Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **x** (xxxx), xxx–xxx.
https://doi.org/10.2298/AADM190427005M

ON A NEW (p, q)-MATHIEU-TYPE POWER SERIES AND ITS APPLICATIONS

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Dedicated to Prof. Gradimir Milovanovic on the occasion on his 70th birthday.

Our aim in this paper, is to establish certain new integral representations for the $(p, q)-$Mathieu–type power series. In particular, we investigate the Mellin–Barnes type integral representations for a particular case of these special function. Moreover, we introduce the notion of the $(p, q)-$Mittag-Leffler functions and we present a relationships between these two functions. Some other applications are proved, in particular two Turán type inequalities for the $(p, q)-$Mathieu–type series are derived.

1. INTRODUCTION

The following familiar infinite series

$$S(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2 + r^2)^2}, \tag{1}$$

is called a Mathieu series. It was introduced and studied by Émile Leonard Mathieu in his book [7] devoted to the elasticity of solid bodies. Bounds for this series are needed for the solution of boundary value problems for the biharmonic equations in a two–dimensional rectangular domain, see [11, Eq. (54), p. 258].

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2010 Mathematics Subject Classification. 31B10, 33C20, 33E20, 33E12.
Keywords and Phrases. $(p, q)-$extended Beta function, $(p, q)-$extended Gaussian hypergeometric function, $(p, q)-$Mittag-Leffler functions, integral representations, $(p, q)-$Mathieu–type series, Mellin-Barnes types integrals.

Several interesting problems and solutions dealing with integral representations and bounds for the following generalization of the Mathieu series, which is so-called generalized Mathieu series with a fractional power can be found in [3, 8, 14, 16]:

$$S_\mu(r) = \sum_{n=1}^{\infty} \frac{2n}{(n^2+r^2)^{\mu+1}} \ \ (\mu > 0,\ r > 0).$$

In [14], the authors derived the following new Laplace type integral representation via Schlömilch series:

$$S_\mu(r) = \frac{\sqrt{\pi}}{2^{\mu-\frac{1}{2}}\Gamma(\mu+1)} \int_0^\infty e^{-rt}\mathcal{K}_\mu(t)dt \ \ (\mu > \frac{3}{2}),$$

where

$$\mathcal{K}_\mu(t) = t^{\mu+\frac{1}{2}} \sum_{k=1}^{\infty} \frac{J_{\mu+\frac{1}{2}}(kt)}{k^{\mu-\frac{1}{2}}},$$

and $J_\mu(z)$ is the Bessel function. Motivated essentially by the works of Cerone and Lenard [1], Srivastava and Tomovski in [13] defined a family of generalized Mathieu series

$$S_\mu^{(\alpha,\beta)}(r;\mathbf{a}) = S_\mu^{(\alpha,\beta)}(r;\{a_k\}_{k=0}^\infty) = \sum_{k=1}^{\infty} \frac{2a_k^\beta}{(a_k^\alpha+r^2)^\mu} \ \ (\alpha,\ \beta,\ \mu,\ r > 0), \tag{2}$$

where it is tacitly assumed that the positive sequence

$$\mathbf{a} = \{a_k\} = \{a_1, a_2, ...\}, \text{ such that } \lim_{k\longrightarrow\infty} a_k = \infty,$$

is so chosen that the infinite series in definition (2) converges, that is, that the following auxiliary series

$$\sum_{k=1}^{\infty} \frac{1}{a_k^{\mu\alpha-\beta}},$$

is convergent.

Definition 1. (see [12, Eq. (6.1), p. 256]) The extended Beta function $B_{p,q}(x,y)$ is defined by

$$B_{p,q}(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}E_{p,q}(t)dt \tag{3}$$

$$\big(x,\ y,\ p,\ q \in \mathbb{C},\ \min(\Re(x),\Re(y)) > 0,\ \min(\Re(p),\Re(q)) \geq 0\big),$$

where $E_{p,q}(t)$ is defined by

$$E_{p,q}(t) = \exp\left(-\frac{p}{t} - \frac{q}{1-t}\right)$$

$$\big(p,\ q \in \mathbb{C},\ \min(\Re(p),\Re(q)) \geq 0\big).$$

In particular, Chaudhry et al. [2, p. 591, Eq. (1.7)], introduced the $p-$extension of Euler's Beta function $B(x, y)$:

$$B_p(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}e^{-\frac{p}{t(1-t)}}dt \ \ (\Re(p)>0),$$

whose special case, when $p=0$ (or $p=q=0$ in (3)), is the familiar Beta integral

$$B(x,y)=\int_0^1 t^{x-1}(1-t)^{y-1}dt \ \ (\Re(x),\Re(y)>0).$$

Definition 2. [5] Assume that $p,q\in\mathbb{C}_>=\{z\in\mathbb{C}:\Re(z)>0\}$, $\lambda,\mu,s\in\mathbb{C}$ and $\nu,a\in\mathbb{C}\setminus\mathbb{Z}_0^-$. The extended Hurwitz–Lerch zeta function is defined by

$$\Phi_{\lambda,\mu,\nu}(z,s,a;p,q)=\sum_{n=0}^{\infty}\frac{(\lambda)_n}{n!}\frac{B_{p,q}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{(a+n)^s} \ \ (|z|<1), \tag{4}$$

where $(\lambda)_n$ denotes the Pochhammer symbol (or the shifted factorial) defined, in terms of Euler's Gamma function, by

$$(\lambda)_\mu=\frac{\Gamma(\lambda+\mu)}{\Gamma(\lambda)}=\begin{cases}1 & (\mu=0;\lambda\in\mathbb{C}\setminus\{0\}),\\ \lambda(\lambda+1)...(\lambda+n-1) & (\mu=n\in\mathbb{N};\lambda\in\mathbb{C}).\end{cases}$$

On the unit circle $|z|=1$, the series (4) converges absolutely if, in addition, we set one of the following conditions [5, Theorem 3]:

$$\Re(s-\lambda)>0, \text{ or } \Re(\nu)>\Re(\mu)>0 \text{ and } \Re(\nu-\mu)>\Re(\lambda-s).$$

Upon setting $\lambda=1$, (4) reduces to

$$\Phi_{\mu,\nu}(z,s,a;p,q)=\sum_{n=0}^{\infty}\frac{B_{p,q}(\mu+n,\nu-\mu)}{B(\mu,\nu-\mu)}\frac{z^n}{(a+n)^s} \ \ (|z|<1).$$

It is easy to observe that

$$\Phi_{\lambda,\mu,\nu}(z,s,a;p,q)=\frac{1}{\Gamma(\lambda)}D_z^{\lambda-1}\{z^{\lambda-1}\Phi_{\mu,\nu}(z,s,a;p,q)\} \ \ (\Re(\lambda)>0), \tag{5}$$

where D_z^λ denotes the well-known Riemann–Liouville fractional derivative operator defined by

$$D_z^\lambda f(z)=\begin{cases}\frac{1}{\Gamma(-\lambda)}\int_0^z(z-t)^{-\lambda-1}f(t)dt & (\Re(\lambda)<0),\\ \frac{d^m}{dz^m}D_z^{\lambda-m}f(z) & (m-1\le\Re(\lambda)<m, \ (m\in\mathbb{N})).\end{cases}$$

In [5, Theorem 3.8] Luo et al. proved the following integral representation for the extended Hurwitz-Lerch zeta funtion $\Phi_{\lambda,\mu,\nu}(z,s,a;p,q)$:

$$\Phi_{\lambda,\mu,\nu}(z,s,a;p,q)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}e^{-at}{}_2F_1\left[\begin{matrix}\lambda,\ \mu\\ \nu\end{matrix};ze^{-t};p,q\right]dt \tag{6}$$

$$\big(|z| < 1,\ p,\ q,\ a,\ s \in \mathbb{C}_{>},\ \lambda,\ \mu \in \mathbb{C},\ \nu \in \mathbb{C} \setminus \mathbb{Z}_0^-\big)$$

where ${}_2F_1\Big[\begin{matrix} a,\, b \\ c \end{matrix}; z; p, q\Big]$ is the extended Gauss hypergeometric function defined by [12, Eq. (6.2)]

$${}_2F_1\Big[\begin{matrix} a,\, b \\ c \end{matrix}; z; p, q\Big] = \sum_{n=0}^{\infty} (a)_n \frac{B_{p,q}(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

$$\big(|z| < 1,\ \Re(c) > \Re(b) > 0,\ \min(\Re(p), \Re(q)) \geq 0\big).$$

When $p = q$, we obtain the extended Gaussian hypergeometric function ${}_2F_1$ defined by [2]:

$${}_2F_1\Big[\begin{matrix} a,\, b \\ c \end{matrix}; z; p\Big] = \sum_{n=0}^{\infty} (a)_n \frac{B_p(b+n, c-b)}{B(b, c-b)} \frac{z^n}{n!}$$

$$\big(|z| < 1,\ \Re(p) \geq 0,\ \Re(c) > \Re(b) > 0\big).$$

The integral representation of the extended Hurwitz-Lerch zeta funtion (6) has an important role to find the integral representation for the extended Mathieu–type power series defined in (8).

The Fox-Wright function ${}_p\Psi_q[.]$ with p numerator parameters $\alpha_1, ..., \alpha_p$ and q denominator parameters $\beta_1, ..., \beta_q$ is defined by

$$(7) \qquad {}_p\Psi_q\Big[\begin{matrix} (\alpha_1,A_1),\ldots,(\alpha_p,A_p) \\ (\beta_1,B_1),\ldots,(\beta_q,B_q) \end{matrix}\Big| z\Big] = {}_p\Psi_q\Big[\begin{matrix} (\alpha_p,A_p) \\ (\beta_q,B_q) \end{matrix}\Big| z\Big] = \sum_{k=0}^{\infty} \frac{\prod_{l=1}^{p} \Gamma(\alpha_l + kA_l)}{\prod_{j=1}^{q} \Gamma(\beta_l + kB_l)} \frac{z^k}{k!}.$$

The defining series in (7) converges in the whole complex $z-$plane when

$$\Delta = \sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j > -1;$$

when $\Delta = 0$, then the series in (7) converges for $|z| < \nabla$, where

$$\nabla = \left(\prod_{j=1}^{p} A_j^{-A_j} \right) \left(\prod_{j=1}^{q} B_j^{B_j} \right).$$

If, in the definition (7) we set

$$A_1 = \cdots = A_p = 1 \quad \text{and} \quad B_1 = \cdots = B_q = 1,$$

we get the relatively more familiar generalized hypergeometric function ${}_pF_q[.]$ given by

$${}_pF_q\Big[\begin{matrix} \alpha_1,\ldots,\alpha_p \\ \beta_1,\ldots,\beta_q \end{matrix}\Big| z\Big] = \frac{\prod_{j=1}^{q} \Gamma(\beta_j)}{\prod_{i=1}^{p} \Gamma(\alpha_i)} {}_p\Psi_q\Big[\begin{matrix} (\alpha_1,1),\ldots,(\alpha_p,1) \\ (\beta_1,1),\ldots,(\beta_q,1) \end{matrix}\Big| z\Big].$$

In terms of the extended Beta function (3) we now define the $(p,q)-$Mathieu–type power series by:

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p,q;z)=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\nu)_nB_{p,q}(\tau+n,\omega-\tau)z^n}{n!B(\tau,\omega-\tau)(a_n^{\alpha}+r^2)^{\mu}} \tag{8}$$

$$\big(r,\ \alpha,\ \beta,\ \nu>0,\ \omega>\tau>0,\ p,q\in\mathbb{C},\ \min(\Re(p),\Re(q))\geq 0,\ |z|\leq 1\big).$$

In particular case when $p=q$, we define the $p-$Mathieu–type power series defined by:

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p;z)=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\nu)_nB_{p}(\tau+n,\omega-\tau)z^n}{n!B(\tau,\omega-\tau)(a_n^{\alpha}+r^2)^{\mu}} \tag{9}$$

$$\big(r,\ \alpha,\ \beta,\ \nu,\ \omega,\ \tau>0,\ p\in\mathbb{C}_{>},\ |z|\leq 1\big).$$

The function $S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p,q;z)$ has many other special cases. If we set $p=q=0$, we get

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};z)=S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};0,0;z)=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\nu)_n(\tau)_nz^n}{n!(\omega)_n(a_n^{\alpha}+r^2)^{\mu}} \tag{10}$$

$$\big(r,\ \alpha,\ \beta,\ \nu,\ \tau,\ \omega>0,\ |z|\leq 1\big).$$

On the other hand, by letting $\tau=\omega$ in (10) we obtain [15, Eq. 5, p. 974]:

$$S^{(\alpha,\beta)}_{\mu,\nu}(r;\mathbf{a};z)=S^{(\alpha,\beta)}_{\mu,\nu,\tau,\tau}(r;\mathbf{a};z)=\sum_{n=1}^{\infty}\frac{2a_n^{\beta}(\nu)_nz^n}{n!(a_n^{\alpha}+r^2)^{\mu}} \tag{11}$$

$$\big(r,\ \alpha,\ \beta,\ \nu>0,\ |z|\leq 1\big).$$

Furthermore, in the special cases when $\nu=z=1$, we get the generalized Mathieu series (2).

The contents of our paper is organized as follows. In section 2, we present new integral representation for the $(p,q)-$Mathieu–type series. In particular, we derive the Mellin-Barnes type integral representations for $(p,q)-$Mathieu–type series. As applications, In Section 3, we introduce the $(p,q)-$Mittag-Leffler functions and we derive some relationships between these two special functions, in particular we derive new series representations for the $(p,q)-$Mathieu–type series. Relationships between the $(p,q)-$ and generalized Mathieu–type series are proved and two Turán type inequalities are established.

2. INTEGRAL REPRESENTATION FOR THE $(p,q)-$MATHIEU TYPE POWER SERIES

In the course of our investigation, one of the main tools is the following result providing the integral representation for the $(p,q)-$Mathieu–type series $S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\{k^{\gamma}\}_{k=0}^{\infty};p,q;z)$.

Theorem 1. *Let* $r,\alpha,\beta,\nu,\mu,\omega,\tau>0,\ p,q\in\mathbb{C}_>$, *such that* $\tau<\omega$ *and* $\gamma(\mu\alpha-\beta)>0$. *Then* $(p,q)-$*Mathieu–type series* $S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\{k^\gamma\}_{k=0}^\infty;p,q;z)$ *possesses the integral representation given by:*

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\{k^\gamma\}_{k=0}^\infty;p,q;z)=\frac{2\nu\tau z}{\omega\Gamma(\mu)}\int_0^\infty t^{\gamma(\mu\alpha-\beta)}e^{-t}K^{(\alpha,\beta)}_{p,q}(\mu,\gamma,\tau,\omega;t)dt, \tag{12}$$

where $K^{(\alpha,\beta)}_{p,q}(\mu,\gamma,\tau,\omega;t)$ *is defined by*

$$K^{(\alpha,\beta)}_{p,q}(\mu,\gamma,\tau,\omega;t)={}_2F_1\left[\begin{matrix}\nu+1,\tau+1\\ \omega+1\end{matrix};ze^{-t};p,q\right]{}_1\Psi_1\left[\begin{matrix}(\mu,1)\\ (\gamma(\mu\alpha-\beta)+1,\gamma\alpha)\end{matrix}\middle|-r^2t^{\gamma\alpha}\right].$$

Proof. By using the definition (8), we can write the extended Mathieu–type series $S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p,q;z)$ in the following form:

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p,q;z)=2\sum_{m=0}^\infty\binom{\mu+m-1}{m}(-r^2)^m\sum_{n=1}^\infty\frac{(\nu)_n}{a_n^{(\mu+m)\alpha-\beta}}\frac{B_{p,q}(\tau+n,\omega-\tau)}{B(\tau,\omega-\tau)}\frac{z^n}{n!}. \tag{13}$$

Putting,

$$\chi^{(\alpha,\beta)}_{n,m}(\mu,\gamma)=(n+1)^{\gamma((\mu+m)\alpha-\beta)+1},$$

$$\Phi^{(\alpha,\beta)}_{m,\mu,\nu}(\tau,\omega;p,q;z)=\Phi_{\nu+1,\tau+1,\omega+1}(z,\gamma[(\mu+m)\alpha-\beta]+1,1;p,q),$$

by (6), we get

$$\begin{aligned}
&S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}\Big(r;\{k^\gamma\}_{k=0}^\infty;p,q;z\Big)=\\
&=2z\sum_{m=0}^\infty\binom{\mu+m-1}{m}(-r^2)^m\sum_{n=0}^\infty\frac{(\nu)_{n+1}}{(n+1)!}\frac{B_{p,q}(\tau+1+n,\omega-\tau)}{B(\tau,\omega-\tau)}\frac{z^n}{\chi^{(\alpha,\beta)}_{n,m}(\mu,\gamma)}\\
&=2\nu z\sum_{m=0}^\infty\binom{\mu+m-1}{m}(-r^2)^m\sum_{n=0}^\infty\frac{(\nu+1)_n}{n!}\frac{B_{p,q}(\tau+1+n,\omega-\tau)}{B(\tau,\omega-\tau)}\frac{z^n}{\chi^{(\alpha,\beta)}_{n,m}(\mu,\gamma)}\\
&=\frac{2\nu zB(\tau+1,\omega-\tau)}{B(\tau,\omega-\tau)}\sum_{m=0}^\infty\binom{\mu+m-1}{m}(-r^2)^m\sum_{n=0}^\infty\frac{(\nu+1)_nB_{p,q}(\tau+1+n,\omega-\tau)z^n}{n!B(\tau+1,\omega-\tau)\chi^{(\alpha,\beta)}_{n,m}(\mu,\gamma)}\\
&=\frac{2\nu zB(\tau+1,\omega-\tau)}{B(\tau,\omega-\tau)}\sum_{m=0}^\infty\binom{\mu+m-1}{m}\Phi^{(\alpha,\beta)}_{m,\mu,\nu}(\tau,\omega;p,q;z)(-r^2)^m\\
&=\frac{2\nu\tau z}{\omega}\int_0^\infty t^{\gamma(\mu\alpha-\beta)}e^{-t}{}_2F_1\left[\begin{matrix}\nu+1,\tau+1\\ \omega+1\end{matrix};ze^{-t};p,q\right]\left(\sum_{m=0}^\infty\frac{\binom{\mu+m-1}{m}(-r^2t^{\gamma\alpha})^m}{\Gamma(\gamma[(\mu+m)\alpha-\beta]+1)}\right)dt\\
&=\frac{2\nu\tau z}{\omega\Gamma(\mu)}\int_0^\infty t^{\gamma(\mu\alpha-\beta)}e^{-t}{}_2F_1\left[\begin{matrix}\nu+1,\tau+1\\ \omega+1\end{matrix};ze^{-t};p,q\right]{}_1\Psi_1\left[\begin{matrix}(\mu,1)\\ (\gamma(\mu\alpha-\beta)+1,\gamma\alpha)\end{matrix}\middle|-r^2t^{\gamma\alpha}\right]dt.
\end{aligned}$$

This completes the proof of Theorem 1. □

Now, in the case $p = q$, theorem 1 reduces to the following corollary.

Corollary 1. *Let $r, \alpha, \beta, \nu, \mu, \tau, \omega > 0$, $p \in \mathbb{C}_>$ such that $\tau < \omega$ and $\gamma(\mu\alpha - \beta) > 0$. Then p–Mathieu–type series $S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r; \{k^\gamma\}_{k=0}^\infty; p; z)$ possesses the integral representation given by:*

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r; \{k^\gamma\}_{k=0}^\infty; p; z) = \frac{2\nu\tau z}{\omega\Gamma(\mu)} \int_0^\infty t^{\gamma(\mu\alpha-\beta)} e^{-t} K_p^{(\alpha,\beta)}(\mu, \gamma, \tau, \omega; t)dt, \tag{14}$$

where $K_p^{(\alpha,\beta)}(\mu, \gamma, \tau, \omega; t)$ is defined by

$$K_p^{(\alpha,\beta)}(\mu, \gamma, \tau, \omega; t) = {}_2F_1\left[\begin{matrix}\nu+1,\tau+1\\ \omega+1\end{matrix}; ze^{-t}; p\right] {}_1\Psi_1\left[\begin{matrix}(\mu,1)\\ (\gamma(\mu\alpha-\beta)+1,\gamma\alpha)\end{matrix}\,\middle|\, -r^2t^{\gamma\alpha}\right].$$

Remark 1. *1. By letting $p = q = 0$ in (14) we deduce that the function $S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r; \{k^\gamma\}_{k=0}^\infty; z)$ possesses the following integral representation:*

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r; \{k^\gamma\}_{k=0}^\infty; z) = \frac{2\nu\tau z}{\omega\Gamma(\mu)} \int_0^\infty t^{\gamma(\mu\alpha-\beta)} e^{-t} K^{(\alpha,\beta)}(\mu, \gamma, \tau, \omega; t)dt, \tag{15}$$

where $K^{(\alpha,\beta)}(\mu, \gamma, \tau, \omega; t)$ is defined by

$$K^{(\alpha,\beta)}(\mu, \gamma, \tau, \omega; t) = {}_2F_1\left[\begin{matrix}\nu+1,\tau+1\\ \omega+1\end{matrix}; ze^{-t}\right] {}_1\Psi_1\left[\begin{matrix}(\mu,1)\\ (\gamma(\mu\alpha-\beta)+1,\gamma\alpha)\end{matrix}\,\middle|\, -r^2t^{\gamma\alpha}\right].$$

2. Setting $\tau = \omega$ in (15) and using the fact that

$${}_2F_1\left[\begin{matrix}a,\, b\\ b\end{matrix}; z\right] = (1-z)^{-a},$$

we obtain the following integral representation for the function $S^{(\alpha,\beta)}_{\mu,\nu}(r; \{k^\gamma\}_{k=0}^\infty; z)$ [15, Theorem 1, (Eq. 8), p. 975]

(16)

$$S^{(\alpha,\beta)}_{\mu,\nu}(r; \{k^\gamma\}_{k=0}^\infty; z) = \frac{2\nu z}{\Gamma(\mu)} \int_0^\infty \frac{t^{\gamma(\mu\alpha-\beta)}e^{-t}}{(1-ze^{-t})^{\nu+1}} {}_1\Psi_1\left[\begin{matrix}(\mu,1)\\ (\gamma(\mu\alpha-\beta)+1,\gamma\alpha)\end{matrix}\,\middle|\, -r^2t^{\gamma\alpha}\right]dt.$$

In the next Theorem we present the Mellin-Barnes integral representation for the alternating Mathieu–type series $S^{(2,1)}_{\mu,\nu,\tau,\omega}\Big(r; \{k\}_{k=0}^\infty; p, q; -z\Big)$.

Theorem 2. *Let $r, \nu, \mu, \tau, \omega > 0$, $\Re(p), \Re(q) \geq 0$. Then the following integral representation*

$$S^{(2,1)}_{\mu,\nu,\tau,\omega}\Big(r; \{k\}_{k=0}^\infty; p, q; -z\Big) =$$

(17)

$$= \frac{-z}{i\pi\Gamma(\nu)} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\nu-s+1)\mathbb{B}_{p,q}(\tau,\omega;s)\left[\Gamma(-s+ir+1)\Gamma(-s-ir+1)\right]^\mu}{z^s\left[\Gamma(-s+ir+2)\Gamma(-s-ir+2)\right]^\mu}ds,$$

holds true for all $|\arg(-z)| < \pi$, where $\mathbb{B}_{p,q}$ is defined by

$$\mathbb{B}_{p,q}(\tau,\omega;s) = \frac{B_{p,q}(\tau-s+1,\omega-\tau)}{B(\tau,\omega-\tau)}.$$

Proof. The contour of integration extends from $c-i\infty$ to $c+i\infty$, such that all the poles of the Gamma function $\Gamma(\nu-s+1)$ at the points $s=k+\nu+1,\ k\in\mathbb{N}$ are separated from the poles of the Gamma function $\Gamma(s)$ at the points $s=-k,\ k\in\mathbb{N}$. Suppose that the poles of the integrand are simple and using the fact that

$$\operatorname{res}[\Gamma,-k]=\lim_{s\longrightarrow -k}(s+k)\Gamma(s)=\frac{(-1)^k}{k!},$$

we find that

$$\frac{z}{i\pi\Gamma(\nu)}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(\nu-s+1)\mathbb{B}_{p,q}(\tau,\omega;s)\left[\Gamma(-s+ir+1)\Gamma(-s-ir+1)\right]^{\mu}}{\left[\Gamma(-s+ir+2)\Gamma(-s-ir+2)\right]^{\mu}}z^{-s}ds$$

$$\begin{aligned}
&=\frac{2z}{\Gamma(\nu)}\sum_{k=0}^{\infty}\lim_{s\longrightarrow -k}\frac{(s+k)\Gamma(s)\mathbb{B}_{p,q}(\tau,\omega;s)\Gamma(\nu-s+1)\left[\Gamma(-s+ir+1)\right]^{\mu}}{\left[\Gamma(-s-ir+1)\right]^{-\mu}\left[\Gamma(-s+ir+2)\Gamma(-s-ir+2)\right]^{\mu}}\\
&=\frac{2z}{\Gamma(\nu)}\sum_{k=0}^{\infty}\frac{(-1)^k}{k!}\frac{B_{p,q}(\tau+k+1,\omega-\tau)\Gamma(\nu+k+1)}{B(\tau,\omega-\tau)((k+1)^2+r^2)^{\mu}}z^k\\
&=-2\sum_{k=1}^{\infty}\frac{(\nu)_k kB_{p,q}(\tau+k,\omega-\tau)}{B(\tau,\omega-\tau)(k^2+r^2)^{\mu}}\frac{(-z)^k}{k!}\\
&=-S_{\mu,\nu,\tau,\omega}^{(2,1)}\Big(r;\{k\}_{k=0}^{\infty};p,q;-z\Big).
\end{aligned}$$

This completes the proof of Theorem 2 ☐

Corollary 2. *Let $r,\nu,\mu,\tau,\omega>0,\ \Re(p)\geq 0$. Then the following integral representation*

$$S_{\mu,\nu,\tau,\omega}^{(2,1)}\Big(r;\{k\}_{k=0}^{\infty};p;-z\Big)=$$

$$=-\frac{z}{i\pi\Gamma(\nu)}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(\nu-s+1)\mathbb{B}_{p,p}(\tau,\omega;s)\left[\Gamma(-s+ir+1)\Gamma(-s-ir+1)\right]^{\mu}}{z^s\left[\Gamma(-s+ir+2)\Gamma(-s-ir+2)\right]^{\mu}}ds, \tag{18}$$

holds true for all $|\arg(-z)|<\pi$, where $\mathbb{B}_{p,p}(\tau,\omega;s)$ is defined by

$$\mathbb{B}_{p,p}(\tau,\omega;s)=\frac{B_p(\tau-s+1,\omega-\tau)}{B(\tau,\omega-\tau)}.$$

Remark 2. *If we set $p=0$ in Corollary 2, then we get the Mellin-Barnes representation of the function $S_{\mu,\nu,\tau,\omega}^{(2,1)}\Big(r;\{k\}_{k=0}^{\infty};-z\Big)$:*

$$S_{\mu,\nu,\tau,\omega}^{(2,1)}\Big(r;\{k\}_{k=0}^{\infty};-z\Big)=$$

$$=-\frac{z\Gamma(\omega)}{i\pi\Gamma(\nu)\Gamma(\tau)}\int_{c-i\infty}^{c+i\infty}\frac{\kappa(\nu,\tau,\omega;s)\left[\Gamma(-s+ir+1)\Gamma(-s-ir+1)\right]^{\mu}}{\left[\Gamma(-s+ir+2)\Gamma(-s-ir+2)\right]^{\mu}}z^{-s}ds, \tag{19}$$

where,

$$\kappa(\nu,\tau,\omega;s)=\frac{\Gamma(s)\Gamma(\nu-s+1)\Gamma(\tau-s+1)}{\Gamma(\omega-s+1)}.$$

In particular, for $\tau=\omega$ *we get*

$$S_{\mu,\nu}^{(2,1)}\Big(r;\{k\}_{k=0}^{\infty};-z\Big)=$$

$$=-\frac{z}{i\pi\Gamma(\nu)}\int_{c-i\infty}^{c+i\infty}\frac{\Gamma(s)\Gamma(\nu-s+1)\left[\Gamma(-s+ir+1)\Gamma(-s-ir+1)\right]^{\mu}}{\left[\Gamma(-s+ir+2)\Gamma(-s-ir+2)\right]^{\mu}}z^{-s}ds.$$

Moreover, if we set $\mu=2$ *and* $\nu=1$ *in the above equation we get the Mellin-Barnes for the alternating Mathieu–type series proved by Saxena el al.* [10, Theorem 3.1].

3. APPLICATIONS

In our first application in this section we present the relationships between the $(p,q)-$Mathieu–type series $S_{2,\nu,\tau,\omega}^{(2,1)}\Big(r;\{k\}_{k=0}^{\infty};p,q;z\Big)$ and the Rieman-Liouvile operator.

3.1 Relationships with $(p,q)-$Mathieu-type series and the Rieman-Liouvile operator

Our first main application is asserted by the following theorem.

Theorem 3. *Let* $r,\mu,\tau,\omega>0$, $\Re(p),\Re(q)\geq 0$ *and* $0\leq\nu<1$. *Then*

$$\begin{aligned}S_{2,\nu,\tau,\omega}^{(2,1)}\Big(r;\{k\}_{k=0}^{\infty};p,q;z\Big) &= \frac{1}{2ir\Gamma(\nu)}\Big[D_z^{\nu-1}\left(z^{\nu-1}\Phi_{\tau,\omega}(z,2,-ir;p,q)\right)\\ &- D_z^{\nu-1}\left(z^{\nu-1}\Phi_{\tau,\omega}(z,2,ir;p,q)\right)\Big].\end{aligned}$$

Proof. By using the definition of the $(p,q)-$Mathieu-type series, we can write the Mathieu-type series
$S_{2,\nu,\tau,\omega}^{(2,1)}\Big(r;\{k\}_{k=0}^{\infty};p,q;z\Big)$ in the following form:

$$S_{2,\nu,\tau,\omega}^{(2,1)}\Big(r;\{k\}_{k=0}^{\infty};p,q;z\Big)=\frac{1}{2ir}\left[\Phi_{\nu,\tau,\omega}(z,2,-ir;p,q)-\Phi_{\nu,\tau,\omega}(z,2,ir;p,q)\right].$$

Combining the above equation with (5) we get the desired result. □

3.2 Relationships with $(p,q)-$Mittag-Leffler function and $(p,q)-$Mathieu–type series. In this section, we introduce the definition of the $(p,q)-$Mittag-Leffler function, we establish an integral representation for this function and we present some relationships with the $(p,q)-$Mathieu–type series. For $\lambda,\tau,\omega,\theta,\sigma,\delta>0$, $\omega>\tau>0$, and $\min(\Re(p),\Re(q))\geq 0$, we define the $(p,q)-$Mittag-Leffler function by

$$E_{\delta,\theta,\sigma;p,q}^{(\lambda,\tau,\omega)}(z)=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{\left[\Gamma(\theta k+\sigma)\right]^{\delta}}\frac{B_{p,q}(\tau+k,\omega-\tau)}{B(\tau,\omega-\tau)}\frac{z^k}{k!}\quad(z\in\mathbb{C}). \tag{20}$$

In the case $p = q$ we define the $p-$Mittag-Leffler function by

$$E^{(\lambda,\tau,\omega)}_{\delta,\theta,\sigma;p}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{[\Gamma(\theta k+\sigma)]^{\delta}} \frac{B_p(\tau+k,\omega-\tau)}{B(\tau,\omega-\tau)} \frac{z^k}{k!} \quad (z \in \mathbb{C}). \tag{21}$$

Namely, when $p = 0$, from (21) we observe

$$E^{(\lambda,\tau,\omega)}_{\delta,\theta,\sigma}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k(\tau)_k}{[\Gamma(\theta k+\sigma)]^{\delta}(\omega)_k} \frac{z^k}{k!} \quad (z \in \mathbb{C}),$$

and then, also with the condition $\omega = \tau$ we can immediately get [15]

$$E^{(\lambda)}_{\delta,\theta,\sigma}(z) = \sum_{k=0}^{\infty} \frac{(\lambda)_k}{[\Gamma(\theta k+\sigma)]^{\delta}} \frac{z^k}{k!} \quad (z \in \mathbb{C}). \tag{22}$$

For $\lambda = 1$ the above series was introduced by S. Gerhold [4].

Lemma 1. *Let* $\tau, \omega, \theta, \sigma, \delta > 0$, *such that* $\tau < \omega$. *Assume that* $\min(\Re(p), \Re(q)) \geq 0$. *Then the following representation*

$$E^{(1,\tau+1,\omega+1)}_{\delta,\theta,\sigma+\theta;p,q}(z) = \frac{\omega}{z\tau}\left[E^{(1,\tau,\omega)}_{\delta,\theta,\sigma;p,q}(z) - \frac{B_{p,q}(\tau,\omega-\tau)}{[\Gamma(\sigma)]^{\delta}B(\tau,\omega-\tau)}\right], \tag{23}$$

holds true.

Proof. By computation, we get

$$\begin{aligned} E^{(1,\tau+1,\omega+1)}_{\delta,\theta,\sigma+\theta;p,q}(z) &= \sum_{k=0}^{\infty} \frac{B_{p,q}(\tau+k+1,\omega-\tau)z^k}{B(\tau+1,\omega-\tau)\,[\Gamma(\theta k+\sigma+\theta)]^{\delta}} \\ &= \frac{B(\tau,\omega-\tau)}{zB(\tau+1,\omega-\tau)} \sum_{k=1}^{\infty} \frac{B_{p,q}(\tau+k,\omega-\tau)z^k}{B(\tau,\omega-\tau)\,[\Gamma(\theta k+\sigma)]^{\delta}} \\ &= \frac{\omega}{z\tau}\left[E^{(1,\tau,\omega)}_{\delta,\theta,\sigma;p,q}(z) - \frac{B_{p,q}(\tau,\omega-\tau)}{[\Gamma(\sigma)]^{\delta}B(\tau,\omega-\tau)}\right]. \end{aligned}$$

The proof of Lemma 1 is completed. □

Theorem 4. *Let* $\lambda, \tau, \omega, \theta, \sigma > 0, \delta \in \mathbb{N}, \omega > \tau > 0,$ *and* $\min(\Re(p), \Re(q)) \geq 0.$ *Then the* $(p, q)-$*Mathieu–type series admits the following series representation:*

$$\begin{aligned} &S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}\Big(r; \{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty}; p, q; z\Big) \\ &= 2\sum_{m=0}^{\infty} \binom{\mu+m-1}{m} (-r^2)^m \left[E^{(\nu,\tau,\omega)}_{\gamma[(\mu+m)\alpha-\beta],\theta,\sigma;p,q}(z) - \frac{\mathbb{B}_{p,q}(\tau,\omega,1)}{[\Gamma(\sigma)]^{\gamma[(\mu+m)\alpha-\beta]}}\right]. \end{aligned} \tag{24}$$

Moreover, the following series representation

$$S^{(\alpha,\beta)}_{\mu,1,\tau,\omega}\Big(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty};p,q;z\Big)=$$

$$=\frac{2z\tau}{\omega}\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2)^m E^{(1,\tau+1,\omega+1)}_{\gamma[(\mu+m)\alpha-\beta],\theta,\sigma+\theta;p,q}(z), \tag{25}$$

holds true.

Proof. In view of the definition of the $(p,q)-$Mittag-Leffler function (20) and the equation (13) we obtain (24). Finally, combining the equation (24) with the relation (23) derived in Lemma 1, we obtain the formula (25). ☐

Taking in (24) the values $\theta=\sigma=1$ we obtain the following representation:

Corollary 3. *Let* $\lambda,\tau,\omega,\theta,\sigma>0,\delta\in\mathbb{N}$ *such that* $\tau<\omega,$. *In addition, assume that* $\min(\Re(p),\Re(q))\geq 0$. *Then the* $(p,q)-$*Mathieu-type series admits the following series representations:*

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}\Big(r;\{(k!)^{\gamma}\}_{k=0}^{\infty};p,q;z\Big)=$$

$$=2\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2)^m\left[E^{(\nu,\tau,\omega)}_{\gamma[(\mu+m)\alpha-\beta],1,1;p,q}(z)-\mathbb{B}_{p,q}(\tau,\omega,1)\right], \tag{26}$$

and

$$S^{(\alpha,\beta)}_{\mu,1,\tau,\omega}\Big(r;\{(k!)^{\gamma}\}_{k=0}^{\infty};p,q;z\Big)=\frac{2z\tau}{\omega}\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2)^m E^{(1,\tau+1,\omega+1)}_{\gamma[(\mu+m)\alpha-\beta],1,2;p,q}(z). \tag{27}$$

Lemma 2. *Let* $p,q>0$ *and* $0<x,y<1/2$. *Then*

$$B_{p,q}(x,y)\leq(2p)^{\frac{2x-1}{2}}(2q)^{\frac{2y-1}{2}}\sqrt{\Gamma(-2x+1,2p)\Gamma(-2y+1,2q)}, \tag{28}$$

where $\Gamma(a,x)$ *is the incomplete Gamma function defined by*

$$\Gamma(a,x)=\int_x^{\infty}t^{a-1}e^{-t}dt,\ \ a>0,\ x\geq 0.$$

Proof. By using the Cauchy-Schwartz inequality we have

$$\begin{aligned}
B_{p,q}(x,y) &\leq \left[\int_0^1(t^{x-1}e^{\frac{-p}{t}})^2dt\right]^{\frac{1}{2}}\left[\int_0^1((1-t)^{y-1}e^{\frac{-q}{1-t}})^2dt\right]^{\frac{1}{2}}\\
&= \left[\int_1^{\infty}t^{-2x}e^{-2pt}dt\right]^{\frac{1}{2}}\left[\int_1^{\infty}t^{-2y}e^{-2qt}dt\right]^{\frac{1}{2}}\\
&= (2p)^{\frac{2x-1}{2}}(2q)^{\frac{2y-1}{2}}\left[\int_{2p}^{\infty}t^{-2x}e^{-t}dt\right]^{\frac{1}{2}}\left[\int_{2q}^{\infty}t^{-2y}e^{-t}dt\right]^{\frac{1}{2}}\\
&= (2p)^{\frac{2x-1}{2}}(2q)^{\frac{2y-1}{2}}\sqrt{\Gamma(-2x+1,2p)\Gamma(-2y+1,2q)},
\end{aligned}$$

which completes the proof of Lemma 2. ☐

Lemma 3. *For* $\lambda, \tau, \omega, \theta, \sigma > 0, \omega > \tau > 0, \delta \in \mathbb{N}$ *and* $\min(\Re(p), \Re(q)) \geq 0$. *Then the* $(p,q)-$*Mittag-Leffler function* $E^{(\lambda,\tau,\omega)}_{\delta,\theta,\sigma;p,q}(z)$ *possesses the following integral representation:*

$$E^{(\lambda,\tau,\omega)}_{\delta,\theta,\sigma;p,q}(z) = \frac{1}{B(\tau, \omega-\tau)} \int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1} E_{p,q}(t) E^{(\lambda)}_{\delta,\theta,\sigma}(zt)dt, \tag{29}$$

holds true.

Proof. By using the definition of the $(p,q)-$Beta function we get

$$\begin{aligned}
&\int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1} E_{p,q}(t) E^{(\lambda)}_{\delta,\theta,\sigma}(zt)dt = \\
&= \int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1} E_{p,q}(t) \left(\sum_{k=0}^{\infty} \frac{(\lambda)_k (zt)^k}{[\Gamma(\theta k+\sigma)]^{\delta}\, k!} \right) dt \\
&= \sum_{k=0}^{\infty} \frac{(\lambda)_k z^k}{[\Gamma(\theta k+\sigma)]^{\delta}\, k!} \int_0^1 t^{\tau+k-1}(1-t)^{\omega-\tau-1} E_{p,q}(t)dt \\
&= B(\tau, \omega-\tau) \sum_{k=0}^{\infty} \frac{(\lambda)_k}{[\Gamma(\theta k+\sigma)]^{\delta}} \frac{B_{p,q}(\tau+k, \omega-\tau)}{B(\tau,\omega-\tau)} \frac{z^k}{k!} \\
&= B(\tau, \omega-\tau) E^{(\lambda,\tau,\omega)}_{\delta,\theta,\sigma;p,q}(z).
\end{aligned}$$

The proof of Lemma 3 is completed.

Theorem 5. *For* $\lambda, \tau, \omega, \theta, \sigma > 0, \omega > \tau > 0,$ *and* $\min(\Re(p), \Re(q)) \geq 0$. *Then the following integral representation*

$$\begin{aligned}
&S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}\Big(r; \{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty}; p, q; z\Big) = \qquad (30)\\
&= \frac{1}{B(\tau,\omega-\tau)} \int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1} E_{p,q}(t) S^{(\alpha,\beta)}_{\mu,\nu}\Big(r; \{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty}; zt\Big)dt \\
&- \frac{2B_{p,q}(\tau,\omega-\tau)}{B(\tau,\omega-\tau)} . \frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2+[\Gamma(\sigma)]^{\gamma\alpha})^{\mu}}
\end{aligned}$$

holds true for all $|z| < 1$. *Moreover, the following integral representation*

$$\begin{aligned}
&S^{(\alpha,\beta)}_{\mu,1,\tau,\omega}\Big(r; \{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty}; p, q; z\Big) = \qquad (31)\\
&= \frac{z\tau}{\omega B(\tau+1,\omega-\tau)} \int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1} E_{p,q}(t) S^{(\alpha,\beta)}_{\mu,1}\Big(r; \{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty}; zt\Big)dt,
\end{aligned}$$

holds true for all $|z| < 1$.

□

Proof. By means of Lemma 3 and the integral representation (24) we get

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}\Big(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty};p,q;z\Big)=2\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2)^m$$

$$\times\left[\frac{1}{B(\tau,\omega-\tau)}\int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1}E_{p,q}(t)E^{(\nu)}_{\gamma[(\mu+m)\alpha-\beta],\theta,\sigma}(zt)dt\right]$$

$$-\frac{2B_{p,q}(\tau,\omega-\tau)}{[\Gamma(\sigma)]^{\gamma(\mu\alpha-\beta)}B(\tau,\omega-\tau)}\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}\left(\frac{-\tau^2}{[\Gamma(\sigma)]^{\gamma\alpha}}\right)^m$$

$$=\frac{2}{B(\tau,\omega-\tau)}$$

$$\times\int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1}E_{p,q}(t)\left(\sum_{m=0}^{\infty}\binom{\mu+m-1}{m}(-r^2)^m E^{(\nu)}_{\gamma[(\mu+m)\alpha-\beta],\theta,\sigma}(zt)\right)dt$$

$$-\frac{2B_{p,q}(\tau,\omega-\tau)}{[\Gamma(\sigma)]^{\gamma(\mu\alpha-\beta)}B(\tau,\omega-\tau)}.\frac{1}{(1+\frac{r^2}{[\Gamma(\sigma)]^{\gamma\alpha}})^{\mu}}$$

$$=\frac{2}{B(\tau,\omega-\tau)}\int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1}E_{p,q}(t)$$

$$\times\left[\sum_{k=0}^{\infty}\frac{(\nu)_k}{[\Gamma(\theta k+\sigma)]^{\gamma(\mu\alpha-\beta)}}\left(\sum_{m=0}^{\infty}\frac{\binom{\mu+m-1}{m}(-r^2)^m}{[\Gamma(\theta k+\sigma)]^{\gamma\alpha m}}\right)\frac{(zt)^k}{k!}\right]dt$$

$$-\frac{2B_{p,q}(\tau,\omega-\tau)}{B(\tau,\omega-\tau)}.\frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2+[\Gamma(\sigma)]^{\gamma\alpha})^{\mu}}$$

$$=\frac{2}{B(\tau,\omega-\tau)}\int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1}E_{p,q}(t)\sum_{k=0}^{\infty}\frac{(\nu)_k}{[\Gamma(\theta k+\sigma)]^{\gamma(\mu\alpha-\beta)}}$$

$$\times\left(1+\frac{r^2}{[\Gamma(\theta k+\sigma)]^{\gamma\alpha}}\right)^{-\mu}\frac{(zt)^k}{k!}dt-\frac{2B_{p,q}(\tau,\omega-\tau)}{B(\tau,\omega-\tau)}.\frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2+[\Gamma(\sigma)]^{\gamma\alpha})^{\mu}}$$

$$=\frac{2}{B(\tau,\omega-\tau)}\int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1}E_{p,q}(t)\sum_{k=0}^{\infty}\frac{(\nu)_k[\Gamma(\theta k+\sigma)]^{\gamma\beta}}{([\Gamma(\theta k+\sigma)]^{\gamma\alpha}+r^2)^{\mu}}\frac{(zt)^k}{k!}dt$$

$$-\frac{2B_{p,q}(\tau,\omega-\tau)}{B(\tau,\omega-\tau)}.\frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2+[\Gamma(\sigma)]^{\gamma\alpha})^{\mu}}$$

$$=\frac{1}{B(\tau,\omega-\tau)}\int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1}E_{p,q}(t)S^{(\alpha,\beta)}_{\mu,\nu}\Big(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty};zt\Big)dt$$

$$-\frac{2B_{p,q}(\tau,\omega-\tau)}{B(\tau,\omega-\tau)}.\frac{[\Gamma(\sigma)]^{\gamma\beta}}{(r^2+[\Gamma(\sigma)]^{\gamma\alpha})^{\mu}},$$

which evidently completes the proof of the representation (30). Finally, combining (25) and (29) and repeating the same calculations as above we get (31). The proof of Theorem 5 is completed. □

Corollary 4. *Let* $\mu,\omega,\theta,\sigma,\alpha>0,\ \min(\Re(p),\Re(q))\geq 0$. *If* $0<\tau<\frac{1}{2}$ *and* $\tau<\omega<\tau+\frac{1}{2}$. *Then, the following inequality*

$$\left|S^{(\alpha,\beta)}_{\mu,1,\tau,\omega}\left(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty};p,q;z\right)\right|\leq\frac{\tau}{\omega}(2p)^{\frac{2\tau-1}{2}}(2q)^{\frac{2(\omega-\tau)-1}{2}}$$

$$\times\sqrt{\Gamma(1-2\tau,2p)\Gamma(-2(\omega-\tau)+1,2q)}\frac{S^{(\alpha,\beta)}_{\mu,1}\left(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty}\right)}{\Gamma(\tau+1,\omega-\tau)},\quad |z|\leq 1,$$

holds true, where $S^{(\alpha,\beta)}_{\mu,1}\left(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty};1\right)=S^{(\alpha,\beta)}_{\mu,1}\left(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty}\right).$

Proof. By using the integral representation (31) we get

$$\begin{aligned}\left|S^{(\alpha,\beta)}_{\mu,1,\tau,\omega}\left(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty};p,q;z\right)\right|&\leq\frac{2\tau S^{(\alpha,\beta)}_{\mu,1}\left(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty}\right)}{\omega\Gamma(\tau+1,\omega-\tau)}\\&\quad\times\int_0^1 t^{\tau-1}(1-t)^{\omega-\tau-1}E_{p,q}(t)dt\\&=\frac{2\tau B_{p,q}(\tau,\omega-\tau)}{\omega\Gamma(\tau+1,\omega-\tau)}S^{(\alpha,\beta)}_{\mu,1}\left(r;\{[\Gamma(\theta k+\sigma)]^{\gamma}\}_{k=0}^{\infty}\right).\end{aligned}$$

Applying Lemma 2 to the above inequality, we obtain the desired result. □

3.3 Turán type inequalities for the $(p,q)-$Mathieu-type series

Theorem 6. *Let* $r,\alpha,\beta,\nu,\mu>0,\omega>\tau>0,\ \min(\Re(p),\Re(q))\geq 0$. *Then the following assertions are true:*

1. The $(p,q)-$*Mathieu-type series considered as a function in* p $($*or* $q)$ *is completely monotonic and log-convex on* $(0,\infty)$. *Furthermore, the following Turán type inequality*

$$\left[S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p+1,q;z)\right]^2\leq S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p,q;z)S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p+2,q;z),\tag{32}$$

holds true for all $z\in(0,1)$.

2. Assume that $r^2+\mathbf{a}\geq 1$. *Then the* $(p,q)-$*Mathieu–type series considered as a function in* μ *is completely monotonic and log-convex on* $(0,\infty)$. *Furthermore, the following Turán type inequality*

$$\left[S^{(\alpha,\beta)}_{\mu+1,\nu,\tau,\omega}(r;\mathbf{a};p,q;z)\right]^2\leq S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p,q;z)S^{(\alpha,\beta)}_{\mu+2,\nu,\tau,\omega}(r;\mathbf{a};p,q;z),\tag{33}$$

holds true for all $z\in(0,1)$ *such that* $r^2+\mathbf{a}\geq 1$.

Proof. 1. In [5, Corollary 2.7], the authors proved that the extended Beta function $p($ or $q)\mapsto B_{p,q}(x,y)$ is completely monotonic function on $(0,\infty)$ and using the fact that sums of completely monotonic functions are completely monotonic too, we deduce that the $p($ or $q)\mapsto S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p,q;z)$ is completely monotonic and log-convex on $(0,\infty)$, since every completely monotonic function is log-convex (see [17, p.167]). Thus, for all $p_1, p_2 > 0$, and $t \in [0,1]$, we obtain

$$S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};tp_1{+}(1{-}t)p_2,q;z) \leq \left[S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p_1,q;z)\right]^t \left[S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p_2,q;z)\right]^{1-t}.$$

Letting $t=\frac{1}{2}$, $p_1 = p$ and $p_2 = p+2$ in the above inequality we get the Turán type inequality (32).

2. We note that the function $\mu \mapsto (r^2+\mathbf{a})^{-\mu}$ is completely monotonic on $(0,\infty)$ such that $r^2+\mathbf{a} \geq 1$, and consequently the function $\mu \mapsto S^{(\alpha,\beta)}_{\mu,\nu,\tau,\omega}(r;\mathbf{a};p,q;z)$ is completely monotonic and log-convex on $(0,\infty)$. ☐

Remark 3. *The condition $r^2+\mathbf{a} \geq 1$ is not necessary to prove the Turán type inequality (33), a similar proof of the Theorem 2 in* [14], we obtain (33).

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(Received 27.04.2019)
(Revised 13.01.2019)

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