ON TRACES OF HOLOMORPHIC FUNCTIONS ON THE UNIT POLYBALL

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ABSTRACT. In this paper we completely describe traces of holomorphic Bergman classes and Bloch-type classes on polyballs and obtain related estimates generalizing classical Bergman projection theorem.

1. INTRODUCTION

Let $\mathbb{C}$ denote the set of complex numbers. Throughout the paper we fix a positive integer n and let $\mathbb{C}^n = \mathbb{C} \times \cdots \times \mathbb{C}$ denote the Euclidean space of complex dimension n . The open unit ball in $\mathbb{C}^n$ is the set $\mathbf{B}^n = \{z \in \mathbb{C}^n \mid |z| < 1\}$. The boundary of $\mathbf{B}^n$ will be denoted by $\mathbf{S}^n$, $\mathbf{S}^n = \{z \in \mathbb{C}^n \mid |z| = 1\}$.

As usual, we denote by $H(\mathbf{B}^n)$ the class of all holomorphic functions on $\mathbf{B}^n$.

For every function $f \in H(\mathbf{B}^n)$ having a series expansion $f(z) = \sum_{|k| \geq 0} a_k z^k$, we define the operator of fractional differentiation by

$$D^\alpha f(z) = \sum_{|k| \geq 0} (|k| + 1)^\alpha a_k z^k,$$

where α is any real number. It is obvious that for any α , D^α operator is acting from $H(\mathbf{B}^n)$ to $H(\mathbf{B}^n)$.

For $z \in \mathbf{B}^n$ and $r > 0$ set $\mathcal{D}(z, r) = \{w \in \mathbf{B}^n : \beta(z, w) < r\}$ where β is a Bergman metric on $\mathbf{B}^n$, $\beta(z, w) = \frac{1}{2} \log \frac{1 + |\varphi_z(w)|}{1 - |\varphi_z(w)|}$ is called the Bergman metric ball at z (see [14]).

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Let $m > 1$ is a natural number, $M \subset \mathbb{C}^n$ and $K \subset \mathbb{C}^{mn}$, $C^{mn} = C^m \times \cdots \times C^m$, be a hyper surface. Let $X(M)$ be a class of functions on M , $Y(K)$ the same. We say $\text{Trace} Y = X$ or in short $\text{Tr} Y = X$, $K = M^m$, $M^m = M \times \cdots \times M$, if for any $f \in Y(K)$, $f(w, \dots, w) \in X(M)$, $w \in M$, and for any $g \in X(M)$, there exist a function $f \in Y(K)$ such that $f(w, \dots, w) = g(w)$, $w \in M$. Traces of various functional spaces in $\mathbb{R}^n$ were described in [6] and [13]. In polydisk this problem is also known as a problem of diagonal map (see [3] and references there).

The intention of this paper is to consider the following natural Trace problem for polyballs. Let M be a unit ball and let K be a polyball (product of m balls) in definition we gave above. Let further $H(\mathbf{B} \times \cdots \times \mathbf{B})$ be a space of all holomorphic functions by each z_j , $z_j \in B$, $j = 1, \dots, m : f(z_1, \dots, z_m)$. Let further Y be a subspace of $H(\mathbf{B} \times \cdots \times \mathbf{B})$. The question we would like to study and solve in this work is the following: Find the complete description of Trace Y in a sense of our definition for several concrete functional classes. We observe that for $n = 1$ this problem completely coincide with the well- known problem of diagonal map. The last problem of description of diagonal of various subspaces of $H(\mathbf{U}^n)$ of spaces of all holomorphic functions in the polydisk was studied by many authors before (see [2], [3], [5], [8], [9], [11], [12] and references there).

The goal of this paper is to give a complete description of traces classical Bergman spaces defined on polyballs and traces of some Bloch type classes in polyballs. Let us note that for $n = 1$ traces of Bergman spaces were completely described previously in [3] and [11] (see also, for example, [12] and reference there). In this paper as in case of polydisk estimates for expanded Bergman projection (the operator of polarization) are playing a crucial role during all our proofs.

Trace theorems even for $n = 1$ (case of polydisk) have numerous applications in the theory of holomorphic functions (see for example [1], [3], [10]).

Throughout the paper, we write C (sometimes with indexes) to denote a positive constant which might be different at each occurrence (even in a chain of inequalities) but is independent of the functions or variables being discussed.

As usual, let $d\nu$ denote the Lebesgue measure on $\mathbf{B}$ normalized such that $\nu(\mathbf{B}) = 1$. For any real number α , let $d\nu_\alpha(z) = c_\alpha(1 - |z|^2)^\alpha d\nu(z)$ for $|z| < 1$. Here, if $\alpha \leq -1$, $c_\alpha = 1$ and if $\alpha > -1$, $c_\alpha = \frac{\Gamma(n+1+\alpha)}{\Gamma(n+1)\Gamma(\alpha+1)}$ is the normalizing constant so that ν_α has unit total mass.

2. BERGMAN CLASSES AND BLOCH TYPE SPACES ON THE POLYBALLS

The following estimate is well-known and will be used often in the paper. For a proof, see [14], Theorem 1.12.

Lemma A. *Suppose $c > 0$ is real and $t > -1$. Then the integral*

$$J_{c,t}(z) = \int_{\mathbf{B}} \frac{(1 - |w|^2)^t d\nu(w)}{|1 - \langle z, w \rangle|^{n+1+t+c}}, \quad z \in \mathbf{B},$$

has the following asymptotic property

$$J_{c,t} \sim (1 - |z|^2)^{-c} \quad \text{as } |z| \rightarrow 1 -.$$

We need the following estimate (see [7]):

Lemma B. *Let $0 \leq t_1 < s < t_0$, then*

$$\begin{aligned} & \int_{\mathbf{B}} \frac{(1 - |\eta|^2)^s}{|1 - \langle z, \eta \rangle|^{n+1+t_0} |1 - \langle \xi, \eta \rangle|^{t_1}} \left(\log^k \frac{2}{1 - |\eta|^2} \right) d\nu(\eta) \\ & \leq \frac{C}{(1 - |z|^2)^{t_0-s} |1 - \langle z, \xi \rangle|^{t_1}} \left(\log^k \frac{2}{1 - |z|^2} \right), \quad z, \xi \in \mathbf{B}, \quad k \in \mathbb{N}. \end{aligned}$$

For any integer $k \geq 1$, positive real numbers $r_1, \dots, r_k$ and function f on $\mathbf{B} \times \dots \times \mathbf{B}$, we define

$$\|f\|_{r_1, \dots, r_k} = \sup_{z_1, \dots, z_k \in \mathbf{B}} \{ |f(z_1, \dots, z_k)| \prod_{j=1}^n (1 - |z_j|^2)^{r_j} \}$$

Let $\Lambda(r_1, \dots, r_k)$ denote the space of all $f \in H(\mathbf{B} \times \dots \times \mathbf{B})$ such that $\|f\|_{r_1, \dots, r_k} < \infty$. It can be checked without difficulties that $\Lambda(r_1, \dots, r_k)$ with the norm $\|f\|_{r_1, \dots, r_k}$ is a Banach space.

Theorem 1. *Let $r_j > 0$, $j = 1, \dots, m$ and $r = r_1 + \dots + r_m$, then $\text{Trace}(\Lambda(r_1, \dots, r_m)) = \Lambda(r)$.*

Proof of Theorem 1. For every positive large enough b_j we have $F(z, \dots, z) = f(z)$, where

$$F(z_1, \dots, z_m) = C \int_{\mathbf{B}} \frac{f(w)(1 - |w|)^{\sum_{j=1}^m b_j - n - 1}}{\prod_{j=1}^m (1 - \langle \bar{w}, z_j \rangle)^{b_j}} dv(w)$$

by Bergman representation formula (see [14]). The proof follows from Hölder's inequality for n - functions and Lemma A.

If $f \in \Lambda(r)$, then $|f(w)| \leq \|f\|_r (1 - |w|^2)^{-r}$. Hence we have by Hölder's inequality

$$\begin{aligned} |F(z_1, \dots, z_m)| &\leq C \int_{\mathbf{B}} \frac{|f(w)|(1 - |w|)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m |1 - \langle z_j, w \rangle|^{b_j}} d\nu(w) \\ &\leq C \|f\|_r \int_{\mathbf{B}} \frac{\prod_{j=1}^m (1 - |w|^2)^{b_j - r_j}}{\prod_{j=1}^m |1 - \langle z_j, w \rangle|^{b_j}} (1 - |w|^2)^{-(n+1)} d\nu(w) \\ &\leq C \|f\|_r \prod_{j=1}^m \left(\int_{\mathbf{B}} \frac{(1 - |w|^2)^{s_j}}{|1 - \langle z_j, w \rangle|^{mb_j}} \right)^{\frac{1}{m}} \leq C \|f\|_r \prod_{j=1}^m (1 - |z_j|^2)^{-r_j}, \end{aligned}$$

where $s_j = m(b_j - r_j) - n - 1$.

Hence $F \in \Lambda(r_1, \dots, r_m)$, $F(z, \dots, z) = f(z)$. The reverse assertion is obvious since if $F \in \Lambda(r_1, \dots, r_m)$, then $F(z, \dots, z) \in \Lambda(r)$. Theorem is proved. $\square$

Let

$$\Lambda_{\log}(r_1, \dots, r_m) = \{f \in H(\mathbf{B}^m) : \sup_{z_j \in \mathbf{B}} |f(z_1, \dots, z_m)| \prod_{j=1}^m \left(\log \frac{1}{1 - |z_j|} \right)^{\frac{-1}{r_j}} (1 - |z_j|)^{\frac{1}{r_j}} < \infty,$$

$$\sum_{j=1}^m \frac{1}{r_j} = 1, \quad r_j > 0\}.$$

Then we have the following theorem. The proof use Lemma B and ideas of Theorem 1.

Theorem 2. *Trace* $(\Lambda_{\log}(r_1, \dots, r_m)) = \Lambda_{\log}(1)$, where

$$\Lambda_{\log}(1) = \{f \in H(\mathbf{B}) : \sup_{z \in \mathbf{B}} |f(z)| \left(\log \frac{1}{1-|z|} \right)^{-1} (1-|z|) < \infty\}.$$

Remark 1. Note that Theorem 1 and Theorem 2 are obvious for $m = 1$.

For each real number α and $p \in (0, \infty)$, the Bergman space A_α^p is the intersection of $H(\mathbf{B})$ with $L^p(\mathbf{B}, d\nu_\alpha)$. It is well-known that A_α^p is a closed subspace of $L^p(\mathbf{B}, d\nu_\alpha)$. See [14], Chapter 2 for more detail.

To the end of the paper, fix an integer $m \geq 1$. For any two n -tuples of real numbers $\mathbf{a} = (a_1, \dots, a_m)$, and $\mathbf{b} = (b_1, \dots, b_m)$, we define the integral operators

$$(T_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m) = \prod_{j=1}^m (1 - |z_j|^2)^{a_j} \int_{\mathbf{B}} \frac{f(w)(1 - |w|^2)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m |1 - \langle z_j, w \rangle|^{a_j+b_j}} d\nu(w),$$

and

$$(S_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m) = \prod_{j=1}^m (1 - |z_j|^2)^{a_j} \int_{\mathbf{B}} \frac{f(w)(1 - |w|^2)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m (1 - \langle z_j, w \rangle)^{a_j+b_j}} d\nu(w),$$

where $z_1, \dots, z_m$ are in $\mathbf{B}$ and f is a function in $L^1(\mathbf{B}, d\nu_{-n-1+\sum_{j=1}^m b_j})$. Note that for such f , the functions $T_{\mathbf{a}, \mathbf{b}} f$ and $S_{\mathbf{a}, \mathbf{b}} f$ are defined on $\mathbf{B}^m$, the product of m copies of $\mathbf{B}$, and we have $|S_{\mathbf{a}, \mathbf{b}} f| \leq T_{\mathbf{a}, \mathbf{b}} |f|$.

We will study the boundedness of $T_{\mathbf{a}, \mathbf{b}}$ and $S_{\mathbf{a}, \mathbf{b}}$ from certain L^p spaces of $\mathbf{B}$ into those of $\mathbf{B}^m$. Consider first the case $1 \leq p < \infty$. Let $s_1, \dots, s_m$ be arbitrary real numbers and put $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. The following proposition gives sufficient conditions for the boundedness of $T_{\mathbf{a}, \mathbf{b}}$ (and hence, the boundedness of $S_{\mathbf{a}, \mathbf{b}}$) from $L^p(\mathbf{B}, d\nu_t)$ into $L^p(\mathbf{B}^m, d\nu_{s_1} \cdots d\nu_{s_m})$.

Proposition 1. *Let $1 \leq p < \infty$ and $s_j > -1$. Suppose for each $j = 1, \dots, m$, we have $-pa_j < s_j + 1$ and $ms_j + 1 < p(mb_j - n) - (m-1)(n+1)$. Then there is a*

constant $C > 0$ such that

$$\begin{aligned} & \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |(T_{a,b}f)(z_1, \dots, z_m)|^p \prod_{j=1}^m (1 - |z_j|^2)^{s_j} d\nu(z_1) \dots d\nu(z_m) \\ & \leq C \int_{\mathbf{B}} |f(w)|^p (1 - |w|^2)^{(m-1)(n+1) + \sum_{j=1}^m s_j} d\nu(w), \end{aligned}$$

for all f in $L^1(\mathbf{B}, d\nu)$.

Proof. The case $p = 1$ follows from Fubini's theorem and the estimates in Lemma A.

Now assume $p > 1$. Let q denote the exponential conjugate of p , that is, $\frac{1}{p} + \frac{1}{q} = 1$.

Choose a positive number such that $p\gamma < \min\{p(mb_j - n) - (m-1)(n+1) - ms_j - 1 : j = 1, \dots, m\}$. Put $\alpha = \frac{1}{m}(\gamma - \frac{1}{q})$ and $\beta = -n - 1 + \sum_{j=1}^m b_j - m\alpha = -n - 1 + \sum_{j=1}^m b_j - \gamma + \frac{1}{q}$. For each j , choose e_j such that

$$\frac{n+1}{mq} + \alpha < e_j < \frac{n+1}{mq} + \alpha + \frac{pa_j + s_j + 1}{p}.$$

It is possible to choose such an e_j since $pa_j + s_j + 1 > 0$. Put $d_j = a_j + b_j - e_j$.

For any measurable function f on $\mathbf{B}$ and $z_1, \dots, z_m$ in $\mathbf{B}$, using Hölder's inequality, we have

$$\begin{aligned} & \int_{\mathbf{B}} \frac{|f(w)|(1 - |w|^2)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m |1 - \langle z_j, w \rangle|^{a_j+b_j}} d\nu(w) \\ & = \int_{\mathbf{B}} \left(\frac{|f(w)|(1 - |w|^2)^\beta}{\prod_{j=1}^m |1 - \langle z_j, w \rangle|^{d_j}} \right) \prod_{j=1}^m \frac{(1 - |w|^2)^\alpha}{|1 - \langle z_j, w \rangle|^{e_j}} d\nu(w) \\ & \leq \left(\int_{\mathbf{B}} \frac{|f(w)|^p (1 - |w|^2)^{p\beta}}{\prod_{j=1}^m |1 - \langle z_j, w \rangle|^{pd_j}} d\nu(w) \right)^{\frac{1}{p}} \prod_{j=1}^m \left(\int_{\mathbf{B}} \frac{(1 - |w|^2)^{mq\alpha}}{|1 - \langle z_j, w \rangle|^{mqe_j}} d\nu(w) \right)^{\frac{1}{mq}}. \end{aligned}$$

For each j , since $mq\alpha = q\gamma - 1 > -1$ and $mqe_j > n + 1 + mq\alpha$, Lemma A shows that

$$\int_{\mathbf{B}} \frac{(1 - |w|^2)^{mq\alpha}}{|1 - \langle z_j, w \rangle|^{mqe_j}} d\nu(w) \leq C(1 - |z_j|^2)^{n+1+mq\alpha-mqe_j},$$

where C is independent of $z_1, \dots, z_m$. Thus we obtain

$$\begin{aligned}
& \int_{\mathbf{B}} \frac{|f(w)|(1-|w|^2)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m |1-\langle z_j, w \rangle|^{a_j+b_j}} d\nu(w) \\
& \leq C \left(\int_{\mathbf{B}} \frac{|f(w)|^p (1-|w|^2)^{p\beta}}{\prod_{j=1}^m |1-\langle z_j, w \rangle|^{pd_j}} d\nu(w) \right)^{\frac{1}{p}} \prod_{j=1}^m (1-|z_j|^2)^{\frac{n+1}{mq}+\alpha-e_j}.
\end{aligned}$$

This implies that

$$\begin{aligned}
& |(T_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m)|^p \\
& \leq C \left(\int_{\mathbf{B}} \frac{|f(w)|^p (1-|w|^2)^{p\beta}}{\prod_{j=1}^m |1-\langle z_j, w \rangle|^{pd_j}} d\nu(w) \right) \prod_{j=1}^m (1-|z_j|^2)^{\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)}.
\end{aligned}$$

Now by Fubini's theorem,

$$\begin{aligned}
(1) \quad & \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |(T_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m)|^p \prod_{j=1}^m (1-|z_j|^2)^{s_j} d\nu(z_1) \cdots d\nu(z_m) \\
& \leq C \int_{\mathbf{B}} \left(\prod_{j=1}^m \int_{\mathbf{B}} \frac{(1-|z_j|^2)^{\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j}}{|1-\langle z_j, w \rangle|^{pd_j}} d\nu(z_j) \right) |f(w)|^p (1-|w|^2)^{p\beta} d\nu(w).
\end{aligned}$$

For each j , by the choice of e_j and γ , we have $\frac{p(n+1)}{mq} + p\alpha - pe_j + pa_j + s_j > -1$ and $n+1 + \frac{p(n+1)}{mq} + p(\alpha - e_j + a_j) + s_j - pd_j < 0$. Applying Lemma A again, we have

$$\begin{aligned}
(2) \quad & \int_{\mathbf{B}} \frac{(1-|z_j|^2)^{\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j}}{|1-\langle z_j, w \rangle|^{pd_j}} d\nu(z_j) \\
& \leq C(1-|w|^2)^{n+1+\frac{p(n+1)}{mq}+p(\alpha-e_j+a_j)+s_j-pd_j} \\
& = C(1-|w|^2)^{\frac{p\gamma-p(mb_j-n)+(m-1)(n+1)+(ms_j+1)}{m}},
\end{aligned}$$

where C independent of w . From (1) and (2) and the fact that

$$\begin{aligned}
& \sum_{j=1}^m \frac{p\gamma - p(mb_j - n) + (m-1)(n+1) + (ms_j + 1)}{m} \\
& = (m-1)(n+1) + \sum_{j=1}^m s_j - p \left(\sum_{j=1}^m b_j - \gamma - n \right) + 1
\end{aligned}$$

$$= (m-1)(n+1) + \sum_{j=1}^m s_j - p\beta,$$

the conclusion of the proposition follows. $\square$

Remark 2. Note that for $m = 1$ our assertion in Proposition 1 is well known and has numerous applications (see [14]).

For any two n -tuples of real numbers $\mathbf{x} = (x_1, \dots, x_m)$ and $\mathbf{y} = (y_1, \dots, y_m)$, we consider the integral operator

$$(R_{\mathbf{x}, \mathbf{y}} g)(w) = (1 - |w|^2)^{-m(n+1) + \sum_{j=1}^m y_j} \times \\ \times \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} g(z_1, \dots, z_m) \left(\prod_{j=1}^m \frac{(1 - |z_j|^2)^{x_j}}{(1 - \langle w, z_j \rangle)^{x_j + y_j}} \right) d\nu(z_1) \cdots d\nu(z_m),$$

for $g \in L^1(\mathbf{B}^m, d\nu_{x_1} \cdots d\nu_{x_m})$ and $w \in \mathbf{B}$. Using Proposition 1, we obtain the following proposition which gives conditions for the boundedness of $R_{\mathbf{x}, \mathbf{y}}$.

Proposition 2. *Let $1 \leq p < \infty$ and $s_j > -1$. Suppose for each j we have $s_j + 1 < p(x_j + 1)$ and $ms_j + 1 > mp(n+1 - y_j) - (m-1)(n+1)$. Then there is a constant $C > 0$ such that*

$$\int_{\mathbf{B}} |(R_{\mathbf{x}, \mathbf{y}} g)(w)|^p (1 - |w|^2)^{(m-1)(n+1) + \sum_{j=1}^m s_j} d\nu(w) \\ \leq C \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |g(z_1, \dots, z_m)|^p \prod_{j=1}^m (1 - |z_j|^2)^{s_j} d\nu(z_1) \cdots d\nu(z_m).$$

Proof. We first consider the case $p = 1$. We have

$$(3) \quad \int_{\mathbf{B}} |(R_{\mathbf{x}, \mathbf{y}} g)(w)| (1 - |w|^2)^{(m-1)(n+1) + \sum_{j=1}^m s_j} d\nu(w) \\ \leq \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |g(z_1, \dots, z_m)| \prod_{j=1}^m (1 - |z_j|^2)^{x_j} \times \\ \times \left(\int_{\mathbf{B}} \frac{(1 - |w|^2)^{-n-1 + \sum_{j=1}^m (y_j + s_j)}}{\prod_{j=1}^m |1 - \langle w, z_j \rangle|^{x_j + y_j}} d\nu(w) \right) d\nu(z_1) \cdots d\nu(z_m).$$

By Hölders inequality,

$$\int_{\mathbf{B}} \frac{(1 - |w|^2)^{-n-1+\sum_{j=1}^m (y_j+s_j)}}{\prod_{j=1}^m |1 - \langle w, z_j \rangle|^{x_j+y_j}} d\nu(w) \leq \left(\prod_{j=1}^m \int_{\mathbf{B}} \frac{(1 - |w|^2)^{-n-1+my_j+ms_j}}{|1 - \langle w, z_j \rangle|^{mx_j+my_j}} d\nu(w) \right)^{\frac{1}{m}}.$$

From the assumption of the proposition, we have $-n-1+my_j+ms_j > -1$ and $mx_j+my_j > (-n-1+my_j+ms_j) + (n+1)$ for each j . Lemma A shows that the above product is less than or equal to $\prod_{j=1}^m (1 - |z_j|^2)^{s_j-x_j}$. From this and (3), the conclusion of the proposition then follows.

Now assume $1 < p < \infty$. Put $\mathbf{s} = (s_1, \dots, s_m)$, and let $\mathbf{a} = \mathbf{x} - \mathbf{s}$ and $\mathbf{b} = \mathbf{y} + \mathbf{s}$. Then

$$(S_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m) = \prod_{j=1}^m (1 - |z_j|^2)^{x_j-s_j} \int_{\mathbf{B}} \frac{f(w)(1 - |w|^2)^{-n-1+\sum_{j=1}^m (y_j+s_j)}}{\prod_{j=1}^m (1 - \langle z_j, w \rangle)^{x_j+y_j}} d\nu(w).$$

By the assumption and Proposition 1, $S_{\mathbf{a}, \mathbf{b}}$ is a bounded operator from $L^q(\mathbf{B}, d\nu_t)$ into $L^q(\mathbf{B}^m, d\nu_{s_1} \cdots d\nu_{s_m})$, where $1 < q < \infty$ is the exponential conjugate of p and $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. On the other hand, it can be checked easily that $S_{\mathbf{a}, \mathbf{b}}^* = R_{\mathbf{x}, \mathbf{y}}$. The conclusion of the proposition follows. $\square$

Remark 3. Note that for $m = 1$ the assertion of Proposition 2 is well known (see [14]).

Proposition 2'. Let $p \in (0, \infty)$, $s_j > -1$, $j = 1, \dots, m$, $m \in \mathbb{N}$. Then the following estimate holds

$$\begin{aligned} \mathcal{J} &= \int_{\mathbf{B}} |g(w, \dots, w)|^p (1 - |w|^2)^{(m-1)(n+1) + \sum_{j=1}^m s_j} d\nu(w) \\ &\leq C \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |g(z_1, \dots, z_m)|^p \prod_{j=1}^m (1 - |z_j|^2)^{s_j} d\nu(z_1) \cdots d\nu(z_m) = \mathcal{J}_1. \end{aligned}$$

Proof. We have by Lemma 2.24 from [14] and properties of r -lattice in Bergman metric (see [14], Theorem 2.23)

$$\mathcal{J} \leq C \sum_{k \geq 0} \sup_{z \in \mathcal{D}(a_k, r)} |g(z, \dots, z)|^p (1 - |a_k|^2)^{(m-1)(n+1) + (\sum_{j=1}^m s_j) + n+1}$$

$$\leq C \sum_{k_1 \geq 0} \cdots \sum_{k_m \geq 0} \sup_{z_1 \in \mathcal{D}(a_{k_1}, r), \dots, z_m \in \mathcal{D}(a_{k_m}, r)} |g(z_1, \dots, z_m)|^p \times \\ \times (1 - |a_{k_1}|^2)^{\frac{\tau_1}{m}} \cdots (1 - |a_{k_m}|^2)^{\frac{\tau_m}{m}} \leq C_1 \mathcal{J}_1,$$

where $\tau_j = (n+1)m + s_j m$. □

Theorem 3. *Suppose $1 \leq p \leq \infty$ and $s_1, \dots, s_m > -1$. Put $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. Then there are bounded operators $S : A^p(\mathbf{B}, d\nu_t) \rightarrow A^p(\mathbf{B}^m, d\nu_{s_1} \cdots d\nu_{s_m})$, and $R : A^p(\mathbf{B}^m, d\nu_{s_1} \cdots d\nu_{s_m}) \rightarrow A^p(\mathbf{B}, d\nu_t)$ such that $(Sf)(z, \dots, z) = f(z)$ and $(Rg)(z) = g(z, \dots, z)$ for all $f \in A^p(\mathbf{B}, d\nu_t)$, all $g \in A^p(\mathbf{B}^m, d\nu_{s_1} \cdots d\nu_{s_m})$ and all $z \in \mathbf{B}$. In other words, the Trace of $A^p(\mathbf{B}^m, d\nu_{s_1} \cdots d\nu_{s_m})$ is $A^p(\mathbf{B}, d\nu_t)$.*

Remark 4. For $n = 1$ Theorem 3 was known before (see [3], [8], [11]).

Proof of Theorem 3. If $p = \infty$, then $A^\infty(\mathbf{B}, d\nu_t) = H^\infty(\mathbf{B})$ and $A^p(\mathbf{B}^m, d\nu_{s_1} \cdots d\nu_{s_m}) = H^\infty(\mathbf{B}^m)$. Define $(Sf)(z_1, \dots, z_m) = f(z_1)$ for $f \in H^\infty(\mathbf{B})$, $z_1, \dots, z_m \in \mathbf{B}$ and $(Rg)(w) = g(w, \dots, w)$ for $g \in H^\infty(\mathbf{B}^m)$ and $w \in \mathbf{B}$. Then $\|S\|, \|R\| \leq 1$ and they satisfy the conclusion of the corollary.

Now suppose $1 \leq p < \infty$. Let $\mathbf{a} = (0, \dots, 0)$, $\mathbf{b} = (b_1, \dots, b_m)$, where b_j is large enough, and $S = CS_{\mathbf{a}, \mathbf{b}}$. It can be checked that $\mathbf{a}$ and $\mathbf{b}$ satisfy the hypothesis of Proposition 1. On the other hand, for $f \in A^p(\mathbf{B}, d\nu_k)$, Sf is holomorphic and for $z \in \mathbf{B}$,

$$(Sf)(z, \dots, z) = C \int_{\mathbf{B}} \frac{f(w)(1 - |w|^2)^{-n-1+\sum_{j=1}^m b_j}}{(1 - \langle z, w \rangle)^{\sum_{j=1}^m b_j}} d\nu(w) \\ = \int_{\mathbf{B}} \frac{f(w)d\nu_{k+1}(w)}{(1 - \langle z, w \rangle)^{n+2+k}} = f(z)$$

by [14], Theorem 2.2.

Let $\mathbf{x} = (x_1, \dots, x_m)$, where x_j is large enough, $\mathbf{y} = (n+1, \dots, n+1)$ and $R = c_{x_1} \cdots c_{x_m} R_{\mathbf{x}, \mathbf{y}}$. Then $\mathbf{x}$ and $\mathbf{y}$ satisfy the hypothesis of Proposition 2. Furthermore,

for $g \in A^p(\mathbf{B}^m, d\nu_{s_1} \cdots d\nu_{s_m})$ and $w \in \mathbf{B}$,

$$\begin{aligned} (Rg)(w) &= \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} \frac{g(z_1, \dots, z_m) d\nu_{x_1}(z_1) \cdots d\nu_{x_m}(z_m)}{\prod_{j=1}^m (1 - \langle w, z_j \rangle)^{x_j + y_j}} \\ &= \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} \frac{g(z_1, \dots, z_m) d\nu_{x_1}(z_1) \cdots d\nu_{x_m}(z_m)}{\prod_{j=1}^m (1 - \langle w, z_j \rangle)^{n+1+x_j}} \\ &= g(w, \dots, w), \end{aligned}$$

by applying [14], Theorem 2.2, m times. Therefore, S and R are the required operators.

It is not difficult to see that to get the second part of the Theorem 3 we can also apply Proposition 2'. $\square$

The next lemma shows that if $s_1 = \cdots = s_m$ and $b_1 = \cdots = b_m$ then the converse of Proposition 1 holds true. We do not know if it is also the case for general $s_1, \dots, s_m$ and $b_1, \dots, b_m$.

Lemma 1. *Let $a_1, \dots, a_m, b_1 = \cdots = b_m = b$ and $s_1, \dots, s_m = s$ be real numbers and let $1 < p < \infty$. Put $t = (m-1)(n+1) + ms$. If $S_{\mathbf{a}, \mathbf{b}}$ is a bounded operator from $L^p(\mathbf{B}, d\nu_t)$ into $L^p(\mathbf{B}^m, (d\nu_s)^m)$, then $-pa_j < s+1$ for all $j = 1, \dots, m$ and $ms+1 < p(mb-n) - (m-1)(n+1)$.*

Proof. Choose N sufficiently large so that the function $f(w) = (1 - |w|^2)^N$ belongs to $L^p(\mathbf{B}, d\nu_t)$. By the rotation-invariant property of the Lebesgue measure, we see that $S_{\mathbf{a}, \mathbf{b}}f$ is a multiple of $\prod_{j=1}^m (1 - |z_j|)^{a_j}$. Since $S_{\mathbf{a}, \mathbf{b}}f$ belongs to $L^p(\mathbf{B}^m, (d\nu_s)^m)$, we conclude that $pa_j + s_j > -1$ for all $j = 1, \dots, m$. Now let $1 < q < \infty$ such that $\frac{1}{p} + \frac{1}{q} = 1$. The boundedness of $S_{\mathbf{a}, \mathbf{b}}$ implies the boundedness of the adjoint $S_{\mathbf{a}, \mathbf{b}}^*$ as an operator from $L^q(\mathbf{B}^m, (d\nu_s)^m)$, into $L^q(\mathbf{B}, d\nu_t)$. One can check without difficulty that for $g \in L^q(\mathbf{B}^m, (d\nu_s)^m)$ the adjoint of $S_{\mathbf{a}, \mathbf{b}}$ equals

$$\begin{aligned} &(1 - |w|^2)^{-n-1-t+\sum_{j=1}^m b_j} \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} g(z_1, \dots, z_m) \prod_{j=1}^m \frac{(1 - |z_j|)^{a_j+s_j}}{(1 - \langle w, z_j \rangle)^{a_j+b_j}} d\nu(z_j) \\ &= (1 - |w|^2)^{-n-1-t+mb} \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} g(z_1, \dots, z_m) \prod_{j=1}^m \frac{(1 - |z_j|)^{a_j+s}}{(1 - \langle w, z_j \rangle)^{a_j+b}} d\nu(z_j) \end{aligned}$$

Letting $g(z_1, \dots, z_m) = \prod_{j=1}^m (1 - |z_j|^2)^M$ for some large M , we see that $S_{\mathbf{a}, \mathbf{b}}^* g$ is a multiple of $(1 - |w|^2)^{-n-1-t+mb}$.

If $1 < p < \infty$, then $1 < q < \infty$. Since $(S_{\mathbf{a}, \mathbf{b}}^* g)$ is in $L^q(\mathbf{B}, d\nu_t)$ we have $q(-n - 1 - t + mb) + t > -1$, which is equivalent to $t + 1 < p(mb - n)$ and hence, $ms + 1 < p(mb - n) - (m - 1)(n + 1)$.

□

Remark 5. Suppose $\mathbf{a} = (a_1, \dots, a_m)$, $\mathbf{b} = (b, \dots, b)$ and $s_1 = \dots = s_m = s$, where $a_1, \dots, a_m, b, s$ are arbitrary real numbers. Put $t = (m - 1)(n + 1) + ms$. Suppose $1 < p < \infty$, $s > -1$. Proposition 1 and Lemma 1 show that the following statements are equivalent.

- (1) The operator $T_{\mathbf{a}, \mathbf{b}}$ is bounded from $L^p(\mathbf{B}, d\nu_t)$ into $L^p(\mathbf{B}^m, (d\nu_s)^m)$.
- (2) The operator $S_{\mathbf{a}, \mathbf{b}}$ is bounded from $L^p(\mathbf{B}, d\nu_t)$ into $L^p(\mathbf{B}^m, (d\nu_s)^m)$.
- (3) $ms + 1 < p(mb - n) - (m - 1)(n + 1)$ and $-pa_j < s + 1$ for $j = 1, \dots, m$.

We now consider the case $0 < p \leq 1$. Let $s_1, \dots, s_m$ be real numbers and let $t = (m - 1)(n + 1) + \sum_{j=1}^m s_j$. The following proposition gives sufficient conditions for the boundedness of $T_{\mathbf{a}, \mathbf{b}}$ from the Bergman space A_t^p into $L^p(\mathbf{B}^m, d\nu_{s_1} \times \dots \times d\nu_{s_m})$.

Proposition 3. *Let $p \in (0, 1]$, $s_j > -1$. Suppose for each $j = 1, \dots, m$, we have $-pa_j < s_j + 1$ and $s_j + 1 < pb_j - n$. Then there is a constant $C > 0$ such that*

$$\begin{aligned} & \int_{\mathbf{B}} \dots \int_{\mathbf{B}} |(T_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m)|^p \prod_{j=1}^m (1 - |z_j|^2)^{s_j} d\nu(z_1) \dots d\nu(z_m) \\ & \leq C \int_{\mathbf{B}} |f(w)|^p (1 - |w|^2)^{(m-1)(n+1) + \sum_{j=1}^m s_j} d\nu(w), \end{aligned}$$

for all f in $H(\mathbf{B}) \cap L^1(\mathbf{B}, d\nu_b)$.

Remark 6. The condition $s_j + 1 < pb_j - n$ is equivalent to $ms_j + 1 < p(mb_j - n) - (m - 1)(n + 1) - n(1 - p)$. This shows that there is an extra summand $(n(1 - p))$ in the condition on the s_j 's compared to that in Proposition 1. This extra summand vanishes when $p = 1$.

Proof. Let $\mathbf{D}(a, r)$ denote the Bergman disk of radius r centered at a for each $a \in \mathbf{B}$. Fix $0 < r \leq 1$ and choose $\{u_k\}_{k=1}^m$ to be any r -lattice in the Bergman metric of $\mathbf{B}$. This means that $\mathbf{B} = \bigcup_{k=1}^{\infty} \mathbf{D}(u_k, r)$, $\mathbf{D}(u_k, r) \cap \mathbf{D}(u_l, r) = \emptyset$ if $k \neq l$ and there is an integer $N \geq 1$ such that each $z \in \mathbf{B}$ belongs to at most N of the sets $\mathbf{D}(u_k, 2r)$. (See [14], Theorem 2.23 and the remark following it for more detail about the existence of such a lattice). For any function $f \in L^1(\mathbf{B}, d\nu_n)$ and any $z_1, \dots, z_m \in \mathbf{B}$, we have

$$\begin{aligned} & |(T_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m)| \\ & \leq \prod_{j=1}^m (1 - |z_j|^2)^{a_j} \sum_{k=1}^{\infty} \int_{\mathbf{D}(u_k, r)} \frac{|f(w)|(1 - |w|^2)^{-n-1+\sum_{j=1}^m b_j}}{\prod_{j=1}^m |1 - \langle z_j, w \rangle|^{a_j+b_j}} d\nu(w). \end{aligned}$$

By [14], Lemma 2.27, there is a constant $C > 0$ so that for each $j = 1, \dots, m$ and $k \geq 1$, $\frac{1}{C} \leq \left| \frac{1 - \langle z_j, w \rangle}{1 - \langle z_j, u_k \rangle} \right| \leq C$, for all $w \in \mathbf{D}(u_k, r)$. Also by [14], Lemma 1.24, $\int_{\mathbf{D}(u_k, r)} (1 - |w|^2)^{-n-1+\sum_{j=1}^m b_j} d\nu(w)$ is comparable with $(1 - |u_k|^2)^{\sum_{j=1}^m b_j}$. Thus we have

$$\begin{aligned} & |(T_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m)| \\ & \leq C \sum_{k=1}^{\infty} \prod_{j=1}^m \frac{(1 - |z_j|^2)^{a_j}}{|1 - \langle z_j, u_k \rangle|^{a_j+b_j}} \int_{\mathbf{D}(u_k, r)} |f(w)|(1 - |w|^2)^{-n-1+\sum_{j=1}^m b_j} d\nu(w) \\ & \leq C \sum_{k=1}^{\infty} \prod_{j=1}^m \frac{(1 - |z_j|^2)^{a_j} (1 - |u_k|^2)^{\sum_{j=1}^m b_j}}{|1 - \langle z_j, u_k \rangle|^{a_j+b_j}} \sup\{|f(w)| : w \in \mathbf{D}(u_k, r)\}. \end{aligned}$$

Now since $0 < p \leq 1$, using the inequality $(x_1 + x_2 + \dots)^p \leq x_1^p + x_2^p + \dots$, which is valid for nonnegative numbers $x_1, x_2, \dots$, we get

$$\begin{aligned} & |(T_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m)|^p \\ & \leq C \sum_{k=1}^{\infty} \prod_{j=1}^m \frac{(1 - |z_j|^2)^{pa_j} (1 - |u_k|^2)^{p(\sum_{j=1}^m b_j)}}{|1 - \langle z_j, u_k \rangle|^{pa_j+pb_j}} \sup\{|f(w)|^p : w \in \mathbf{D}(u_k, r)\}. \end{aligned}$$

Integrating with respect to $d\nu_{s_1}(z_1) \cdots d\nu_{s_m}(z_m)$ and using Lemma A (note that by assumption, $pa_j + s_j > -1$ and $pa_j + pb_j > n + 1 + pa_j + s_j$), we obtain

$$\begin{aligned}
& \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |(T_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m)|^p \prod_{j=1}^m (1 - |z_j|^2)^{s_j} d\nu(z_1) \cdots d\nu(z_m) \\
& \leq C \sum_{k=1}^{\infty} \left(\prod_{j=1}^m (1 - |u_k|^2)^{n+1+s_j-pb_j} \right) (1 - |u_k|^2)^{p(\sum_{j=1}^m b_j)} \sup\{|f(w)|^p : w \in \mathbf{D}(u_k, r)\} \\
& \leq C \sum_{k=1}^{\infty} (1 - |u_k|^2)^{m(n+1)+\sum_{j=1}^m s_j} \sup\{|f(w)|^p : w \in \mathbf{D}(u_k, r)\}.
\end{aligned}$$

From [14], Lemma 2.20, $(1 - |u_k|^2)$ is comparable with $(1 - |w|^2)$ when $w \in \mathbf{D}(u_k, r)$. This together with [14], Lemma 2.24 implies that, if f is holomorphic on $\mathbf{B}$, then

$$\begin{aligned}
& \int_{\mathbf{B}} \cdots \int_{\mathbf{B}} |(T_{\mathbf{a}, \mathbf{b}} f)(z_1, \dots, z_m)|^p \prod_{j=1}^m (1 - |z_j|^2)^{s_j} d\nu(z_1) \cdots d\nu(z_m) \\
& \leq C \sum_{k=1}^{\infty} \sup\{|f(w)|^p (1 - |w|^2)^{m(n+1)+\sum_{j=1}^m s_j} : w \in \mathbf{D}(u_k, r)\} \\
& \leq C \sum_{k=1}^{\infty} \int_{\mathbf{D}(u_k, 2r)} |f(w)|^p (1 - |w|^2)^{(m-1)(n+1)+\sum_{j=1}^m s_j} d\nu(w) \\
& \leq C \int_{\mathbf{B}} |f(w)|^p (1 - |w|^2)^{(m-1)(n+1)+\sum_{j=1}^m s_j} d\nu(w).
\end{aligned}$$

To derive the last inequality, we have used the fact that each $z \in \mathbf{B}$ belongs to at most N of the sets $\mathbf{D}(u_k, 2r)$. $\square$

Remark 7. Proposition 3 for $m = 1$ is obvious, for $m > 1$, $n = 1$ it was proved in [3].

The following theorem follows directly from Proposition 3 and Proposition 2'.

Theorem 4. Let $p \in (0, 1]$, $s_1 > -1, \dots, s_m > -1$, $t = (m-1)(n+1) + \sum_{j=1}^m s_j$. Then $\text{Trace} A^p(\mathbf{B}^m, d\nu_{s_1}, \dots, d\nu_{s_m}) = A^p(\mathbf{B}, d\nu_t)$.

Remark 8. For $n = 1$ Theorem 4 was known before (see [3], [8], [11]).

Remark 9. It is not difficult to notice that some our assertions proved above ($p \leq 1$ case) are true even under general assumption that f is a subharmonic function in the unit ball $\mathbf{B}$.

Remark 10. Using approaches we develop in this paper and the previous remark (not sharp) assertions of the type $\text{Trace}X \subset Y$ or $Y \subset \text{Trace}X$ for H^p Hardy classes, weighted Hardy classes, some mixed norm spaces and so-called Bergman-Nevanlinna classes (see [4] Chapter 4) can be also obtained.

Remark 11. Traces of mixed norm type analogues of Bergman type classes on polyballs we considered in this paper:

$$\|f\|_{p_1, \dots, p_m} = \left(\int_{\mathbf{B}} \cdots \left(\int_{\mathbf{B}} |f(w_1, \dots, w_m)|^{p_1} (1 - |w_1|)^{\alpha_1} d\nu \right)^{\frac{p_2}{p_1}} \cdots \right)^{\frac{p_m}{p_{m-1}}}$$

can be also described with the help of approaches we develop in this note with some restrictions on $p_1, \dots, p_m$.

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