

INTERVAL OSCILLATION CRITERIA FOR IMPULSIVE CONFORMABLE PARTIAL DIFFERENTIAL EQUATIONS

G.E. Chatzarakis, K. Logarasi, T. Raja and V. Sadhasivam*

In this paper, we present some sufficient conditions for the oscillation of all solutions of forced impulsive delay conformable partial differential equations. We consider two factors, namely impulse and delay that jointly affect the interval qualitative properties of the solutions of those equations. The results obtained in this paper extend and generalize some of the known results for forced impulsive conformable partial differential equations. An example illustrating the results is also given.

1. INTRODUCTION

Fractional differential equations are generalizations of the classical differential equations of integer order. They have gained considerable popularity and importance during the past three decades or so, due mainly to their demonstrated applications in numerous diverse and widespread fields of science and engineering. Nowadays the number of scientific and engineering problems involving fractional calculus is already very large and still growing. It has been found that various, especially interdisciplinary applications can be elegantly modeled, using fractional derivatives. Fractional differentials and integrals provide more accurate models of the systems under consideration. The present areas of application of fractional calculus include fluid flow, rheology, dynamical processes in self-similar and porous structures, diffusive transport akin to diffusion, electrical networks, probability and statistics,

*Corresponding author. T. Raja

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control theory of dynamical systems, viscoelasticity, electrochemistry of corrosion, chemical physics, optics and signal processing, economics and others; see, for example [5, 6, 14, 20–22, 27] and the references cited therein. For the theory and applications of fractional differential equations, we refer the reader to the monographs [14, 26]. Though, the most common definitions are based on the integration along a singular kernel which is nonlocal: the Riemann -Liouville derivative and the Caputo derivative. Moreover, for this type of derivative, the useful product rule and chain rules, are not applicable. But in 2014 Khalil et. al [13] introduced a new fractional derivative called the conformable derivative which closely resembles the classical derivative. The basic theory of conformable derivatives of the class of non differentiable(classical sense) functions was discussed in [1–3, 32].

The oscillation of fractional differential equations as a new research field has received significant attention and some interesting results have already been obtained. We refer the reader to [8–11, 19, 23, 36] and the references quoted therein.

Impulsive differential equations provide a natural formalism for describing observed evolutionary processes and several real world problems in the applied sciences. Detailed treatises of the theory and application of impulsive differential equations, can be found in the monographs [7, 16, 17, 30, 34, 37] and reference cited therein.

Of particular interest is the work of Q.L. Li and W.S. Cheng [18] that established interval oscillation criteria, for a second order differential equation under impulse effects of the form

$$\begin{aligned} (p(t)x'(t))' + q(t)x(t - \tau) + \sum_{i=1}^n q_i(t)\Phi_{\alpha_i}(x(t - \tau)) &= e(t), \quad t \geq t_0, \\ x(t_k^+) &= a_k x(t_k), \quad x'(t_k^+) = b_k x'(t_k), \quad k = 1, 2, \dots, \end{aligned}$$

where $0 \leq t_0 < t_1 < \dots < t_k < \dots$, $\lim_{k \rightarrow \infty} t_k = \infty$, $p, q, q_i, e \in PLC([t_0, \infty), \mathbb{R})$. Using the Riccati equation, they obtained some interesting results.

In the last decades, interval oscillation of impulsive differential equations we have seen a strong activity and considerable work, in the interval oscillation of impulsive differential equations by several researchers, see [12, 24, 28, 29, 31, 33, 35, 38] and the references cited therein. Though, the majority of the existing literature has concentrated on interval oscillation criteria, for the case without delay. Only a very small fraction of papers, have studied the case with delay. To the best of our knowledge, there has been no published work on oscillation criteria, for impulsive conformable partial differential equations.

The lack of work in this problem, has been our motivation, for considering

the following model

$$(1) \quad \left\{ \begin{array}{l} \frac{\partial^\alpha}{\partial t^\alpha} \left(\frac{\partial^\alpha}{\partial t^\alpha} u(x, t) \right) + q(x, t) f(u(x, t - \varphi)) + \sum_{i=1}^n q_i(x, t) f_i(u(x, t - \varphi)) \\ = a(t) \Delta u(x, t) + \int_a^b b(t, \xi) \Delta u(x, \theta(t, \xi)) d\eta(\xi) + F(x, t), \quad \text{for } t \neq t_k, \\ u(x, t_k^+) = \alpha_k(x, t_k, u(x, t_k)), \\ \frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k^+) = \beta_k \left(x, t_k, \frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k) \right), \quad \text{for } k = 1, 2, \dots, \\ \text{and } (x, t) \in \Omega \times \mathbb{R}_+ \equiv G, \end{array} \right.$$

where Ω is a bounded domain in \mathbb{R}^N with a piecewise smooth boundary $\partial\Omega$, Δ is the Laplacian in the Euclidean space \mathbb{R}^N and $\frac{\partial^\alpha}{\partial t^\alpha}$ denotes the conformable partial derivative of order α , $0 < \alpha \leq 1$. Moreover, we consider the following boundary condition:

$$(2) \quad \frac{\partial}{\partial \gamma} u(x, t) + \mu(x, t) u(x, t) = 0, \quad (x, t) \in \partial\Omega \times \mathbb{R}_+,$$

where γ is the unit exterior normal vector to $\partial\Omega$, $\mu(x, t) \in C(\partial\Omega \times \mathbb{R}_+, \mathbb{R}_+)$ and $\mathbb{R}_+ = [0, +\infty)$.

We assume the following conditions to hold through the rest of the paper.

$$(A_1) \quad q(x, t), q_i(x, t) \in C(\bar{G}, \mathbb{R}_+), q(t) = \min_{x \in \Omega} q(x, t), q_i(t) = \min_{x \in \Omega} q_i(x, t), i = 1, 2, \dots, n, \\ f, f_i \in C(\mathbb{R}, \mathbb{R}) \text{ are convex in } \mathbb{R}_+ \text{ with } uf(u) > 0, uf_i(u) > 0 \text{ and } \frac{f(u)}{u} \geq c > 0, \\ \frac{f_i(u)}{u} \geq c_i > 0 \text{ for } u \neq 0, i = 1, 2, \dots, n, t - \varphi < t, \lim_{t \rightarrow +\infty} t - \varphi = +\infty \text{ and } \\ F \in C(\bar{G}, \mathbb{R}).$$

$$(A_2) \quad b(t, \xi) \in C(\mathbb{R}_+ \times \mathbb{R}_+), \theta(t, \xi) \in C(\mathbb{R}_+ \times [a, b], \mathbb{R}), \theta(t, \xi) \leq t \text{ for } \xi \in [a, b], \\ \theta(t, \xi) \text{ is non-decreasing with respect to } t \text{ and } \xi \text{ respectively and } \liminf_{t \rightarrow +\infty, \xi \in [a, b]} \theta(t, \xi) = \\ +\infty, \eta(\xi) : [a, b] \rightarrow \mathbb{R} \text{ is nondecreasing and the integral is a stieltjes inte-} \\ \text{gral in (1), } a(t) \in PC(\mathbb{R}_+, \mathbb{R}_+) \text{ where } PC \text{ represents the class of functions} \\ \text{which are piecewise continuous in } t \text{ with discontinuities of first kind only at} \\ t = t_k, k = 1, 2, \dots, \text{ and left continuous at } t = t_k, k = 1, 2, \dots.$$

$$(A_3) \quad u(x, t) \text{ and its derivative } \frac{\partial^\alpha}{\partial t^\alpha} u(x, t) \text{ are piecewise continuous in } t \text{ with dis-} \\ \text{continuities of first kind only at } t = t_k, k = 1, 2, \dots, \text{ and left continuous at} \\ t = t_k, u(x, t_k) = u(x, t_k^-), \frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k) = \frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k^-), k = 1, 2, \dots.$$

$$(A_4) \quad \alpha_k, \beta_k \in PC(\bar{\Omega} \times \mathbb{R}_+ \times \mathbb{R}, \mathbb{R}) \text{ for } k = 1, 2, \dots, \text{ and there exist constants}$$

a_k, a_k^*, b_k, b_k^* such that for $k = 1, 2, \dots$,

$$a_k^* \leq \frac{\alpha_k(x, t_k, u(x, t_k))}{u(x, t_k)} \leq a_k, \quad b_k^* \leq \frac{\beta_k\left(x, t_k, \frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k)\right)}{\frac{\partial^\alpha}{\partial t^\alpha} u(x, t_k)} \leq b_k.$$

(A₅) For any $T \geq 0$ there exist intervals $[c_1, d_1]$ and $[c_2, d_2]$ contained in $[T, \infty)$ such that $c_1 < d_1 \leq d_1 + \varphi \leq c_2 < d_2$, $c_j, d_j \notin \{t_k\}$, $j = 1, 2$, $k = 1, 2, \dots$, $q(t) \geq 0$, $q_i(t) \geq 0$, $i = 1, 2, \dots, n$ for $t \in [c_1 - \varphi, d_1] \cup [c_2 - \varphi, d_2]$ and $F(t)$ has different signs in $[c_1 - \varphi, d_1]$ and $[c_2 - \varphi, d_2]$, for instance, let

$$F(t) \leq 0 \quad \text{for } t \in [c_1 - \varphi, d_1], \quad \text{and} \quad F(t) \geq 0 \quad \text{for } t \in [c_2 - \varphi, d_2].$$

We denote

$$J(s) := \max\{j : t_0 < t_j < s\}, \quad j = 1, 2;$$

$$J_p(c_j, d_j) = \{p \in C^1[c_j, d_j], \quad p(t) \neq 0, \quad p(c_j) = p(d_j) = 0, \quad j = 1, 2\}.$$

This paper is organized as follows: In Section 2, we present some definitions and results that will be needed to establish our main results. In Section 3, we present our main results and in the final section, we provide an example to illustrate our results.

2. PRELIMINARIES

In this section, we present some definitions and review important results, from the literature that we will use throughout the paper.

Definition 0.1. *By a solution of (1)-(2) we mean a function $u(x, t) \in C^{2\alpha}(\bar{\Omega} \times [t_1, +\infty), \mathbb{R}) \cap C^\alpha(\bar{\Omega} \times [\hat{t}_1, +\infty), \mathbb{R})$ which satisfies (1), where*

$$t_1 := \min\left\{0, \min_{\xi \in [c, d]} \left\{\inf_{t \geq 0} \theta(t, \xi)\right\}\right\} \quad \text{and} \quad \hat{t}_1 := \min\left\{0, \left\{\inf_{t \geq 0} t - \varphi\right\}\right\}.$$

Definition 0.2. *The solution u of the problem (1)-(2) is said to be oscillatory in the domain G , if it has arbitrary large zeros. Otherwise it is non-oscillatory.*

We use the following definition introduced by R.R. Khalil et al. [13].

Definition 0.3. *Let $f : [0, \infty) \rightarrow \mathbb{R}$. Then the “conformable derivative” of f of order α is defined by*

$$T_\alpha(f)(t) = \lim_{\varepsilon \rightarrow 0} \frac{f(t + \varepsilon t^{1-\alpha}) - f(t)}{\varepsilon}$$

for all $t > 0, \alpha \in (0, 1]$.

If f is α -differentiable in some $(0, a)$, $a > 0$ and $\lim_{t \rightarrow 0^+} f^{(\alpha)}(t)$ exists, then we define

$$T_\alpha f(0) = \lim_{t \rightarrow 0^+} T_\alpha f(t).$$

Definition 0.4. $I_\alpha^a(f)(t) = I_1^a(t^{\alpha-1}f) = \int_a^t \frac{f(x)}{x^{1-\alpha}} dx$, where the integral is the usual Riemann improper integral and $\alpha \in (0, 1)$.

Some key properties of the conformable derivative are summarized in the following Theorem.

Theorem 0.5. Let $\alpha \in (0, 1]$ and f, g be α -differentiable at a point $t > 0$. Then

- (i) $T_\alpha(af + bg) = aT_\alpha(f) + bT_\alpha(g)$, for all $a, b \in \mathbb{R}$.
- (ii) $T_\alpha(t^p) = pt^{p-\alpha}$, for all $p \in \mathbb{R}$.
- (iii) $T_\alpha(\lambda) = 0$, for all constant functions $f(t) = \lambda$.
- (iv) $T_\alpha(fg) = fT_\alpha(g) + gT_\alpha(f)$.
- (v) $T_\alpha\left(\frac{f}{g}\right) = \frac{gT_\alpha(f) - fT_\alpha(g)}{g^2}$.
- (vi) If f is differentiable, then $T_\alpha(f)(t) = t^{1-\alpha} \frac{df}{dt}(t)$.

Definition 0.6. [4] Let f be a function with n variables x_1, x_2, \dots, x_n . Then the conformable partial derivative of f of order $0 < \alpha \leq 1$ in x_i is defined as follows:

$$\frac{\partial^\alpha}{\partial x_i^\alpha} f(x_1, x_2, \dots, x_n) = \lim_{\varepsilon \rightarrow 0} \frac{f(x_1, x_2, \dots, x_{i-1}, x_i + \varepsilon x_i^{1-\alpha}, \dots, x_n) - f(x_1, x_2, \dots, x_n)}{\varepsilon}$$

For convenience, we introduce the following notation: $U(t) = \frac{1}{|\Omega|} \int_\Omega u(x, t) dx$, where $|\Omega| = \int_\Omega dx$.

3. MAIN RESULTS

In this section, we establish some new interval oscillation criteria for the problem (1)-(2), using the Riccati transformation method. In the following Lemma 0.7, we also assume that the function u is differentiable in classical sense with respect to t .

Lemma 0.7. If the impulsive conformable differential inequality

$$(3) \quad \begin{cases} T_\alpha(T_\alpha(U(t))) + cq(t)U(t - \varphi) + \sum_{i=1}^n c_i q_i(t)U(t - \varphi) \leq F(t), & t \neq t_k \\ a_k^* \leq \frac{U(t_k^+)}{U(t_k)} \leq a_k, & b_k^* \leq \frac{T_\alpha(U(t_k^+))}{T_\alpha(U(t_k))} \leq b_k, & k = 1, 2, \dots, \end{cases}$$

has no eventually positive solution, then every solution of the problem (1)-(2) is oscillatory in G .

Proof. Assume that $u(x, t)$ is a non-oscillatory solution of (1)-(2) and $u(x, t) > 0$. By the assumption that there exists a $t_1 > t_0 > 0$ such that $t - \varphi \geq t_0$ and $\theta(t, \xi) \geq t_0$ for $(t, \xi) \in [t_1, +\infty) \times [a, b]$, we have that $u(x, t - \varphi) > 0$ for $(x, t - \varphi) \in \Omega \times [t_1, +\infty)$, $u(x, \theta(t, \xi)) > 0$ for $x \in \Omega$, $t \in [t_1, +\infty]$, $\xi \in [a, b]$.

For $t \geq t_0$ and $t \neq t_k$ for $k = 1, 2, \dots$, we multiply both sides of inequality (1) by $\frac{1}{|\Omega|}$ and integrate with respect to x over the domain Ω to attain

$$(4) \quad \begin{cases} t^{1-\alpha} \frac{d}{dt} \left(t^{1-\alpha} \frac{d}{dt} \left(\frac{1}{|\Omega|} \int_{\Omega} u(x, t) dx \right) \right) + \frac{1}{|\Omega|} \int_{\Omega} q(x, t) f(u(x, t - \varphi)) dx \\ + \frac{1}{|\Omega|} \sum_{i=1}^n \int_{\Omega} q_i(x, t) f_i(u(x, t - \varphi)) dx - a(t) \frac{1}{|\Omega|} \int_{\Omega} \Delta u(x, t) dx \\ - \frac{1}{|\Omega|} \int_{\Omega} \int_a^b b(t, \xi) \Delta u(x, \theta(t, \xi)) d\eta(\xi) dx \leq \frac{1}{|\Omega|} \int_{\Omega} F(x, t) dx, \quad t \neq t_k. \end{cases}$$

Using Green's formula and boundary condition (2), we get

$$(5) \quad \int_{\Omega} \Delta u(x, t) dx = \int_{\partial\Omega} \frac{\partial}{\partial\gamma} u(x, t) dS = - \int_{\partial\Omega} \mu(x, t) u(x, t) dS \leq 0,$$

and

$$(6) \quad \begin{aligned} \int_{\Omega} \Delta u(x, \theta(t, \xi)) d\eta(\xi) dx &= \int_{\partial\Omega} \frac{\partial}{\partial\gamma} u(x, \theta(t, \xi)) d\eta(\xi) dS \\ &= - \int_{\partial\Omega} \mu(x, \theta(t, \xi)) u(x, \theta(t, \xi)) d\eta(\xi) dS \leq 0, \quad t \geq t_0 \end{aligned}$$

where dS is the surface element on $\partial\Omega$. Moreover, using Jensen's Inequality and (A_1) , we have that

$$(7) \quad \frac{1}{|\Omega|} \int_{\Omega} q(x, t) f(u(x, t - \varphi)) dx \geq cq(t) \frac{1}{|\Omega|} \int_{\Omega} u(x, t - \varphi) dx \geq cq(t) U(t - \varphi).$$

For $i = 1, 2, \dots, n$, we get

$$(8) \quad \frac{1}{|\Omega|} \int_{\Omega} q_i(x, t) f_i(u(x, t - \varphi)) dx \geq c_i q_i(t) U(t - \varphi).$$

Also we have

$$(9) \quad F(t) = \frac{1}{|\Omega|} \int_{\Omega} F(x, t) dx \leq 0.$$

Combining (4)-(9) we get

$$(10) \quad T_{\alpha}(T_{\alpha}(U(t))) + cq(t)U(t - \varphi) + \sum_{i=1}^n c_i q_i(t)U(t - \varphi) \leq F(t), \quad t \neq t_k.$$

For $t \geq t_0$, $t = t_k$, $k = 1, 2, \dots$, we multiply both sides of the above inequality by $\frac{1}{|\Omega|}$ and integrate with respect to x over the domain Ω , to obtain

$$(11) \quad a_k^* \leq \frac{U(t_k^+)}{U(t_k)} \leq a_k, \quad b_k^* \leq \frac{T_\alpha(U(t_k^+))}{T_\alpha(U(t_k))} \leq b_k, \quad k = 1, 2, \dots$$

Therefore (10) and (11), show that $U(t) > 0$ is a positive solution of the impulsive conformable differential inequality (3). This is a contradiction. The proof is complete. \square

Theorem 0.8. *Assume that conditions $(A_1) - (A_5)$ hold and for any $T \geq 0$, there exist c_j, d_j satisfying $T \leq c_1 < d_1$, $T \leq c_2 < d_2$ and $p(t) \in J_p(c_1, d_1)$ such that*

$$(12) \quad \begin{aligned} & \int_{c_j}^{t_{J(c_j)+1}} \left[(p'(t))^2 t^{2-2\alpha} - Q(t) p^2(t) M_{J(c_j)}^j(t) \right] dt \\ & - \sum_{k=J(c_j)+1}^{J(d_j)-1} \int_{t_k}^{t_{k+1}} \left[(p'(t))^2 t^{2-2\alpha} - Q(t) p^2(t) M_{J(c_j)}^j(t) \right] dt \\ & - \int_{t_{J(d_j)}}^{d_j} \left[(p'(t))^2 t^{2-2\alpha} - Q(t) p^2(t) M_{J(d_j)}^j(t) \right] dt \\ & + \int_{c_j}^{d_j} w(t) p^2(t) (1-\alpha) t^{-\alpha} dt \leq \Lambda(p, c_j, d_j) \end{aligned}$$

where $Q(t) = cq(t) + \sum_{i=1}^n c_i q_i(t)$, $\Lambda(p, c_j, d_j) = 0$ for $J(c_j) = J(d_j)$ and

$$\begin{aligned} \Lambda(p, c_j, d_j) = \alpha \left\{ p^2(t_{J(c_j)+1}) t_{J(c_j)+1}^{1-\alpha} \frac{a_{J(c_j)+1}^* - b_{J(c_j)+1}}{a_{J(c_j)+1}^* (t_{J(c_j)+1}^\alpha - c_j^\alpha)} \right. \\ \left. + \sum_{k=J(c_j)+2}^{J(d_j)} p^2(t_k) t_k^{1-\alpha} \frac{a_k^* - b_k}{a_k^* (t_k^\alpha - t_{k-1}^\alpha)} \right\}. \end{aligned}$$

Moreover, for $J(c_j) < J(d_j)$, $j = 1, 2$

$$M_k^j(t) = \begin{cases} \frac{\varphi \alpha}{\varphi \alpha a_k + b_k (t^\alpha - t_k^\alpha)} \frac{(t - \varphi)^\alpha - (t_k - \varphi)^\alpha}{t_k^\alpha - (t_k - \varphi)^\alpha}, & t \in (t_k, t_k + \varphi) \\ \frac{(t - \varphi)^\alpha - t_k^\alpha}{t^\alpha - t_k^\alpha}, & t \in [t_k + \varphi, t_{k+1}), \end{cases}$$

then every solution of the problem (1)-(2) is oscillatory in G .

Proof. Assume to the contrary that $u(x, t)$ is a non-oscillatory solution of (1)-(2). Without loss of generality we may assume that $u(x, t)$ is an eventually positive

solution of (1). Then there exists $t_1 \geq t_0$ such that $u(x, t) > 0$ for $t \geq t_1$. Therefore, from (3), it follows that

(13)

$$T_\alpha(T_\alpha(U(t))) \leq F(t) - cq(t)U(t - \varphi) - \sum_{i=1}^n c_i q_i(t)U(t - \varphi), \quad \text{for } t \in [t_1, \infty).$$

We define the Riccati transformation

$$(14) \quad w(t) := \frac{T_\alpha(U(t))}{U(t)}.$$

From (13), it follows that $w(t)$ satisfies

$$T_\alpha(w(t)) \leq \frac{F(t)}{U(t)} - \left[cq(t) + \sum_{i=1}^n c_i q_i(t) \right] \frac{U(t - \varphi)}{U(t)} - w^2(t).$$

From assumption (A_5) , we can choose $c_1, d_1 \geq t_0$ such that $q(t) \geq 0$ and $q_i(t) \geq 0$ for $t \in [c_1 - \varphi, d_1]$, $i = 1, 2, \dots, n$ and $F(t) \leq 0$ for $t \in [c_1 - \varphi, d_1]$. Then, from (13), we can easily see that

$$(15) \quad T_\alpha(w(t)) \leq -w^2(t) - Q(t) \frac{U(t - \varphi)}{U(t)}.$$

For $t = t_k$, $k = 1, 2, \dots$, we have

$$(16) \quad w(t_k^+) = \frac{T_\alpha(U(t_k^+))}{U(t_k^+)} \leq \frac{b_k}{a_k^*} w(t_k).$$

At first, we consider the case $J(c_1) < J(d_1)$. In this case, all the impulsive moments in $[c_1, d_1]$ are $t_{J(c_1)+1}, t_{J(c_1)+2}, \dots, t_{J(d_1)}$. We choose an $p(t) \in J_p(c_1, d_1)$ and multiply both sides of (15) by $p^2(t)$. Then, integrating both sides of the resulting inequality, from c_1 to d_1 , we obtain

$$\begin{aligned} & \int_{c_1}^{t_{J(c_1)+1}} p^2(t) t^{1-\alpha} w'(t) dt + \int_{t_{J(c_1)+1}}^{t_{J(c_1)+2}} p^2(t) t^{1-\alpha} w'(t) dt + \dots + \int_{t_{J(d_1)}}^{d_1} p^2(t) t^{1-\alpha} w'(t) dt \\ & \leq - \int_{c_1}^{t_{J(c_1)+1}} p^2(t) w^2(t) dt - \int_{t_{J(c_1)+1}}^{t_{J(c_1)+2}} p^2(t) w^2(t) dt - \dots - \int_{t_{J(d_1)}}^{d_1} p^2(t) w^2(t) dt \\ & \quad - \int_{c_1}^{t_{J(c_1)+1}} p^2(t) Q(t) \frac{U(t - \varphi)}{U(t)} dt - \int_{t_{J(c_1)+1}}^{t_{J(c_1)+1} + \varphi} p^2(t) Q(t) \frac{U(t - \varphi)}{U(t)} dt \\ & \quad - \int_{t_{J(c_1)+1} + \varphi}^{t_{J(c_1)+2}} p^2(t) Q(t) \frac{U(t - \varphi)}{U(t)} dt - \dots - \int_{t_{J(d_1)-1} + \varphi}^{t_{J(d_1)}} p^2(t) Q(t) \frac{U(t - \varphi)}{U(t)} dt \\ & \quad - \int_{t_{J(d_1)}}^{d_1} p^2(t) Q(t) \frac{U(t - \varphi)}{U(t)} dt. \end{aligned}$$

Using integration by parts on the left-hand side and noting the condition $p(c_1) = p(d_1) = 0$, we get

$$\begin{aligned}
& \sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} [w(t_k) - w(t_k^+)] \leq - \int_{c_1}^{d_1} [p'(t) t^{1-\alpha} - p(t) w(t)]^2 dt \\
& - \int_{c_1}^{t_{J(c_1)+1}} p^2(t) Q(t) \frac{U(t-\varphi)}{U(t)} dt \\
& - \sum_{k=J(c_1)+1}^{J(d_1)-1} \left[\int_{t_k}^{t_{k+\varphi}} p^2(t) Q(t) \frac{U(t-\varphi)}{U(t)} dt + \int_{t_{k+\varphi}}^{t_{k+1}} p^2(t) Q(t) \frac{U(t-\varphi)}{U(t)} dt \right] \\
& - \int_{t_{J(d_1)}}^{d_1} p^2(t) Q(t) \frac{U(t-\varphi)}{U(t)} dt \\
(17) \quad & + \int_{c_1}^{d_1} t^{2-2\alpha} (p'(t))^2 dt + \int_{c_1}^{d_1} (1-\alpha) t^{-\alpha} p^2(t) w(t) dt.
\end{aligned}$$

There are several cases to consider to estimate $\frac{U(t-\varphi)}{U(t)}$.

Case 1: For $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$. If $t \in (t_k, t_{k+1}] \subset [c_1, d_1]$, since $t_{k+1} - t_k > \varphi$, we consider two sub cases:

Case 1.1: If $t \in [t_k + \varphi, t_{k+1}]$, then $t - \varphi \in [t_k, t_{k+1} - \varphi]$ and there are no impulsive moments in $(t - \varphi, t)$. Then, for any $t \in [t_k + \varphi, t_{k+1}]$, we have

$$U(t) - U(t_k^+) = T_\alpha(U(\xi)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right), \quad \xi \in (t_k, t).$$

Since $U(t_k^+) > 0$ and $T_\alpha(U(\xi)) \geq T_\alpha(U(t))$, the above relation implies that

$$U(t) \geq T_\alpha(U(t)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right).$$

or

$$\frac{T_\alpha(U(t))}{U(t)} < \frac{\alpha}{t^\alpha - t_k^\alpha}.$$

Integrating the above inequality, from $t - \varphi$ to t , we have

$$\frac{U(t-\varphi)}{U(t)} > \frac{(t-\varphi)^\alpha - t_k^\alpha}{t^\alpha - t_k^\alpha}.$$

Case 1.2: If $t \in (t_k, t_k + \varphi)$ then $t - \varphi \in (t_k - \varphi, t_k)$ and there is an impulsive moment t_k in $(t - \varphi, t)$. Similar to Case 1.1, we obtain

$$U(t) - U(t_k - \varphi) = T_\alpha(U(\xi_1)) \left(\frac{t^\alpha - (t_k - \varphi)^\alpha}{\alpha} \right), \quad \xi_1 \in (t_k - \varphi, t_k]$$

or

$$\frac{T_\alpha(U(t))}{U(t)} < \frac{\alpha}{t^\alpha - (t_k - \varphi)^\alpha}.$$

Integrating the above inequality, from $t - \varphi$ to t , we get

$$(18) \quad \frac{U(t - \varphi)}{U(t_k)} > \frac{(t - \varphi)^\alpha - (t_k - \varphi)^\alpha}{t_k^\alpha - (t_k - \varphi)^\alpha} > 0, \quad t \in (t_k, t_k + \varphi).$$

For any $t \in (t_k, t_k + \varphi)$, we have

$$U(t) - U(t_k^+) < T_\alpha(U(t_k^+)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right), \quad \xi_2 \in (t_k, t).$$

Using the impulsive conditions in equation (3), we get

$$\begin{aligned} U(t) - a_k U(t_k) &< b_k T_\alpha(U(t_k)) \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right) \\ \frac{U(t)}{U(t_k)} &< b_k \frac{T_\alpha(U(t_k))}{U(t_k)} \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right) + a_k. \end{aligned}$$

Using $\frac{T_\alpha(U(t_k))}{U(t_k)} < \frac{1}{\varphi}$, we obtain

$$\frac{U(t)}{U(t_k)} < a_k + \frac{b_k}{\varphi} \left(\frac{t^\alpha - t_k^\alpha}{\alpha} \right).$$

That is,

$$(19) \quad \frac{U(t_k)}{U(t)} > \frac{\varphi \alpha}{\varphi \alpha a_k + b_k (t^\alpha - t_k^\alpha)}.$$

From (18) and (19), we get

$$\frac{U(t - \varphi)}{U(t)} > \frac{\varphi \alpha}{\varphi \alpha a_k + b_k (t^\alpha - t_k^\alpha)} \frac{(t - \varphi)^\alpha - (t_k - \varphi)^\alpha}{t_k^\alpha - (t_k - \varphi)^\alpha} \geq 0.$$

Case 2: If $t \in [c_1, t_{J(c_1)+1}]$, we consider three sub-cases.

Case 2.1: If $t_{J(c_1)} > c_1 - \varphi$ and $t \in [t_{J(c_1)} + \varphi, t_{J(c_1)+1}]$ then $t - \varphi \in [t_{J(c_1)}, t_{J(c_1)+1} - \varphi]$ and there are no impulsive moments in $(t - \varphi, t)$. By a similar analysis, as in Case 1.1 and using the Mean-Value Theorem on $(t_{J(c_1)}, t_{J(c_1)+1}]$, we get

$$\frac{U(t - \varphi)}{U(t)} > \frac{(t - \varphi)^\alpha - t_{J(c_1)}^\alpha}{t^\alpha - t_{J(c_1)}^\alpha} > 0, \quad t \in [t_{J(c_1)} + \varphi, t_{J(c_1)+1}].$$

Case 2.2: If $t_{J(c_1)} > c_1 - \varphi$ and $t \in [c_1, t_{J(c_1)} + \varphi)$, then $t - \varphi \in [c_1 - \varphi, t_{J(c_1)})$ and there is an impulsive moment $t_{J(c_1)}$ in $(t - \varphi, t)$. By a similar analysis, as in

Case 1.2, we have

$$\frac{U(t-\varphi)}{U(t)} > \frac{\varphi\alpha}{\varphi\alpha a_{J(c_1)} + b_{J(c_1)}(t^\alpha - t_{J(c_1)}^\alpha)} \frac{(t-\varphi)^\alpha - (t_{J(c_1)} - \varphi)^\alpha}{t_{J(c_1)}^\alpha - (t_{J(c_1)} - \varphi)^\alpha} \geq 0, \\ t \in (c_1, t_{J(c_1)} + \varphi).$$

Case 2.3: If $t_{J(c_1)} < c_1 - \varphi$, then for any $t \in [c_1, t_{J(c_1)+1}]$, $t - \varphi \in [c_1 - \varphi, t_{J(c_1)+1} - \varphi]$ and there are no impulsive moments in $(t - \varphi, t)$. By a similar analysis, as in Case 1.1, we obtain

$$\frac{U(t-\varphi)}{U(t)} > \frac{(t-\varphi)^\alpha - t_{J(c_1)}^\alpha}{t^\alpha - t_{J(c_1)}^\alpha} > 0, \quad t \in [c_1, t_{J(c_1)+1}].$$

Case 3: For $t \in (t_{J(d_1)}, d_1]$, there are three sub-cases:

Case 3.1: If $t_{J(d_1)} + \varphi < d_1$ and $t \in [t_{J(d_1)} + \varphi, d_1]$ then $t - \varphi \in [t_{J(d_1)}, d_1 - \varphi]$ and there are no impulsive moments in $(t - \varphi, t)$. By a similar analysis, as in Case 2.1, we have

$$\frac{U(t-\varphi)}{U(t)} > \frac{(t-\varphi)^\alpha - t_{J(d_1)}^\alpha}{t^\alpha - t_{J(d_1)}^\alpha} > 0, \quad t \in [t_{J(d_1)} + \varphi, d_1].$$

Case 3.2: If $t_{J(d_1)} + \varphi < d_1$ and $t \in [t_{J(d_1)}, t_{J(d_1)} + \varphi]$, then $t - \varphi \in [t_{J(d_1)} - \varphi, t_{J(d_1)})$ and there is an impulsive moment $t_{J(d_1)}$ in $(t - \varphi, t)$. By a similar analysis, as in Case 2.2, we obtain

$$\frac{U(t-\varphi)}{U(t)} > \frac{\varphi\alpha}{\varphi\alpha a_{J(d_1)} + b_{J(d_1)}(t^\alpha - t_{J(d_1)}^\alpha)} \frac{(t-\varphi)^\alpha - (t_{J(d_1)} - \varphi)^\alpha}{t_{J(d_1)}^\alpha - (t_{J(d_1)} - \varphi)^\alpha} \geq 0.$$

Case 3.3: If $t_{J(d_1)} + \varphi \geq d_1$, then for any $t \in (t_{J(d_1)}, d_1]$, we get $t - \varphi \in (t_{J(d_1)} - \varphi, d_1 - \varphi]$ and there is an impulsive moment $t_{J(d_1)}$ in $(t - \varphi, t)$. By a similar analysis, as in Case 3.2, we get

$$\frac{U(t-\varphi)}{U(t)} > \frac{\varphi\alpha}{\varphi\alpha a_{J(d_1)} + b_{J(d_1)}(t^\alpha - t_{J(d_1)}^\alpha)} \frac{(t-\varphi)^\alpha - (t_{J(d_1)} - \varphi)^\alpha}{t^\alpha - (t_{J(d_1)} - \varphi)^\alpha} \geq 0.$$

Combining all these cases, we have

$$\frac{U(t-\varphi)}{U(t)} > \begin{cases} M_{J(c_1)}^1(t) & \text{for } t \in [c_1, t_{J(c_1)+1}] \\ M_k^1(t) & \text{for } t \in (t_k, t_{k+1}], \quad k = J(c_1) + 1, \dots, J(d_1) - 1 \\ M_{J(d_1)}^1(t) & \text{for } t \in (t_{J(d_1)+1}, d_1]. \end{cases}$$

Hence by (17), we have

$$\begin{aligned}
& \sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} [w(t_k) - w(t_k^+)] \\
& \leq \int_{c_1}^{t_{J(c_1)+1}} [(p'(t))^2 t^{2-2\alpha} - p^2(t) Q(t) M_{J(c_1)}^1(t)] dt \\
& \quad + \sum_{k=J(c_1)+1}^{J(d_1)-1} \int_{t_k}^{t_{k+1}} [(p'(t))^2 t^{2-2\alpha} - p^2(t) Q(t) M_k^1(t)] dt \\
& \quad + \int_{t_{J(d_1)}}^{d_1} [(p'(t))^2 t^{2-2\alpha} - p^2(t) Q(t) M_{J(d_1)}^1(t)] dt \\
(20) \quad & + \int_{c_1}^{d_1} (1 - \alpha) t^{-\alpha} p^2(t) w(t) dt.
\end{aligned}$$

Since $T_\alpha(U(t))$ is non-increasing in $(c_1, t_{J(c_1)+1}]$, we have

$$U(t) > U(t) - U(c_1) = T_\alpha(U(\xi_3)) \left(\frac{t^\alpha - c_1^\alpha}{\alpha} \right) \geq T_\alpha(U(t)) \left(\frac{t^\alpha - c_1^\alpha}{\alpha} \right), \quad \xi_3 \in (c_1, t).$$

Letting $t \rightarrow t_{J(c_1)+1}^-$, it follows that

$$(21) \quad w(t_{J(c_1)+1}) < \frac{\alpha}{t_{J(c_1)+1}^\alpha - c_1^\alpha}.$$

Similarly, we can prove that on $(t_{k-1}, t_k]$, $k = J(c_1) + 2, \dots, J(d_1)$,

$$(22) \quad w(t_k) < \frac{\alpha}{t_k^\alpha - t_{k-1}^\alpha}.$$

Hence, from (21) and (22), we have

$$\begin{aligned}
& \sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} w(t_k) \left[\frac{a_k^* - b_k}{a_k^*} \right] \\
& \geq \alpha \left[p^2(t_{J(c_1)+1}) t_{J(c_1)+1}^{1-\alpha} \frac{a_{J(c_1)+1}^* - b_{J(c_1)+1}}{a_{J(c_1)+1}^*} \frac{1}{t_{J(c_1)+1}^\alpha - c_1^\alpha} \right. \\
& \quad \left. + \sum_{k=J(c_1)+2}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} \frac{a_k^* - b_k}{a_k^*} \frac{1}{t_k^\alpha - t_{k-1}^\alpha} \right] \\
& \geq \Lambda(p, c_1, d_1).
\end{aligned}$$

Thus we have

$$\sum_{k=J(c_1)+1}^{J(d_1)} p^2(t_k) t_k^{1-\alpha} w(t_k) \left[\frac{a_k^* - b_k}{a_k^*} \right] \geq \Lambda(p, c_1, d_1).$$

Therefore, from (20), we get

$$\begin{aligned} & \int_{c_1}^{t_{J(c_1)+1}} \left[(p'(t))^2 t^{2-2\alpha} - p^2(t) Q(t) M_{J(c_1)}^1(t) \right] dt \\ & + \sum_{k=J(c_1)+1}^{J(d_1)-1} \int_{t_k}^{t_{k+1}} \left[(p'(t))^2 t^{2-2\alpha} - p^2(t) Q(t) M_k^1(t) \right] dt \\ & + \int_{t_{J(d_1)}}^{d_1} \left[(p'(t))^2 t^{2-2\alpha} - p^2(t) Q(t) M_{J(d_1)}^1(t) \right] dt \\ & + \int_{c_1}^{d_1} (1-\alpha) t^{-\alpha} p^2(t) w(t) dt > \Lambda(p, c_1, d_1) \end{aligned}$$

which contradicts (12).

If $J(c_1) = J(d_1)$ then $\Lambda(p, c_1, d_1) = 0$ and there are no impulsive moments in $[c_1, d_1]$. Similarly to the proof of (20), we obtain

$$\int_{c_1}^{d_1} \left[(p'(t))^2 t^{2-2\alpha} - p^2(t) Q(t) M_{J(c_1)}^1(t) + p^2(t) (1-\alpha) t^{-\alpha} w(t) \right] dt > 0.$$

This again contradicts our assumption. Finally if $U(t)$ is eventually negative, we can consider $[c_2, d_2]$ and reach a similar contradiction. The proof of theorem is complete. \square

Following [15] and [25], we introduce a new class of functions. Let $D = \{(t, s) : t_0 \leq s \leq t\}$, then the functions $H_1, H_2 \in C(D, \mathbb{R})$ are said to belong to the class \mathcal{H} if

$$(A_6) \quad H_1(t, t) = H_2(t, t) = 0, \quad H_1(t, s) > 0, \quad H_2(t, s) > 0 \quad \text{for } t > s \text{ and}$$

$$(A_7) \quad H_1 \text{ and } H_2 \text{ have partial derivatives } \frac{\partial H}{\partial t} \text{ and } \frac{\partial H}{\partial s} \text{ on } D \text{ such that}$$

$$\frac{\partial H_1}{\partial t} = h_1(t, s) H_1(t, s), \quad \frac{\partial H_2}{\partial s} = -h_2(t, s) H_2(t, s)$$

where $h_1, h_2 \in L_{loc}(D, \mathbb{R})$.

For two constants $c, d \notin \{t_k\}$ with $c < d$ and a function $\Phi \in C([c, d], \mathbb{R})$, we define the operator $\Xi : C([c, d], \mathbb{R}) \rightarrow \mathbb{R}$ by

$$\Xi_c^d[\Phi] = \begin{cases} 0, & J(c) = J(d) \\ \Phi(t_{J(c)+1})\tau(c) + \sum_{k=J(c)+2}^{J(d)} \Phi(t_k)\sigma(t_k), & J(c) < J(d), \end{cases}$$

where

$$\tau(c) = t_{J(c)+1}^{1-\alpha} \frac{a_{J(c)+1}^* - b_{J(c)+1}}{a_{J(c)+1}^* (t_{J(c)+1}^\alpha - c^\alpha)}, \quad \sigma(t_k) = t_k^{1-\alpha} \frac{a_k^* - b_k}{a_k^* (t_k^\alpha - t_{k+1}^\alpha)}.$$

$$\begin{aligned} \Omega_{1,j} &= \int_{c_j}^{t_{J(c_j)+1}} H_1(t, c_j) Q(t) M_{J(c_j)}^j(t) dt \\ &+ \sum_{k=J(c_j)+1}^{J(\lambda_j)-1} \int_{t_k}^{t_{k+1}} H_1(t, c_j) Q(t) M_k^j(t) dt + \int_{t_{J(\lambda_j)}}^{\lambda_j} H_1(t, c_j) Q(t) M_{J(d_j)}^j(t) dt \\ &+ \int_{c_j}^{\lambda_j} H_1(t, c_j) [w^2(t) - w(t)t^{1-\alpha} h_1(t, c_j) - (1-\alpha)t^{-\alpha} w(t)] dt \end{aligned}$$

and

$$\begin{aligned} \Omega_{2,j} &= \int_{\lambda_j}^{t_{J(\lambda_j)+1}} H_2(d_j, t) Q(t) M_{J(\lambda_j)}^j(t) dt \\ &+ \sum_{k=J(\lambda_j)+1}^{J(d_j)-1} \int_{t_k}^{t_{k+1}} H_2(d_j, t) Q(t) M_k^j(t) dt + \int_{t_{J(d_j)}}^{d_j} H_2(d_j, t) Q(t) M_{J(d_j)}^j(t) dt \\ &+ \int_{\lambda_j}^{d_j} H_2(d_j, t) [w^2(t) + w(t)t^{1-\alpha} h_2(d_j, t) - (1-\alpha)t^{-\alpha} w(t)] dt. \end{aligned}$$

Theorem 0.9. *Assume that conditions (A₁) – (A₅) hold. Furthermore, for any $T \geq 0$ there exist c_j, d_j satisfying (A₆), (H₇) with $c_1 < \lambda_1 < d_1 \leq c_2 < \lambda_2 < d_2$. If there exists $H_1, H_2 \in \mathcal{H}$ such that*

$$(23) \quad \frac{1}{H_1(\lambda_1, c_1)} \Omega_{1,1} + \frac{1}{H_2(d_1, \lambda_1)} \Omega_{2,1} > \Lambda(H_1, H_2; c_j, d_j)$$

where

$$\Lambda(H_1, H_2; c_j, d_j) = - \left\{ \frac{1}{H_1(\lambda_j, c_j)} \Xi_{c_j}^{\lambda_j} [H_1(\cdot, c_j)] + \frac{1}{H_2(d_j, \lambda_j)} \Xi_{\lambda_j}^{d_j} [H_2(d_j, \cdot)] \right\},$$

then every solution of the problem (1)-(2) is oscillatory in G .

Proof. Suppose to the contrary that there is a non-oscillatory solution $u(x, t)$ of (1)-(2). Notice that regardless of whether there exist impulsive moments in $[c_1, \lambda_1]$ and $[\lambda_1, d_1]$, or not, we should consider the following cases $J(c_1) < J(\lambda_1) < J(d_1)$, $J(c_1) = J(\lambda_1) < J(d_1)$, $J(c_1) < J(\lambda_1) = J(d_1)$ and $J(c_1) = J(\lambda_1) = J(d_1)$. Moreover, the impulsive moments of $U(t-\varphi)$ having the following two cases: $t_{J(\lambda_j)} + \varphi > \lambda_j$ and $t_{J(\lambda_j)} + \varphi \leq \lambda_j$. Consider the case $J(c_1) < J(\lambda_1) < J(d_1)$, with $t_{J(\lambda_j)} + \varphi > \lambda_j$. For this case, the impulsive moments are $t_{J(\lambda_1)+1}, t_{J(\lambda_1)+2}, \dots, t_{J(d_1)}$ in $[\lambda_1, d_1]$.

Multiplying both sides of (15) by $H_1(t, c_1)$ and integrating from c_1 to λ_1 , we obtain

$$\int_{c_1}^{\lambda_1} H_1(t, c_1) T_\alpha(w(t)) dt \leq - \int_{c_1}^{\lambda_1} H_1(t, c_1) Q(t) \frac{U(t-\varphi)}{U(t)} dt - \int_{c_1}^{\lambda_1} H_1(t, c_1) w^2(t) dt.$$

Applying integration by parts, on the L.H.S, we get,

$$\begin{aligned} & \sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k, c_1) t_k^{1-\alpha} [w(t_k) - w(t_k^+)] - H_1(\lambda_1, c_1) \lambda_1^{1-\alpha} w(\lambda_1) \\ & - \int_{c_1}^{\lambda_1} w(t) [h_1(t, c_1) H_1(t, c_1) t^{1-\alpha} + H_1(t, c_1) (1-\alpha) t^{-\alpha}] dt \\ (24) \quad & \leq - \int_{c_1}^{\lambda_1} H_1(t, c_1) Q(t) \frac{U(t-\varphi)}{U(t)} dt - \int_{c_1}^{\lambda_1} w^2(t) H_1(t, c_1) dt. \end{aligned}$$

By Theorem 0.8, we can divide the interval $[c_1, \lambda_1]$ into several subintervals. Calculating the function $\frac{U(t-\varphi)}{U(t)}$, we obtain

$$\begin{aligned} \int_{c_1}^{\lambda_1} H_1(t, c_1) Q(t) \frac{U(t-\varphi)}{U(t)} dt & \geq \int_{c_1}^{t_{J(c_1)+1}} H_1(t, c_1) Q(t) M_{J(c_1)}^1(t) dt \\ & + \sum_{k=J(c_1)+1}^{J(\lambda_1)-1} \int_{t_k}^{t_{k+1}} H_1(t, c_1) Q(t) M_{k(t)}^1 dt \\ (25) \quad & + \int_{t_{J(\lambda_1)}}^{\lambda_1} H_1(t, c_1) Q(t) M_{J(\lambda_1)}^1(t) dt. \end{aligned}$$

From (24) and (25), we obtain

$$\begin{aligned} & \int_{c_1}^{t_{J(c_1)+1}} H_1(t, c_1) Q(t) M_{J(c_1)}^1(t) dt + \sum_{k=J(c_1)+1}^{J(\lambda_1)-1} \int_{t_k}^{t_{k+1}} H_1(t, c_1) Q(t) M_{k(t)}^1 dt \\ & + \int_{t_{J(\lambda_1)}}^{\lambda_1} H_1(t, c_1) Q(t) M_{J(\lambda_1)}^1(t) dt \\ & + \int_{c_1}^{\lambda_1} H_1(t, c_1) [w^2(t) - t^{1-\alpha} h_1(t, c_1) w(t) - (1-\alpha) t^{-\alpha} w(t)] dt \\ (26) \quad & \leq - \sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k, c_1) t_k^{1-\alpha} \left[\frac{a_k^* - b_k}{a_k^*} \right] w(t_k) - H_1(\lambda_1, c_1) \lambda_1^{1-\alpha} w(\lambda_1). \end{aligned}$$

On the other hand, multiplying both sides of (15) by $H_2(d_1, t)$, integrating from λ_1

to d_1 and proceeding as above, we get

$$\begin{aligned}
& \int_{\lambda_1}^{t_{J(\lambda_1)+1}} H_2(d_1, t) Q(t) M_{J(\lambda_1)}^1(t) dt + \sum_{k=J(\lambda_1)+1}^{J(d_1)-1} \int_{t_k}^{t_{k+1}} H_2(d_1, t_k) Q(t) M_{k(t)}^1 dt \\
& + \int_{t_{J(d_1)}}^{d_1} H_2(d_1, t) Q(t) M_{J(d_1)}^1(t) dt + \int_{\lambda_1}^{d_1} H_2(d_1, t) [w^2(t) + t^{1-\alpha} h_2(d_1, t) w(t) - w(t)(1-\alpha)t^{-\alpha}] dt \\
(27) \quad & \leq - \sum_{k=J(\lambda_1)+1}^{J(d_1)} H_2(d_1, t_k) t_k^{1-\alpha} \left[\frac{a_k^* - b_k}{a_k^*} \right] w(t_k) + H_2(d_1, \lambda_1) \lambda_1^{1-\alpha} w(\lambda_1).
\end{aligned}$$

Dividing (26) and (27) by $H_1(\lambda_1, c_1)$ and $H_2(d_1, \lambda_1)$ respectively and adding them together, we get

$$\begin{aligned}
& \frac{1}{H_1(\lambda_1, c_1)} \Omega_{1,1} + \frac{1}{H_2(d_1, \lambda_1)} \Omega_{2,1} \\
& \leq - \left[\frac{1}{H_1(\lambda_1, c_1)} \sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k, c_1) t_k^{1-\alpha} \left[\frac{a_k^* - b_k}{a_k^*} \right] w(t_k) \right. \\
(28) \quad & \left. + \frac{1}{H_2(d_1, \lambda_1)} \sum_{k=J(\lambda_1)+1}^{J(d_1)} H_2(d_1, t_k) t_k^{1-\alpha} \left[\frac{a_k^* - b_k}{a_k^*} \right] w(t_k) \right].
\end{aligned}$$

Using a similar procedure as in the derivation of (22), we obtain

$$(29) \quad \left. \begin{aligned}
& - \sum_{k=J(c_1)+1}^{J(\lambda_1)} H_1(t_k, c_1) t_k^{1-\alpha} \left[\frac{a_k^* - b_k}{a_k^*} \right] w(t_k) \leq -\Xi_{c_1}^{\lambda_1} [H_1(\cdot, c_1)] \\
& - \sum_{k=J(\lambda_1)+1}^{J(d_1)} H_2(d_1, t_k) t_k^{1-\alpha} \left[\frac{a_k^* - b_k}{a_k^*} \right] w(t_k) \leq -\Xi_{\lambda_1}^{d_1} [H_2(d_1, \cdot)].
\end{aligned} \right\}$$

From (28) and (29), we obtain

$$\begin{aligned}
& \frac{1}{H_1(\lambda_1, c_1)} \Omega_{1,1} + \frac{1}{H_2(d_1, \lambda_1)} \Omega_{2,1} \\
& \leq - \left\{ \frac{r_1}{H_1(\lambda_1, c_1)} \Xi_{c_1}^{\lambda_1} [H_1(\cdot, c_1)] + \frac{r_1}{H_2(d_1, \lambda_1)} \Xi_{\lambda_1}^{d_1} [H_2(d_1, \cdot)] \right\} \\
(30) \quad & \leq \Lambda(H_1, H_2; c_1, d_1)
\end{aligned}$$

which is a contradiction to condition (23). Suppose $u(x, t) < 0$. Then, we take the interval $[c_2, d_2]$, for equation (1). The proof is similar and hence omitted. The proof is complete. \square

4. EXAMPLE

In this section we provide an example to illustrate our results.

Example 0.10. Consider the following impulsive conformable differential equations

$$(31) \quad \left. \begin{aligned} & \frac{\partial^{1/2}}{\partial t^{1/2}} \left(\frac{\partial^{1/2}}{\partial t^{1/2}} (u(x, t)) \right) + \frac{m}{2} u(x, t - \frac{\pi}{8}) + \frac{m}{2} u(x, t - \frac{\pi}{8}) \\ & = t \Delta u(x, t) + \frac{1}{2} \int_0^\pi \Delta u(x, t - \xi) d\xi + F(x, t), \quad t \neq 2k\pi \pm \frac{\pi}{4}, \\ & u(x, t_k^+) = 4u(x, t_k), \quad \frac{\partial^{1/2}}{\partial t^{1/2}} u(x, t_k^+) = 5 \frac{\partial^{1/2}}{\partial t^{1/2}} u(x, t_k), \quad k = 1, 2, \dots, \end{aligned} \right\}$$

for $(x, t) \in (0, \pi) \times \mathbb{R}_+$, with the boundary condition

$$(32) \quad u_x(0, t) + u(0, t) = u_x(\pi, t) + u(\pi, t), \quad t \neq 2k\pi \pm \frac{\pi}{4}, \quad k = 1, 2, \dots$$

Here $\alpha = \frac{1}{2}$, $a_k = 4$, $b_k = 5$, $q(t) = \frac{m}{4}$, $q_1(t) = \frac{m}{2}$, $f(u) = 2u$, $f_1(u) = u$, $i = 1$, $a(t) = t$, $b(t, \xi) = \frac{1}{2}$, $[a, b] = [0, \pi]$, $\theta(t, \xi) = t - \xi$,

$$F(x, t) = e^{-x} \left(-2t - \frac{3}{2} \sin t + m \cos(t - \frac{\pi}{8}) \right)$$

and m is a positive constant. Also $\varphi = \frac{\pi}{8}$, $t_{k+1} - t_k = \pi/2 > \pi/8$. For any $T > 0$, we choose k large enough such that $T < c_1 = 4k\pi - \frac{\pi}{2} < d_1 = 4k\pi$ and $c_2 = 4k\pi + \frac{\pi}{8} < d_2 = 4k\pi + \frac{\pi}{2}$, $k = 1, 2, \dots$. Then there is an impulsive movement $t_k = 4k\pi - \frac{\pi}{4}$ in $[c_1, d_1]$ and an impulsive moment $t_{k+1} = 4k\pi + \frac{\pi}{4}$ in $[c_2, d_2]$. For $c = 2$, $c_1 = 1$ we have $Q(t) = m$, and we take $p(t) = \sin 8t \in J_p(c_j, d_j)$, $j = 1, 2$, $t_{J(c_1)} = 4k\pi - \frac{7\pi}{4}$, $t_{J(d_1)} = 4k\pi - \frac{\pi}{4}$, then by a straightforward calculation, the left

hand side of equation (12) becomes

$$\begin{aligned}
& \int_{c_j}^{t_{J(c_j)+1}} \left[(p'(t))^2 t^{2-2\alpha} - Q(t) p^2(t) M_{J(c_j)}^j(t) \right] dt \\
& \sum_{k=J(c_j)+1}^{J(d_j)-1} \int_{t_k}^{t_{k+1}} \left[(p'(t))^2 t^{2-2\alpha} - Q(t) p^2(t) M_{J(c_j)}^j(t) \right] dt \\
& \int_{t_{J(d_j)}}^{d_j} \left[(p'(t))^2 t^{2-2\alpha} - Q(t) p^2(t) M_{J(d_j)}^j(t) \right] dt + \int_{c_j}^{d_j} w(t) p^2(t) (1-\alpha) t^{-\alpha} dt \\
& \leq \int_{4k\pi - \frac{\pi}{2}}^{4k\pi} \frac{1 - t^{\frac{1}{2}} \sin t}{2 \cos t} t^{-\frac{1}{2}} \sin^2(8t) dt \\
& \quad + \int_{4k\pi - \frac{\pi}{2}}^{4k\pi - \frac{\pi}{4}} \left(t(8 \cos 8t)^2 - m \sin^2(8t) \left(\frac{(t - \frac{\pi}{8})^{\frac{1}{2}} - (4k\pi - \frac{7\pi}{4})^{\frac{1}{2}}}{t^{\frac{1}{2}} - (4k\pi - \frac{7\pi}{4})^{\frac{1}{2}}} \right) \right) dt \\
& \quad + \int_{4k\pi - \frac{\pi}{4}}^{4k\pi - \frac{\pi}{8}} \left(t(8 \cos 8t)^2 - m \sin^2(8t) \left(\frac{\frac{\pi}{16}}{\frac{\pi}{4} + 5 \left(t^{\frac{1}{2}} - (4k\pi - \frac{\pi}{4})^{\frac{1}{2}} \right)} \right) \right) \\
& \quad \times \left(\frac{(t - \frac{\pi}{8})^{\frac{1}{2}} - (4k\pi - \frac{\pi}{4} - \frac{\pi}{8})^{\frac{1}{2}}}{(4k\pi - \frac{\pi}{4})^{\frac{1}{2}} - (4k\pi - \frac{\pi}{4} - \frac{\pi}{8})^{\frac{1}{2}}} \right) \\
& \quad + \int_{4k\pi - \frac{\pi}{8}}^{4k\pi} \left(t(8 \cos 8t)^2 - m \sin^2(8t) \left(\frac{(t - \frac{\pi}{8})^{\frac{1}{2}} - (4k\pi - \frac{\pi}{4})^{\frac{1}{2}}}{t^{\frac{1}{2}} - (4k\pi - \frac{\pi}{4})^{\frac{1}{2}}} \right) \right) dt \\
& \simeq 593.01435 - m(0.44497)
\end{aligned}$$

for m large enough. On the other hand, note that $J(c_1) = k - 1$, $J(d_1) = k$, we have $\Lambda(p, c_i, d_i) = 0$. Therefore, condition (12) is satisfied in $[c_1, d_1]$. Similarly, we can prove that for $t \in [c_2, d_2]$. Hence by Theorem 0.8, every solution of (31)–(32) is oscillatory. In fact $u(x, t) = e^{-x} \cos t$ is one such solution of the boundary problem (31)–(32).

Conclusion:

In this article, the authors have mainly focused on deriving some new sufficient conditions for the oscillation of certain classes of impulsive conformable partial differential equations by using Riccati technique. This present paper can be regarded as an extension work of without impulse effect. The results obtained are essentially new and improved some of the results already prevailing in the existing literature in the classical case. We also illustrated the effectiveness of our main results with a suitable example.

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G.E. Chatzarakis

Department of Electrical and
Electronic Engineering Educators
School of Pedagogical and
Technological Education (ASPETE),
14121, Athens, Greece
E-mail: geaxatz@otenet.gr

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K. Logaarasi,

PG and Research Department of Mathematics
Thiruvalluvar Government Arts College
(Affiliated to Periyar University, Salem - 636 011)
Rasipuram - 637 401,
Namakkal (Dt), Tamil Nadu, India.
E-mail: logajoni@gmail.com,

T. Raja

PG and Research Department of Mathematics
Thiruvalluvar Government Arts College
(Affiliated to Periyar University, Salem - 636 011)
Rasipuram - 637 401,
Namakkal (Dt), Tamil Nadu, India.
E-mail: trmaths19@gmail.com,

V. Sadhasivam

PG and Research Department of Mathematics
Thiruvalluvar Government Arts College
(Affiliated to Periyar University, Salem - 636 011)
Rasipuram - 637 401,
Namakkal (Dt), Tamil Nadu, India.
E-mail: ovsadha@gmail.com