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CONVEXITY AND MONOTONICITY FOR ELLIPTIC INTEGRALS OF THE FIRST KIND AND APPLICATIONS

*Zhen-Hang Yang and Jing-Feng Tian**

In the paper, by virtue of two new tools, the authors present the monotonicity and convexity of certain combinations of the complete elliptic integrals of the first kind, and obtain new sharp bounds and inequalities for them.

1. INTRODUCTION

For $r \in (0,1)$, Legendre's complete elliptic integrals $\mathcal{K}(r)$ and $\mathcal{E}(r)$ of the first kind and second kind are defined [**1**] by

$$\begin{aligned} \mathcal{K}(r) &= \int_0^{\pi/2} \frac{dt}{\sqrt{1-r^2\sin^2 t}}, \\ \mathcal{E}(r) &= \int_0^{\pi/2} \sqrt{1-r^2\sin^2 t}dt, \end{aligned}$$

respectively. The Legendre's complete elliptic integrals play a very important and basic role in different branches of modern mathematics such as classical analysis, number theory, geometric function theory, quasiconformal mappings and analysis. Motivated by the importance of elliptic integrals, many monotonicity and convexity properties of $\mathcal{K}(r)$ and $\mathcal{E}(r)$ have been obtained in, for example, [**2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13**].

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* Corresponding author. Jing-Feng Tian
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As is well known, Legendre's complete elliptic integrals cannot be represented by elementary transcendental functions. Therefore, there is a need for sharp computable bounds for the family of integrals. The goal of this paper is to study monotonicity and convexity of certain combinations of the complete elliptic integrals of the first kind, and establish some new bounds and inequalities for it.

The elliptic integrals occur in the formulas for the moduli of plane ring domains in theory of quasiconformal mappings. In particular, for $r \in (0,1)$, the special combination

$$\mu(r) = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)},$$

the so-called modulus of the Grötzsch extremal ring, is indispensable in the study of quasiconformal distortion [**14, 15**].

It is known that the complete elliptic integrals are the particular cases of the Gaussian hypergeometric function

$$F(a,b;c;x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!} \quad (-1 < x < 1), \tag{0.1}$$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ and Γ is the classical gamma function [**16, 17**]. Indeed, we have

$$\mathcal{K}(r) = \frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right) \quad \text{and} \quad \mathcal{E}(r) = \frac{\pi}{2}F\left(\frac{1}{2},-\frac{1}{2};1;r^2\right).$$

The hypergeometric function $F(a,b;c;x)$ has the simple differentiation formula

$$F'(a,b;c;x) = \frac{ab}{c}F(a+1,b+1;c+1;x). \tag{0.2}$$

The behavior of the hypergeometric function near $x=1$ in the three cases $a+b<c$, $a+b=c$, and $a+b>c$, $a,b,c>0$, is given by

$$\begin{cases} F(a,b;c;1) = \dfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)} & \text{if } c > a+b, \\[2ex] F(a,b;c;x) = \dfrac{R(a,b) - \ln(1-x)}{B(a,b)} + O((1-x)\ln(1-x)) & \text{if } c = a+b, \\[2ex] F(a,b;c;x) = (1-x)^{c-a-b}F(c-a,c-b;c;x) & \text{if } c < a+b, \end{cases} \tag{0.3}$$

where $B(a,b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$ is the beta function and

$$R(a,b) = -2\gamma - \psi(a) - \psi(b), \tag{0.4}$$

here $\psi(z) = \frac{\Gamma'(z)}{\Gamma(z)}$, $\Re(z) > 0$ and γ is the Euler-Mascheroni constant (see [**18**]). In particular, from the second formula in (0.3) it is easy to obtain the following asymptotic formula

$$\mathcal{K}(r)=\frac{\pi}{2}F\left(\frac{1}{2},\frac{1}{2};1;r^2\right)\sim\ln\frac{4}{r'},\quad \text{as } r\to 1^-,$$

where and in what follows $r'=\sqrt{1-r^2}$.

Motivated by the asymptotic formula for $\mathcal{K}(r)$, many comparison inequalities for $\mathcal{K}(r)$ with $\ln(4/r')$ were established, for example, in [**8, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29**]. Some of new generalizations can be found in [**30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40**].

In particular, Anderson, Vamanamurthy and Vuorinen [**21**, Theorem 2.2] showed that the ratio $\frac{\mathcal{K}(r)}{\ln(c/r')}$ is strictly decreasing if and only if $0<c\le 4$ (strictly speaking, c should be in $[1,4]$) and strictly increasing if and only if $c\ge e^2$. It is natural to ask that what are the conditions such that this function is concave or convex on $(0,1)$? The first aim of this paper is to give an answer for concavity.

Theorem 1. *Let* $c\ge 1$. *The function*

$$x\mapsto Q_1(x)=\frac{\mathcal{K}(\sqrt{x})}{\ln\left(c/\sqrt{1-x}\right)} \tag{0.5}$$

is strictly concave on $(0,1)$ *if and only if* $c=e^{4/3}$.

Remark 1. According to [**21**, Theorem 2.2], Q_1 is decreasing on $(0,1)$ for $c=e^{4/3}\in(0,4]$. Also, by the properties of positive concave functions, we easily see that Q_1 is log-concave and $1/Q_1$ is convex on $(0,1)$.

In 1992, Anderson, Vamanamurthy and Vuorinen [**22**] conjectured that the inequality

$$\mathcal{K}(r)<\ln\left(1+\frac{4}{r'}\right)-\left(\ln 5-\frac{\pi}{2}\right)(1-r) \tag{0.6}$$

holds for $r\in(0,1)$, which was proved in [**28**] by Qiu, Vamanamurthy and Vuorinen. We remark that as an approximation for $\mathcal{K}(r)$, the function $\ln(1+4/r')$ is obviously better than $\ln(4/r')$. Therefore, the second aim of this paper is to further vestigate the monotonicity of the ratio $\mathcal{K}(r)/\ln(1+4/r')$ and the convexity of the difference $\mathcal{K}(r)-\ln(1+4/r')$. We will prove the following two theorems.

Theorem 2. *The function*

$$r\mapsto Q_2(r)=\frac{\mathcal{K}(r)}{\ln(1+4/r')} \tag{0.7}$$

is strictly increasing from $(0,1)$ *onto* $(\pi/\ln 25,1)$. *Consequently, the double inequality*

$$\frac{\pi}{2\ln 5}\ln\left(1+\frac{4}{r'}\right)<\mathcal{K}(r)<\ln\left(1+\frac{4}{r'}\right)$$

holds with the best constants $\pi/\ln 25=0.97599\ldots$ *and* 1.

Theorem 3. *The function*

$$D(x) = \mathcal{K}\left(\sqrt{x}\right) - \ln\left(1 + \frac{4}{\sqrt{1-x}}\right) \tag{0.8}$$

is strictly convex on $(0,1)$.

2. LEMMAS

To prove our main results, we need the following lemmas. In order to state the first lemma, we introduce a useful auxiliary function $H_{f,g}$. For $-\infty \leq a < b \leq \infty$, let f and g be differentiable on (a,b) and $g' \neq 0$ on (a,b). Then the function $H_{f,g}$ is defined by

$$H_{f,g} := \frac{f'}{g'}g - f. \tag{0.9}$$

The function $H_{f,g}$ has some well properties [**41**, Properties 1, 2]. In particular, if f and g are twice differentiable on (a,b), then we have

$$\left(\frac{f}{g}\right)' = \frac{g'}{g^2}\left(\frac{f'}{g'}g - f\right) = \frac{g'}{g^2}H_{f,g}, \tag{0.10}$$

$$H'_{f,g} = \left(\frac{f'}{g'}\right)' g. \tag{0.11}$$

Lemma 1 ([**42**, Theorem 1], [**43**, Remark 2])**.** *Let* $A(t) = \sum_{k=0}^{\infty} a_k t^k$ *and* $B(t) = \sum_{k=0}^{\infty} b_k t^k$ *be two real power series converging on* $(-r,r)$ *and* $b_k > 0$ *for all* k*. Suppose that for certain* $m \in \mathbb{N}$*, the sequences* $\{a_k/b_k\}_{0\leq k\leq m}$ *and* $\{a_k/b_k\}_{k\geq m}$ *both are non-constant, and they are increasing (decreasing) and decreasing (increasing), respectively. Then the function* A/B *is strictly increasing (decreasing) on* $(0,r)$ *if and only if* $H_{A,B}(r^-) \geq (\leq)\, 0$*. If* $H_{A,B}(r^-) < (>)\, 0$*, then there exists* $t_0 \in (0,r)$ *such that the function* A/B *is strictly increasing (decreasing) on* $(0,t_0)$ *and strictly decreasing (increasing) on* (t_0,r)*.*

Remark 2. Lemma 1 is a powerful tool to deal with the monotonicity of the ratio of power series in the case when the sequence $\{a_k/b_k\}_{k\geq 0}$ is piecewise monotonic. Recently, it was applied preliminarily to certain special functions, for example, trigonometric and hyperbolic functions [**44**],Toader-Qi mean [**45**] (see also [**46**], [**47**]), generalized elliptic integrals [**43, 48**], hypergeometric functions [**49**], Bessel functions [**50, 51**].

The following lemma offers a simple criterion to determine the sign of a class of special series. A polynomial version appeared in [**52**], and another series version converging on $(0,\infty)$ can see [**53**].

Lemma 2. *Let* $\{a_k\}_{k=0}^{\infty}$ *be a nonnegative real sequence with* $a_m > 0$ *and* $\sum_{k=m+1}^{\infty} a_k > 0$ *and let*

$$S(t) = -\sum_{k=0}^{m} a_k t^k + \sum_{k=m+1}^{\infty} a_k t^k$$

be a convergent power series on the interval $(0,r)$ $(r>0)$. *(i) If* $S(r^-)\leq 0$, *then* $S(t)<0$ *for all* $t\in(0,r)$. *(ii) If* $S(r^-)>0$, *then there is a unique* $t_0\in(0,r)$ *such that* $S(t)<0$ *for* $t\in(0,t_0)$ *and* $S(t)>0$ *for* $t\in(t_0,r)$.

Proof. We prove the desired assertions by mathematical induction for the negative integral m.

For $m=0$, we have $S'(t)=\sum_{k=m+1}^{\infty}ka_kt^{k-1}>0$, which together with $S(0^+)=-a_0<0$ and $S(r^-)\leq(>)\,0$ yields the desired assertions.

Suppose that the desired assertions are true for $m=n$. We prove they are also true for $m=n+1$ by distinguishing two cases.

Case 1: $S'(r^-)\leq 0$. By inductive hypothesis we have $S'(t)<0$ for all $t\in(0,r)$. Then $S(t)<S(0^+)=-a_0\leq 0$ for all $t\in(0,r)$.

Case 2: $S'(r^-)>0$. By inductive hypothesis it is deduced that there is a $t_1\in(0,r)$ such that $S'(t)<0$ for $t\in(0,t_1)$ and $S'(t)>0$ for $t\in(t_1,r)$. If $S(r^-)\leq 0$, then we have $S(t)<S(r^-)\leq 0$ for $t\in(t_1,r)$ and $S(t)<S(0^+)=-a_0\leq 0$ for $t\in(0,t_1)$, which implies that $S(t)\leq 0$ for all $t\in(0,r)$. If $S(r^-)>0$, then since $S(t)<S(0^+)=-a_0\leq 0$ for $t\in(0,t_1)$, there is a $t_0\in(t_1,r)$ such that $S(t)<0$ for $t\in(0,t_0)$ and $S(t)>0$ for $t\in(t_0,r)$, which completes the proof. □

Lemma 3. *We have the following asymptotic formulas:*

$$\begin{aligned} F\left(\frac{1}{2},\frac{1}{2};1;x\right) &= \frac{\ln(16/t)}{\pi}+\frac{t}{4\pi}\left(\ln\frac{16}{t}-2\right)+O\left(t^2\ln t\right), \\ F\left(\frac{1}{2},\frac{1}{2};2;x\right) &= \frac{4}{\pi}-\frac{t}{\pi}\left(\ln\frac{16}{t}-3\right)+O\left(t^2\ln t\right), \end{aligned}$$

as $t\to 0^+$, *where* $t=1-x$.

Proof. It was listed in [**18**, p. 559] that

$$F(a,b;a+b;x)=\frac{1}{B(a,b)}\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{n!^2}\left[2\psi(n+1)-\psi(n+a)-\psi(n+b)-\ln t\right]t^n,$$

and for $m=1,2,3,...$,

$$F(a,b;a+b+m;x)=\frac{\Gamma(m)\Gamma(a+b+m)}{\Gamma(a+m)\Gamma(b+m)}\sum\nolimits_{n=0}^{m-1}\frac{(a)_n(b)_n}{n!(1-m)_n}t^n$$

$$-\frac{\Gamma(a+b+m)}{\Gamma(a)\Gamma(b)}(-t)^m\times\sum\nolimits_{n=0}^{\infty}\left[\frac{(a+m)_n(b+m)_n}{n!(n+m)!}t^n\times\right.$$

$$\left.(\ln t-\psi(n+1)-\psi(n+m+1)+\psi(a+n+m)+\psi(b+n+m))\right],$$

where $t=1-x\in(0,1)$. Letting $a=b=1/2$ and taking $m=1,2$ yield the desired asymptotic formulas. □

To state Lemma 4, we use W_n to denote the Wallis ratio, that is,

$$W_n = \frac{\Gamma(n+1/2)}{\Gamma(1/2)\Gamma(n+1)}. \tag{0.12}$$

It is easy to see that W_n satisfies the recurrence relation

$$W_{n+1} = \frac{n+1/2}{n+1} W_n. \tag{0.13}$$

Lemma 4. *Let the sequence* $\{\beta_n\}$ *be defined by*

$$\beta_n = \sum_{k=0}^{n-1} \left(\frac{(11k-17) W_k^2}{(k+1)(k+2)(k+3)} \frac{1}{n-k} \right). \tag{0.14}$$

Then β_n *satisfies the recurrence formula*

$$\beta_{n+1} - \lambda_n \beta_n = -\frac{1}{9} \frac{(2n+1)\left(880n^4 + 2404n^3 - 7319n^2 - 20\,301n - 10\,404\right)}{(11n-17)(n+1)^2(n+2)(n+3)(n+4)} W_n^2, \tag{0.15}$$

where

$$\lambda_n = \frac{1}{4} \frac{(11n-6)(2n+1)^2}{(n+4)(11n-17)(n+1)}. \tag{0.16}$$

Proof. Denote by

$$\beta_{k,n-k} = \frac{(11k-17) W_k^2}{(k+1)(k+2)(k+3)} \frac{1}{n-k}.$$

Then we have

$$\begin{aligned} \beta_{n+1} - \lambda_n \beta_n &= \sum_{k=0}^{n} \beta_{k,n+1-k} - \lambda_n \sum_{k=0}^{n-1} \beta_{k,n-k} \\ &= \beta_{0,n+1} + \sum_{k=0}^{n-1} \left(\beta_{k+1,n-k} - \lambda_n \beta_{k,n-k} \right) \end{aligned}$$

$$\begin{aligned} = \quad & -\frac{17}{6(n+1)} + \sum_{k=0}^{n-1} \left[\frac{11k-6}{(k+4)(k+2)(k+3)} \left(\frac{k+1/2}{k+1} \right)^2 \right. \\ & \left. - \frac{(11n-6)(2n+1)^2}{4(n+4)(11n-17)(n+1)} \frac{11k-17}{(k+1)(k+2)(k+3)} \right] \frac{W_k^2}{n-k} \end{aligned}$$

$$= -\frac{17}{6(n+1)} - \frac{1}{4(n+1)(n+4)(11n-17)} \sum_{k=0}^{n-1} \frac{\phi(k,n)}{(k+1)^2(k+2)(k+3)(k+4)} W_k^2,$$

where

$$\begin{aligned}\phi(k,n) &= 11\left(132n^2-151n-266\right)k^2 \\ &\quad -\left(1661n^2+3252n+1132\right)k-2\left(1463n^2+566n-319\right)\end{aligned}$$

$$\begin{aligned}&= 11\left(132n^2-151n-266\right)\left(k+\frac{1}{2}\right)^2 \\ &\quad -(283n+299)(11n-6)\left(k+\frac{1}{2}\right)-\frac{315}{4}(2n+1)(11n-6)\,.\end{aligned}$$

Then
(0.17)

$$\beta_{n+1}-\lambda_n\beta_n=-\frac{17}{6(n+1)}-\frac{11}{4}\frac{\left(132n^2-151n-266\right)}{(n+1)(n+4)(11n-17)}\phi_2(n)$$

$$+\frac{1}{4}\frac{(283n+299)(11n-6)}{(n+1)(n+4)(11n-17)}\phi_1(n)+\frac{315}{16}\frac{(2n+1)(11n-6)}{(n+1)(n+4)(11n-17)}\phi_0(n)\,,$$

where

$$\begin{aligned}\phi_2(n) &= \sum_{k=0}^{n-1}\frac{(k+1/2)^2W_k^2}{(k+1)^2(k+2)(k+3)(k+4)}, \\ \phi_1(n) &= \sum_{k=0}^{n-1}\frac{(k+1/2)W_k^2}{(k+1)^2(k+2)(k+3)(k+4)}, \\ \phi_0(n) &= \sum_{k=0}^{n-1}\frac{W_k^2}{(k+1)^2(k+2)(k+3)(k+4)}.\end{aligned}$$

Note that $\phi_i(n)$ $(i=2,1,0)$ can be written as

$$\begin{aligned}\phi_2(n) &= \frac{1}{3!}\left(\sum_{k=1}^{n}\frac{\left(\frac{1}{2}\right)_k^2}{(4)_k\,k!}-1\right),\ \phi_1(n)=-\frac{2}{3!}\left(\sum_{k=1}^{n}\frac{\left(-\frac{1}{2}\right)_k\left(\frac{1}{2}\right)_k}{(4)_k\,k!}-1\right), \\ \phi_0(n) &= \frac{4}{3!}\left(\sum_{k=1}^{n}\frac{\left(-\frac{1}{2}\right)_k^2}{(4)_k\,k!}-1\right),\end{aligned}$$

by comparing the coefficients of power series of the following formulas

$$\begin{aligned}\frac{1}{1-x}F\left(\frac{1}{2},\frac{1}{2};4;x\right) &= (1-x)^2F\left(\frac{7}{2},\frac{7}{2};4;x\right), \\ \frac{1}{1-x}F\left(-\frac{1}{2},\frac{1}{2};4;x\right) &= (1-x)^3F\left(\frac{9}{2},\frac{7}{2};4;x\right), \\ \frac{1}{1-x}F\left(-\frac{1}{2},-\frac{1}{2};4;x\right) &= (1-x)^4F\left(\frac{9}{2},\frac{9}{2};4;x\right),\end{aligned}$$

which follows from the second formula of (0.3), we can obtain

$$\phi_2(n) = \frac{1}{225}\frac{(2n+1)^2(32n^2+168n+225)}{(n+1)(n+2)(n+3)}W_n^2 - \frac{1}{6},$$

$$\phi_1(n) = \frac{1}{3} - \frac{2}{525}\frac{(2n+1)(128n^3+736n^2+1236n+525)}{(n+1)(n+2)(n+3)}W_n^2,$$

$$\phi_0(n) = \frac{4}{3675}\frac{2048n^4+12\,800n^3+25\,664n^2+18\,288n+3675}{(n+1)(n+2)(n+3)}W_n^2 - \frac{2}{3},$$

here we omit the details. Substituting $\phi_i(n)$ $(i = 2, 1, 0)$ into (0.17) and simplifying prove the recurrence relation (0.15).

The proof of this lemma is done. □

3. PROOF OF THEOREM 1

Now we are in a position to prove our the first result.

Proof of Theorem 1. Differentiation yields

$$\frac{2}{\pi}Q_1'(x) = \frac{\frac{1}{4}F\left(\frac{3}{2},\frac{3}{2};2;x\right)\ln\left(c/\sqrt{1-x}\right) - F\left(\frac{1}{2},\frac{1}{2};1;x\right)\frac{1}{2(1-x)}}{\left[\ln\left(c/\sqrt{1-x}\right)\right]^2} := \frac{f_1(x)}{g_1(x)},$$

$$\begin{aligned}\frac{2}{\pi}Q_1''(x) &= \frac{\frac{9}{32}F\left(\frac{5}{2},\frac{5}{2};3;x\right)\ln\left(c/\sqrt{1-x}\right) - F\left(\frac{1}{2},\frac{1}{2};1;x\right)\frac{1}{2(1-x)^2}}{\left[\ln\left(c/\sqrt{1-x}\right)\right]^2} \\ &\quad - \frac{\left[\frac{1}{4}F\left(\frac{3}{2},\frac{3}{2};2;x\right)\ln\left(c/\sqrt{1-x}\right) - F\left(\frac{1}{2},\frac{1}{2};1;x\right)\frac{1}{2(1-x)}\right]\frac{1}{1-x}}{\left[\ln\left(c/\sqrt{1-x}\right)\right]^3}.\end{aligned}$$

The necessity easily follows from the inequality

$$\frac{2}{\pi}Q_1''(0^+) = \frac{1}{32}\frac{(3\ln c - 4)^2}{\ln^3 c} \leq 0.$$

To prove the sufficiency, it suffices to prove $(2/\pi)Q_2' = f_1/g_1$ is strictly decreasing on $(0,1)$ for $c = e^{4/3}$. Differentiations by (0.2) and simplifications by the third formula of (0.3) yield

$$\begin{aligned}\frac{f_1'(x)}{g_1'(x)} &= \frac{\frac{9}{32}F\left(\frac{5}{2},\frac{5}{2};3;x\right)\left(\frac{4}{3} - \ln\sqrt{1-x}\right) - F\left(\frac{1}{2},\frac{1}{2};1;x\right)\frac{1}{2(1-x)^2}}{\left(\frac{4}{3} - \ln\sqrt{1-x}\right)/(1-x)} \\ &= \frac{\frac{9}{16}F\left(\frac{1}{2},\frac{1}{2};3;x\right)\left(\frac{4}{3} - \ln\sqrt{1-x}\right) - F\left(\frac{1}{2},\frac{1}{2};1;x\right)}{2(1-x)\left(\frac{4}{3} - \ln\sqrt{1-x}\right)} := \frac{f_2(x)}{g_2(x)}.\end{aligned}$$

Since $g_2' = \ln(1-x) - 5/3 < 0$, if we prove $H_{f_2,g_2} = (f_2'/g_2') g_2 - f_2 > 0$ for $x \in (0,1)$, then $(f_1'/g_1')' = (f_2/g_2)' = (g_2'/g_2^2) H_{f_2,g_2} < 0$. This yields $H_{f_1,g_1}' = (f_1'/g_1')' g_1 < 0$, and so

$$H_{f_1,g_1}(x) < H_{f_1,g_1}(0) = \frac{f_1'(0)}{g_1'(0)} g_1(0) - f_1(0) = \left(-\frac{3}{32}\right)\left(\frac{4}{3}\right)^2 - \left(-\frac{1}{6}\right) = 0.$$

Then $(f_1/g_1)' = (g_1'/g_1^2) H_{f_1,g_1} < 0$, which proves $Q_1''(x) < 0$.

Differentiation yields

$$f_2'(x) = \frac{3}{64} F\left(\frac{3}{2}, \frac{3}{2}; 4; x\right)\left(\frac{4}{3} - \ln\sqrt{1-x}\right) + \frac{9}{32}\frac{1}{1-x} F\left(\frac{1}{2}, \frac{1}{2}; 3; x\right) - \frac{1}{4} F\left(\frac{3}{2}, \frac{3}{2}; 2; x\right),$$

$$\frac{g_2(x)}{g_2'(x)} = \frac{2(1-x)\left(4/3 - \ln\sqrt{1-x}\right)}{\ln(1-x) - 5/3} = -(1-x)\frac{8/3 - \ln(1-x)}{5/3 - \ln(1-x)}.$$

We first claim that $f_2'(x) > 0$ for $x \in (0,1)$. Due to the first item is clearly positive, it suffices to show that the sum of the second and third ones is also positive, which is equivalent to

$$\begin{aligned} f_3(x) \quad : &= (1-x)\left[\frac{9}{32}\frac{1}{1-x} F\left(\frac{1}{2}, \frac{1}{2}; 3; x\right) - \frac{1}{4} F\left(\frac{3}{2}, \frac{3}{2}; 2; x\right)\right] \\ &= \frac{9}{32} F\left(\frac{1}{2}, \frac{1}{2}; 3; x\right) - \frac{1}{4} F\left(\frac{1}{2}, \frac{1}{2}; 2; x\right) > 0. \end{aligned}$$

Expanding in power series leads to

$$f_3(x) = \frac{1}{32} - \frac{1}{16}\sum_{n=1}^{\infty} \frac{4n-1}{(n+2)(n+1)} W_n^2 x^n,$$

where W_n is defined by (0.12). And, we easily get that $f_3(1^-) = 0$. It follows from Lemma 2 that $f_3(x) > 0$ for $x \in (0,1)$.

Second, it is obvious that

$$\frac{g_2(x)}{g_2'(x)} = -(1-x)\frac{8/3 - \ln(1-x)}{5/3 - \ln(1-x)} > -\frac{8}{5}(1-x).$$

It is then obtained that

$$H_{f_2,g_2}(x) = f_2'(x)\frac{g_2(x)}{g_2'(x)} - f_2(x) > -\frac{8}{5}(1-x) f_2'(x) - f_2(x) := f_4(x), \tag{0.18}$$

and it is enough to prove $f_4(x) > 0$ for $x \in (0,1)$. To do this, we use series expansion to get

$$\begin{aligned} f_4(x) \quad = \quad & \left[-\frac{3}{40}(1-x) F\left(\frac{3}{2}, \frac{3}{2}; 4; x\right) - \frac{9}{16} F\left(\frac{1}{2}, \frac{1}{2}; 3; x\right)\right]\left(\frac{4}{3} - \ln\sqrt{1-x}\right) \\ & -\frac{8}{5}\left[\frac{9}{32} F\left(\frac{1}{2}, \frac{1}{2}; 3; x\right) - \frac{1}{4} F\left(\frac{1}{2}, \frac{1}{2}; 2; x\right)\right] + F\left(\frac{1}{2}, \frac{1}{2}; 1; x\right) \end{aligned}$$

$$= \left(\sum_{n=0}^{\infty}\frac{9}{40}\frac{11n-17}{(n+3)(n+2)(n+1)}W_n^2x^n\right)\left(\frac{4}{3}+\frac{1}{2}x\sum_{n=0}^{\infty}\frac{x^n}{n+1}\right)$$
$$+\sum_{n=0}^{\infty}\frac{1}{10}\frac{\left(10n^2+34n+19\right)W_n^2}{(n+1)(n+2)}x^n$$

$$= \sum_{n=0}^{\infty}\frac{1}{5}\frac{\left(5n^3+32n^2+77n+3\right)}{(n+1)(n+2)(n+3)}W_n^2x^n$$
$$+x\sum_{n=0}^{\infty}\sum_{k=0}^{n}\left(\frac{1}{20}\frac{(11k-17)W_k^2}{(k+3)(k+2)(k+1)}\frac{1}{n-k+1}\right)x^n = \frac{1}{10}+\sum_{n=1}^{\infty}\alpha_nx^n,$$

where

$$\alpha_n = \frac{1}{5}\frac{5n^3+32n^2+77n+3}{(n+1)(n+2)(n+3)}W_n^2+\frac{9}{80}\beta_n,$$

here β_n is defined by (0.14).

Straightforward computations give

$$\alpha_1 = -\frac{3}{40},\ \alpha_2 = -\frac{9}{640},\ \alpha_3 = -\frac{31}{20\,480},\ \alpha_4 = \frac{243}{163\,840} > 0.$$

On the other hand, by Lemma 4 and recurrence relation (0.13), we have

$$\alpha_{n+1}-\lambda_n\alpha_n = \frac{1}{5}\frac{5n^3+47n^2+156n+117}{(n+2)(n+3)(n+4)}\left(\frac{n+1/2}{n+1}\right)^2W_n^2$$
$$-\frac{1}{4}\frac{(11n-6)(2n+1)^2}{(n+4)(11n-17)(n+1)}\frac{1}{5}\frac{5n^3+32n^2+77n+3}{(n+1)(n+2)(n+3)}W_n^2+\frac{9}{80}\left(\beta_{n+1}-\lambda_n\beta_n\right)$$

$$= \frac{1}{20}\frac{(2n+1)^2\left(110n^3+262n^2-936n-1971\right)}{(11n-17)(n+1)^2(n+2)(n+3)(n+4)}W_n^2$$
$$-\frac{1}{80}\frac{(2n+1)\left(880n^4+2404n^3-7319n^2-20\,301n-10\,404\right)}{(11n-17)(n+1)^2(n+2)(n+3)(n+4)}W_n^2$$
$$= \frac{3}{80}\frac{(2n+1)\left(44n^3+293n^2+263n+840\right)}{(11n-17)(n+1)^2(n+2)(n+3)(n+4)}W_n^2 > 0,$$

for $n\geq 2$. This in combination with $\alpha_4>0$ reveals that $\alpha_n>0$ for $n\geq 4$. Hence, we derive that

$$f_4(x) = \frac{1}{10}+\sum_{n=1}^{\infty}\alpha_nx^n > \frac{1}{10}+\sum_{n=1}^{3}\alpha_nx^n \tag{0.19}$$
$$= \frac{1}{10}-\frac{3}{40}x-\frac{9}{640}x^2-\frac{31}{20\,480}x^3 := f_5(x).$$

Since $f_5(1) = 193/20\,480 > 0$, by Lemma 2 we get that $f_5(x) > 0$. Taking into account (0.18) and (0.19) proves $H_{f_2,g_2}(x) > 0$ for $x \in (0,1)$, which completes the proof. $\square$

Note that

$$\mathcal{K}\left(\frac{1}{\sqrt{2}}\right) = \frac{\Gamma(1/4)^2}{4\sqrt{\pi}}.$$

By the properties of the concave functions we find that

(0.20)

$$\sqrt{Q_1(t)\,Q_1(1-t)} \le \frac{Q_1(t) + Q_1(1-t)}{2} \le Q_1\left(\frac{1}{2}\right) = \frac{3\Gamma(1/4)^2}{2(3\ln 2 + 8)\sqrt{\pi}} := \theta_0,$$

which, by putting $t = r^2 \in (0,1)$, gives the following assertions.

Corollary 1. *The inequality*

$$\frac{\mathcal{K}(r)}{\ln\left(e^{4/3}/r'\right)} + \frac{\mathcal{K}(r')}{\ln\left(e^{4/3}/r\right)} \le 2\theta_0,$$

or equivalently,

$$\mu(r) \le \pi\theta_0 \frac{\ln\left(e^{4/3}/r\right)}{\mathcal{K}(r)} - \frac{\pi}{2}\frac{\ln\left(e^{4/3}/r\right)}{\ln\left(e^{4/3}/r'\right)} \tag{0.21}$$

holds for $r \in (0,1)$, where $\theta_0 = 1.1037...$ given in (0.20) is the best constant.

Corollary 2. *The inequality*

$$\mathcal{K}(r)\,\mathcal{K}(r') \le \theta_0^2 \ln\left(\frac{e^{4/3}}{r}\right)\ln\left(\frac{e^{4/3}}{r'}\right) \tag{0.22}$$

holds for $r \in (0,1)$ with the best constant $\theta_0 = 1.1037...$ given in (0.20).

Corollary 3. *The function $(1-x)^{1/4}\,\mathcal{K}(\sqrt{x})$ is strictly log-concave on $(0,1)$, and therefore, we have*

$$\sqrt{rr'}\mathcal{K}(r)\,\mathcal{K}(r') \le \frac{1}{\sqrt{2}}\mathcal{K}\left(\frac{1}{\sqrt{2}}\right)^2 = \frac{\Gamma(1/4)^4}{16\sqrt{2}\pi} = 2.4307.... \tag{0.23}$$

Proof. We easily see that $Q_1(x)$ is also log-concave on $(0,1)$ by Theorem 1. Moreover, since

$$\frac{d^2}{dx^2}\left((1-x)^{1/4}\ln\frac{e^{4/3}}{\sqrt{1-x}}\right) = \frac{3}{32}\frac{\ln(1-x)}{(1-x)^{7/4}} < 0,$$

the function $(1-x)^{1/4}\ln\left(e^{4/3}/\sqrt{1-x}\right)$ is positive and concave on $(0,1)$, and is also log-concave on $(0,1)$. Consequently, the function

$$(1-x)^{1/4}\,\mathcal{K}\left(\sqrt{x}\right) = (1-x)^{1/4}\ln\left(\frac{e^{4/3}}{\sqrt{1-x}}\right) \times Q_1(x)$$

is log-concave on $(0,1)$. Therefore, we obtain

$$(1-x)^{1/4}\mathcal{K}\left(\sqrt{x}\right)x^{1/4}\mathcal{K}\left(\sqrt{1-x}\right)\leq\left[\left(\frac{1}{2}\right)^{1/4}\mathcal{K}\left(\frac{1}{\sqrt{2}}\right)\right]^2,$$

which proves the inequality (0.23). □

Remark 3. It was proven in [**22**, Corollary 3.13] that for $r\in(0,1)$,

$$\frac{2}{\pi}rr'\mathcal{K}(r)\mathcal{K}(r')<\min\left(r\ln\frac{4}{r},r'\ln\frac{4}{r'}\right)=\begin{cases} r\ln\dfrac{4}{r} & \text{if } r\in\left(0,\frac{1}{\sqrt{2}}\right), \\ r'\ln\dfrac{4}{r'} & \text{if } r\in\left(\frac{1}{\sqrt{2}},1\right). \end{cases} \tag{0.24}$$

We write inequality (0.22) as

$$\frac{2}{\pi}rr'\mathcal{K}(r)\mathcal{K}(r')\leq\frac{2}{\pi}\theta_0^2 rr'\ln\left(\frac{e^{4/3}}{r}\right)\ln\left(\frac{e^{4/3}}{r'}\right). \tag{0.25}$$

Numeric computation shows that the upper bounds in (0.24) and in (0.25) are not comparable. But the minimum upper bound given in (0.24) equals $\left(5\sqrt{2}\ln 2\right)/4=1.2253...$, which is greater than one given in (0.25), because

$$\frac{d^2}{dx^2}\sqrt{x}\ln\left(\frac{e^{4/3}}{\sqrt{x}}\right)=-\frac{8+3\ln(1/x)}{24x^{3/2}}<0,$$

$x\mapsto\sqrt{x}\ln\left(e^{4/3}/\sqrt{x}\right)$ is concave, which also implies that it is log-concave on $(0,1)$, and so

$$\frac{2}{\pi}\theta_0^2 rr'\ln\left(\frac{e^{4/3}}{r}\right)\ln\left(\frac{e^{4/3}}{r'}\right)\leq\frac{2}{\pi}\theta_0^2\left[\sqrt{1/2}\ln\left(\frac{e^{4/3}}{\sqrt{1/2}}\right)\right]^2=\frac{\Gamma(1/4)^4}{16\pi^2}=1.0942....$$

Remark 4. It was proven in [**21**, Theorem 2.2 (3)] that $r'\mathcal{K}(r)^2$ is strictly decreasing on $[0,1)$. This in combination with Corollary 3 shows that $(1-x)^{1/4}\mathcal{K}(\sqrt{x})$ is strictly decreasing and log-concave on $(0,1)$.

Remark 5. Very recently, Yang [**13**, Proposition 1] showed that for $p\in(0,1/2]$, $(r')^p\mathcal{K}(r)=\sum_{n=0}^{\infty}a_n(p)r^{2n}$ with $a_0(p)=1$, $a_1(p)\leq 0$, $a_n(p)<0$ for $n\geq 2$. This further indicates that $(1-x)^{1/4}\mathcal{K}(\sqrt{x})$ is strictly decreasing and concave on $(0,1)$.

Moreover, it is clear that the functions $\frac{Q_1(t)-Q_1(0)}{t}$ and $\frac{Q_1(1)-Q_1(t)}{1-t}$ are decreasing on $(0,1)$, which yield the following corollary.

Corollary 4. *Both of the functions*

$$r^2\mapsto\frac{1}{r^2}\left(\frac{\mathcal{K}(r)}{\ln\left(e^{4/3}/r'\right)}-\frac{3\pi}{8}\right),$$

$$r^2\mapsto\frac{1}{(r')^2}\left(1-\frac{\mathcal{K}(r)}{\ln\left(e^{4/3}/r'\right)}\right)$$

are strictly decreasing on $(0,1)$. *Consequently, the double inequality*

$$1+\left(\frac{3\pi}{8}-1\right)(r')^2<\frac{\mathcal{K}(r)}{\ln\left(e^{4/3}/r'\right)}<\frac{21\pi}{64}+\frac{3\pi}{64}(r')^2 \tag{0.26}$$

holds for $r\in(0,1)$. *The lower and upper bounds are sharp.*

Remark 6. Inequalities (0.26) offer a new type of sharp lower and upper bounds for $\mathcal{K}(r)$. Similar inequalities can be found in [**21**], [**25**], [**27**].

4. PROOFS OF THEOREM 2 AND 3

Proof of Theorem 2. Let $f(x)=F\left(\frac{1}{2},\frac{1}{2};1;x\right)$ and $g(x)=\ln\left(1+4/\sqrt{1-x}\right)$, where $x=r^2\in(0,1)$. Differentiations yield

$$\begin{aligned} f'(x) &= \frac{1}{4}F\left(\frac{3}{2},\frac{3}{2};2;x\right)=\frac{1}{4}\frac{1}{1-x}F\left(\frac{1}{2},\frac{1}{2};2;x\right), \\ g'(x) &= \frac{2}{\left(4+\sqrt{1-x}\right)(1-x)}=\frac{2\left(4-\sqrt{1-x}\right)}{(15+x)(1-x)}, \end{aligned}$$

and then, we have

$$\frac{f'(x)}{g'(x)}=\frac{\frac{1}{4}\frac{1}{1-x}F\left(\frac{1}{2},\frac{1}{2};2;x\right)}{\frac{2\left(4-\sqrt{1-x}\right)}{(15+x)(1-x)}}=\frac{(15+x)F\left(\frac{1}{2},\frac{1}{2};2;x\right)}{8\left(4-\sqrt{1-x}\right)}:=\frac{f_1(x)}{g_1(x)}.$$

Expanding in power series leads to

$$\frac{f_1(x)}{g_1(x)}=\frac{15+\sum_{n=1}^{\infty}\frac{64n^2-56n+15}{(2n-1)^2(n+1)}W_n^2x^n}{24+\sum_{n=1}^{\infty}\frac{4\Gamma(n-1/2)}{\Gamma(1/2)\Gamma(n+1)}x^n}:=\frac{\sum_{n=0}^{\infty}a_nx^n}{\sum_{n=0}^{\infty}b_nx^n},$$

where

$$\begin{aligned} a_n &= \begin{cases} 15 & \text{for } n=0, \\ \dfrac{64n^2-56n+15}{(2n-1)^2(n+1)}W_n^2 & \text{for } n\geq 1, \end{cases} \\ b_n &= \begin{cases} 24 & \text{for } n=0, \\ \dfrac{8}{2n-1}W_n & \text{for } n\geq 1, \end{cases} \end{aligned}$$

here W_n is the Wallis ratio defined by (0.12). It is easy to check that the sequence $\{a_n/b_n\}_{n\geq 0}$ is increasing for $n=0,1$ and decreasing for $n\geq 1$. In fact, we have

$$\frac{a_n}{b_n}=\begin{cases} \dfrac{5}{8} & \text{for } n=0, \\ \dfrac{1}{8}\dfrac{64n^2-56n+15}{(2n-1)(n+1)}W_n & \text{for } n\geq 1. \end{cases}$$

It then follows that
$$\frac{a_1}{b_1}-\frac{a_0}{b_0}=\frac{23}{32}-\frac{5}{8}=\frac{3}{32}>0,$$
and for $n\geq 1$,
$$\frac{a_{n+1}}{b_{n+1}}\Big/\frac{a_n}{b_n}-1=-\frac{1}{2}\frac{64n^2-168n+83}{(n+2)\left(64n^2-56n+15\right)}<0,$$
which proves the piecewise monotonicity of the sequence $\{a_n/b_n\}_{n\geq 0}$.

On the other hand, we find that
$$\begin{aligned}H_{f_1,g_1}(x) &= \frac{f_1'(x)}{g_1'(x)}g_1(x)-f_1(x)\\ &= \frac{F\left(\frac{1}{2},\frac{1}{2};2;x\right)+(15+x)\frac{1}{8}F\left(\frac{3}{2},\frac{3}{2};3;x\right)}{4/\sqrt{1-x}}\\ &\quad \times 8\left(4-\sqrt{1-x}\right)-(15+x)F\left(\frac{1}{2},\frac{1}{2};2;x\right)\\ &\to -\frac{64}{\pi}<0 \text{ as } x\to 1^-.\end{aligned}$$
From Lemma 1 it is deduced that there is an $x_1\in(0,1)$ such that $f_1/g_1=f'/g'$ is increasing on $(0,x_1)$ and decreasing on $(x_1,1)$. Using the formulas (0.10) and (0.11) together with $g>0$ we find that $H_{f,g}'>0$ for $x\in(0,x_1)$ and $H_{f,g}'<0$ for $x\in(x_1,1)$. Now, due to $g'>0$, if we show that $H_{f,g}(0^+)\geq 0$ and $H_{f,g}(1^-)\geq 0$, then we have $(f/g)'=\left(g'/g^2\right)H_{f,g}>0$, and the proof is done. A simple computation yields
$$\begin{aligned}H_{f,g}(x) &= \frac{f'(x)}{g'(x)}g(x)-f(x)\\ &= \frac{(15+x)F\left(\frac{1}{2},\frac{1}{2};2;x\right)}{8\left(4-\sqrt{1-x}\right)}\ln\left(1+\frac{4}{\sqrt{1-x}}\right)-F\left(\frac{1}{2},\frac{1}{2};1;x\right)\\ &\to \frac{15}{24}\ln 5-1>0 \text{ as } x\to 0^+.\end{aligned}$$
Application of asymptotic formulas given in Lemma 3 gives
$$\begin{aligned}H_{f,g}(x) &\quad sim \quad \frac{(16-t)\left[\frac{4}{\pi}-\frac{t}{\pi}(\ln(16/t)-3)\right]}{8\left(4-\sqrt{t}\right)}\ln\left(1+\frac{4}{\sqrt{t}}\right)\\ &\quad -\left[\frac{\ln(16/t)}{\pi}+\frac{t}{4\pi}\left(\ln\frac{16}{t}-2\right)\right]\\ &\to 0 \text{ as } t=1-x\to 0^+.\end{aligned}$$
This completes the proof. □

Using the monotonicity of Q_2, we obtain the following corollary.

Corollary 5. *The inequality*

$$\mu = \frac{\pi}{2}\frac{\mathcal{K}(r')}{\mathcal{K}(r)} > \frac{\pi}{2}\frac{\ln(1+4/r)}{\ln(1+4/r')}$$

holds for $r \in (0, 1/\sqrt{2})$*. It is reversed for* $r \in (1/\sqrt{2}, 1)$*.*

Proof of Theorem 3. Differentiations yield

$$\begin{aligned} D'(x) &= \frac{\pi}{8}F\left(\frac{3}{2},\frac{3}{2};2;x\right) - \frac{2\left(4-\sqrt{1-x}\right)}{(15+x)(1-x)}, \\ D''(x) &= \frac{\pi}{8}\frac{9}{8}F\left(\frac{5}{2},\frac{5}{2};3;x\right) - \frac{16(x+7)-(3x+13)\sqrt{1-x}}{(x+15)^2(1-x)^2} \\ &:= \frac{h(x)}{(x+15)^2(1-x)^2}, \end{aligned}$$

where

$$h(x) = \frac{9\pi}{64}(x+15)^2 F\left(\frac{1}{2},\frac{1}{2};3;x\right) + (3x+13)\sqrt{1-x} - 16(x+7).$$

Expanding in power series yields

$$\begin{aligned} h(x) &= \frac{9}{32}(x+15)^2\sum_{n=0}^{\infty}\frac{W_n^2}{(n+1)(n+2)}x^n - (3x+13)\sum_{n=0}^{\infty}\frac{W_n}{2n-1}x^n - 16(x+7) \\ &= \left(\frac{2025}{64}\pi - 99\right) + \left(\frac{1755}{256}\pi - \frac{39}{2}\right)x \\ &\quad + \sum_{n=2}^{\infty}\left(\frac{9}{32}\frac{4096n^4 - 14\,848n^3 + 17\,984n^2 - 8672n + 2025}{(2n-1)^2(2n-3)^2(n+1)(n+2)}W_n^2 \right. \\ &\quad \left. - \frac{32n-39}{(2n-1)(2n-3)}W_n\right)x^n := \sum_{n=0}^{\infty}c_n x^n, \end{aligned}$$

where

$$c_0 = \frac{2025}{64}\pi - 99 > 0, \;\; c_1 = \frac{1755}{256}\pi - \frac{39}{2} > 0,$$

and for $n \geq 2$,

$$c_n = \frac{(32n-39)W_n}{(2n-1)(2n-3)}(d_n - 1),$$

here

$$d_n = \frac{9\sqrt{\pi}}{32}\frac{4096n^4 - 14\,848n^3 + 17\,984n^2 - 8672n + 2025}{(2n-1)(2n-3)(n+1)(n+2)(32n-39)}\frac{\Gamma(n+1/2)}{\Gamma(n+1)}.$$

Clearly, in view of $\lim_{n\to\infty}(d_n-1)=-1$, we see that all coefficients c_n do not keep the same sign. However, if we prove that there is an $n_0\geq 2$ such that $(d_n-1)\geq 0$ for $2\leq n\leq n_0$ and $(d_n-1)<0$ for $n>n_0$, then we have that $c_n>0$ for $0\leq n\leq n_0$ and $c_n<0$ for $n>n_0$. Also, if $h(1^-)\geq 0$, then by Lemma 2 we obtain that $h(x)\geq 0$ for all $x\in(0,1)$, and so is $D''(x)$. Now, since the numerator of the second quotient of the expression of d_n can be written as

$$16n^2(16n-29)^2+(4528n-8672)n+2025,$$

which is clearly positive for $n\geq 2$, we easily check that

$$\frac{d_{n+1}}{d_n}-1=-\frac{196\,608\times P_5(n)}{(32n-7)(n+3)(4096n^4-14\,848n^3+17\,984n^2-8672n+2025)}<0,$$

because

$$\begin{aligned}P_5(n) &= n^5-\frac{427}{96}n^4+\frac{1823}{256}n^3-\frac{33\,203}{6144}n^2+\frac{4831}{2048}n-\frac{51\,165}{131\,072}\\ &= (n-2)^5+\frac{533}{96}(n-2)^4+\frac{8861}{768}(n-2)^3\\ &\quad+\frac{64\,957}{6144}(n-2)^2+\frac{23\,729}{6144}(n-2)+\frac{67\,235}{131\,072}\end{aligned}$$

is positive for $n\geq 2$. This shows that the sequence $\{d_n\}_{n\geq 2}$ is strictly decreasing, and so is $\{d_n-1\}_{n\geq 2}$, which together with the facts

$$d_2-1=\frac{10\,107}{25\,600}\pi-1>0 \quad\text{and}\quad \lim_{n\to\infty}(d_n-1)=-1$$

reveals that there is an $n_0\geq 2$ such that $(d_n-1)\geq 0$ for $2\leq n\leq n_0$ and $(d_n-1)<0$ for $n>n_0$.

On the other hand, it is easy to verify that

$$\begin{aligned}h(x) &= \frac{9\pi}{64}(x+15)^2F\left(\frac{1}{2},\frac{1}{2};3;x\right)+(3x+13)\sqrt{1-x}-16(x+7)\\ &\to \frac{9\pi}{64}(1+15)^2\frac{32}{9\pi}-16(1+7)=0 \text{ as } x\to 1^-.\end{aligned}$$

We thus complete the proof. □

Applying the properties of the convex functions to $D(x)$, we can obtain the following consequences.

Corollary 6. *We have*

$$\frac{\mathcal{K}(r)+\mathcal{K}(r')}{2}>\frac{\ln(1+4/r')+\ln(1+4/r)}{2}+\mathcal{K}\left(\frac{1}{\sqrt{2}}\right)-\ln\left(1+4\sqrt{2}\right)$$

holds for $r\in(0,1)$.

Corollary 7. *The functions*

$$r^2 \mapsto \frac{\mathcal{K}(r) - \ln(1+4/r') - (\ln 5 - \pi/2)}{r^2},$$
$$r^2 \mapsto \frac{\mathcal{K}(r) - \ln(1+4/r')}{(r')^2}$$

are strictly increasing and decreasing on $(0,1)$*, respectively. Consequently, the double inequality*

$$\ln\left(1+\frac{4}{r'}\right) - \left(\ln 5 - \frac{\pi}{2}\right) + pr^2 < \mathcal{K}(r) < \ln\left(1+\frac{4}{r'}\right) - \left(\ln 5 - \frac{\pi}{2}\right) + qr^2 \tag{0.27}$$

holds for $r \in (0,1)$ *if and only if* $p \le p_0 = \pi/8 - 2/5$ *and* $q \ge p_1 = \ln 5 - \pi/2$.

Remark 7. From the right hand side inequality of (0.27) for $q = p_1 = \ln 5 - \pi/2$, we obtain

$$\mathcal{K}(r) < \ln\left(1+\frac{4}{r'}\right) - \left(\ln 5 - \frac{\pi}{2}\right)\left(1-r^2\right) < \ln\left(1+\frac{4}{r'}\right) - \left(\ln 5 - \frac{\pi}{2}\right)(1-r),$$

which gives a refinement of the inequality (0.6). Moreover, the double inequality (0.27) gives a companion one.

Remark 8. Due to $D'(0) = \pi/8 - 2/5 < 0$, $D'(1^-) = \infty$, there is an $x_0 \in (0,1)$ such that $D'(x) < 0$ for $x \in (0, x_0)$ and $D'(x) > 0$ for $x \in (x_0, 1)$.

5. CONCLUDING REMARKS

In this paper, we investigate the convexity of two functions $Q_1(x) = \frac{\mathcal{K}(\sqrt{x})}{\ln\left(c/\sqrt{1-x}\right)}$ and $D(x) = \mathcal{K}(\sqrt{x}) - \ln\left(1+4/\sqrt{1-x}\right)$, and monotonicity of $Q_2(r) = \frac{\mathcal{K}(r)}{\ln(1+4/r')}$. It is known that Q_1 is decreasing (increasing) on $(0,1)$ if and only if $c \in [1,4]$ ($[e^2,\infty)$), but Q_1 is concave on $(0,1)$ if and only if $c = e^{4/3}$, which is an interesting result. We also show that Q_2 and D are strictly increasing and convex on $(0,1)$, respectively. These results give some new sharp inequalities for $\mathcal{K}(r)$, $\mathcal{K}(r')$ and their combinations $\mu(r)$. Moreover, it should be noted that these functions mentioned above are simple but it is somewhat difficult to prove their properties, and fortunately, we have Lemmas 1 and 2 as powerful tools.

Finally, we close this paper by posing the following problem and conjectures.

Problem 1. *Determine the best parameter* c *such that* $Q_1(x) = \frac{\mathcal{K}(\sqrt{x})}{\ln\left(c/\sqrt{1-x}\right)}$ *is convex on* $(0,1)$*, or* $1/Q_1$ *is concave on* $(0,1)$.

Conjecture 1. *The function* $x \mapsto (1-x)^p \mathcal{K}(\sqrt{x})$ *is log-concave on* $(0,1)$ *if and only if* $p \ge 7/32$.

Conjecture 2. *The function* $Q_2(x) = \frac{\mathcal{K}(\sqrt{x})}{\ln\left(1+4/\sqrt{1-x}\right)}$ *is convex on* $(0,1)$*, or* $1/Q_2$ *is concave on* $(0,1)$.

Conjecture 3. *Let* $D(x) = \mathcal{K}(\sqrt{x}) - \ln\left(1 + 4/\sqrt{1-x}\right)$. *Then* D'' *is absolute monotonic on* $(0,1)$.

Remark 9. As pointed out in [**26**], the monotonicity of a function normally yields constant bounds for this function, while the convexity or concavity yield better estimates. If these problem and conjectures are proved to be true, then it will yields some new better estimates for $\mathcal{K}(r)$, $\mathcal{K}(r')$ and their combinations. For example, if $1/Q_2$ is concave on $(0,1)$, then we have

$$\sqrt{\frac{\ln(1+4/r')}{\mathcal{K}(r)}\frac{\ln(1+4/r)}{\mathcal{K}(r')}} \leq \frac{1}{2}\left(\frac{\ln(1+4/r')}{\mathcal{K}(r)} + \frac{\ln(1+4/r)}{\mathcal{K}(r')}\right) \leq \frac{\ln\left(1+4\sqrt{2}\right)}{\mathcal{K}\left(1/\sqrt{2}\right)},$$

$$1 - \frac{2\ln 5}{\pi} < \frac{1}{r^2}\left(\frac{\ln(1+4/r')}{\mathcal{K}(r)} - \frac{2\ln 5}{\pi}\right) < \frac{2}{\pi}\left(\frac{2}{5} - \frac{1}{4}\ln 5\right).$$

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Zhen-Hang Yang
North China Electric Power University
282 Rui Xiang Street,
Baoding 071051,
P. R. China
E-mail: *yzhkm@163.com*

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Jingfeng Tian
North China Electric Power University
282 Rui Xiang Street,
Baoding 071051,
P. R. China
E-mail: *tianjf@ncepu.edu.cn*