

ASYMPTOTIC EXPANSIONS FOR CERTAIN MATHEMATICAL CONSTANTS AND SPECIAL FUNCTIONS

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For fixed real $b > 1$ and $\alpha > 0$, let $S_b^{[\alpha]}(n) = \sum_{k=1}^n b^k k^{-\alpha}$. Abel proved that $S_b^{[\alpha]}(n) \sim b^n \sum_{k=0}^{\infty} c_k n^{-(k+\alpha)}$ ($n \rightarrow \infty$), and gave an explicit formula for determining the coefficients $c_k \equiv c_k(b, \alpha)$ in terms of Stirling numbers of the second kind. We here provide a recurrence relation for determining the coefficients c_k , without Stirling numbers. We also consider asymptotic expansions concerning Somos' quadratic recurrence constant, Glaisher-Kinkelin constant, Choi-Srivastava constants, and the Barnes G -function.

1. INTRODUCTION

Abel [1] derived a complete asymptotic expansion for a sequence of the following sum

$$(1) \quad S_b^{[\alpha]}(n) = \sum_{k=1}^n \frac{b^k}{k^\alpha}$$

as $n \rightarrow \infty$, for fixed real $b > 1$ and $\alpha > 0$. More precisely, Abel [1] established the following asymptotic expansion:

$$(2) \quad S_b^{[\alpha]}(n) \sim b^n \sum_{k=0}^{\infty} \binom{\alpha + k - 1}{k} \frac{a_k(b)}{n^{\alpha+k}} \quad (n \rightarrow \infty),$$

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where $a_k(b)$ are given by

$$(3) \quad a_k(b) = b \sum_{j=0}^k \frac{j! \sigma(k, j)}{(b-1)^{j+1}},$$

and $\sigma(k, j)$ denote Stirling numbers of the second kind. Stirling numbers of the second kind can be computed by the formula (see, e.g., [54, p. 99])

$$\sigma(k, j) = \frac{1}{j!} \sum_{i=1}^j (-1)^{j-i} \binom{j}{i} i^k.$$

Remark 1. *Hassani [34] proposed the following problem: Prove that*

$$\sum_{k=1}^n \frac{2^k - 1}{k} = \frac{2^{n+1}}{n} (1 + R_n), \quad \text{where } R_n \sim \frac{1}{n} \quad (n \rightarrow \infty).$$

This problem has been proved by Giuliano [31] Simic [51]. Abel [1] gave a complete asymptotic expansion for the sequence $\{S_b(n)\}_{n \in \mathbb{N}}$ of sums

$$S_b(n) = \sum_{k=1}^n \frac{b^k - 1}{k} \sim b^n \sum_{k=0}^{\infty} \frac{a_k(b)}{n^{k+1}} \quad (n \rightarrow \infty),$$

as $n \rightarrow \infty$, for fixed real $b > 1$. The choice $b = 2$ yields a solution of the problem [34] by Hassani. Also in [1], Abel derived a complete asymptotic expansion for $S_b^{[\alpha]}(n)$, and pointed out that $S_b(n)$ is asymptotically equivalent to $S_b^{[1]}(n)$ as $n \rightarrow \infty$.

Alzer et al. [3] applied a classical series identity involving the psi function with a view to deriving series representations for a number of known mathematical constants. Chen and Srivastava [18] obtained several properties associated with inequalities and the logarithmically complete monotonicity of functions related to the gamma and psi functions and the Barnes G -function. Chen et al. [21] presented some properties associated with the monotonicity and the complete monotonicity of the psi function and establish the higher-order estimate for the familiar Euler-Mascheroni constant. Chen and Srivastava [19] established new analytical representations for the Euler-Mascheroni constant in terms of the psi function. Chen and Srivastava [20] established several further analytical representations for the Euler-Mascheroni constant in terms of the psi function.

In this paper, we aim to provide a recurrence relation for determining the coefficients of $n^{-(\alpha+k)}$ in Abel's expansion (2), without help of Stirling numbers of the second kind. We also consider asymptotic expansions concerning Somos' quadratic recurrence constant, Glaisher-Kinkelin constant, Choi-Srivastava constants, and the Barnes G -function.

2. COEFFICIENTS OF ABEL'S EXPANSION

Theorem 1. For reals $\alpha > 0$ and $b > 1$, the following asymptotic formula holds

$$(4) \quad S_b^{[\alpha]}(n) \sim b^n \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \quad (n \rightarrow \infty),$$

where the coefficients $c_k \equiv c_k(b, \alpha)$ are given by the recurrence relation

$$(5) \quad c_0 = \frac{b}{b-1}, \quad c_k = \frac{1}{b-1} \sum_{j=0}^{k-1} c_j \binom{k+\alpha-1}{k-j} \quad (k \geq 1).$$

Namely,

$$(6) \quad S_b^{[\alpha]}(n) \sim \frac{b^{n+1}}{n^\alpha} \left\{ \frac{1}{b-1} + \frac{\alpha}{(b-1)^2 n} + \frac{\alpha(1+\alpha)(b+1)}{2(b-1)^3 n^2} + \frac{\alpha(1+\alpha)(2+\alpha)(b^2+4b+1)}{6(b-1)^4 n^3} + \dots \right\} \quad (n \rightarrow \infty).$$

Proof. Denote

$$S_n \equiv S_b^{[\alpha]}(n) \quad \text{and} \quad T_n = b^n \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}}.$$

In view of (2), we can let $S_n \sim T_n$ and

$$(7) \quad \Delta S_n := S_n - S_{n-1} \sim T_n - T_{n-1} =: \Delta T_n$$

as $n \rightarrow \infty$, where c_k are real numbers to be determined.

It is easy to see that

$$(8) \quad \Delta S_n = b^n n^{-\alpha}.$$

Direct computation yields

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{c_k}{(n-1)^{k+\alpha}} &= \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \left(1 - \frac{1}{n}\right)^{-k-\alpha} = \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \sum_{j=0}^{\infty} (-1)^j \binom{-k-\alpha}{j} \frac{1}{n^j} \\ &= \sum_{k=0}^{\infty} \frac{c_k}{n^{k+\alpha}} \sum_{j=0}^{\infty} \binom{k+\alpha+j-1}{j} \frac{1}{n^j} = \sum_{k=0}^{\infty} \sum_{j=0}^k c_j \binom{k+\alpha-1}{k-j} \frac{1}{n^{k+\alpha}}. \end{aligned}$$

We then obtain

$$(9) \quad \begin{aligned} \Delta T_n &= \sum_{k=0}^{\infty} \left\{ b^n c_k - \sum_{j=0}^k b^{n-1} c_j \binom{k+\alpha-1}{k-j} \right\} \frac{1}{n^{k+\alpha}} \\ &= \frac{b^n (1 - \frac{1}{b}) c_0}{n^\alpha} + b^n \sum_{k=1}^{\infty} \left\{ c_k - \sum_{j=0}^k \frac{c_j}{b} \binom{k+\alpha-1}{k-j} \right\} n^{-(k+\alpha)}. \end{aligned}$$

Equating coefficients of the term $n^{-(k+\alpha)}$ ($k = 0, 1, 2, \dots$) on the right-hand sides of (8) and (9) yields $c_0 = \frac{b}{b-1}$ and

$$c_k = \sum_{j=0}^k \frac{c_j}{b} \binom{k+\alpha-1}{k-j} = \sum_{j=0}^{k-1} \frac{c_j}{b} \binom{k+\alpha-1}{k-j} + \frac{c_k}{b} \quad (k \geq 1),$$

which gives the desired formula (5). The proof is complete.

3. SOMOS' QUADRATIC RECURRENCE CONSTANT

Somos' quadratic recurrence constant is defined by

$$(10) \quad \sigma = \sqrt{1\sqrt{2\sqrt{3\cdots}}} = \prod_{n=1}^{\infty} n^{1/2^n} = 1.66168794\dots$$

The constant σ arises in the study of the asymptotic behaviour of the sequence

$$(11) \quad g_0 = 1, \quad g_n = ng_{n-1}^2, \quad n \in \mathbb{N} := \{1, 2, 3, \dots\},$$

with first few terms

$$g_0 = 1, \quad g_1 = 1, \quad g_2 = 2, \quad g_3 = 12, \quad g_4 = 576, \quad g_5 = 1658880, \dots$$

This sequence behaves as follows (see [30, p. 446]):

$$(12) \quad g_n \sim \frac{\sigma^{2^n}}{n} \left(1 + \frac{2}{n} - \frac{1}{n^2} + \frac{4}{n^3} - \frac{21}{n^4} + \frac{138}{n^5} - \frac{1091}{n^6} + \dots \right)^{-1} \quad (n \rightarrow \infty).$$

The constant σ appears in important problems from pure and applied analysis, which is a motivation of a large number of papers (see, e.g., [12, 14, 16, 33, 35, 39, 40, 41, 43, 47, 48, 52]).

Nemes [43] studied the coefficients in the asymptotic expansion (12) and developed some recurrence relations. Chen [12] presented the following asymptotic expansion:

$$(13) \quad g_n \sim \frac{\sigma^{2^n}}{n} \left(1 - \frac{2}{n} + \frac{5}{n^2} - \frac{16}{n^3} + \frac{66}{n^4} - \frac{348}{n^5} + \dots \right),$$

which yields

$$(14) \quad g_n \sim \frac{\sigma^{2^n}}{n} \exp \left\{ -\frac{2}{n} + \frac{3}{n^2} - \frac{26}{3n^3} + \frac{75}{2n^4} - \frac{1082}{5n^5} + \dots \right\}.$$

In this section, we provide a recurrence relation for determining the coefficients of n^{-k} in the expansion (14), asserted in Theorem 2.

Theorem 2. *As $n \rightarrow \infty$, the following asymptotic expansion holds*

$$(15) \quad g_n \sim \frac{\sigma^{2^n}}{n} \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{n^k}\right),$$

where the coefficients p_k are given by the recurrence relation

$$(16) \quad p_1 = -2, \quad p_k = (-1)^k \left\{ \frac{2}{k} + \sum_{j=1}^{k-1} (-1)^j p_j \binom{k-1}{k-j} \right\} \quad (k \geq 2).$$

Proof. In view of (14), we can let

$$g_n \sim \frac{\sigma^{2^n}}{n} \exp\left(\sum_{k=1}^{\infty} \frac{p_k}{n^k}\right) \quad \text{as } n \rightarrow \infty,$$

where p_k ($k \in \mathbb{N}$) are real numbers to be determined. Denote

$$U_n = \sum_{k=1}^n \frac{\ln k}{2^k} + \frac{\ln n}{2^n} - \ln \sigma \quad \text{and} \quad V_n = \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{p_k}{n^k}.$$

Noting that $g_n = \prod_{k=1}^n k^{2^{n-k}}$, we can let $U_n \sim V_n$ and

$$(17) \quad \Delta U_n := U_{n+1} - U_n \sim V_{n+1} - V_n =: \Delta V_n \quad \text{as } n \rightarrow \infty.$$

It is easy to see that

$$(18) \quad \Delta U_n = \frac{1}{2^n} \ln\left(1 + \frac{1}{n}\right) = \frac{1}{2^n} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} n^{-k}.$$

Direct computation yields

$$(19) \quad \begin{aligned} \sum_{k=1}^{\infty} \frac{p_k}{(n+1)^k} &= \sum_{k=1}^{\infty} \frac{p_k}{n^k} \left(1 + \frac{1}{n}\right)^{-k} = \sum_{k=1}^{\infty} \frac{p_k}{n^k} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{1}{n^j} \\ &= \sum_{k=1}^{\infty} \frac{p_k}{n^k} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{1}{n^j} \\ &= \sum_{k=1}^{\infty} \sum_{j=1}^k p_j (-1)^{k-j} \binom{k-1}{k-j} \frac{1}{n^k}. \end{aligned}$$

We then obtain

$$(20) \quad \Delta V_n = \frac{1}{2^n} \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^k \frac{p_j}{2} (-1)^{k-j} \binom{k-1}{k-j} - p_k \right\} \frac{1}{n^k}.$$

Equating coefficients of the term n^{-k} on the right-hand sides of (18) and (20) yields

$$\frac{(-1)^{k-1}}{k} = \sum_{j=1}^k \frac{p_j}{2} (-1)^{k-j} \binom{k-1}{k-j} - p_k \quad (k \geq 1).$$

For $k = 1$ we obtain $p_1 = -2$, and for $k \geq 2$ we have

$$\frac{(-1)^{k-1}}{k} = \sum_{j=1}^{k-1} \frac{p_j}{2} (-1)^{k-j} \binom{k-1}{k-j} - \frac{1}{2} p_k \quad (k \geq 2),$$

which gives the desired formula (16). The proof is complete.

4. BARNES G -FUNCTION

The double gamma function Γ_2 and the multiple gamma functions Γ_n were introduced and investigated by Barnes in a series of papers [4, 5, 6, 7]. Barnes applied these functions in the theory of elliptic functions and theta functions. Nonetheless, except possibly for the citations of Γ_2 in the exercises by Whittaker and Watson [57, p.264] and also by Gradshteyn and Ryzhik [32, p.661, Entry 6.441 (4); p.937, Entry 8.333], these functions were revived only in about the middle of the 1980s in the study of determinants of Laplacians on the n -dimensional unit sphere S^n (see, e.g., [24, 38, 44, 49, 55, 56]). The theory of the double gamma function has indeed found interesting applications in many other recent investigations (see, for details, [53, 54]).

Barnes [4] defined the double gamma function (or Barnes G -function) $\Gamma_2 = 1/G$ satisfying each of the following properties:

- (i) $G(z+1) = \Gamma(z)G(z)$, for all complex z ,
- (ii) $G(1) = 1$,
- (iii) As $n \rightarrow \infty$,

$$\begin{aligned} \ln G(z+n+2) &= \frac{n+1+z}{2} \ln(2\pi) + \left(\frac{n^2}{2} + n + \frac{5}{12} + \frac{z^2}{2} + (n+1)z \right) \ln n \\ (21) \quad &\quad - \frac{3n^2}{4} - n - nz - \ln A + \frac{1}{12} + O(n^{-1}), \end{aligned}$$

where Γ is the gamma function and A is called the Glaisher-Kinkelin constant defined by

$$(22) \quad \ln A = \lim_{n \rightarrow \infty} \left\{ \ln \left(\prod_{k=1}^n k^k \right) - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} \right\},$$

the numerical value of A being 1.28242713...

The Glaisher-Kinkelin constant A can be expressed as follows (see [29])

$$(23) \quad A = \lim_{n \rightarrow \infty} n^{-n^2/2 - n/2 - 1/12} e^{n^2/4} \prod_{k=1}^n k^k,$$

$$(24) \quad \frac{e^{1/12}}{A} = \lim_{n \rightarrow \infty} \frac{G(n+1)}{n^{n^2/2 - 1/12} (2\pi)^{n/2} e^{-3n^2/4}},$$

and (see [23, p. 129, Eq. (3.22)])

$$(25) \quad A = e^{\frac{1}{12} - \zeta'(-1)} = (2\pi)^{1/12} [e^{\gamma\pi^2/6 - \zeta'(2)}]^{1/(2\pi^2)},$$

where $\zeta'(z)$ is the derivative of the Riemann zeta function $\zeta(z)$ (see [27]). The constant A has drawn attention in many works (for example) [13, 17, 23, 26, 27] as well as in [4]. Finch shared a section in his book [30, pp. 135–138] to introduce this Glaisher-Kinkelin constant A . The Glaisher-Kinkelin constant A plays an important role in the study of Barnes G -function (for details, see, e.g., [54, Section 1.4]).

The following integral representation for Barnes G -function was established by Ferreira and López [29, Theorem 1]: For $|\text{Arg}(z)| < \pi$,

$$(26) \quad \ln G(z+1) = \frac{1}{4}z^2 + z \ln \Gamma(z+1) - \left(\frac{1}{2}z^2 + \frac{1}{2}z + \frac{1}{12} \right) \ln z - \ln A \\ + \sum_{k=1}^{N-1} \frac{B_{2k+2}}{2k(2k+1)(2k+2)z^{2k}} + R_N(z) \quad (N = 1, 2, \dots),$$

where B_{2k+2} are the Bernoulli numbers. The remainder $R_N(z)$ is for $\Re(z) > 0$ given by

$$(27) \quad R_N(z) = \int_0^\infty \left(\frac{t}{e^t - 1} - \sum_{k=0}^{2N} \frac{B_k}{k!} t^k \right) \frac{e^{-zt}}{t^3} dt.$$

Estimates for $|R_N(z)|$ are also found by Ferreira and López [29], showing that the expansion is indeed an asymptotic expansion of $\ln G(z+1)$ in sectors of the complex plane cut along the negative real axis. Pedersen [45, Theorem 1.1] proved that for any $N \geq 1$, the function $x \mapsto (-1)^N R_N(x)$ is completely monotonic on $(0, \infty)$. Other asymptotic relations (avoiding the $\ln \Gamma$ term) has been obtained by Ruijsenaars [50] and investigated by Pedersen [46], Koumandos [36] and Koumandos and Pedersen [37]. Some upper and lower bounds for the double gamma function were derived in terms of the gamma, psi and polygamma functions, see [8, 9, 10, 18]. Chen [11] and Mortici [42] established the inequalities and asymptotic expansions for $\ln A$ in (22). Chen and Lin [17] and Chen [13] presented a class of asymptotic expansions related to Glaisher-Kinkelin constant and the Barnes G -function.

As $x \rightarrow \infty$, the Stirling formula for Barnes G -function can be found (see [53, p. 26]):

$$(28) \quad \ln G(x+1) = \frac{x}{2} \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12} \right) \ln x + O(x^{-1}).$$

In this section, we develop the formula (28) to produce a complete asymptotic expansion given by Theorem 3.

Theorem 3. *As $x \rightarrow \infty$, the following asymptotic expansion holds*

$$(29) \quad \ln G(x+1) = \frac{x}{2} \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12} \right) \ln x + \sum_{k=1}^{\infty} \frac{q_k}{x^k},$$

where the coefficients q_k are given by the recurrence relation

$$(30) \quad q_1 = 0, \quad q_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=1}^{k-1} q_j (-1)^j \binom{k}{k-j+1} + \frac{(-1)^k B_{k+2}}{(k+1)(k+2)} + \frac{(k+6)(k-1)}{12(k+3)(k+2)(k+1)} \right\} \quad \text{for } k \geq 2.$$

Here B_n denote Bernoulli numbers. Namely,

$$(31) \quad \ln G(x+1) \sim \frac{x}{2} \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12} \right) \ln x - \frac{1}{240x^2} + \frac{1}{1008x^4} - \frac{1}{1440x^6} + \frac{1}{1056x^8} - \frac{691}{327600x^{10}} + \dots$$

Proof. Denote

$$P(x) = \ln G(x+1) - \left\{ \frac{x}{2} \ln(2\pi) - \ln A + \frac{1}{12} - \frac{3x^2}{4} + \left(\frac{x^2}{2} - \frac{1}{12} \right) \ln x \right\}$$

and

$$Q(x) = \sum_{k=1}^{\infty} \frac{q_k}{x^k}.$$

In view of (28), we can let $P(x) \sim Q(x)$ and

$$(32) \quad \Delta P(x) := P(x+1) - P(x) \sim Q(x+1) - Q(x) =: \Delta Q(x)$$

as $x \rightarrow \infty$, where q_k are real numbers to be determined.

Noting that $\ln G(x+1) - \ln G(x) = \ln \Gamma(x)$ and

$$(33) \quad \ln \Gamma(x+1) \sim x \ln x - x + \ln \sqrt{2\pi x} + \sum_{k=1}^{\infty} \frac{B_{k+1}}{k(k+1)x^k}.$$

(see [2, p. 257, Eq. (6.1.40)]), we obtain

$$(34) \quad \begin{aligned} \Delta P(x) &= \ln \Gamma(x+1) - x \ln x + x - \ln \sqrt{2\pi x} \\ &\quad + \frac{2x+3}{4} - \left(\frac{x^2}{2} + x + \frac{5}{12} \right) \ln \left(1 + \frac{1}{x} \right) \\ &\sim \sum_{k=3}^{\infty} \left\{ \frac{B_{k+1}}{k(k+1)} + \frac{(-1)^{k-1}(k+5)(k-2)}{12(k+2)(k+1)k} \right\} \frac{1}{x^k}. \end{aligned}$$

By (19), we have

$$(35) \quad \begin{aligned} \Delta Q(x) &= \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^k q_j (-1)^{k-j} \binom{k-1}{k-j} - q_k \right\} \frac{1}{x^k} \\ &= \sum_{k=1}^{\infty} \left\{ \sum_{j=1}^{k-1} q_j (-1)^{k-j} \binom{k-1}{k-j} \right\} \frac{1}{x^k} \\ &= -\frac{q_1}{x^2} + \sum_{k=3}^{\infty} \left\{ \sum_{j=1}^{k-1} q_j (-1)^{k-j} \binom{k-1}{k-j} \right\} \frac{1}{x^k}. \end{aligned}$$

Equating coefficients of the term x^{-k} on the right-hand sides of (34) and (35) yields $q_1 = 0$ and for $k \geq 3$,

$$\frac{B_{k+1}}{k(k+1)} + \frac{(-1)^{k-1}(k+5)(k-2)}{12(k+2)(k+1)k} = \sum_{j=1}^{k-1} q_j (-1)^{k-j} \binom{k-1}{k-j},$$

$$\frac{B_{k+1}}{k(k+1)} + \frac{(-1)^{k-1}(k+5)(k-2)}{12(k+2)(k+1)k} = \sum_{j=1}^{k-2} q_j (-1)^{k-j} \binom{k-1}{k-j} - (k-1)q_{k-1},$$

$$q_{k-1} = \frac{(-1)^k}{k-1} \left\{ \sum_{j=1}^{k-2} q_j (-1)^j \binom{k-1}{k-j} + \frac{(-1)^{k+1} B_{k+1}}{k(k+1)} + \frac{(k+5)(k-2)}{12(k+2)(k+1)k} \right\},$$

which can be written as (30). The proof is complete.

5. GLAISHER-KINKELIN AND CHOI-SRIVASTAVA CONSTANTS

Choi and Srivastava (see [26, p.102] and [27]) introduced two mathematical constants B and C (analogous to the Glaisher-Kinkelin constant A) defined by

$$(36) \quad \ln B = \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \ln n + \frac{n^3}{9} - \frac{n}{12} \right\}$$

and

$$(37) \quad \ln C = \lim_{n \rightarrow \infty} \left\{ \sum_{k=1}^n k^3 \ln k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \right\}$$

for which the approximate numerical values are given by

$$B = 1.03091675\dots \quad \text{and} \quad C = 0.97955746\dots$$

Like the expression of the Glaisher-Kinkelin constant A in (25), the constants B and C are also known to be expressible in terms of special values of the derivative of the Riemann zeta function $\zeta(s)$ as follows (see [27] and [28, Eq. (1.9)]):

$$(38) \quad \ln B = -\zeta'(-2) \quad \text{and} \quad \ln C = -\frac{11}{720} - \zeta'(-3).$$

As the Euler-Mascheroni constant γ is involved in the classical gamma function Γ , the constants A , B and C have appeared naturally in the theory of the multiple gamma functions Γ_n (see, e.g., [54, Section 1.4]) and play their respective roles, for example ([53, p. 39, p. 247], [25, p. 523, Eq. (2.50)], [23]).

Chen [11] established the asymptotic expansions related to the Glaisher-Kinkelin constant A and Choi-Srivastava constants B and C . Mortici [42] dealt with the same problem. Recently, Cheng and Chen [22] and Chen and Choi [15] established new asymptotic expansions of the Glaisher-Kinkelin and Choi-Srivastava constants. For example, by using Bernoulli numbers, Chen [11] established the asymptotic expansions related to the constants A , B and C .

In this section, we provide a recurrence relation for determining the coefficients of each asymptotic expansion related to the constants A , B and C , without Bernoulli numbers.

Theorem 4. *As $n \rightarrow \infty$, the following asymptotic expansion holds*

$$(39) \quad \sum_{k=1}^n k \ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} \sim \ln A - \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k},$$

where the coefficients λ_k are given by the recurrence relation

$$(40) \quad \lambda_2 = -\frac{1}{720}, \quad \lambda_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=2}^{k-1} \lambda_j (-1)^j \binom{k}{k-j+1} + \frac{k(k-1)}{12(k+1)(k+2)(k+3)} \right\}$$

for $k \geq 3$. Namely,

$$(41) \quad \sum_{k=1}^n k \ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} \\ \sim \ln A + \frac{1}{720n^2} - \frac{1}{5040n^4} + \frac{1}{10080n^6} - \frac{1}{9504n^8} + \dots$$

Proof. Denote

$$I_n = \sum_{k=1}^n k \ln k - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln n + \frac{n^2}{4} - \ln A \quad \text{and} \quad J_n = - \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k}.$$

We can let $I_n \sim J_n$ and

$$\Delta I_n := I_{n+1} - I_n \sim J_{n+1} - J_n =: \Delta J_n$$

as $n \rightarrow \infty$, where λ_k are real numbers to be determined.

We obtain, after some elementary transformations, that

$$(42) \quad \Delta I_n = \frac{n}{2} + \frac{1}{4} - \left(\frac{n^2}{2} + \frac{n}{2} + \frac{1}{12} \right) \ln \left(1 + \frac{1}{n} \right) = - \sum_{k=3}^{\infty} \frac{(-1)^{k-1} (k-1)(k-2)}{12k(k+1)(k+2)} n^{-k}.$$

Direct computation yields

$$\begin{aligned} \sum_{k=2}^{\infty} \frac{\lambda_k}{(n+1)^k} &= \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k} \left(1 + \frac{1}{n} \right)^{-k} = \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k} \sum_{j=0}^{\infty} \binom{-k}{j} \frac{1}{n^j} \\ &= \sum_{k=2}^{\infty} \frac{\lambda_k}{n^k} \sum_{j=0}^{\infty} (-1)^j \binom{k+j-1}{j} \frac{1}{n^j} = \sum_{k=2}^{\infty} \sum_{j=2}^k \lambda_j (-1)^{k-j} \binom{k-1}{k-j} \frac{1}{n^k}. \end{aligned}$$

We then obtain

$$(43) \quad \begin{aligned} \Delta J_n &= - \sum_{k=2}^{\infty} \left\{ \sum_{j=2}^k \lambda_j (-1)^{k-j} \binom{k-1}{k-j} - \lambda_k \right\} \frac{1}{n^k} \\ &= - \sum_{k=3}^{\infty} \left\{ \sum_{j=2}^{k-1} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} \right\} n^{-k}. \end{aligned}$$

Equating coefficients of the term n^{-k} on the right-hand sides of (42) and (43) yields

$$\frac{(-1)^{k-1} (k-1)(k-2)}{12k(k+1)(k+2)} = \sum_{j=2}^{k-1} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} \quad (k \geq 3).$$

For $k=3$ we obtain $\lambda_2 = -\frac{1}{720}$, and for $k \geq 4$ we have

$$\frac{(-1)^{k-1} (k-1)(k-2)}{12k(k+1)(k+2)} = \sum_{j=2}^{k-2} \lambda_j (-1)^{k-j} \binom{k-1}{k-j} - (k-1)\lambda_{k-1} \quad (k \geq 4),$$

$$\lambda_{k-1} = \frac{(-1)^k}{k-1} \left\{ \sum_{j=2}^{k-2} \lambda_j (-1)^j \binom{k-1}{k-j} + \frac{(k-1)(k-2)}{12k(k+1)(k+2)} \right\} \quad (k \geq 4),$$

which can be written as (40). The proof is complete.

Following the same method as was used in the proof of Theorem 4, we can prove the following Theorems 5 and 6. We here omit the proofs.

Theorem 5. As $n \rightarrow \infty$, the following asymptotic expansion holds

$$(44) \quad \sum_{k=1}^n k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \ln n + \frac{n^3}{9} - \frac{n}{12} \sim \ln B + \sum_{k=1}^{\infty} \frac{\mu_k}{n^k},$$

where the coefficients μ_k are given by the recurrence relation

$$(45) \quad \mu_1 = -\frac{1}{360}, \quad \mu_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=1}^{k-1} \mu_j (-1)^j \binom{k}{k-j+1} - \frac{k}{6(k+2)(k+3)(k+4)} \right\}$$

for $k \geq 2$. Namely,

$$(46) \quad \sum_{k=1}^n k^2 \ln k - \left(\frac{n^3}{3} + \frac{n^2}{2} + \frac{n}{6} \right) \ln n + \frac{n^3}{9} - \frac{n}{12} \\ \sim \ln B - \frac{1}{360n} + \frac{1}{7560n^3} - \frac{1}{25200n^5} + \frac{1}{33264n^7} - \frac{691}{16216200n^9} + \dots$$

Theorem 6. As $n \rightarrow \infty$, the following asymptotic expansion holds

$$(47) \quad \sum_{k=1}^n k^3 \ln k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \sim \ln C - \sum_{k=2}^{\infty} \frac{\nu_k}{n^k},$$

where the coefficients ν_k are given by the recurrence relation

$$(48) \quad \nu_2 = \frac{1}{5040} \quad \text{and} \\ \nu_k = \frac{(-1)^{k+1}}{k} \left\{ \sum_{j=2}^{k-1} \nu_j (-1)^j \binom{k}{k-j+1} - \frac{k(k-1)(k+13)}{120(k+1)(k+3)(k+4)(k+5)} \right\}$$

for $k \geq 3$. Namely,

$$(49) \quad \sum_{k=1}^n k^3 \ln k - \left(\frac{n^4}{4} + \frac{n^3}{2} + \frac{n^2}{4} - \frac{1}{120} \right) \ln n + \frac{n^4}{16} - \frac{n^2}{12} \\ \sim \ln C - \frac{1}{5040n^2} + \frac{1}{33600n^4} - \frac{1}{66528n^6} + \frac{691}{43243200n^8} - \frac{1}{34320n^{10}} + \dots$$

REFERENCES

1. U. ABEL: *A complete asymptotic expansion for a sequence of certain sums*. Appl. Math. Comput., **217** (2010), 4302–4305.
2. M. ABRAMOWITZ, I.A. STEGUN (EDS.): *Handbook of mathematical functions with formulas, graphs, and mathematical tables*. National Bureau of Standards, Applied Mathematics Series, vol. 55. 9th printing, Dover, New York, 1972.
3. H. ALZER, D. KARAYANAKIS, H.M. SRIVASTAVA: *Series representations for some mathematical constants*. J. Math. Anal. Appl. **320** (2006), 145–162.
4. E. W. BARNES: *The theory of G-function*. Quart. J. Math., **31** (1899), 264–314.
5. E. W. BARNES: *Genesis of the double gamma function*. Proc. Lond Math. Soc., **31** (1900), 358–381.
6. E. W. BARNES: *The theory of the double gamma function*. Philos. Trans. R. Soc. Lond. Ser. A, **196** (1901), 265–388.
7. E. W. BARNES: *On the theory of the multiple gamma functions*. Trans. Cambridge Philos. Soc., **19** (1904), 374–439.
8. N. BATIR: *Inequalities for the double gamma function*. J. Math. Anal. Appl., **351** (2009), 182–185.
9. N. BATIR, M. CANCAN: *A double inequality for the double gamma function*. Int. J. Math. Anal., **2** (2008), 329–335.
10. C.-P. CHEN: *Inequalities associated with Barnes G-function*. Expo. Math., **29** (2011), 119–125.
11. C.-P. CHEN: *Glaisher-Kinkelin constant*. Integral Transforms Spec. Funct., **23** (2012), 785–792.
12. C.-P. CHEN: *New asymptotic expansions related to Somos' quadratic recurrence constant*. C. R. Acad. Sci. Paris, Ser. I, **351** (2013), 9–12.
13. C.-P. CHEN: *Asymptotic expansions for Barnes G-function*. J. Number Theory, **135** (2014), 36–42.
14. C.-P. CHEN: *Sharp inequalities and asymptotic series related to Somos' quadratic recurrence constant*. J. Number Theory, **172** (2017), 145–159.
15. C.-P. CHEN, J. CHOI: *Unified treatment of several asymptotic expansions concerning some mathematical constants*. Appl. Math. Comput., **305** (2017), 348–363.
16. C.-P. CHEN, X.-F. HAN: *On Somos' quadratic recurrence constant*. J. Number Theory, **166** (2016), 31–40.
17. C.-P. CHEN, L. LIN: *Asymptotic expansions related to Glaisher-Kinkelin constant based on the Bell polynomials*. J. Number Theory, **133** (2013), 2699–2705.
18. C.-P. CHEN, H. M. SRIVASTAVA: *Some inequalities and monotonicity properties associated with the gamma and psi functions and the Barnes G-function*. Integral Transforms Spec. Funct., **22** (2011), 1–15.
19. C.-P. CHEN, H. M. SRIVASTAVA: *New representations for the Lugo and Euler-Mascheroni constants*. Appl. Math. Lett. **24** (2011), 1239–1244.
20. C.-P. CHEN, H. M. SRIVASTAVA: *New representations for the Lugo and Euler-Mascheroni constants. II*. Appl. Math. Lett. **25** (2012), 333–338.

21. C.-P. CHEN, H. M. SRIVASTAVA, L. LI, S. MANYAMA: *Inequalities and monotonicity properties for the psi (or digamma) function and estimates for the Euler–Mascheroni constant*. Integral Transforms Spec. Funct. **22** (2011), 681–693.
22. J.-X. CHENG, C.-P. CHEN: *Asymptotic expansions of the Glaisher–Kinkelin and Choi–Srivastava constants*. J. Number Theory, **144** (2014), 105–110.
23. J. CHOI: *Some mathematical constants*. Appl. Math. Comput., **187** (2007), 122–140.
24. J. CHOI: *Determinant of Laplacian on S^3* . Math. Japon., **40** (1994), 155–166.
25. J. CHOI, Y.J. CHO, H.M. SRIVASTAVA: *Series involving the zeta function and multiple Gamma functions*. Appl. Math. Comput., **159** (2004), 509–537.
26. J. CHOI, H.M. SRIVASTAVA: *Certain classes of series involving the zeta function*. J. Math. Anal. Appl., **231** (1999), 91–117.
27. J. CHOI, H.M. SRIVASTAVA: *Certain classes of series associated with the zeta function and multiple Gamma functions*. J. Comput. Appl. Math., **118** (2000), 87–109.
28. J. CHOI, H. M. SRIVASTAVA: *Asymptotic formulas for the triple gamma function Γ_3 by means of its integral representation*. Appl. Math. Comput., **218** (2011), 2631–2640.
29. C. FERREIRA, J. L. LÓPEZ: *An asymptotic expansion of the double gamma function*. J. Approx. Theory, **111** (2001), 298–314.
30. S. R. FINCH: *Mathematical Constants*. Cambridge Univ. Press, 2003.
31. R. GIULIANO: *A possible solution to the problem of Hassani*. RGMIA mailing list, 2008.
32. I. S. GRADSHTEYN, I. M. RYZHIK: *Tables of Integrals, Series, and Products (Corrected and Enlarged Edition prepared by A. Jeffrey)*. Academic Press, New York, 1980.
33. J. GUILLERA, J. SONDOW: *Double integrals and infinite products for some classical constants via analytic continuations of Lerch’s transcendent*. Ramanujan J., **16** (2008), 247–270.
34. M. HASSANI: *Problem*. RGMIA mailing list, 2008.
35. M.D. HIRSCHHORN: *A note on Somos’ quadratic recurrence constant*. J. Number Theory, **131** (2011), 2061–2063.
36. S. KOUMANDOS: *On Ruijsenaars’ asymptotic expansion of the logarithm of the double gamma function*. J. Math. Anal. Appl., **341** (2008), 1125–1132.
37. S. KOUMANDOS, H. L. PEDERSEN: *Completely monotonic functions of positive order and asymptotic expansions of the logarithm of Barnes double gamma function and Euler’s gamma function*. J. Math. Anal. Appl., **355** (2009), 33–40.
38. H. KUMAGAI: *The determinant of the Laplacian on the n -sphere*. Acta Arith., **91** (1999), 199–208.
39. V. LAMPRET: *Approximation of Sondow’s generalized-Euler-constant function on the interval $[-1, 1]$* . Ann. Univ. Ferrara, **56** (2010), 65–76.
40. D. LU, Z. SONG: *Some new continued fraction estimates of the Somos’ quadratic recurrence constant*. J. Number Theory, **155** (2015), 36–45.
41. C. MORTICI: *Estimating the Somos’ quadratic recurrence constant*. J. Number Theory, **130** (2010), 2650–2657.

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42. C. MORTICI: *Approximating the constants of Glaisher-Kinkelin type*. J. Number Theory, **133** (2013), 2465–2469.
43. G. NEMES: *On the coefficients of an asymptotic expansion related to Somos' quadratic recurrence constant*. Appl. Anal. Discrete Math., **5** (2011), 60–66.
44. B. OSGOOD, R. PHILLIPS, P. SARNAK: *Extremals of determinants of Laplacians*. J. Funct. Anal., **80** (1988), 148–211.
45. H. L. PEDERSEN: *On the remainder in an asymptotic expansion of the double gamma function*. Mediterr. J. Math., **2** (2005), 171–178.
46. H. L. PEDERSEN: *The remainder in Ruijsenaars' asymptotic expansion of Barnes double gamma function*. Mediterr. J. Math., **4** (2007), 419–433.
47. K.H. PILEHROOD, T.H. PILEHROOD: *Vacca-type series for values of the generalized Euler constant function and its derivative*. J. Integer Sequences, **13** (2010), Article 10.7.3
48. K.H. PILEHROOD, T.H. PILEHROOD: *Arithmetical properties of some series with logarithmic coefficients*. Math. Z., **255** (2007), 117–131.
49. J. R. QUINE, J. CHOI: *Zeta regularized products and functional determinants on spheres*. Rocky Mountain J. Math., **26** (1996), 719–729.
50. S. N. M. RUIJSENAARS: *On Barnes' multiple zeta and gamma functions*. Adv. Math., **156** (2000), 107–132.
51. S. SIMIC: *A simpler solution of the recent problem*. RGMIA mailing list, 2008.
52. J. SONDOW, P. HADJICOSTAS: *The generalized-Euler-constant function $\gamma(z)$ and a generalization of Somos's quadratic recurrence constant*. J. Math. Anal. Appl., **332** (2007), 292–314.
53. H. M. SRIVASTAVA, J. CHOI: *Series Associated with the zeta and related functions*. Kluwer Academic Publishers, Dordrecht, 2001.
54. H. M. SRIVASTAVA AND J. CHOI: *Zeta and q-zeta functions and associated series and integrals*. Elsevier Science Publishers, Amsterdam, London and New York, 2012.
55. I. VARDI: *Determinants of Laplacians and multiple gamma functions*. SIAM J. Math. Anal., **19** (1988), 493–507.
56. A. VOROS: *Special functions, spectral functions and the Selberg Zeta function*. Comm. Math. Phys., **110** (1987), 439–465.
57. E. T. WHITTAKER, G. N. WATSON: *A Course of Modern Analysis: An Introduction to the General Theory of Infinite Processes and of Analytic Functions; With an Account of the Principal Transcendental Functions*, 4th ed. (Reprinted), Cambridge University Press, Cambridge, London and New York, 1963.

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