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A Reverse Mulholland-Type Inequality in the Whole Plane with Multi-Parameters

Michael Th. Rassias and Bicheng Yang*

By introducing multi-parameters, and applying weight coefficients, we prove a reverse Mulholland-type inequality in the whole plane with a best possible constant factor. Moreover, the equivalent forms and a few particular cases are also considered.

1. INTRODUCTION

Let $p > 1, \frac{1}{p} + \frac{1}{q} = 1, a_m, b_n \geq 0$,

$$0 < \sum_{m=1}^{\infty} a_m^p < \infty \quad \text{and} \quad 0 < \sum_{n=1}^{\infty} b_n^q < \infty.$$

We have the following Hardy-Hilbert inequality (cf. [**1**]):

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{m=1}^{\infty} a_m^p \right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q \right)^{1/q}, \tag{1.0.1}$$

with the best possible constant factor $\frac{\pi}{\sin(\pi/p)}$. The Mulholland inequality with the same best possible constant factor $\frac{\pi}{\sin(\pi/p)}$ was provided as follows (cf. Theorem

*Corresponding author. Michael Th. Rassias
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343 of [**1**], replacing $\frac{a_m}{m}, \frac{b_n}{n}$ by a_m, b_n):

$$\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{a_m b_n}{\ln mn} < \frac{\pi}{\sin(\pi/p)}\left(\sum_{m=2}^{\infty}\frac{a_m^p}{m^{1-p}}\right)^{1/p}\left(\sum_{n=2}^{\infty}\frac{b_n^q}{n^{1-q}}\right)^{1/q}. \tag{1.0.2}$$

The inequalities (1.0.1)-(1.0.2) are important in the field of analysis and its applications (cf. [**1**], [**2**]).

In 2007, the following Hilbert-type integral inequality in the whole plane was presented (cf. [**3**]):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{f(x)g(y)}{(1+e^{x+y})^{\lambda}}dxdy$$

$$< \quad B(\frac{\lambda}{2},\frac{\lambda}{2})\left(\int_{-\infty}^{\infty}e^{-\lambda x}f^2(x)dx\int_{-\infty}^{\infty}e^{-\lambda y}f^2(y)dy\right)^{\frac{1}{2}}, \tag{1.0.3}$$

where, the constant factor $B(\frac{\lambda}{2},\frac{\lambda}{2})$ $(\lambda>0, B(u,v)$ is the Beta function) is the best possible. Some extensions of (1.0.1)-(1.0.3) as well as other types of inequalities were established in [**4**]-[**29**].

In 2016, Yang and Chen [**30**] obtained the following extension of (1.0.1) in the whole plane with multi-parameters:

$$\sum_{|n|=1}^{\infty}\sum_{|m|=1}^{\infty}\frac{a_m b_n}{(|m-\xi|+|n-\eta|)^{\lambda}} < 2B(\lambda_1,\lambda_2)$$

$$\times\left[\sum_{|m|=1}^{\infty}|m-\xi|^{p(1-\lambda_1)-1}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=1}^{\infty}|n-\eta|^{q(1-\lambda_2)-1}b_n^q\right]^{\frac{1}{q}}, \tag{1.0.4}$$

where the constant factor $2B(\lambda_1,\lambda_2)$ $(0<\lambda_1,\lambda_2\le 1, \lambda_1+\lambda_2=\lambda, \xi,\eta\in[0,\frac{1}{2}])$ is the best possible. Another result on this kind of inequalities was provided by Xin et al. [**31**].

In the present paper, by introducing multi-parameters and applying weight coefficients, we obtain a reverse Mulholland-type inequality in the whole plane with a best possible constant factor similar to the reverse of (1.0.4). Moreover, the equivalent forms and a few particular cases are also considered.

2. SOME LEMMAS

In the sequel, we shall assume that $0<p<1$ $(q<0)$, $\frac{1}{p}+\frac{1}{q}=1$, $\sigma\in\mathbf{R}=(-\infty,\infty)$, $-\sigma<\lambda_1,\lambda_2\le 1-\sigma$, $\lambda_1+\lambda_2=\lambda, \rho\ge 1$,

$$\arccos\sqrt{1-\frac{1}{\rho}}\le\gamma\le\pi-\arccos\sqrt{1-\frac{1}{\rho}}\,(\gamma=\alpha,\beta), \tag{2.0.5}$$

and $\zeta \in (-1,1)$, satisfying

$$\frac{1}{\rho(1-\cos\gamma)} - 1 \le \zeta \le 1 - \frac{1}{\rho(1+\cos\gamma)} \tag{2.0.6}$$

$((\zeta,\gamma) = (\xi,\alpha)$ or $(\eta,\beta))$. In particular, for $\rho = 1$, it follows that $\alpha = \beta = \frac{\pi}{2}$ and $\xi = \eta = 0$.

Remark 1. *In view of (2.0.6) we have*

$$\rho(1 \pm \zeta)(1 \mp \cos\gamma) \ge 1, (\zeta,\gamma) = (\xi,\alpha) \text{ (or } (\eta,\beta)).$$

We define

$$h_\gamma(\lambda_1) := \frac{2(\lambda + 2\sigma)\csc^2\gamma}{(\lambda_1+\sigma)(\lambda_2+\sigma)} \in \mathbf{R}_+ = (0,\infty) \ (\gamma = \alpha,\beta). \tag{2.0.7}$$

For $|t| > 1$ $(t = x, y), (\zeta,\gamma,t) = (\xi,\alpha,x)$ (or (η,β,y)), we set the following functions

$$A_{\zeta,\gamma}(t) := |t-\zeta| + (t-\zeta)\cos\gamma,$$

and

$$H(x,y) := \frac{[\min\{\ln \rho A_{\xi,\alpha}(x), \ln \rho A_{\eta,\beta}(y)\}]^\sigma}{[\max\{\ln \rho A_{\xi,\alpha}(x), \ln \rho A_{\eta,\beta}(y)\}]^{\lambda+\sigma}}.$$

Lemma 2.1. *Define two weight coefficients as follows:*

$$\omega(\lambda_2, m) := \sum_{|n|=2}^{\infty} \frac{H(m,n)}{A_{\eta,\beta}(n)} \frac{\ln^{\lambda_1} \rho A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2} \rho A_{\eta,\beta}(n)}, |m| \in \mathbf{N}\backslash\{1\}, \tag{2.0.8}$$

$$\varpi(\lambda_1, n) := \sum_{|m|=2}^{\infty} \frac{H(m,n)}{A_{\xi,\alpha}(m)} \frac{\ln^{\lambda_2} \rho A_{\eta,\beta}(n)}{\ln^{1-\lambda_1} \rho A_{\xi,\alpha}(m)}, |n| \in \mathbf{N}\backslash\{1\}, \tag{2.0.9}$$

where

$$\sum_{|j|=2}^{\infty} \cdots = \sum_{j=-2}^{-\infty} + \cdots + \sum_{j=2}^{\infty} \cdots \quad (j = m, n).$$

Lemma 2.2. *(cf. [**31**]) Let us assume that $g(t)$ (> 0) is strictly decreasing in $(1,\infty)$, satisfying*

$$\int_1^\infty g(t)dt \in \mathbf{R}_+.$$

We have

$$\int_2^\infty g(t)dt < \sum_{n=2}^{\infty} g(n) < \int_1^\infty g(t)dt. \tag{2.0.10}$$

If $(-1)^i g(t) > 0$ $(i = 0, 1, 2; t \in (\frac{3}{2}, \infty))$,

$$\int_{\frac{3}{2}}^{\infty} g(t)dt \in \mathbf{R}_+,$$

*then we have the following Hermite-Hadamard inequality (cf. [**32**]):*

$$\sum_{n=2}^{\infty} g(n) < \int_{\frac{3}{2}}^{\infty} g(t)dt. \tag{2.0.11}$$

Lemma 2.3. *For* $\lambda_1 > -\sigma, -\sigma < \lambda_2 \leq 1 - \sigma$, *the following inequalities are valid:*

$$h_\beta(\lambda_1)(1 - \theta(\lambda_2, m)) < \omega(\lambda_2, m) < h_\beta(\lambda_1), |m| \in \mathbf{N}\backslash\{1\}, \tag{2.0.12}$$

where

$$\begin{aligned} \theta(\lambda_2, m) &:= \frac{(\lambda_1 + \sigma)(\lambda_2 + \sigma)}{\lambda + 2\sigma} \int_0^{\frac{\ln \rho(2+\eta)(1+\cos\beta)}{\ln \rho A_{\xi,\alpha}(m)}} \frac{(\min\{1, u\})^\sigma u^{\lambda_2 - 1}}{(\max\{1, u\})^{\lambda+\sigma}} du \\ &= O\left(\frac{1}{\ln^{\lambda_2+\sigma} \rho A_{\xi,\alpha}(m)}\right) \in (0, 1). \end{aligned} \tag{2.0.13}$$

Proof. For $|m| \in \mathbf{N}\backslash\{1\}$, we set

$$\begin{aligned} H^{(1)}(m, y) &:= \frac{[\min\{\ln \rho A_{\xi,\alpha}(m), \ln \rho(y - \eta)(\cos\beta - 1)\}]^\sigma}{[\max\{\ln \rho A_{\xi,\alpha}(m), \ln \rho(y - \eta)(\cos\beta - 1)\}]^{\lambda+\sigma}}, y < -1, \\ H^{(2)}(m, y) &:= \frac{[\min\{\ln \rho A_{\xi,\alpha}(m), \ln \rho(y - \eta)(\cos\beta + 1)\}]^\sigma}{[\max\{\ln \rho A_{\xi,\alpha}(m), \ln \rho(y - \eta)(\cos\beta + 1)\}]^{\lambda+\sigma}}, y > 1, \end{aligned}$$

wherefrom,

$$H^{(1)}(m, -y) = \frac{[\min\{\ln \rho A_{\xi,\alpha}(m), \ln \rho(y + \eta)(1 - \cos\beta)\}]^\sigma}{[\max\{\ln \rho A_{\xi,\alpha}(m), \ln \rho(y + \eta)(1 - \cos\beta)\}]^{\lambda+\sigma}}, y > 1.$$

Then we deduce that

$$\begin{aligned} \omega(\lambda_2, m) &= \sum_{n=-2}^{-\infty} \frac{H^{(1)}(m, n)}{(n - \eta)(\cos\beta - 1)} \frac{\ln^{\lambda_1} \rho A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2} \rho(n - \eta)(\cos\beta - 1)} \\ &\quad + \sum_{n=2}^{\infty} \frac{H^{(2)}(m, n)}{(n - \eta)(\cos\beta + 1)} \frac{\ln^{\lambda_1} \rho A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2} \rho(n - \eta)(\cos\beta + 1)} \\ &= \frac{\ln^{\lambda_1} \rho A_{\xi,\alpha}(m)}{1 - \cos\beta} \sum_{n=2}^{\infty} \frac{H^{(1)}(m, -n)}{(n + \eta) \ln^{1-\lambda_2} \rho(n + \eta)(1 - \cos\beta)} \\ &\quad + \frac{\ln^{\lambda_1} \rho A_{\xi,\alpha}(m)}{1 + \cos\beta} \sum_{n=2}^{\infty} \frac{H^{(2)}(m, n)}{(n - \eta) \ln^{1-\lambda_2} \rho(n - \eta)(1 + \cos\beta)}. \end{aligned} \tag{2.0.14}$$

By the fact that $\lambda_1 > -\sigma$, $-\sigma < \lambda_2 \leq 1-\sigma$ $(\lambda_1+\lambda_2=\lambda)$, we obtain that for $y>1, i=1,2,$

$$\frac{H^{(i)}(m,(-1)^i y)}{[y-(-1)^i\eta]\ln^{1-\lambda_2}\rho[y-(-1)^i\eta][1+(-1)^i\cos\beta]}$$

$$= \begin{cases} \dfrac{1}{[y-(-1)^i\eta](\ln\rho A_{\xi,\alpha}(m))^{\lambda+\sigma}\ln^{1-\lambda_2-\sigma}\rho[y-(-1)^i\eta][1+(-1)^i\cos\beta]}, \\ \qquad \dfrac{1}{\rho}<[y-(-1)^i\eta][1+(-1)^i\cos\beta]\leq A_{\xi,\alpha}(m) \\ \dfrac{(\ln\rho A_{\xi,\alpha}(m))^{\sigma}}{[y-(-1)^i\eta]\ln^{1+\lambda_1+\sigma}\rho[y-(-1)^i\eta][1+(-1)^i\cos\beta]}, \\ \qquad [y-(-1)^i\eta][1+(-1)^i\cos\beta]>A_{\xi,\alpha}(m) \end{cases}$$

are strictly decreasing in $(1,\infty)$. By (2.0.14) and (2.0.10) we get

$$\omega(\lambda_2,m) < \frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1-\cos\beta}\int_1^\infty \frac{H^{(1)}(m,-y)dy}{(y+\eta)\ln^{1-\lambda_2}\rho(y+\eta)(1-\cos\beta)}$$
$$+\frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1+\cos\beta}\int_1^\infty \frac{H^{(2)}(m,y)dy}{(y-\eta)\ln^{1-\lambda_2}\rho(y-\eta)(1+\cos\beta)}.$$

Setting

$$u=\frac{\ln\rho(y+\eta)(1-\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)} \quad \left(u=\frac{\ln\rho(y-\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}\right)$$

in the above first (second) integral, in view of Remark 1, by simplifications, we obtain

$$\begin{aligned}\omega(\lambda_2,m) &< \left(\frac{1}{1-\cos\beta}+\frac{1}{1+\cos\beta}\right)\int_0^\infty \frac{(\min\{1,u\})^{\sigma}u^{\lambda_1-1}}{(\max\{1,u\})^{\lambda+\sigma}}du \\ &= 2\csc^2\beta\left(\int_0^1 u^{\lambda_1+\sigma-1}du+\int_1^\infty \frac{u^{\lambda_1-1}}{u^{\lambda+\sigma}}du\right)=h_\beta(\lambda_1).\end{aligned}$$

By (2.0.14) and (2.0.10), similarly, we have

$$\begin{aligned}\omega(\lambda_2,m) &> \frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1-\cos\beta}\int_2^\infty \frac{H^{(1)}(m,-y)dy}{(y+\eta)\ln^{1-\lambda_2}\rho(y+\eta)(1-\cos\beta)} \\ &\quad +\frac{\ln^{\lambda_1}\rho A_{\xi,\alpha}(m)}{1+\cos\beta}\int_2^\infty \frac{H^{(2)}(m,y)dy}{(y-\eta)\ln^{1-\lambda_2}\rho(y-\eta)(1+\cos\beta)}\end{aligned}$$

$$\begin{aligned}
&\geq 2\csc^2\beta\int_{\frac{\ln\rho(2+\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}}^{\infty}\frac{(\min\{1,u\})^{\sigma}u^{\lambda_1-1}}{(\max\{1,u\})^{\lambda+\sigma}}du\\
&= h_{\beta}(\lambda_1)-2\csc^2\beta\int_0^{\frac{\ln\rho(2+\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}}\frac{(\min\{1,u\})^{\sigma}u^{\lambda_1-1}}{(\max\{1,u\})^{\lambda+\sigma}}du\\
&= h_{\beta}(\lambda_1)(1-\theta(\lambda_2,m))>0,
\end{aligned}$$

where $\theta(\lambda_2,m)$ is as in (2.0.13). It follows that $\theta(\lambda_2,m)<1$ and for

$$A_{\xi,\alpha}(m)\geq(2+\eta)(1+\cos\beta),$$

we have

$$\begin{aligned}
0 &< \theta(\lambda_2,m)=\frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln\rho(2+\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}}u^{\lambda_2+\sigma-1}du\\
&= \frac{\lambda_1+\sigma}{\lambda+2\sigma}\left[\frac{\ln\rho(2+\eta)(1+\cos\beta)}{\ln\rho A_{\xi,\alpha}(m)}\right]^{\lambda_2+\sigma}.
\end{aligned}$$

Hence, (2.0.12) and (2.0.13) are valid. This completes the proof of the lemma. □

Similarly, we obtain the lemma below:

Lemma 2.4. *For $\lambda_2>-\sigma,-\sigma<\lambda_1\leq 1-\sigma$, the following inequalities are valid:*

$$h_{\alpha}(\lambda_1)(1-\widetilde{\theta}(\lambda_1,n))<\varpi(\lambda_1,n)<h_{\alpha}(\lambda_1),|n|\in\mathbf{N}\backslash\{1\}, \tag{2.0.15}$$

where

$$\begin{aligned}
\widetilde{\theta}(\lambda_1,n) &:= \frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln\rho(2+\xi)(1+\cos\alpha)}{\ln\rho A_{\eta,\beta}(n)}}\frac{(\min\{1,u\})^{\sigma}u^{\lambda_1-1}}{(\max\{1,u\})^{\lambda+\sigma}}du\\
&= O\left(\frac{1}{\ln^{\lambda_1+\sigma}\rho A_{\eta,\beta}(n)}\right)\in(0,1).
\end{aligned} \tag{2.0.16}$$

Lemma 2.5. *If $(\zeta,\gamma,k)=(\xi,\alpha,m)$ (or (η,β,n)), then for $\varepsilon>0$, we have*

$$H_{\varepsilon}(\zeta,\gamma):=\sum_{|k|=2}^{\infty}\frac{\ln^{-1-\varepsilon}\rho A_{\zeta,\gamma}(k)}{A_{\zeta,\gamma}(k)}=\frac{1}{\varepsilon}(2\csc^2\gamma+o(1))\qquad(\varepsilon\to 0^+). \tag{2.0.17}$$

Proof. By (2.0.11), we get that

$$\begin{aligned}
H_{\varepsilon}(\zeta,\gamma) &= \sum_{k=-2}^{-\infty}\frac{\ln^{-1-\varepsilon}\rho(k-\zeta)(\cos\gamma-1)}{(k-\zeta)(\cos\gamma-1)}\\
&\quad+\sum_{k=2}^{\infty}\frac{\ln^{-1-\varepsilon}\rho(k-\zeta)(\cos\gamma+1)}{(k-\zeta)(\cos\gamma+1)}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{k=2}^{\infty}\left[\frac{\ln^{-1-\varepsilon}\rho(k+\zeta)(1-\cos\gamma)}{(k+\zeta)(1-\cos\gamma)}+\frac{\ln^{-1-\varepsilon}\rho(k-\zeta)(\cos\gamma+1)}{(k-\zeta)(\cos\gamma+1)}\right]\\
&< \int_{\frac{3}{2}}^{\infty}\left[\frac{\ln^{-1-\varepsilon}\rho(y+\zeta)(1-\cos\gamma)}{(y+\zeta)(1-\cos\gamma)}+\frac{\ln^{-1-\varepsilon}\rho(y-\zeta)(\cos\gamma+1)}{(y-\zeta)(\cos\gamma+1)}\right]dy\\
&= \frac{1}{\varepsilon}\left[\frac{\ln^{-\varepsilon}\rho(\frac{3}{2}+\zeta)(1-\cos\gamma)}{1-\cos\gamma}+\frac{\ln^{-\varepsilon}\rho(\frac{3}{2}-\zeta)(1+\cos\gamma)}{1+\cos\gamma}\right]\\
&= \frac{1}{\varepsilon}\left(\frac{1}{1-\cos\gamma}+\frac{1}{1+\cos\gamma}+o_1(1)\right)(\varepsilon\to 0^+).
\end{aligned}
$$

By (2.0.10), we derive that

$$
\begin{aligned}
H_\varepsilon(\zeta,\gamma) &= \sum_{n=2}^{\infty}\left[\frac{\ln^{-1-\varepsilon}\rho(n+\zeta)(1-\cos\gamma)}{(n+\zeta)(1-\cos\gamma)}+\frac{\ln^{-1-\varepsilon}\rho(n-\zeta)(\cos\gamma+1)}{(n-\zeta)(\cos\gamma+1)}\right]\\
&> \int_{2}^{\infty}\left[\frac{\ln^{-1-\varepsilon}\rho(y+\zeta)(1-\cos\gamma)}{(y+\zeta)(1-\cos\gamma)}+\frac{\ln^{-1-\varepsilon}\rho(y-\zeta)(\cos\gamma+1)}{(y-\zeta)(\cos\gamma+1)}\right]dy\\
&= \frac{1}{\varepsilon}\left[\frac{\ln^{-\varepsilon}\rho(2+\zeta)(1-\cos\gamma)}{1-\cos\gamma}+\frac{\ln^{-\varepsilon}\rho(2-\zeta)(1+\cos\gamma)}{1+\cos\gamma}\right]\\
&= \frac{1}{\varepsilon}\left(\frac{1}{1-\cos\gamma}+\frac{1}{1+\cos\gamma}+o_2(1)\right)(\varepsilon\to 0^+).
\end{aligned}
$$

Hence, (2.0.17) is valid. This completes the proof of the lemma. □

3. MAIN RESULTS AND SOME PARTICULAR CASES

Setting the constant

$$
k_{\alpha,\beta}(\lambda_1) := h_\beta^{1/p}(\lambda_1)h_\alpha^{1/q}(\lambda_1) = \frac{2(\lambda+2\sigma)\csc^{2/p}\beta\csc^{2/q}\alpha}{(\lambda_1+\sigma)(\lambda_2+\sigma)}, \tag{3.0.18}
$$

and using the kernel

$$
H(m,n) = \frac{[\min\{\ln\rho A_{\xi,\alpha}(m),\ln\rho A_{\eta,\beta}(n)\}]^\sigma}{[\max\{\ln\rho A_{\xi,\alpha}(m),\ln\rho A_{\eta,\beta}(n)\}]^{\lambda+\sigma}},
$$

it follows that

Theorem 3.6. *If* $a_m, b_n \geq 0$ $(|m|,|n| \in \mathbf{N}\backslash\{1\})$, *satisfying*

$$
\begin{aligned}
0 &< \sum_{|m|=2}^{\infty}\frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p < \infty,\\
0 &< \sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q < \infty,
\end{aligned}
$$

then we have the following reverse equivalent inequalities:

$$\begin{aligned} I &:= \sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty} H(m,n)a_m b_n \\ &> k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=2}^{\infty}(1-\theta(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}} \\ &\quad \times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}}, \end{aligned} \tag{3.0.19}$$

$$\begin{aligned} J_1 &:= \left[\sum_{|n|=2}^{\infty}\frac{\ln^{p\lambda_2-1}\rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)}\left(\sum_{|m|=2}^{\infty}H(m,n)a_m\right)^p\right]^{\frac{1}{p}} \\ &> k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=2}^{\infty}(1-\theta(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}}, \end{aligned} \tag{3.0.20}$$

$$\begin{aligned} J_2 &:= \left[\sum_{|m|=2}^{\infty}\frac{\ln^{q\lambda_1-1}\rho A_{\xi,\alpha}(m)}{(1-\theta(\lambda_2,m))^{q-1}A_{\xi,\alpha}(m)}\left(\sum_{|n|=2}^{\infty}H(m,n)b_n\right)^q\right]^{\frac{1}{q}} \\ &> k_{\alpha,\beta}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{1/q}. \end{aligned} \tag{3.0.21}$$

In particular, (i) for $\alpha = \beta = \frac{\pi}{2}$, $\frac{1}{\rho} - 1 \leq \xi, \eta \leq 1 - \frac{1}{\rho}$, setting

$$\begin{aligned} \theta_1(\lambda_2, m) &:= \frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln\rho(2+\eta)}{\ln\rho|m-\xi|}}\frac{(\min\{1,u\})^{\sigma}u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du \\ &= O\left(\frac{1}{\ln^{\lambda_2+\sigma}\rho|m-\xi|}\right) \in (0,1), \end{aligned}$$

we have the following reverse equivalent inequalities:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{(\min\{\ln\rho|m-\xi|,\ln\rho|n-\eta|\})^{\sigma}}{(\max\{\ln\rho|m-\xi|,\ln\rho|n-\eta|\})^{\lambda+\sigma}}a_m b_n$$
$$> \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\left[\sum_{|m|=2}^{\infty}(1-\theta_1(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}\rho|m-\xi|}{|m-\xi|^{1-p}}a_m^p\right]^{\frac{1}{p}}$$
$$\times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho\,|n-\eta|}{|n-\eta|^{1-q}}b_n^q\right]^{\frac{1}{q}}, \tag{3.0.22}$$

$$\left\{\sum_{|n|=2}^{\infty}\frac{\ln^{p\lambda_2-1}\rho\,|n-\eta|}{|n-\eta|}\left[\sum_{|m|=2}^{\infty}\frac{(\min\{\ln\rho|m-\xi|,\ln\rho|n-\eta|\})^{\sigma}a_m}{(\max\{\ln\rho|m-\xi|,\ln\rho|n-\eta|\})^{\lambda+\sigma}}\right]^{p}\right\}^{\frac{1}{p}}$$
$$\frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\left[\sum_{|m|=2}^{\infty}(1-\theta_1(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}\rho\,|m-\xi|}{|m-\xi|^{1-p}}a_m^p\right]^{\frac{1}{p}}. \tag{3.0.23}$$

$$\left\{\sum_{|m|=2}^{\infty}\frac{\ln^{q\lambda_1-1}\rho|m-\xi|}{(1-\theta_1(\lambda_2,m))^{q-1}|m-\xi|}\right.$$
$$\left.\times\left[\sum_{|n|=2}^{\infty}\frac{(\min\{\ln\rho|m-\xi|,\ln\rho|n-\eta|\})^{\sigma}b_n}{(\max\{\ln\rho|m-\xi|,\ln\rho|n-\eta|\})^{\lambda+\sigma}}\right]^{q}\right\}^{\frac{1}{q}}$$
$$> k_{\alpha,\beta}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho|n-\eta|}{|n-\eta|^{1-q}}b_n^q\right]^{\frac{1}{q}}. \tag{3.0.24}$$

(ii) For $\xi=\eta=0,\ \arccos\sqrt{1-\frac{1}{\rho}}\le\alpha,\beta\le\pi-\arccos\sqrt{1-\frac{1}{\rho}}$, setting

$$\begin{aligned}\theta_2(\lambda_2,m) &:= \frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln 2\rho(1+\cos\beta)}{\ln\rho(|m|+m\cos\alpha)}}\frac{(\min\{1,u\})^{\sigma}u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du\\ &= O\left(\frac{1}{\ln^{\lambda_2+\sigma}\rho(|m|+m\cos\alpha)}\right)\in(0,1),\end{aligned}$$

we have the following reverse equivalent inequalities:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{[\min\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+n\cos\beta)\}]^{\sigma}a_m b_n}{[\max\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+\cos\beta)\}]^{\lambda+\sigma}}$$

$$> \quad k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=2}^{\infty}(1-\theta_2(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}\rho(|m|+m\cos\alpha)}{(|m|+m\cos\alpha)^{1-p}}a_m^p\right]^{\frac{1}{p}}$$

$$(3.0.25) \qquad \times\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho(|n|+\cos\beta)}{(|n|+\cos\beta)^{1-q}}b_n^q\right]^{\frac{1}{q}},$$

$$\left\{\sum_{|n|=2}^{\infty}\frac{\ln^{p\lambda_2-1}\rho(|n|+\cos\beta)}{|n|+\cos\beta}\right.$$

$$\left.\times\left[\sum_{|m|=2}^{\infty}\frac{[\min\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+n\cos\beta)\}]^{\sigma}a_m}{[\max\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+\cos\beta)\}]^{\lambda+\sigma}}\right]^p\right\}^{\frac{1}{p}}$$

$$(3.0.26) \qquad > k_{\alpha,\beta}(\lambda_1)\left[\sum_{|m|=2}^{\infty}(1-\theta_2(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}\rho(|m|+m\cos\alpha)}{(|m|+m\cos\alpha)^{1-p}}a_m^p\right]^{\frac{1}{p}},$$

and

$$\left\{\sum_{|m|=2}^{\infty}\frac{\ln^{q\lambda_1-1}\rho(|m|+m\cos\alpha)}{(1-\theta_2(\lambda_2,m))^{q-1}(|m|+m\cos\alpha)}\right.$$

$$\left.\times\left[\sum_{|n|=2}^{\infty}\frac{[\min\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+n\cos\beta)\}]^{\sigma}b_n}{[\max\{\ln\rho(|m|+m\cos\alpha),\ln\rho(|n|+\cos\beta)\}]^{\lambda+\sigma}}\right]^q\right\}^{\frac{1}{q}}$$

$$(3.0.27) > \quad k_{\alpha,\beta}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho(|n|+n\cos\beta)}{(|n|+n\cos\beta)^{1-q}}b_n^q\right]^{\frac{1}{q}}.$$

Proof. By the reverse Hölder inequality with weight (cf. **[33]**) and (2.0.9), we obtain that

$$\begin{aligned}
&\left(\sum_{|m|=2}^{\infty} H(m,n)a_m\right)^p \\
&= \left\{\sum_{|m|=2}^{\infty} H(m,n)\left[\frac{(A_{\xi,\alpha}(m))^{1/q}\ln^{(1-\lambda_1)/q}\rho A_{\xi,\alpha}(m)}{\ln^{(1-\lambda_2)/p}\rho A_{\eta,\beta}(n)}a_m\right]\right. \\
&\quad \left.\times\left[\frac{\ln^{(1-\lambda_2)/p}\rho A_{\eta,\beta}(n)}{(A_{\xi,\alpha}(m))^{1/q}\ln^{(1-\lambda_1)/q}\rho A_{\xi,\alpha}(m)}\right]\right\}^p \\
&\geq \sum_{|m|=2}^{\infty} H(m,n)\frac{(A_{\xi,\alpha}(m))^{p/q}\ln^{(1-\lambda_1)p/q}\rho A_{\xi,\alpha}(m)}{\ln^{1-\lambda_2}\rho A_{\eta,\beta}(n)}a_m^p \\
&\quad \times\left[\sum_{|m|=2}^{\infty} H(m,n)\frac{\ln^{(1-\lambda_2)q/p}\rho A_{\eta,\beta}(n)}{A_{\xi,\alpha}(m)\ln^{1-\lambda_1}\rho A_{\xi,\alpha}(m)}\right]^{p-1} \\
&= \frac{(\varpi(\lambda_1,n))^{p-1}A_{\eta,\beta}(n)}{\ln^{p\lambda_2-1}\rho A_{\eta,\beta}(n)}\sum_{|m|=2}^{\infty}\frac{H(m,n)(A_{\xi,\alpha}(m))^{\frac{p}{q}}\ln^{(1-\lambda_1)\frac{p}{q}}\rho A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n)\ln^{1-\lambda_2}\rho A_{\eta,\beta}(n)}a_m^p.
\end{aligned}$$

By (2.0.15), for $0 < p < 1$, it follows that $(\varpi(\lambda_1,n))^{p-1} > h_\alpha^{p-1}(\lambda_1)$ and

$$\begin{aligned}
J_1 &> h_\alpha^{1/q}(\lambda_1)\left[\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{H(m,n)(A_{\xi,\alpha}(m))^{p/q}\ln^{(1-\lambda_1)p/q}\rho A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n)\ln^{1-\lambda_2}\rho A_{\eta,\beta}(n)}a_m^p\right]^{\frac{1}{p}} \\
&= h_\alpha^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\sum_{|n|=2}^{\infty}\frac{H(m,n)(A_{\xi,\alpha}(m))^{p/q}\ln^{(1-\lambda_1)p/q}\rho A_{\xi,\alpha}(m)}{A_{\eta,\beta}(n)\ln^{1-\lambda_2}\rho A_{\eta,\beta}(n)}a_m^p\right]^{\frac{1}{p}} \\
&= h_\alpha^{1/q}(\lambda_1)\left[\sum_{|m|=2}^{\infty}\omega(\lambda_2,m)\frac{\ln^{p(1-\lambda_1)-1}\rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}}a_m^p\right]^{\frac{1}{p}}. \qquad (3.0.28)
\end{aligned}$$

By (2.0.12) and (3.0.18) we have (3.0.20).

Using again the reverse Hölder inequality, we have

$$\begin{aligned}
I &= \sum_{|n|=2}^{\infty}\left[\frac{\ln^{\lambda_2-(1/p)}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1/p}}\sum_{|m|=2}^{\infty}H(m,n)a_m\right]\left[\frac{\ln^{(1/p)-\lambda_2}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{-1/p}}b_n\right] \\
&\geq J_1\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}\rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}}b_n^q\right]^{\frac{1}{q}}, \qquad (3.0.29)
\end{aligned}$$

and then by (3.0.20) we deduce (3.0.19).
On the other hand, assuming that (3.0.19) is valid, we set

$$b_n := \frac{\ln^{p\lambda_2 - 1} \rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \left(\sum_{|m|=2}^{\infty} H(m,n) a_m \right)^{p-1}, \quad |n| \in \mathbf{N} \backslash \{1\},$$

and find

$$J_1 = \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} \rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q \right]^{\frac{1}{p}}.$$

By (3.0.28), it follows that $J_1 > 0$. If $J_1 = \infty$, then (3.0.20) is trivially valid; if $J_1 < \infty$, then we have

$$\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} \rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q = J_1^p = I > k_{\alpha,\beta}(\lambda_1)$$

$$\times \left[\sum_{|m|=2}^{\infty} (1 - \theta(\lambda_2, m)) \frac{\ln^{p(1-\lambda_1)-1} \rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{p}}$$

$$\times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} \rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q \right]^{\frac{1}{q}},$$

$$\begin{aligned} J_1 &= \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} \rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q \right]^{\frac{1}{p}} \\ &> k_{\alpha,\beta}(\lambda_1) \left[\sum_{|m|=2}^{\infty} (1 - \theta(\lambda_2, m)) \frac{\ln^{p(1-\lambda_1)-1} \rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{p}}. \end{aligned}$$

Hence, (3.0.20) is valid, and thus (3.0.19) and (3.0.20) are equivalent. Next we will show that inequalities (3.0.19) and (3.0.21) are equivalent, which will show that all the inequalities (3.0.19), (3.0.20) and (3.0.21) are equivalent.
Having proved that (3.0.19) is valid, we then set

$$a_m := \frac{\ln^{q\lambda_1 - 1} \rho A_{\xi,\alpha}(m)}{(1 - \theta(\lambda_2, m))^{q-1} A_{\xi,\alpha}(m)} \left(\sum_{|n|=2}^{\infty} H(m,n) b_n \right)^{q-1}, \quad |m| \in \mathbf{N} \backslash \{1\},$$

and find

$$J_2 = \left[\sum_{|m|=2}^{\infty} (1 - \theta(\lambda_2, m)) \frac{\ln^{p(1-\lambda_1)-1} \rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{q}}.$$

If $J_2 = 0$, then (3.0.21) is impossible, namely $J_2 > 0$. If $J_2 = \infty$, then (3.0.21) is trivially valid; if $J_2 < \infty$, then we have

$$\sum_{|m|=2}^{\infty} (1-\theta(\lambda_2,m)) \frac{\ln^{p(1-\lambda_1)-1} \rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p = J_2^q = I$$

$$\begin{aligned} &> k_{\alpha,\beta}(\lambda_1) \left[\sum_{|m|=2}^{\infty} (1-\theta(\lambda_2,m)) \frac{\ln^{p(1-\lambda_1)-1} \rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{p}} \\ &\quad \times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} \rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q \right]^{\frac{1}{q}}, \end{aligned}$$

$$\begin{aligned} J_2 &= \left[\sum_{|m|=2}^{\infty} (1-\theta(\lambda_2,m)) \frac{\ln^{p(1-\lambda_1)-1} \rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{q}} \\ &> k_{\alpha,\beta}(\lambda_1) \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} \rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} b_n^q \right]^{\frac{1}{q}}. \end{aligned}$$

Hence, (3.0.21) holds true.

On the other-hand, assuming that (3.0.21) is valid, using the reverse Hölder inequality, we have

$$\begin{aligned} I &= \sum_{|m|=2}^{\infty} \left[\frac{\ln^{(1/q)-\lambda_1} \rho A_{\xi,\alpha}(m)}{(1-\theta(\lambda_2,m))^{-1/p}(A_{\xi,\alpha}(m))^{-1/q}} a_m \right] \\ &\quad \times \left[\frac{\ln^{\lambda_1-(1/q)} \rho A_{\xi,\alpha}(m)}{(1-\theta(\lambda_2,m))^{1/p}(A_{\xi,\alpha}(m))^{1/q}} \sum_{|n|=2}^{\infty} H(m,n) b_n \right] \\ &\geq \left[\sum_{|m|=2}^{\infty} (1-\theta(\lambda_2,m)) \frac{\ln^{p(1-\lambda_1)-1} \rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} a_m^p \right]^{\frac{1}{p}} J_2, \qquad (3.0.30) \end{aligned}$$

and then by (3.0.21) we have (3.0.19), which is equivalent to (3.0.21).

Therefore, inequalities (3.0.19), (3.0.20) and (3.0.21) are equivalent. This completes the proof of the theorem. □

Theorem 3.7. *With respect to the assumptions of Theorem 1, the constant factor $k_{\alpha,\beta}(\lambda_1)$ in (3.0.19), (3.0.20) and (3.0.21) is the best possible.*

Proof. For $0 < \varepsilon < p(\lambda_1 + \sigma)$, we set $\widetilde{\lambda}_1 = \lambda_1 - \frac{\varepsilon}{p} (\in (-\sigma, 1-\sigma)), \widetilde{\lambda}_2 = \lambda_2 + \frac{\varepsilon}{p}$ $(> -\sigma)$, and

$$\widetilde{a}_m := \frac{\ln^{\lambda_1 - (\varepsilon/p) - 1} \rho A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} = \frac{\ln^{\widetilde{\lambda}_1 - 1} \rho A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} \ (|m| \in \mathbf{N} \backslash \{1\}),$$

$$\widetilde{b}_n := \frac{\ln^{\lambda_2 - (\varepsilon/q) - 1} \rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} = \frac{\ln^{\widetilde{\lambda}_2 - \varepsilon - 1} \rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \ \ (|n| \in \mathbf{N} \backslash \{1\}).$$

By (2.0.17) and (2.0.15) we find

$$\begin{aligned}
\widetilde{I}_1 &:= \left[\sum_{|m|=2}^{\infty} (1 - \theta(\lambda_2, m)) \frac{\ln^{p(1-\lambda_1)-1} \rho A_{\xi,\alpha}(m)}{(A_{\xi,\alpha}(m))^{1-p}} \widetilde{a}_m^p \right]^{\frac{1}{p}} \\
&\quad \times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{q(1-\lambda_2)-1} \rho A_{\eta,\beta}(n)}{(A_{\eta,\beta}(n))^{1-q}} \widetilde{b}_n^q \right]^{\frac{1}{q}} \\
&= \left[\sum_{|m|=2}^{\infty} \frac{\ln^{-1-\varepsilon} \rho A_{\xi,\alpha}(m)}{A_{\xi,\alpha}(m)} - \sum_{|m|=2}^{\infty} \frac{O(\ln^{-1-(\lambda_2+\sigma+\varepsilon)} \rho A_{\xi,\alpha}(m))}{A_{\xi,\alpha}(m)} \right]^{\frac{1}{p}} \\
&\quad \times \left[\sum_{|n|=2}^{\infty} \frac{\ln^{-1-\varepsilon} \rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \right]^{\frac{1}{q}} \\
&= \frac{1}{\varepsilon} (2 \csc^2 \alpha + o(1) - \varepsilon O(1))^{1/p} (2 \csc^2 \beta + \widetilde{o}(1))^{1/q} \ (\varepsilon \to 0^+),
\end{aligned}$$

$$\begin{aligned}
\widetilde{I} &:= \sum_{|m|=2}^{\infty} \sum_{|n|=2}^{\infty} H(m,n) \widetilde{a}_m \widetilde{b}_n \\
&= \sum_{|n|=2}^{\infty} \sum_{|m|=2}^{\infty} H(m,n) \frac{\ln^{\widetilde{\lambda}_1 - 1} \rho A_{\xi,\alpha}(m) \ln^{\widetilde{\lambda}_2 - \varepsilon - 1} \rho A_{\eta,\beta}(n)}{A_{\xi,\alpha}(m) A_{\eta,\beta}(n)} \\
&= \sum_{|n|=2}^{\infty} \varpi(\widetilde{\lambda}_1, n) \frac{\ln^{-1-\varepsilon} \rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} < h_\alpha(\widetilde{\lambda}_1) \sum_{|n|=2}^{\infty} \frac{\ln^{-1-\varepsilon} \rho A_{\eta,\beta}(n)}{A_{\eta,\beta}(n)} \\
&= \frac{1}{\varepsilon} h_\alpha(\lambda_1 - \frac{\varepsilon}{p})(2 \csc^2 \beta + \widetilde{o}(1)).
\end{aligned}$$

If there exists a positive number $K \geq k_{\alpha,\beta}(\lambda_1)$, such that (3.0.19) is still satisfied when replacing $k_{\alpha,\beta}(\lambda_1)$ by K, then in particular, we have

$$\varepsilon \widetilde{I} = \varepsilon \sum_{|m|=2}^{\infty} \sum_{|n|=2}^{\infty} H(m,n) \widetilde{a}_m \widetilde{b}_n > \varepsilon K \widetilde{I}_1.$$

In view of the above results, it follows that

$$\begin{aligned} & h_\alpha(\lambda_1 - \frac{\varepsilon}{p})(2\csc^2\beta + \widetilde{o}(1)) \\ > \; & K \cdot (2\csc^2\alpha + o(1) - \varepsilon O(1))^{1/p}(2\csc^2\beta + \widetilde{o}(1))^{1/q}, \end{aligned}$$

and then

$$\frac{4(\lambda + 2\sigma)\csc^2\alpha}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\csc^2\beta \geq 2K\csc^{2/p}\alpha\csc^{2/q}\beta \; (\varepsilon \to 0^+),$$

namely,

$$k_{\alpha,\beta}(\lambda_1) = \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\csc^{2/p}\beta\csc^{2/q}\alpha \geq K.$$

Hence, $K = k_{\alpha,\beta}(\lambda_1)$ is the best possible constant factor in (3.0.19).

The constant factor $k_{\alpha,\beta}(\lambda_1)$ in (3.0.20) ((3.0.21)) is still the best possible. Otherwise we would reach a contradiction by (3.0.29) ((3.0.30)) that the constant factor in (3.0.19) is not the best possible. This completes the proof of the theorem. □

Remark 2. *(i) For $\rho = 1, \xi = \eta = 0$ in (3.0.22), setting*

$$\begin{aligned} \widetilde{\theta}_1(\lambda_2, m) &:= \frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln 2}{\ln|m|}} \frac{(\min\{1,u\})^\sigma u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du \\ &= O\left(\frac{1}{\ln^{\lambda_2+\sigma}|m|}\right) \in (0,1), \end{aligned}$$

we have the following inequality:

$$\sum_{|n|=2}^{\infty}\sum_{|m|=2}^{\infty}\frac{(\min\{\ln|m|,\ln|n|\})^\sigma a_m b_n}{(\max\{\ln|m|,\ln|n|\})^{\lambda+\sigma}} > \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}$$

(3.0.31)

$$\times \left[\sum_{|m|=2}^{\infty}(1-\widetilde{\theta}_1(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}|m|}{|m|^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{|n|=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}|n|}{|n|^{1-q}}b_n^q\right]^{\frac{1}{q}},$$

It follows that (3.0.21) is an extension of (3.0.31).

(ii) If $a_{-m} = a_m$ and $b_{-n} = b_n$ $(m, n \in \mathbf{N}\backslash\{1\})$, setting

$$\begin{aligned} \widehat{\theta}_1(\lambda_2, m) &:= \frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln\rho(2+\eta)}{\ln\rho(m-\xi)}} \frac{(\min\{1,u\})^\sigma u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du \\ &= O\left(\frac{1}{\ln^{\lambda_2+\sigma}\rho(m-\xi)}\right) \in (0,1), \end{aligned}$$

$$\begin{aligned}\widetilde{\theta}_1(\lambda_2,m) &:= \frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln\rho(2+\eta)}{\ln\rho(m+\xi)}}\frac{(\min\{1,u\})^\sigma u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du\\ &= O\left(\frac{1}{\ln^{\lambda_2+\sigma}\rho(m+\xi)}\right)\in(0,1),\end{aligned}$$

then (3.0.22) reduces to

$$\begin{aligned}&\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\left\{\frac{[\min\{\ln\rho(m-\xi),\ln\rho(n-\eta)]^\sigma}{[\max\{\ln\rho(m-\xi),\ln\rho(n-\eta)]^{\lambda+\sigma}}\right.\\ &+\frac{[\min\{\ln\rho(m-\xi),\ln\rho(n+\eta)]^\sigma}{[\max\{\ln\rho(m-\xi),\ln\rho(n+\eta)]^{\lambda+\sigma}}+\frac{[\min\{\ln\rho(m+\xi),\ln\rho(n-\eta)]^\sigma}{[\max\{\ln\rho(m+\xi),\ln\rho(n-\eta)]^{\lambda+\sigma}}\\ &\left.+\frac{[\min\{\ln\rho(m+\xi),\ln\rho(n+\eta)]^\sigma}{[\max\{\ln\rho(m+\xi),\ln\rho(n+\eta)]^{\lambda+\sigma}}\right\}a_m b_n\\ &> \frac{2(\lambda+2\sigma)}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\left\{\sum_{m=2}^{\infty}\left[(1-\widehat{\theta}_1(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}\rho(m-\xi)}{(m-\xi)^{1-p}}\right.\right.\\ &\left.\left.+(1-\widetilde{\theta}_1(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}\rho(m+\xi)}{(m+\xi)^{1-p}}\right]a_m^p\right\}^{\frac{1}{p}}\\ &\times\left\{\sum_{n=2}^{\infty}\left[\frac{\ln^{q(1-\lambda_2)-1}\rho(n-\eta)}{(n-\eta)^{1-q}}+\frac{\ln^{q(1-\lambda_2)-1}\rho(n+\eta)}{(n+\eta)^{1-q}}\right]b_n^q\right\}^{\frac{1}{q}}.\end{aligned}\tag{3.0.32}$$

In particular, for $\rho=1,\xi=\eta=0$, setting

$$\begin{aligned}\overline{\theta}_1(\lambda_2,m) &:= \frac{(\lambda_1+\sigma)(\lambda_2+\sigma)}{\lambda+2\sigma}\int_0^{\frac{\ln 2}{\ln m}}\frac{(\min\{1,u\})^\sigma u^{\lambda_2-1}}{(\max\{1,u\})^{\lambda+\sigma}}du\\ &= O\left(\frac{1}{\ln^{\lambda_2+\sigma}m}\right)\in(0,1),\end{aligned}$$

we have the following reverse Mulholland-type inequality:

$$\begin{aligned}&\sum_{n=2}^{\infty}\sum_{m=2}^{\infty}\frac{(\min\{\ln m,\ln n\})^\sigma}{(\max\{\ln m,\ln n\})^{\lambda+\sigma}}a_m b_n>\frac{\lambda+2\sigma}{(\lambda_1+\sigma)(\lambda_2+\sigma)}\\ &\times\left[\sum_{m=2}^{\infty}(1-\overline{\theta}_1(\lambda_2,m))\frac{\ln^{p(1-\lambda_1)-1}m}{m^{1-p}}a_m^p\right]^{\frac{1}{p}}\left[\sum_{n=2}^{\infty}\frac{\ln^{q(1-\lambda_2)-1}n}{n^{1-q}}b_n^q\right]^{\frac{1}{q}}.\end{aligned}\tag{3.0.33}$$

Competing interests.
The authors declare that they have no competing interests.

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Michael Th. Rassias
Institute of Mathematics,
University of Zurich, CH-8057,
Zurich, Switzerland
& Moscow Institute of Physics and Technology,
141700 Dolgoprudny,
Institutskiy per, d. 9,
Russia
& Institute for Advanced Study,
Program in Interdisciplinary Studies,
1 Einstein Dr, Princeton,
NJ 08540, USA.
E-mail: *michail.rassias@math.uzh.ch*

Bicheng Yang
Department of Mathematics,
Guangdong University of Education,
Guangzhou, Guangdong 510303,
P. R. China.
E-mail: *bcyang@gdei.edu.cn*
bcyang818@163.com