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CHROMATIC ZEROS AND GENERALIZED FIBONACCI NUMBERS

Saeid Alikhani, Yee-hock Peng

In this article we consider the problem whether generalized Fibonacci constants can be zeros of chromatic polynomials. We prove that all $2n$-anacci numbers and all their natural powers cannot be zeros of any chromatic polynomial. Also we investigate $(2n+1)$-anacci numbers as chromatic zeros.

1. INTRODUCTION

Let G be a simple graph and $\lambda \in \mathbb{N}$. A mapping $f: V(G) \rightarrow \{1, 2, \ldots, \lambda\}$ is called a λ-colouring of G if $f(u) \neq f(v)$ whenever the vertices u and v are adjacent in G. The number of distinct λ-colourings of G, denoted by $P(G, \lambda)$ is called the *chromatic polynomial* of G. A zero of $P(G, \lambda)$ is called a *chromatic zero* of G. An interval is called a *zero-free interval* for a chromatic polynomial $P(G, \lambda)$ if G has no chromatic zero in this interval. It is well-known that $(-\infty, 0)$ and $(0, 1)$ are two maximal zero-free intervals for the family of all graphs (see [**6**]). Jackson [**6**] showed that $(1, \frac{32}{27}]$ is another maximal zero-free interval for the family of all graphs and the value $\frac{32}{27}$ is best possible. A graph is *planar* if it can be drawn in the plane with no edges crossing. *Near-triangulation graph* is plane graph with at most one non-triangular face. A near- triangulation with 3-face is a *triangulation.*

We recall that a complex number ζ is called an *algebraic number* (respectively, *algebraic integer*) if it is a zero of some monic polynomial with rational (respectively, integer) coefficients (see [**11**]). Corresponding to any algebraic number ζ, there is a unique monic polynomial p with rational coefficients, called the *minimal polynomial* of ζ (over the rationals), with the property that p divides every polynomial with rational coefficients having ζ as a zero. (The minimal polynomial of ζ has integer coefficients if and only if ζ is an algebraic integer.)

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Since the chromatic polynomial $P(G,\lambda)$ is a monic polynomial in λ with integer coefficients, its zeros are, by definition, algebraic integers. This naturally raises the question: *Which* algebraic integers can occur as zeros of chromatic polynomials? Clearly those lying in $(-\infty,0)\cup(0,1)\cup(1,\frac{32}{27}]$ are forbidden. The n-th *Beraha number* is given by $B_n = 2+2\cos(\frac{2\pi}{n})$ (see [**2**]). Tutte [**13**] proved that the Beraha number $B_5=\frac{3+\sqrt{5}}{2}$ cannot be a chromatic zero, for otherwise $B_5^*=\frac{3-\sqrt{5}}{2}$ would also be a chromatic zero, which is impossible since $B_5^*\in(0,1)$. Salas and Sokal in [**10**] extended this result to show that the generalized Beraha numbers $B_n^{(k)} = 4\cos^2(k\pi/n)$ for $n=5,7,8,9$ and $n\geq 11$, with k coprime to n, are never chromatic zeros. For $n=10$ they showed the weaker result that $B_{10}=\frac{5+\sqrt{5}}{2}$ and $B_{10}^*=\frac{5-\sqrt{5}}{2}$ are not chromatic zeros of any plane near-triangulation.

In this paper we would like to prove some further results of this kind.

In Section 2, we prove that $2n$-anacci numbers and all their natural powers, cannot be zeros of any chromatic polynomials. In Section 3, we study $(2n+1)$-anacci. Finally, in the last section, we introduce numbers which are related to $2n$-anacci constants, and ask a question similar to the Beraha question [**2**].

2. Chromatic zeros and $2n$-anacci constants

In this section, we investigate the $2n$-anacci constant (defined below) as a chromatic zero. We show that $2n$-anacci, and their natural powers, cannot be chromatic zeros.

Definition 1. *An n-step ($n\geq 2$) Fibonacci sequence $F_k^{(n)}$, $k=1,2,3,\ldots$ is defined by letting $F_1^{(n)}=F_2^{(n)}=\ldots=F_n^{(n)}=1$ and other terms according to the linear recurrence equation $F_k^{(n)}=\sum_{i=1}^{k-1}F_{k-i}^{(n)}$, $(k>2)$. The limit $\varphi_n = \lim_{k\to\infty}\dfrac{F_k^{(n)}}{F_{k-1}^{(n)}}$ is called the n-anacci constant.*

It is easy to see that φ_n is the real positive zero of $f_n(x)=x^n-x^{n-1}-\ldots-x-1$, and this polynomial is the minimal polynomial of φ_n over $\mathbb{Z}[x]$. It is obvious that φ_n is a zero of $g_n(x)=x^n(2-x)-1$. Note that $\varphi_2=\tau$, where $\tau=\frac{1+\sqrt{5}}{2}$ is the golden ratio, and $\lim_{n\to\infty}\varphi_n=2$ (See [**8**], [**14**]).

We need the following two theorems to show our main results in this section.

Theorem 1. ([**8**]) *The polynomial $f_n(x)=x^n-x^{n-1}-\ldots-x-1$ is an irreducible polynomial over $\mathbb{Q}$.*

Theorem 2. ([**5**]) *Let G be a graph with n vertices and k connected components. Then the chromatic polynomial of G is of the form*

$$P(G,\lambda)=a_n\lambda^n+a_{n-1}\lambda^{n-1}+\ldots+a_k\lambda^k$$

with $a_n, a_{n-1}, \ldots, a_k$ integers, $a_n = 1$, and $(-1)^{n-l}a_l > 0$ for $k \leq l < n$. Furthermore, if G has at least one edge, then

$$P(G, 1) = a_n + a_{n-1} + \ldots + a_k = 0.$$

Now we prove that the $2n$-anacci numbers cannot be zeros of any chromatic polynomials.

Theorem 3. *For every integer $n \geq 1$, the $2n$-anacci number φ_{2n} cannot be zero of any chromatic polynomial.*

Proof. We know that φ_{2n} is a zero of $f_{2n}(x) = x^{2n} - x^{2n-1} - \ldots - x - 1$, which is minimal polynomial for this root. It is obvious that $f_{2n}(x)$ is not a chromatic polynomial (see Theorem 2). Now suppose that there exists a chromatic polynomial $P(x)$ such that $P(\varphi_n) = 0$. By Theorem 1, $f_{2n}(x)|P(x)$. Since $f_{2n}(0) = -1 < 0$, and $f_{2n}(-1) = 1 > 0$, by the intermediate value theorem, $f_{2n}(x)$ and therefore $P(x)$ has a root in $(-1, 0)$ and this is a contradiction. □

With arguments similar to the proof of Theorem 3, we prove that all natural powers of $2n$-anacci constant cannot be chromatic zero.

Theorem 4. *All natural powers of φ_{2n} cannot be chromatic zeros.*

Proof. Suppose that $\varphi_{2n}^m (m \in \mathbb{N})$ is a chromatic zero, that is there exists a chromatic polynomial

$$P(G, \lambda) = \lambda^k + a_{k-1}\lambda^{k-1} + \ldots + a_1\lambda$$

such that $P(G, \varphi_{2n}^m) = 0$. Therefore,

$$\varphi_{2n}^{mk} + a_{k-1}\varphi_{2n}^{m(k-1)} + \ldots + a_1\varphi_{2n}^m = 0.$$

So we can say that φ_{2n} is a zero of the polynomial,

$$Q(\lambda) = \lambda^{mk} + a_{k-1}\lambda^{mk-m} + \ldots + a_1\lambda^m$$

in $\mathbb{Z}[x]$. But we know that $f_{2n}(\lambda) = \lambda^{2n} - \lambda^{2n-1} - \ldots - \lambda - 1$ is the minimal polynomial of φ_{2n} over $\mathbb{Z}[x]$. Therefore $f_{2n}(\lambda)|Q(\lambda)$. Since $f_{2n}(0) = -1 < 0$ and $f_{2n}(-1) = 1 > 0$, $f_{2n}(\lambda)$ and so $Q(\lambda)$ have a zero say α, in $(-1, 0)$. Therefore, α^m is a root of $P(G, \lambda)$. Since $\alpha^m \in (-1, 0) \cup (0, 1)$, we have a contradiction. □

Corollary 1. *$\tau^n (n \in \mathbb{N})$ cannot be zero of any chromatic polynomials, where $\tau = \frac{1+\sqrt{5}}{2}$ is the golden ratio.*

Proof. Since $\tau = \varphi_2$, then we have the result by Theorems 3 and 4. □

Remark. We have obtained this result in [**1**] with a different approach.

3. Chromatic zeros and $(2n + 1)$-anacci constants

We think that $(2n+1)$-anacci numbers and all their natural powers also cannot be chromatic zeros, but we are not able to prove it. In this section we study and obtain a result for $(2n+1)$-anacci numbers as chromatic zeros.
Now we obtain the following result related to φ_{2n+1}.

Theorem 5. *For every natural n, φ_{2n+1} cannot be a root of the chromatic polynomial of a connected graph G with $|V(G)| \leq 4n+2$.*

Proof. We know that φ_{2n+1} is a zero of $f_{2n+1}(x) = x^{2n+1} - x^{2n} - \ldots - x - 1$, which is the minimal polynomial for this root. It is obvious that $f_{2n+1}(x)$ is not a chromatic polynomial (see Theorem 2). Now suppose that there exists a chromatic polynomial

$$P(x) = a_m x^m + a_{m-1}x^{m-1} + \ldots + a_1 x + a_0, \quad (m > 2n+1)$$

such that $P(\varphi_{2n+1}) = 0$, then $f_{2n+1}(x)|P(x)$. So $P(x) = f_{2n+1}(x)g(x)$. Let $g(x) = b_{m-2n-1}x^{m-2n-1} + b_{m-2n-2}x^{m-2n-2} + \ldots + b_1 x + b_0$. Therefore the following equalities hold between the coefficients of $g(x)$ and $P(x)$, when $m-2n-1 \leq 2n+1$, i.e. $m \leq 4n+2$:

$$\begin{array}{rcl} a_0 &=& b_0 = 0, \\ a_1 &=& -b_1, \\ a_2 &=& -b_2 - b_1, \\ a_3 &=& -b_3 - b_2 - b_1, \\ &\vdots& \\ a_{m-2n-1} &=& -b_{m-2n-1} - b_{m-2n-2} - \ldots - b_1 \end{array}$$

By Theorem 2, since $P(x)$ is a chromatic polynomial, we have $(x-1)|P(x)$. Therefore we have

$$(x-1)|b_{m-2n-1}x^{m-2n-1} + b_{m-2n-2}x^{m-2n-2} + \cdots + b_1 x + b_0,$$

hence $b_{m-2n-1}+b_{m-2n-2}+\ldots+b_1 = 0$. By the above equalities, we have $a_{m-2n-1} = 0$, contradicts the fact that the coefficients $a_m, a_{m-1}, \ldots, a_1$ of a chromatic polynomial of a connected graph are different from zero. Since the above equalities between the coefficients of $P(x)$ and $g(x)$ are valid provided that $m \leq 4n+2$, φ_{2n+1} cannot be chromatic zero of any graph G with $|V(G)| \leq 4n+2$. $\square$

Some questions and remarks

This section discusses results and conjectures going in the opposite direction to those in the rest of the paper, namely, while the rest of the paper is concerned with proving that $2n$-anacci constants and all their natural powers cannot be chromatic roots; here we are interested in showing that those numbers are accumulation points of (real) chromatic roots.
Thomassen [**12**] stated the following significant result.

n	$A_{n,2} = 2 + \varphi_n$	$B_n = 2 + 2\cos(\frac{2\pi}{n})$
2	$2 + \tau$	0
3	3.839286755	1
4	3.927561975	2
5	3.965948237	$\tau^2 = 2.618033989$
6	3.983582843	3
7	3.991964197	3.246979604
8	3.996031180	3.414213562
9	3.998029470	3.532088886
10	3.999018633	$2+\tau$=3.618033989
$\vdots$	$\vdots$	$\vdots$

TABLE 1. $A_{n,2}$ and B_n

Theorem 6. *If G is a graph of order n and $1 \le \lambda \le \frac{32}{27}$, then $|P(G,\lambda)| \ge \lambda(1-\lambda)^{n-1}$. Moreover, if $\lambda_0 > \frac{32}{27}, \varepsilon > 0$, then there exists a graph G such that $P(G,\lambda)$ has a root in $(\lambda_0 - \varepsilon, \lambda_0 + \varepsilon)$.*

Theorem 6 provides information on the general case when $\lambda > \frac{32}{27}$. But the problem has been considered for some families of graphs as well. One of this families is the family of triangulation graphs, and there are some open problems for it. We recall the Beraha question, which says:

Question 1. (Beraha's question [**2**]) *Is it true that for every $\varepsilon > 0$, there exists a plane triangulation G such that $P(G,\lambda)$ has a root in $(B_n - \varepsilon, B_n + \varepsilon)$, where $B_n = 2 + 2cos(\frac{2\pi}{n})$ is called the n-th Beraha constant (or number)?*

Beraha *et al.* [**4**] proved that $B_5 = \tau^2 = 1 + \tau = \frac{3+\sqrt{5}}{2} \approx 2.61803...$, is an accumulation point of real chromatic roots of certain plane triangulations (namely $4_{per} \times n_{free}$ pieces of the triangulation lattice with an extra vertex at top and bottom to complete the triangulation). Jacobsen *et al.* [**7**] extended this to show that B_7, B_8, B_9 and B_{10} are likewise accumulation points of real chromatic roots of plane triangulations (namely, $m_{per} \times n_{free}$ pieces of the triangular lattice of width m up to 12). Finally, Royle [**9**] has recently exhibited a family of plane triangulations with chromatic roots converging to 4. We know that the sequence B_n has a limit 4. If the answer to Beraha's question is positive, then there exist planar graphs whose chromatic roots are arbitrarily close to 4. Note that by exhibition of Royle [**9**], we have a family of plane triangulations with chromatic roots converging to 4. Of course, it is an open question which other numbers in the interval $(\frac{32}{27}, 4)$ can be accumulation points of real chromatic roots of planar graphs. The following conjecture of Thomassen is one possible answer.

Conjecture 1. *The set of chromatic roots of the family of planar graphs consists of* $0, 1$ *and a dense subset of* $(\frac{32}{27}, 4)$. (See [**12**]).

Now, let $A_{n,i} = i + \varphi_n$, where $i \in \{0, 1, 2\}$. In Table 1, we compare $A_{n,2}$ with B_n. Here we ask the following question that is analogous to Beraha's question.

Question 2. *Is it true that, for any $\varepsilon > 0$, there exists a plane triangulation graph G such that $P(G, \lambda)$ has a root in $(A_{n,i} - \varepsilon, A_{n,i} + \varepsilon)$ $(n \geq 2, i \in \{0, 1, 2\})$?*

Note that $A_{2,2} = B_{10} = 2+\tau$. Beraha, Kahane and Reid [**3**] proved that the answer to Question 2 (or respectively, Beraha's question) is positive for $i = 2$, $n = 2$ (or $n = 10$).

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