

SOME IDENTITIES RELATED TO EULERIAN POLYNOMIALS AND INVOLVING THE STIRLING NUMBERS

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In the paper, the authors establish two identities, which can be regarded as nonlinear differential equations, for the generating function of Eulerian polynomials, find two identities for the Stirling numbers of the second kind, present two identities for Eulerian polynomials and higher order Eulerian polynomials, and pose two open problems about summability of two finite sums involving the Stirling numbers of the second kind. Some of these conclusions meaningfully and significantly simplify several known results.

1. MOTIVATIONS

In [6, 7], Kims stated that Eulerian polynomials $A_n(t)$ for $n \geq 0$ can be generated by

$$\frac{1-t}{e^{x(t-1)}-t} = \sum_{n=0}^{\infty} A_n(t) \frac{x^n}{n!}, \quad t \neq 1$$

and that higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ for integers $n \geq 0$ and real numbers $\alpha > 0$ can be generated by

$$\left[\frac{1-t}{e^{x(t-1)}-t} \right]^{\alpha} = \sum_{n=0}^{\infty} A_n^{(\alpha)}(t) \frac{x^n}{n!}, \quad t \neq 1.$$

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This generation of $A_n(t)$ is same as the one in [1, p. 2], but different from the one

$$\frac{1-u}{e^{t(u-1)}-u} = 1 + \sum_{n=1}^{\infty} \frac{A_n(u)}{u} \frac{t^n}{n!}$$

in [2, p. 244, Eq. [5j]].

In [7, Theorem 1], Kims established inductively and recurrently that the generating function

$$(1) \quad F(t, x) = \frac{1}{e^{x(t-1)} - t}, \quad t \neq 1$$

satisfies the nonlinear ordinary differential equation

$$(2) \quad \frac{\partial^n F(t, x)}{\partial x^n} = (1-t)^n \sum_{i=1}^{n+1} a_{i-1}(n, t) F^i(t, x), \quad n \in \{0\} \cup \mathbb{N},$$

where

$$(3) \quad a_0(n, t) = a_0(n-1, t) = \cdots = a_0(1, t) = a_0(0, t) = 1$$

and

$$(4) \quad a_i(n, t) = it \sum_{j=0}^{n-i} (i+1)^j a_{i-1}(n-j-1, t), \quad 1 \leq j \leq n.$$

In [7, Theorems 2 and 3], Kims presented that

$$A_{k+n}(t) = (1-t)^{n+1} \sum_{i=1}^{n+1} a_{i-1}(n, t) \frac{A_k^{(i)}(t)}{(1-t)^i}$$

and

$$\sum_{j=0}^{\infty} t^j (j+1)^{k+n} = \frac{1}{(1-t)^k} \sum_{i=1}^{n+1} a_{i-1}(n, t) \frac{A_k^{(i)}(t)}{(1-t)^i}$$

for $k, n \in \{0\} \cup \mathbb{N}$. From (3) and (4), Kims derived inductively that

$$(5) \quad a_i(n, t) = i! t^i \sum_{j_{i-1}=0}^{n-i} \sum_{j_{i-2}=0}^{n-j_{i-1}-i} \cdots \sum_{j_1=0}^{n-j_{i-1}-\cdots-j_2-i} (i+1)^{j_{i-1}} \\ \times i^{j_{i-2}} \cdots 3^{j_1} (2^{n-j_{i-1}-j_{i-2}-\cdots-j_1-i+1} - 1)$$

for $1 \leq i \leq n$.

It is clear that the above formulas (4) and (5) for $a_i(n, t)$ cannot be computed easily either by hand or by computer software. Can one find a simple expression for the quantities $a_i(n, t)$? For supplying a solution to this problem, the first and third authors of this paper obtained in [29] the following three theorems.

Theorem 1 ([29, Theorem 1]). *Eulerian polynomials $A_n(t)$ and higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ can be computed by*

$$(6) \quad A_n(t) = \sum_{k=0}^n k! S(n, k) (t-1)^{n-k}$$

and

$$(7) \quad A_n^{(\alpha)}(t) = \frac{1}{\Gamma(\alpha)} \sum_{k=0}^n \Gamma(k+\alpha) S(n, k) (t-1)^{n-k},$$

where $n \geq 0$ is an integer, $\alpha > 0$ is a real number, $S(n, k)$, which can be generated by the exponential function

$$\frac{(e^x - 1)^k}{k!} = \sum_{n=k}^{\infty} S(n, k) \frac{x^n}{n!}$$

and can be computed by the explicit formula

$$S(n, k) = \frac{1}{k!} \sum_{\ell=1}^k (-1)^{k-\ell} \binom{k}{\ell} \ell^n,$$

stand for the Stirling numbers of the second kind, and $\Gamma(z)$ denotes the classical Euler gamma function.

Theorem 2 ([29, Theorem 2]). *The generating function $F(t, x)$ satisfies the non-linear ordinary differential equations*

$$(8) \quad \frac{\partial^n F(t, x)}{\partial x^n} = (t-1)^n \sum_{i=0}^n \left[\sum_{k=i}^n (-1)^k k! S(n, k) \binom{k}{i} \right] t^i F^{i+1}(t, x)$$

and, generally,

$$(9) \quad \frac{\partial^n F^\alpha(t, x)}{\partial x^n} = \frac{(t-1)^n}{\Gamma(\alpha)} \sum_{i=0}^n \left[\sum_{k=i}^n (-1)^k \Gamma(k+\alpha) S(n, k) \binom{k}{i} \right] t^i F^{\alpha+i}(t, x),$$

where $n \geq 0$ is an integer and $\alpha > 0$ is a real number.

Theorem 3 ([29, Theorem 3]). *For $n \in \mathbb{N}$ and $\alpha > 0$, higher order Eulerian polynomials $A_n^{(\alpha)}(t)$ satisfy the recurrence relation*

$$(10) \quad \sum_{k=0}^n \binom{n}{k} \left[\sum_{\ell=0}^{n-k} S(n-k, \ell) \frac{\langle \alpha \rangle_\ell}{(1-t)^\ell} \right] \frac{A_k^{(\alpha)}(t)}{(t-1)^k} = 0,$$

where

$$\langle \alpha \rangle_n = \prod_{k=0}^{n-1} (\alpha - k) = \begin{cases} \alpha(\alpha-1) \cdots (\alpha-n+1), & n \geq 1 \\ 1, & n = 0 \end{cases}$$

is called the falling factorial. In particular, when $\alpha = 1$,

$$\sum_{k=0}^n (-1)^k \binom{n}{k} \left[\frac{S(n-k, 0)}{(1-t)^{k-1}} + \frac{S(n-k, 1)}{(1-t)^k} \right] A_k(t) = 0.$$

It is easy to see that the equation (8) simplifies the one (2). As stated in [29, Remarks 1 and 2], comparing the equation (2) with (8) reveals that

$$(11) \quad a_i(n, t) = \left[\sum_{k=i}^n (-1)^k k! S(n, k) \binom{k}{i} \right] t^i, \quad 0 \leq i \leq n.$$

This expression is simpler, more significant, and more meaningful than the one in (5). It is easier to compute the quantities in brackets of the nonlinear ordinary differential equations (8) and (9) than to compute the quantity $a_i(n, t)$ in (5).

In [6, Theorem 1], it was obtained inductively and recursively that the nonlinear differential equations

$$(12) \quad n! t^n (1-t)^n F^{n+1}(t, x) = \sum_{i=0}^n a_i(n) (1-t)^{n-i} \frac{\partial^i F(t, x)}{\partial x^i}, \quad n \in \mathbb{N}$$

have a solution $F(t, x)$ defined by (1) for $t \neq 1$, where $a_0(n) = (-1)^n n!$,

$$(13) \quad a_i(n) = (-1)^{n-i} n! H_{n,i}, \quad 1 \leq i \leq n,$$

with $H_{n,0} = 1$ for $n \in \mathbb{N}$, $H_{n,1} = H_n = \sum_{k=1}^n \frac{1}{k}$ for $n \in \mathbb{N}$, and $H_{n,i} = \sum_{k=i}^n \frac{H_{k-1,i-1}}{k}$ for $2 \leq i \leq n$. For more information on $H_{n,i}$, please refer to [43, Remark 1] and closely related references therein. Therefore, the following results were derived in [6, Theorems 2 and 3]:

1. For $n, k \geq 0$,

$$n! t^n A_k^{(n+1)}(t) = \sum_{i=0}^n a_i(n) (1-t)^{n-i} A_{k+i}(t).$$

2. For $k \geq 1$ and $m \geq k+n$,

$$n! t^n A_k^{(n+1)}(t) = \sum_{m=0}^{k+n-1} \sum_{i=0}^n \sum_{\ell=0}^m (-1)^\ell \binom{n+k+1}{\ell} (m-\ell+1)^{k+i} a_i(n) t^m$$

and

$$\sum_{i=0}^n \sum_{\ell=0}^m (-1)^\ell \binom{n+k+1}{\ell} (m-\ell+1)^{k+i} a_i(n) = 0$$

3. For $k = 0$ and $m \geq n+1$,

$$n! t^n A_0^{(n+1)}(t) = \sum_{m=0}^n \sum_{i=0}^n \sum_{\ell=0}^m (-1)^\ell \binom{n+1}{\ell} (m-\ell+1)^i a_i(n) t^m$$

and

$$\sum_{i=0}^n \sum_{\ell=0}^m (-1)^\ell \binom{n+1}{\ell} (m-\ell+1)^i a_i(n) = 0.$$

We observe that the equations (2) and (12) can be rewritten respectively as

$$\frac{1}{(1-t)^n} \frac{\partial^n F(t, x)}{\partial x^n} = \sum_{i=0}^n a_i(n, t) F^{i+1}(t, x), \quad n \in \{0\} \cup \mathbb{N}$$

and

$$(14) \quad n!t^n F^{n+1}(t, x) = \sum_{i=0}^n a_i(n) \frac{1}{(1-t)^i} \frac{\partial^i F(t, x)}{\partial x^i}, \quad n \in \{0\} \cup \mathbb{N}.$$

Consequently, the equations (2) and (12) are essentially inversive to each other. This motivates us to consider two questions:

1. can one simplify the expression of the quantities $a_i(n)$ significantly and meaningfully?
2. what are the inversive ones of the equations (8) and (9)?

2. MAIN RESULTS AND THEIR PROOFS

Now we are in a position to state and prove our main results.

Theorem 4. *For $n \geq 0$, the function $F(t, x)$ defined by (1) satisfies nonlinear differential equations*

$$(15) \quad F^{n+1}(t, x) = \frac{1}{n!t^n} \sum_{i=0}^n \frac{s(n+1, i+1)}{(1-t)^i} \frac{\partial^i F(t, x)}{\partial x^i},$$

where $s(n, k)$, which can be generated by

$$\frac{[\ln(1+x)]^k}{k!} = \sum_{n=k}^{\infty} s(n, k) \frac{x^n}{n!}, \quad |x| < 1,$$

stand for the Stirling numbers of the first kind.

Proof. In [32] and [43], it was obtained that

$$(16) \quad (-1)^{n+k} s(n, k) = (n-1)! H_{n-1, k-1}, \quad n \geq k \geq 1.$$

Substituting the relations (13) and (16) into (12) or (14) yields

$$n!t^n F^{n+1}(t, x) = \sum_{i=0}^n (-1)^{n-i} n! H_{n, i} \frac{1}{(1-t)^i} \frac{\partial^i F(t, x)}{\partial x^i}$$

$$= \sum_{i=0}^n s(n+1, i+1) \frac{1}{(1-t)^i} \frac{\partial^i F(t, x)}{\partial x^i}$$

for $n \geq 0$. The proof of Theorem 4 is complete. \square

Theorem 5. For $n \geq 0$, the function $F(t, x)$ defined by (1) satisfies nonlinear differential equations

$$(17) \quad \frac{\partial^n F(t, x)}{\partial x^n} = (1-t)^n \sum_{k=0}^n S(n+1, k+1) k! t^k F^{k+1}(t, x).$$

The Stirling numbers of the second kind $S(n, k)$ satisfy the identities

$$(18) \quad \sum_{\ell=k}^n (-1)^\ell \binom{\ell}{k} \ell! S(n, \ell) = (-1)^n k! S(n+1, k+1), \quad n \geq k \geq 0$$

and

$$(19) \quad \sum_{\ell=k+1}^n (-1)^\ell \binom{\ell-1}{k} \ell! S(n, \ell) = (-1)^n (k+1)! S(n, k+1), \quad n > k \geq 0.$$

Proof. Rewriting the equations in (15) as

$$\begin{pmatrix} F(t, x) \\ tF^2(t, x) \\ \vdots \\ n! t^n F^{n+1}(t, x) \end{pmatrix} = M_{(n+1) \times (n+1)} \begin{pmatrix} F(t, x) \\ \frac{1}{1-t} \frac{\partial F(t, x)}{\partial x} \\ \vdots \\ \frac{1}{(1-t)^n} \frac{\partial^n F(t, x)}{\partial x^n} \end{pmatrix},$$

where the $(n+1) \times (n+1)$ matrix

$$M_{(n+1) \times (n+1)} = \begin{pmatrix} s(1, 1) & 0 & 0 & \cdots & 0 \\ s(2, 1) & s(2, 2) & 0 & \cdots & 0 \\ s(3, 1) & s(3, 2) & s(3, 3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ s(n+1, 1) & s(n+1, 2) & s(n+1, 3) & \cdots & s(n+1, n+1) \end{pmatrix}.$$

Since the inverse matrix

$$M_{(n+1) \times (n+1)}^{-1} = \begin{pmatrix} S(1, 1) & 0 & 0 & \cdots & 0 \\ S(2, 1) & S(2, 2) & 0 & \cdots & 0 \\ S(3, 1) & S(3, 2) & S(3, 3) & \cdots & 0 \\ \cdots & \cdots & \cdots & \ddots & \cdots \\ S(n+1, 1) & S(n+1, 2) & S(n+1, 3) & \cdots & S(n+1, n+1) \end{pmatrix},$$

see [2, p. 213, eq. [5c]], it follows immediately that

$$\begin{pmatrix} F(t, x) \\ \frac{1}{1-t} \frac{\partial F(t, x)}{\partial x} \\ \frac{1}{(1-t)^2} \frac{\partial^2 F(t, x)}{\partial x^2} \\ \vdots \\ \frac{1}{(1-t)^n} \frac{\partial^n F(t, x)}{\partial x^n} \end{pmatrix} = M_{(n+1) \times (n+1)}^{-1} \begin{pmatrix} F(t, x) \\ tF^2(t, x) \\ 2!t^2F^3(t, x) \\ \vdots \\ n!t^n F^{n+1}(t, x) \end{pmatrix},$$

that is,

$$\frac{1}{(1-t)^n} \frac{\partial^n F(t, x)}{\partial x^n} = \sum_{k=1}^{n+1} S(n+1, k)(k-1)!t^{k-1}F^k(t, x), \quad n \geq 0.$$

This is equivalent to (17). Comparing (17) with (8) leads to (18).

In [47, p. 118, Eq. (9.18)], it was proved that

$$(20) \quad \sum_{j=\alpha}^n (-1)^{n-j} \binom{j-1}{\alpha-1} j! S(n, j) = \alpha! S(n, \alpha), \quad n \geq \alpha \geq 0.$$

By virtue of (18) and (20), we obtain

$$\sum_{\ell=k}^n (-1)^\ell \left[\binom{\ell}{k} - \binom{\ell-1}{k-1} \right] \ell! S(n, \ell) = (-1)^n k! [S(n+1, k+1) - S(n, k)].$$

Since the recurrence relations

$$\binom{x}{j} = \binom{x-1}{j} + \binom{x-1}{j-1}, \quad x \in \mathbb{C}, \quad j \in \{0\} \cup \mathbb{N}$$

and

$$S(n+1, k) = kS(n, k) + S(n, k-1), \quad 0 \leq k-1 \leq n,$$

see [47, p. 8, Eq. (1.27)] and [47, p. 114, Eq. (9.1)], we arrive at the identity (19) straightforwardly. The proof of Theorem 5 is complete. \square

Theorem 6. For $n \geq 0$, Eulerian polynomials $A_n(t)$ and higher order Eulerian polynomials $A_k^{(\alpha)}(t)$ satisfy

$$(21) \quad \sum_{k=0}^n \frac{s(n, k)}{(t-1)^k} A_k(t) = \frac{n!}{(t-1)^n}$$

and

$$(22) \quad \sum_{k=0}^n \frac{s(n, k)}{(t-1)^k} A_k^{(\alpha)}(t) = \frac{\Gamma(n+\alpha)}{\Gamma(\alpha)} \frac{1}{(t-1)^n}.$$

Proof. Theorem 12.1 in [47, p. 171] reads that, if b_α and a_k are a collection of constants independent of n , then

$$a_n = \sum_{\alpha=0}^n S(n, \alpha) b_\alpha \quad \text{if and only if} \quad b_n = \sum_{k=0}^n s(n, k) a_k.$$

The identity (6) can be rearranged as

$$\frac{A_n(t)}{(t-1)^n} = \sum_{k=0}^n S(n, k) \frac{k!}{(t-1)^k}.$$

Consequently, it follows that

$$\frac{n!}{(t-1)^n} = \sum_{k=0}^n s(n, k) \frac{A_k(t)}{(t-1)^k}$$

which can be rewritten as (21).

Similarly, the identity (7) can also be reformulated as

$$\frac{\Gamma(\alpha) A_n^{(\alpha)}(t)}{(t-1)^n} = \sum_{k=0}^n S(n, k) \frac{\Gamma(k+\alpha)}{(t-1)^k}$$

and, consequently,

$$\frac{\Gamma(n+\alpha)}{(t-1)^n} = \sum_{k=0}^n s(n, k) \frac{\Gamma(\alpha) A_k^{(\alpha)}(t)}{(t-1)^k}$$

which can be rearranged as (22). The required proof is complete. \square

3. REMARKS

In this section, among other things, we give several remarks about our main results and pose two open problems.

Remark 1. The expressions (11) and (13) for $a_i(n, t)$ and $a_i(n)$ can be meaningfully and significantly simplified as

$$a_i(n, t) = (-1)^n i! S(n+1, i+1) t^i, \quad 0 \leq i \leq n$$

and

$$a_i(n) = s(n+1, i+1).$$

As a result, those main results in [6, 7], mentioned in the first section, can be meaningfully and significantly simplified. For the sake of saving the space and shortening the length of this paper, we do not rewrite them in details here. This answers the first question posed in the first section of this paper.

Remark 2. The identity (15) in Theorem 4 partially answers the second question posed in the first section of this paper.

Remark 3. Theorem 3 is a simplified and reformulated version of Theorem 3 in [29].

Remark 4. The identity (17) in Theorem 5 simplifies the one (8) in Theorem 2.

Remark 5. To the best of our knowledge, identities (18) and (19) are new.

Remark 6. Motivated by identities (18), (19), and (20), we naturally pose another open problem: for $n \geq k \geq 0$ and $\alpha > 0$, is the finite sum

$$\sum_{k=i}^n (-1)^k \binom{k}{i} \Gamma(k + \alpha) S(n, k)$$

in the bracket of the identity (9) summable? The solution of this problem can be used to partially answer the second question posed in the first section of this paper.

Remark 7. Theorem 12.2 in [47, p. 171] states that, if b_j and a_k are a collection of constants which are independent of n and if α is a nonnegative integer such that $\alpha \geq n$, then

$$a_n = \sum_{j=0}^{\alpha} S(j, n) b_j \quad \text{if and only if} \quad b_n = \sum_{k=0}^{\alpha} s(k, n) a_k.$$

Motivated by the above mentioned Theorems 12.1 and 12.2 in [47, p. 171] and by identities (18), (19), and (20), we naturally pose an open problem more: is the finite sum

$$\sum_{\ell=0}^{n-k} S(n-k, \ell) \langle \alpha \rangle_{\ell} t^{\ell}$$

in the bracket of the identity (10), or equivalently,

$$\sum_{\ell=0}^n S(n, \ell) \langle \alpha \rangle_{\ell} t^{\ell},$$

summable?

Remark 8. For some new results on the gamma function $\Gamma(z)$, please refer to [8, 18, 28, 30, 36] and closely related references therein.

Remark 9. For some recent development of the Stirling numbers of the first and second kinds, please refer to [3, 4, 5, 9, 12, 13, 14, 15, 17, 26, 27, 41] and closely related references therein.

Remark 10. The motivation of this paper is same as the one in [10, 11, 16, 19, 20, 21, 22, 23, 24, 25, 27, 29, 31, 32, 33, 34, 37, 38, 39, 40, 42, 43, 44, 45, 46, 48] and closely related references therein.

Remark 11. This paper is a slightly revised version of the preprint [35].

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