ denote its indicator function. Now, if χ is a nontrivial character of $\underline{\Omega}$, then, using (4), we find that the Fourier transform $\hat{I}_A(\chi)$ of I_A is equal to 0. If χ_0 is the trivial character of $\underline{\Omega}$, then, putting $\hat{I}_A(\chi_0) = u$, we obtain that, for every character χ of $\underline{\Omega}$,

$$\hat{I}_A(\chi) = \hat{u}(\chi).$$

From this it easily follows that $m_H(A) = 1$ or $m_H(A) = 0$, that is, the transformation $\underline{\Phi}_{\underline{h}}$ is ergodic. □

3. LIMIT THEOREMS

Denote by $H(D)$, $D = \{s \in \mathbb{C} : \frac{1}{2} < \sigma < 1\}$, the space of analytic functions on D endowed with the topology of uniform convergence on compacta. This section is devoted to limit theorems for

$$P_{N,\underline{h}}(A) \stackrel{def}{=} \frac{1}{N+1} \#\{0 \leq k \leq N : \underline{\zeta}(s + ik\underline{h}, \underline{\alpha}; \mathfrak{a}) \in A\}, \; A \in \mathcal{B}\left(H^{\kappa}(D)\right)$$

as $N \to \infty$, where $\underline{\alpha} = (\alpha_1, \dots, \alpha_r)$, $\underline{\mathfrak{a}} = (\mathfrak{a}_{11}, \dots, \mathfrak{a}_{1l_1}, \dots, \mathfrak{a}_{r1}, \dots, \mathfrak{a}_{rl_r})$, $\kappa = l_1 + \cdots l_r + 1$, and

$$\begin{aligned}\underline{\zeta}(s + ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}) \quad = \quad & \big(\zeta(s+ikh), \zeta(s+ikh_1, \alpha_1; \mathfrak{a}_{11}), \dots, \zeta(s+ikh_1, \alpha_1; \mathfrak{a}_{1l_1}), \dots, \\ & \zeta(s+ikh_r, \alpha_r; \mathfrak{a}_{r1}), \dots, \zeta(s+ikh_r, \alpha_r; \mathfrak{a}_{rl_r})\big).\end{aligned}$$

On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$, define the $H^\kappa(D)$-valued random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ by

$$\begin{aligned}\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \quad = \quad & \big(\zeta(s, \hat{\omega}), \zeta(s, \alpha_1, \omega_1; , \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \\ & \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big),\end{aligned}$$

where

$$\zeta(s, \hat{\omega}) = \prod_p \left(1 - \frac{\hat{\omega}(p)}{p^s}\right)^{-1}$$

and

$$\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl}) = \sum_{m=0}^{\infty} \frac{a_{mjl}\omega_j(m)}{(m+\alpha_j)^s}, \quad j = 1, \dots, r, \quad l = 1, \dots, l_j.$$

Let $P_{\underline{\zeta}}$ be the distribution of the random element $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$, that is,

$$P_{\underline{\zeta}}(A) = m_H \left\{\underline{\omega} \in \underline{\Omega} : \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\right\}, \quad A \in \mathcal{B}\left(H^\kappa(D)\right).$$

Theorem 4. *Suppose that the set* $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi)$ *is linearly independent over* $\mathbb{Q}$ *and that* $\operatorname{rank}(B_j) = l_j$, $j = 1, \dots, r$. *Then* $P_{N,\underline{h}}$ *converges weakly to* $P_{\underline{\zeta}}$ *as* $N \to \infty$.

The outline of the proof of Theorem 4 is the following. First of all, using Lemma 1, we prove a limit theorem for a collection $\underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}})$ consisting from absolutely convergent Dirichlet series. To pass from $\underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}})$ to $\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}})$, we aproximate $\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}})$ by $\underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}})$ in the mean. This approximation result, a limit theorem for $\underline{\zeta}_n(s, \underline{\alpha}; \underline{\mathfrak{a}})$ and properties of weak convergence of probability measures including the Prokhorov theory ([**2**], Section 6.1) allows to obtain a limit theorem for $\underline{\zeta}(s, \underline{\alpha}; \underline{\mathfrak{a}})$. The last part of the proof is devoted to identification of the limit measure, and is based on Lemma 2. For clarity of the proof, we divide it into several lemmas.

Thus, we start with a limit theorem for the probability measure defined by means of absolutely convergent Dirichlet series. For a fixed number $\theta > \frac{1}{2}$, define

$$v_n(m) = \exp\left\{-\left(\frac{m}{n}\right)^\theta\right\}, \quad m, n \in \mathbb{N},$$

and

$$v_n(m, \alpha_j) = \exp\left\{-\left(\frac{m+\alpha_j}{n+\alpha_j}\right)^\theta\right\}, \quad m \in \mathbb{N}_0, \quad n \in \mathbb{N}.$$

Then we have (see [**8**]) that the series

$$\zeta_n(s)=\sum_{m=0}^{\infty}\frac{v_n(m)}{m^s}$$

and

$$\zeta_n(s,\alpha_j;\mathfrak{a}_{jl})=\sum_{m=1}^{\infty}\frac{a_{mjl}v_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r,\quad l=1,\dots,l_j,$$

are absolutely convergent for $\sigma>\frac{1}{2}$. Extend the function $\hat{\omega}(p)$ to the set $\mathbb{N}$ by

$$\hat{\omega}(p)=\prod_{\substack{p^l\mid m\\ p^{l+1}\nmid m}}\omega^l(p),\quad m\in\mathbb{N},$$

and define

$$\zeta_n(s,\hat{\omega})=\sum_{m=1}^{\infty}\frac{\hat{\omega}(m)v_n(m)}{m^s}$$

and

$$\zeta_n(s,\alpha_j,\omega_j;\mathfrak{a}_{jl})=\sum_{m=0}^{\infty}\frac{a_{mjl}\omega_j(m)v_n(m,\alpha_j)}{(m+\alpha_j)^s},\quad j=1,\dots,r,\quad l=1,\dots,l_j.$$

Since $|\hat{\omega}(m)|=1$ and $|\omega_j(m)|=1$, the latter series are also absolutely convergent for $\sigma>\frac{1}{2}$. We set

$$\begin{aligned}\underline{\zeta}_n(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}) \;=\; &\big(\zeta_n(s,\hat{\omega}),\zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{11}),\dots,\zeta_n(s,\alpha_1,\omega_1;\mathfrak{a}_{1l_1}),\dots,\\ &\zeta_n(s,\alpha_r,\omega_r;\mathfrak{a}_{r1}),\dots,\zeta(s,\alpha_r,\omega_r;\mathfrak{a}_{rl_r})\big).\end{aligned}$$

Let the function $u_n:\Omega\to H^\kappa(D)$ be given by

$$u_n(\underline{\omega})=\underline{\zeta}_n(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}}).$$

Since the series constituting $\underline{\zeta}_n(s,\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$ are absolutely convergent for $\sigma>\frac{1}{2}$, we have that the functions u_n are continuous. Therefore, they are $(\mathcal{B}(\underline{\Omega}),\mathcal{B}(H^\kappa(D)))$-measurable, and hence the measure m_H induces the unique probability measure $P_n=m_Hu_n^{-1}$ on $(H^\kappa(D),\mathcal{B}(H^\kappa(D)))$, where

$$P_n(A)=m_Hu_n^{-1}(A)=m_H(u_n^{-1}A),\quad A\in\mathcal{B}(H^\kappa(D)).$$

Moreover, if

$$\begin{aligned}\underline{\zeta}_n(s+ik\underline{h},\underline{\alpha};,\underline{\mathfrak{a}}) \;=\; &\big(\zeta_n(s+ikh),\zeta_n(s+ikh_1,\alpha_1;\mathfrak{a}_{11}),\dots,\zeta_n(s+ikh_1,\alpha_1;\mathfrak{a}_{1l_1}),\\ &\dots,\zeta_n(s+ikh_r,\alpha_r;\mathfrak{a}_{r1}),\dots,\zeta_n(s+ikh_r,\alpha_r;\mathfrak{a}_{rl_r})\big)\end{aligned}$$

and

$$P_{N,n,\underline{h}}(A)=\frac{1}{N+1}\#\{0\le k\le N:\underline{\zeta}_n(s+ik\underline{h},\underline{\alpha};\underline{\mathfrak{a}})\in A\},\quad A\in\mathcal{B}\left(H^\kappa(D)\right),$$

then we have that

$$u_n\left(\left(p^{-ikh} : p \in \mathbb{P}\right), \left((m+\alpha_1)^{-ikh_1} : m \in \mathbb{N}_0\right), \dots, \left((m+\alpha_r)^{-ikh_r} : m \in \mathbb{N}_0\right)\right) = \underline{\zeta}_n(s+ik\underline{h}, \underline{\alpha}; \mathfrak{a}),$$

and thus $P_{N,n,\underline{h}} = Q_{N,\underline{h}} u_n^{-1}$. This, Lemma 1, the continuity of u_n, and Theorem 5.1 of [**2**] (If P_n, $n \in \mathbb{N}$, and P are probability measures on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$, $u : \mathbb{X} \to \mathbb{X}_1$ is a continuous mapping, and P_n converges weakly to P, then also $P_n u^{-1}$ converges weakly to Pu^{-1} as $n \to \infty$) lead to the following statement.

Lemma 3. *Suppose that the set $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi)$ is linearly independent over $\mathbb{Q}$. Then $P_{N,n,\underline{h}}$ converges weakly to P_n as $N \to \infty$.*

Similarly, we obtain a limit theorem for

$$P_{N,n,\underline{h},\underline{\omega}}(A) = \frac{1}{N+1} \#\{0 \le k \le N : \underline{\zeta}_n(s+ik\underline{h}, \underline{\alpha}, \underline{\omega}; \mathfrak{a}) \in A\}, \quad A \in \mathcal{B}\left(H^\kappa(D)\right),$$

where, for $\underline{\omega} \in \underline{\Omega}$,

$$\begin{aligned}\underline{\zeta}_n(s+ik\underline{h}, \underline{\alpha}, \underline{\omega}; \mathfrak{a}) = \big(&\zeta_n(s+ikh, \hat{\omega}), \zeta_n(s+ikh_1, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots,\\ &\zeta_n(s+ikh_1, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \zeta_n(s+ikh_r, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots,\\ &\zeta_n(s+ikh_r, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big).\end{aligned}$$

Lemma 4. *Under the hypotheses of Lemma 3, $P_{N,n,\underline{h},\underline{\omega}}$ also converges weakly to P_n as $N \to \infty$.*

Proof. We follow the proof of Lemma 3 and apply the invariance of the Haar measure m_H.

□

Furthermore, we will approximate $\underline{\zeta}(s, \underline{\alpha}; \mathfrak{a})$ by $\underline{\zeta}_n(s, \underline{\alpha}; \mathfrak{a})$. For this, we need a metric on $H(D)$ inducing its topology. It is well known that there exists a sequence of compact subsets $\{K_v : v \in \mathbb{N}\} \subset D$ such that

$$D = \bigcup_{v=1}^{\infty} K_v,$$

$K_v \subset K_{v+1}$ for all $v \in \mathbb{N}$, and if $K \subset D$ is a compact set, then $K \subset K_v$ for some $v \in \mathbb{N}$. For $g_1, g_2 \in H(D)$, define

$$\varrho(g_1, g_2) = \sum_{v=1}^{\infty} 2^{-v} \frac{\sup\limits_{s \in K_v} |g_1(s) - g_2(s)|}{1 + \sup\limits_{s \in K_v} |g_1(s) - g_2(s)|}.$$

Then ϱ is a metric on $H(D)$ inducing its topology of convergence on compacta. Now, let

$$\underline{g}_m = (g_{m0}, g_{m11}, \dots, g_{m1l_1}, \dots, g_{mr1}, \dots, g_{mrl_r}) \in H(D), \quad m = 1, 2.$$

Then

$$\varrho_\kappa(\underline{g}_1, \underline{g}_2) = \max\left(\varrho(g_{10}, g_{20}), \max_{1\le j\le r}\max_{1\le l\le l_j}\varrho(g_{1jl}, g_{2jl})\right)$$

is a metric on $H^\kappa(D)$ inducing its topology.

Lemma 5. *We have*

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varrho_\kappa\left(\underline{\zeta}(s+ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}), \underline{\zeta}_n(s+ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}})\right) = 0$$

for all $\underline{h}$, $\underline{\alpha}$, *and* $\underline{\mathfrak{a}}$.

Proof. From the definition of the metric ϱ_κ we have that it suffices that

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varrho\left(\zeta(s+ikh), \zeta_n(s+ikh)\right) = 0$$

and

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varrho\left(\zeta(s+ikh_j, \alpha_j; \mathfrak{a}_{jl}), \zeta_n(s+ikh_j, \alpha_j; \mathfrak{a}_{jl})\right) = 0$$

for all $j = 1, \dots, r$, $l = 1, \dots, l_j$. However, the first equality was obtained in the proof of Lemma 3 in [**4**], and the second equality is Theorem 4.1 of [**18**]. □

We also need an analogue of Lemma 5 for $\underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$ and $\underline{\zeta}_n(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})$. Let

$$\begin{aligned}\underline{\zeta}(s+ik\underline{h}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \;=\; &\big(\zeta(s+ikh, \hat{\omega}), \zeta(s+ikh_1, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots,\\ &\zeta(s+ikh_1, \alpha_1, \omega_1; \mathfrak{a}_{1l_1}), \dots, \zeta(s+ikh_r, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots,\\ &\zeta(s+ikh_r, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})\big).\end{aligned}$$

Lemma 6. *Suppose that the set* $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi)$ *is linearly independent over* $\mathbb{Q}$. *Then, for almost all* $\underline{\omega} \in \underline{\Omega}$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varrho_\kappa\left(\underline{\zeta}(s+ik\underline{h}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}), \underline{\zeta}_n(s+ik\underline{h}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}})\right) = 0.$$

Proof. In the proof of Lemma 6 of [**4**], it is obtained that, for almost all $\hat{\omega} \in \hat{\Omega}$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varrho\left(\zeta(s+ikh, \hat{\omega}), \zeta_n(s+ikh, \hat{\omega})\right) = 0. \tag{5}$$

Moreover, the linear independence of the set $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi)$ implies that of the sets $\{h_j \log(m+\alpha_j) : m \in \mathbb{N}\}$, $j = 1, \dots, r$. Therefore, repeating the proof of Lemma 4 of [**5**], we find that, for almost all $\omega_j \in \Omega_j$,

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\varrho\left(\zeta(s+ikh_j, \alpha_j, \omega_j; \mathfrak{a}_{jl}), \zeta_n(s+ikh_j, \alpha_j, \omega_j; \mathfrak{a}_{jl})\right) = 0$$

for all $j = 1, \dots, r$, $l = 1, \dots, l_j$. This and (5) prove the lemma. □

To obtain the weak convergence for $P_{n,\underline{h}}$, Theorem 4.2 of [**2**] will be applied. For convenience of a reader, we state it as a separate lemma. Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution.

Lemma 7. *Suppose that the space* $(\mathbb{X}, \varrho)$ *is separable, and* $Y_n, X_{1n}, X_{2n}, \ldots$ *are* $\mathbb{X}$-*valued random elements defined on the probability space with the measure* μ. *Let* $X_{kn} \xrightarrow[n\to\infty]{\mathcal{D}} X_k$ *for each* $k \in \mathbb{N}$, *and* $X_k \xrightarrow[k\to\infty]{\mathcal{D}} X$. *If, for every* $\epsilon > 0$,

$$\lim_{k\to\infty} \limsup_{n\to\infty} \mu\left\{\varrho\left(X_{kn}, Y_n\right) \geq \epsilon\right\} = 0,$$

then $Y_n \xrightarrow[n\to\infty]{\mathcal{D}} X$.

Lemmas 3 and 5 are two important ingredients for application of Lemma 7, however, we need the third one, i.e., the weak convergence of P_n as $n \to \infty$, where P_n is the limit measure in Lemma 3. For this, we apply the Prokhorov theory connecting the notions of tightness and relative compactness of families of probability measures. We recall that the family of probability measures $\{P\}$ on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ is tight if, for every $\epsilon > 0$, there exists a compact subset $K = K(\epsilon) \subset \mathbb{X}$ such that $P(K) > 1 - \epsilon$ for all $P \in \{P\}$, and $\{P\}$ is relatively compact if every sequence $\{P_n\} \subset \{P\}$ contains a subsequence $\{P_{n_m}\}$ such that P_{n_m} converges weakly to certain probability measure P on $(\mathbb{X}, \mathcal{B}(\mathbb{X}))$ as $m \to \infty$. The Prokhorov theorem says that if the family $\{P\}$ is tight, then $\{P\}$ is relatively compact. Thus, it is sufficient to prove the tightness for $\{P_n : n \in \mathbb{N}\}$. On the other hand, the tightness of $\{P_n\}$ is closely related to discrete mean squares of the functions $\zeta_n(s)$ and $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$. Therefore, we state the Gallagher lemma which connects continuous and discrete mean squares of certain functions.

Lemma 8. *Let* T_0 *and* $T \geq \delta > 0$ *be real numbers, and* $\mathcal{T}$ *be a finite set of points in the interval* $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$, *and*

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1.$$

Suppose that f *is a complex-valued continuous function on* $[T_0, T + T_0]$ *that has a continuous derivative on* $(T_0, T + T_0)$. *Then*

$$\sum_{t\in\mathcal{T}} N_\delta^{-1}(t)|f(t)|^2 \leq \frac{1}{\delta}\int_{T_0}^{T_0+T} |f(x)|^2\,dx + \left(\int_{T_0}^{T_0+T} |f(x)|^2\,dx \int_{T_0}^{T_0+T} |f'(x)|^2\,dx\right)^{\frac{1}{2}}.$$

The proof of the lemma is given in [**23**], Lemma 1.4.

Now, we are in position to prove the tightness of $\{P_n\}$.

Lemma 9. *Suppose that the set* $L\left(\mathbb{P}, \alpha_1, \ldots, \alpha_r, h, h_1, \ldots, h_r, \pi\right)$ *is linearly independent over* $\mathbb{Q}$. *Then the family* $\{P_n : n \in \mathbb{N}\}$ *is tight.*

Proof. By Lemma 3, we have that $P_{N,n,\underline{h}}$ converges weakly to P_n as $N \to \infty$. Let $\underline{X}_n = \underline{X}_n(s) = (X_n(s), X_{n,1,1}(s), ..., X_{n,1,l_1}(s), ..., X_{n,r,1}(s), ..., X_{n,r,l_r}(s))$ be an $H^\kappa(D)$-valued random element having the distribution P_n, and let θ_N be a discrete random variable defined on a certain probability space with the measure μ and distribution

$$\mu\{\theta_N = k\} = \frac{1}{N+1}, \quad k = 0, 1, \dots, N.$$

Similarly to $\underline{\zeta}_n(s + ik\underline{h}, \underline{\alpha}; \underline{\mathfrak{a}})$, define the $H^\kappa(D)$-valued random element

$$\begin{aligned} \underline{X}_{N,n,\underline{h}} &= \underline{X}_{N,n,\underline{h}}(s) \\ &= (X_{N,n,h}(s), X_{N,n,h_1,1}(s), ..., X_{N,n,h_1,l_1}(s), ..., X_{N,n,h_r,1}(s), ..., X_{N,n,h_r,l_r}(s)) \\ &= \underline{\zeta}_n(s + i\theta_N \underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}). \end{aligned}$$

Then Lemma 3 implies the relation

$$\underline{X}_{N,n,\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} \underline{X}_n. \tag{6}$$

Let M_v and M_{vjl} be positive numbers, and let K_v be the compact set from the definition of the metric ϱ. Then we have

$$\begin{aligned} &\mu\left\{\sup_{s\in K_v} |X_{N,n,h}(s)| > M_v \quad \text{or} \quad \exists j, l : \sup_{s\in K_v} |X_{N,n,h_j,l}(s)| > M_{vjl}\right\} \leq \\ &\mu\left\{\sup_{s\in K_v} |X_{N,n,h}(s)| > M_v\right\} + \sum_{j=1}^{r}\sum_{l=1}^{l_j} \mu\left\{\sup_{s\in K_v} |X_{N,n,h_j,l}(s)| > M_{vjl}\right\} \leq \\ &\frac{1}{(N+1)M_v} \sum_{k=0}^{N} \sup_{s\in K_v} |\zeta_n(s + ikh)| + \\ &+ \sum_{j=1}^{r}\sum_{l=1}^{l_j} \frac{1}{(N+1)M_{vjl}} \sum_{k=0}^{N} \sup_{s\in K_v} |\zeta_n(s + ikh_j, \alpha_j; \mathfrak{a}_{jl})|. \end{aligned} \tag{7}$$

The series for $\zeta_n(s)$ and $\zeta_n(s, \alpha_j; \mathfrak{a}_{jl})$ are absolutely convergent for $\sigma > \frac{1}{2}$, therefore, for $\frac{1}{2} < \sigma < 1$, the mean squares

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \int_0^T \left|\zeta_n^{(m)}(\sigma + it)\right|^2 \mathrm{d}t$$

and

$$\sup_{n\in\mathbb{N}} \limsup_{T\to\infty} \int_0^T \left|\zeta_n^{(m)}(\sigma + it, \alpha_j; \mathfrak{a}_{jl})\right|^2 \mathrm{d}t,$$

$m = 0, 1$, are bounded. This and Lemma 8 show that the discrete mean squares

$$\sup_{n\in\mathbb{N}} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} |\zeta_n(\sigma + ikh)|^2$$

and

$$\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}|\zeta_n(\sigma+ikh_j,\alpha_j;\mathfrak{a}_{jl})|^2$$

are bounded for $\frac{1}{2}<\sigma<1$ as well. Hence, by the integral Cauchy formula, we find that

$$\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K_v}|\zeta_n(\sigma+ikh)|\le C_v<\infty$$

and

$$\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{k=0}^{N}\sup_{s\in K_v}|\zeta_n(\sigma+ikh_j,\alpha_j;\mathfrak{a}_{jl})|\le C_{vjl}<\infty.$$

Let ϵ be an arbitrary positive number, $M_v=C_v2^{v+1}\epsilon^{-1}$ and $M_{vjl}=C_{vjl}2^{v+1}(\kappa-1)\epsilon^{-1}$. Then, in view of (7),

$$\sup_{n\in\mathbb{N}}\limsup_{N\to\infty}\mu\left\{\sup_{s\in K_v}|X_{N,n,h}(s)|>M_v \text{ or } \exists j,l:\sup_{s\in K_v}|X_{N,n,h_j,l}(s)|>M_{vjl}\right\}\le \frac{\epsilon}{2^{v+1}}+\frac{\epsilon}{2^{v+1}}=\frac{\epsilon}{2^v}.$$

Therefore, (6) gives

$$\sup_{n\in\mathbb{N}}\mu\left\{\sup_{s\in K_v}|X_N(s)|>M_v \quad\text{or}\quad \exists j,l:\sup_{s\in K_v}|X_{n,h_j,l}(s)|>M_{vjl}\right\}\le\frac{\epsilon}{2^v}. \tag{8}$$

Define the set

$$H_\epsilon^\kappa = \big\{(g,g_{11},...,g_{1l_1},...,g_{r1},...,g_{rl_r})\in H^\kappa(D): \sup_{s\in K_v}|g(s)|\le M_v,\ \sup_{s\in K_v}|g_{jl}(s)|\le M_{vjl},\ j=1,...,r, l=1,...,l_j, v\in\mathbb{N}\big\}.$$

Then H_ϵ^κ is a compact set in $H^\kappa(D)$, and by (8),

$$\mu\{\underline{X}_n\in H_\epsilon^\kappa\}\ge 1-\epsilon\sum_{v=1}^{\infty}\frac{1}{2^v}=1-\epsilon$$

for all $n\in\mathbb{N}$, or, by the definition of X_n,

$$P_n(H_\epsilon^\kappa)\ge 1-\epsilon$$

for all $n\in\mathbb{N}$. □

As we have mentioned above, the tightness of $\{P_n\}$ together with Lemmas 3 and 5 is sufficient for proving the weak convergence for $P_{N,\underline{h}}$. However, having in mind the identification of the limit measure, we define, for $\underline{\omega}\in\underline{\Omega}$,

$$P_{N,\underline{h},\underline{\omega}}(A)=\frac{1}{N+1}\#\{0\le k\le N:\underline{\zeta}(s+ik\underline{h},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})\in A\},\quad A\in\mathcal{B}(H^\kappa(D)),$$

where $\underline{\zeta}(s+ik\underline{h},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$ is defined similarly to $\underline{\zeta}_n(s+ik\underline{h},\underline{\alpha},\underline{\omega};\underline{\mathfrak{a}})$, and prove the following lemma.

Lemma 10. *Suppose that the set $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, h_1, \dots, h_r, \pi)$ is linearly independent over $\mathbb{Q}$. Then, on $(H^\kappa(D), \mathcal{B}(H^\kappa(D)))$, there exists a probability measure P such that $P_{N,\underline{h}}$ and $P_{N,\underline{h},\underline{\omega}}$ both converge weakly to P as $N \to \infty$.*

Proof. By Lemma 9 and the Prokhorov theorem, the sequence $\{P_n\}$ is relatively compact. Therefore, there exists a subsequence $\{P_{n_m}\} \subset \{P_n\}$ such that P_{n_m} converges weakly to a certain probability measure P on $(H^\kappa(D), \mathcal{B}(H^\kappa(D)))$ as $m \to \infty$. In other words,

$$\underline{X}_{n_m} \xrightarrow[m\to\infty]{\mathcal{D}} P. \tag{9}$$

Define the $H^\kappa(D)$-valued random element

$$\underline{X}_{N,\underline{h}} = \underline{X}_{N,\underline{h}}(s) = \underline{\zeta}(s + i\theta_N \underline{h}, \underline{\alpha}; \underline{\mathfrak{a}}).$$

Then, using Lemma 5 and a Chebyshev-type inequality, we obtain that, for every $\epsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu \left\{ \varrho_\kappa \left(\underline{X}_{N,\underline{h}}, \underline{X}_{N,n,\underline{h}} \right) \geq \epsilon \right\} = 0.$$

This, (6) and (9) show that all hypotheses of Lemma 7 are satified, therefore,

$$\underline{X}_{N,\underline{h}} \xrightarrow[N\to\infty]{\mathcal{D}} P,$$

that is, $P_{N,\underline{h}}$ converges weakly to P as $N \to \infty$. The latter relation also shows that the measure P is independent of the subsequence $\{P_{n_m}\}$. Therefore, we obtain that

$$\underline{X}_{n} \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{10}$$

This means that, as $N \to \infty$, $P_{N,\underline{h}}$ converges weakly to the limit measure P of P_n as $n \to \infty$.

It remains to prove the weak convergence for $P_{N,\underline{h},\underline{\omega}}$. Let

$$\underline{X}_{N,n,\underline{h},\underline{\omega}} = \underline{X}_{N,n,\underline{h},\underline{\omega}}(s) = \underline{\zeta}_n(s + i\theta_N \underline{h}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}).$$

and

$$\underline{X}_{N,\underline{h},\underline{\omega}} = \underline{X}_{N,\underline{h},\underline{\omega}}(s) = \underline{\zeta}(s + i\theta_N \underline{h}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}).$$

Then, using Lemmas 4 and 6, and the relation (10), and repeating the above arguments, we obtain that $P_{N,\underline{h},\underline{\omega}}$ also converges weakly to P as $N \to \infty$. □

Proof of Theorem 4. In virtue of Lemma 10, it remains to show that $P = P_{\underline{\zeta}}$. For this, we will use Lemma 2 which ensure the application of the classical Birkhoff-Khintchine ergodic theorem (see, e.g., [**27**]). Moreover, Lemma 10 for $P_{N,\underline{h},\underline{\omega}}$, and the equivalent of weak convergence of probability measures in terms of continuity sets (see, Theorem 2.1 of [**2**]) will be employed.

On the probability space $(\underline{\Omega}, \mathcal{B}(\underline{\Omega}), m_H)$, define the random variable ξ by

$$\xi(\underline{\omega}) = \begin{cases} 1 & \text{if} \quad \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A, \\ 0 & \text{otherwise}, \end{cases}$$

where A is the continuity set of the measure P, that is, $P(\partial A) = 0$, where ∂A denotes the boundary of A. Clearly, we have that the expectation

$$\mathbb{E}(\xi) = \int\limits_{\underline{\Omega}} \xi \mathrm{d} m_H = P_{\underline{\zeta}}(A). \tag{11}$$

Since $P_{N,\underline{h},\underline{\omega}}$ converges weakly to P, we have that

$$\lim_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \underline{\zeta}(s + ik\underline{h}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\} = P(A). \tag{12}$$

From the Birkhoff–Khintchine theorem, we find that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{k=0}^{N} \xi(\underline{\Phi}_{\underline{h}}^k(\underline{\omega})) = \mathbb{E}(\xi). \tag{13}$$

Moreover, by the definitions of ξ and $\underline{\Phi}_{\underline{h}}$,

$$\frac{1}{N+1} \sum_{k=0}^{N} \xi(\underline{\Phi}_{\underline{h}}^k(\underline{\omega})) = \frac{1}{N+1} \#\{0 \le k \le N : \underline{\zeta}(s + ik\underline{h}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\}.$$

This, (13), and (11) show that

$$\lim_{N\to\infty} \frac{1}{N+1} \#\{0 \le k \le N : \underline{\zeta}(s + ik\underline{h}, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}}) \in A\} = P_{\underline{\zeta}}(A),$$

and by (12) we obtain that $P = P_{\underline{\zeta}}$. □

For proving of universality, we additionally need the explicit form of the support of the measure $P_{\underline{\zeta}}$. We recall that the support of $P_{\underline{\zeta}}$ is a minimal closed set $S_{\underline{\zeta}} \subset H^\kappa(D)$ such that $P_{\underline{\zeta}}(S_{\underline{\zeta}}) = 1$. If $\underline{g} \in S_{\underline{\zeta}}$, then, for every open neighbourhood G of $\underline{g}$, the inequality $P_{\underline{\zeta}}(G)$ is satisfied.

Let $S = \{g \in H(D) : \ g(s) \neq 0 \text{ or } g(s) \equiv 0\}$.

Lemma 11. *Under hypotheses of Theorem 4, the support of $P_{\underline{\zeta}}$ is the set $S \times H^{\kappa-1}(D)$.*

Proof. We reduce the support of $P_{\underline{\zeta}}$ to the product of supports of marginal measures on $(H(D), \mathcal{B}(H(D)))$ and $(H^{l_j}(D), \mathcal{B}(H^{l_j}(D)))$, $j = 1, ..., r$. We write

$$H^\kappa(D) = H(D) \times H^{l_1}(D) \times \cdots \times H^{l_r}(D).$$

The space $H(D)$ is separable, and therefore [**2**] the σ-field $\mathcal{B}(H^\kappa(D))$ coincides with

$$\mathcal{B}(H(D)) \times \mathcal{B}(H^{l_1}(D)) \times \cdots \times \mathcal{B}(H^{l_r}(D)),$$

that is, it coincides with the σ-field generated by the sets

$$A = A_0 \times A_1 \times \cdots \times A_r,$$

where $A_0 \in \mathcal{B}(H(D))$ and $A_j \in \mathcal{B}(H^{l_j}(D))$, $j = 1, \dots, r$. Let $\hat{m}_H$ be the probability Haar measure on $(\hat{\Omega}, \mathcal{B}(\hat{\Omega}))$, and let m_{jH} denote the probability Haar measure on $(\Omega_j, \mathcal{B}(\Omega_j))$, $j = 1, \dots, r$. Then we have that the measure m_H is the product of the measures $\hat{m}_H, m_{1H}, \dots, m_{rH}$. Therefore,

$$\begin{aligned} P_{\underline{\zeta}}(A) &= m_H \left\{ \underline{\omega} \in \underline{\Omega} : \underline{\zeta}(s, \underline{\alpha}, \underline{\omega}; \underline{\mathfrak{a}} \in A \right\} \\ &= \hat{m}_H \big\{ \hat{\omega} \in \hat{\Omega} : \zeta(s, \hat{\omega}) \in A_0 \big\} \times \\ &\quad m_{1H} \big\{ \omega_1 \in \Omega_1 : (\zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{11}), \dots, \zeta(s, \alpha_1, \omega_1; \mathfrak{a}_{1l_1})) \in A_1 \big\} \times \cdots \times \\ &\quad m_{rH} \big\{ \omega_r \in \Omega_r : (\zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{r1}), \dots, \zeta(s, \alpha_r, \omega_r; \mathfrak{a}_{rl_r})) \in A_r \big\}. \end{aligned} \tag{14}$$

It is known [**8**] that the support of

$$P_\zeta(A_0) = \hat{m}_H \big\{ \hat{\omega} \in \hat{\Omega} : \zeta(s, \hat{\omega}) \in A_0 \big\}$$

is the set S. Since $\operatorname{rank}(B_j) = l_j$ and the set $\{\log(m + \alpha_j) : m \in \mathbb{N}\}$ is linearly independent over $\mathbb{Q}$, by Lemma 11 of [**15**] the support of the measure

$$P_{\zeta, \alpha_j}(A_j) = m_{jH} \big\{ \omega_j \in \Omega_j : (\zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{j1}), \dots, \zeta(s, \alpha_j, \omega_j; \mathfrak{a}_{jl_j})) \in A_j \}$$

is the whole $H^{l_j}(D)$, $j = 1, \dots, r$. Therefore, by (14) and the minimality property of the support we obtain that the support of $P_{\underline{\zeta}}$ is the set

$$S \times H^{l_1}(D) \times \cdots \times H^{l_r}(D) = S \times H^{\kappa - 1}(D).$$

□

We observe that, in the case $h = h_1 = \ldots = h_r$, the proof of Theorem 4 is simpler, and we do not need Lammas 2, 4, and 6. Indeed, in [**8**], under the hypothesis that the numbers $\alpha_1, \dots, \alpha_r$ are algebraically independent over $\mathbb{Q}$, it is proved that

$$\frac{1}{T} \operatorname{meas} \left\{ \tau \in [0; T] : \underline{\zeta}(s + i\tau, \underline{\alpha}; \underline{\mathfrak{a}}) \in A \right\}, \quad A \in \mathcal{B}\left(H^\kappa(D)\right),$$

as $T \to \infty$, converges weakly to the limit measure P of $\hat{P}_n$ as $n \to \infty$, P coincides with $P_{\underline{\zeta}}$, and the support of $P_{\underline{\zeta}}$ is the set $S \times H^{\kappa-1}(D)$. However, the algebraic independence of the numbers $\alpha_1, \dots, \alpha_r$ is only used to prove the linear independence of the set

$$\{(\log p : p \in \mathbb{P}), (\log(m + \alpha_j) : m \in \mathbb{N}_0, j = 1, \dots, r)\}.$$

Clearly, the linear independence of the set $L(\mathbb{P}, \alpha_1, \dots, \alpha_r, h, \pi)$ implies that of the former. Therefore, P_N also converges weakly to $P_{\underline{\zeta}}$ as $N \to \infty$, and the support of $P_{\underline{\zeta}}$ is the set $S \times H^{\kappa-1}(D)$.

4. PROOF OF UNIVERSALITY

Proof of Theorem 3. By the Mergelyan theorem [**20**] on the approximation of analytic functions by polynomials there exist polynomials $p(s)$ and $p_{jl}(s)$ such that

$$\sup_{s\in K}|f(s)-e^{p(s)}|<\frac{\epsilon}{2} \tag{15}$$

and

$$\sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|f_{jl}(s)-p_{jl}(s)|<\frac{\epsilon}{2}. \tag{16}$$

In view of Lemma 11, $\left(e^{p(s)},p_{11},\dots,p_{1l_1},\dots,p_{r1},\dots,p_{rl_r}\right)$ is an element of the support of the measure $P_{\underline{\zeta}}$. Define

$$\begin{aligned} G_{\frac{\epsilon}{2}} = \Big\{ & g_0,g_{11},\dots,g_{1l_1},\dots,g_{r1},\dots,g_{rl_r}\in H(D): \\ & \sup_{s\in K}|g_0(s)-e^{p(s)}|<\frac{\epsilon}{2}, \sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|g_{jl_j}(s)-p_{jl}(s)|<\frac{\epsilon}{2}\Big\}. \end{aligned}$$

Then $P_{\underline{\zeta}}(G_{\frac{\epsilon}{2}})>0$. In the definition of G_ϵ, let us take $f(s)$ instead of $e^{p(s)}$ and $f_{jl}(s)$ instead of $p_{jl}(s)$ and denote the obtained set by $\hat{G}_\epsilon$. Then, by Theorem 4,

$$\begin{aligned} \liminf_{N\to\infty}\frac{1}{N+1}\#\Big\{ & 0\le k\le N: \sup_{s\in K}|\zeta(s+ikh)-f(s)|<\epsilon, \\ & \sup_{1\le j\le r}\sup_{1\le l\le l_j}\sup_{s\in K_{jl}}|\zeta(s+ikh_j,\alpha_j;\mathfrak{a}_{jl})-f_{jl}(s)|<\epsilon\Big\} \\ & \ge P_{\underline{\zeta}}(\hat{G}_\epsilon)\ge P_{\underline{\zeta}}(G_{\frac{\epsilon}{2}})>0 \end{aligned}$$

because $G_{\frac{\epsilon}{2}}\subset\hat{G}_\epsilon$ by (15) and (16). □

Remark 1. *The function $\zeta(s)$ in Theorem 3 can be replaced by Dirichlet L-functions, zeta-functions of Hecke eigen forms, L-functions from the Selberg class, Matsumoto zeta-functions, or periodic zeta-functions with multiplicative coefficients.*

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Antanas Laurinčikas
Institute of Mathematics,
Faculty of Mathematics and Informatics,
Vilnius University.
Naugarduko str. 24, LT-03225 Vilnius,
Lithuania
E-mail: *antanas.laurincikas@mif.vu.lt*

Renata Macaitienė
Research Institute,
Šiauliai University.
P. Višinskio str. 25,
LT-76351 Šiauliai, Lithuania
E-mail: *renata.macaitiene@su.lt*