SOME RESULTS ON MATCHING AND TOTAL DOMINATION IN GRAPHS

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Let $G$ be a graph. A set $S$ of vertices of $G$ is called a total dominating set of $G$ if every vertex of $G$ is adjacent to at least one vertex in $S$. The total domination number $\gamma_t(G)$ and the matching number $\alpha'(G)$ of $G$ are the cardinalities of the minimum total dominating set and the maximum matching of $G$, respectively. In this paper, we will introduce an upper bound of the difference between $\gamma_t(G)$ and $\alpha'(G)$. We will also characterize every tree $T$ with $\gamma_t(T) \leq \alpha'(T)$, and give a family of graphs with $\gamma_t(G) \leq \alpha'(G)$.

1. Introduction

Domination and its variants in graphs have been being well-studied in the past decade. The literature on this subject has been surveyed thoroughly in the two books by Haynes, Hedetniemi and Slater [4,5].

Let $G = (V, E)$ be a simple graph of order $n$. A matching $M$ in a graph $G$ is a set of independent edges in $G$. A vertex $v$ of $G$ is saturated by $M$ if it is the endpoint of an edge of $M$; otherwise, vertex $v$ is unsaturated by $M$. An induced matching $M$ is a matching where no two edges of $M$ are joined by an edge of $G$. The matching number $\alpha'(G)$ and the induced matching number $\text{im}(G)$ are the cardinalities of a maximum matching and a maximum induced matching of $G$, respectively. It is obvious that any induced matching is a matching. So $\alpha'(G) \geq \text{im}(G)$.

Let $V$ be the set of vertices of $G$. A set $S \subseteq V$ is called a dominating set of $G$ if every vertex in $V - S$ is adjacent to at least one vertex in $S$. A total dominating set, which was introduced by Cockayne, Dawes, and Hedetniemi [2], is a dominating set of $G$ containing no isolated vertices. The total domination number $\gamma_t(G)$ of $G$ is the cardinality of a minimum total dominating set.

We in general follow the notation and graph terminology in [4,5]. Specifically, the degree, neighborhood and closed neighborhood of a vertex $v$ in the graph $G$ are denoted by $d(v)$, $N(v)$ and $N[v] = N(v) \cup \{v\}$, respectively. For a subset $S$
For every integer 

Let 

For every claw-free graph 

For every 

δ maximum degree, order and induced matching number. 

domination number and the matching number in terms of the minimum degree, 

numbers at most their matching numbers. 

will characterize all trees and give a family of graphs with the total domination 

number. 

with minimum degree at least three that have equal total domination and matching 

number. 

of graphs with total domination number at most their matching number. 

Henning et al. [8] investigated the relationships between the matching and total 

domination number of a graph. They showed that the matching number and the 
total domination number of a graph are incomparable, even for an arbitrarily large, 

but fixed (with respect to the order of the graph), minimum degree. 

It is obvious that \( \gamma_t(G) \leq 2\alpha'(G) \). That is, \( \gamma_t(G) - \alpha'(G) \leq \alpha'(G) \leq \frac{n}{2} \). We shall improve this bound in Section 2. First, we prove that, for any connected graph \( G \), \( \gamma_t(G) = \left( \frac{\Delta + 1}{\Delta + \delta} \right) im(G) \leq \left( \frac{\Delta + 1}{\Delta + \delta} \right) \left( \frac{n}{2} \right) \), and characterize the extremal graphs. Then, we work out an upper bound on the difference between the total domination number and the matching number. 

**Theorem 1.1 ( [8] ).** For every integer \( \delta \geq 2 \), there exist graphs \( G \) and \( H \) with \( \delta(G) = \delta(H) = \delta \) satisfying \( \gamma_t(G) > \alpha'(G) \) and \( \gamma_t(H) < \alpha'(H) \). 

Furthermore, Henning et al. raised the following question: Find other families of graphs with total domination number at most their matching number. 

Recently, Henning and Yeo [7] characterized the connected claw-free graphs with minimum degree at least three that have equal total domination and matching number. 

In this paper, we will obtain an upper bound on the difference between the total domination number and the matching number in Section 2. In Section 3, we will characterize all trees and give a family of graphs with the total domination numbers at most their matching numbers.

**2. An upper bound on the difference between the total domination number and the matching number**

In this section, we present an upper bound on the difference between the total domination number and the matching number in terms of the minimum degree, maximum degree, order and induced matching number. 

**Lemma 2.1 ( [6] ).** Let \( G \) be a bipartite graph with partite sets \( (X, Y) \) whose vertices in \( Y \) are of degree at least \( \delta \geq 1 \). Then there exists a set \( A \subseteq X \) of cardinality at most \( \frac{1}{\delta}(|X| + \frac{|Y|}{\delta}) \) dominating \( Y \).
Theorem 2.2. For any connected graph $G$,

$$\gamma_t(G) - \left( \frac{\delta(\Delta + \delta) + \delta - 1}{\delta(\Delta + \delta)} \right) \text{im}(G) \leq \left( \frac{\Delta + 1}{\Delta + \delta} \right) \left( \frac{n}{2} \right).$$

Furthermore the equality holds if and only if $G$ is isomorphic to either $P_2$ or $C_5$.

Proof. Let $M$ be a maximum induced matching of $G$ and let $S_1$ be the set of saturated vertices by $M$. Define $S_2 = N(S_1) - S_1$ and $S_3 = V - N[S_1]$. It is obvious that $|S_1| + |S_2| + |S_3| = n$.

Suppose $S_3 = \emptyset$. Then $\gamma_t(G) \leq |S_1| = 2 \text{im}(G)$ and $\text{im}(G) \leq \frac{\Delta - 1}{\Delta + 1} \frac{n}{2}$. Hence

$$\gamma_t(G) - \left( \frac{\delta(\Delta + \delta) + \delta - 1}{\delta(\Delta + \delta)} \right) \text{im}(G) \leq \left( \frac{\delta(\Delta + \delta) - \delta + 1}{\delta(\Delta + \delta)} \right) \text{im}(G)$$

$$= \left( \frac{(\Delta + \delta) - 1 + \frac{1}{\Delta}}{(\Delta + \delta)} \right) \text{im}(G)$$

$$\leq \left( \frac{\Delta + 1}{\Delta + \delta} \right) \left( \frac{n}{2} \right).$$

If the equality holds, then $\text{im}(G) = \frac{n}{2}$ and $\delta = 1$. Hence $G \cong P_2$.

Suppose $S_3 \neq \emptyset$ and there is an edge $e = uv \in E(G[S_3])$. Since both $u$ and $v$ are at distance at least 2 from $S_1$, it follows that $M \cup \{e\}$ is an induced matching of $G$ larger than $M$, which is a contradiction. Therefore, $S_3$ is an independent set.

Let $H$ be the bipartite subgraph of $G$ with partite sets $(S_3, N(S_3))$ and with the edge set defined by $G[S_3, N(S_3)]$. Then each vertex in $S_3$ has degree at least $\delta \geq 1$ in $H$. By Lemma 2.1, there exists a set $A \subseteq N(S_3)$ of cardinality at most $\frac{1}{2}(|S_3| + \frac{|N(S_3)|}{\delta})$ dominating $S_3$. Since $N(S_3) \subseteq S_2$, it follows that $|A| \leq \frac{1}{2}(|S_3| + \frac{|S_2|}{\delta})$.

As the number of edges joining $S_1 \cup S_3$ and $S_2$ satisfies $(\delta - 1)|S_1| + \delta |S_3| \leq |G[S_1 \cup S_3, S_2]| \leq \Delta |S_2|$, we have $|S_1| + |S_3| \leq \frac{\Delta n + |S_1|}{\Delta + \delta}$. 
Moreover, $S_1 \cup A$ is a total dominating set of $G$, so it follows that

$$\gamma_t(G) \leq |S_1 \cup A|$$

$$\leq |S_1| + \frac{1}{2}(|S_4| + \frac{|S_5|}{\delta})$$

$$= |S_1| + \frac{|S_4|}{2} + \frac{n - |S_1| - |S_3|}{2\delta}$$

$$= \frac{n}{2\delta} + \frac{|S_4|}{2} + \delta - \frac{1}{2\delta} (|S_1| + |S_3|)$$

$$\leq \frac{n}{2\delta} + \frac{|S_4|}{2} + \left(\frac{\delta - 1}{2\delta}\right) \left(\frac{\Delta + 1}{\Delta + \delta}\right)$$

$$= \left(\frac{\delta(\Delta + \delta) + \delta - 1}{\delta(\Delta + \delta)}\right) im(G) + \left(\frac{\Delta + 1}{\Delta + \delta}\right) \left(\frac{n}{2}\right)$$

That is $\gamma_t(G) = \left(\frac{\delta(\Delta + \delta) + \delta - 1}{\delta(\Delta + \delta)}\right) im(G) \leq \left(\frac{\Delta + 1}{\Delta + \delta}\right) \left(\frac{n}{2}\right)$.

Suppose the equality holds. Then all inequalities in the previous proof become equalities. It follows that $S_2$ is an independent set and $N(S_3) = S_2$. Furthermore, $d(v) = \Delta$ for each vertex $v \in S_2$ and $d(u) = \delta$ for each vertex $u \in S_1 \cup S_3$. We have the following claims.

**Claim 1:** For each vertex $v \in S_2$, $|N(v) \cap S_3| = 1$.

Suppose to the contrary that there exists a vertex $v \in S_2$ such that $|N(v) \cap S_3| \geq 2$. Let $S_4 = S_3 - N(v) \cap S_3$ and $S_5 = S_3 - \{v\}$ and let $H_1$ be the bipartite subgraph of $G$ with the partite sets $(S_4, S_5)$ and the edge set defined by $G[S_4, S_5]$. By Lemma 2.1, there exists a set $B \subseteq S_5$ of cardinality at most $\frac{1}{2}(|S_4| + \frac{|S_5|}{\delta})$ that dominates $S_4$. Since $S_1 \cup B \cup \{v\}$ is a total dominating set of $G$, it follows that

$$\gamma_t(G) \leq |S_1 \cup B \cup \{v\}|$$

$$\leq |S_1| + \frac{1}{2}(|S_4| + \frac{|S_5|}{\delta}) + 1$$

$$\leq |S_1| + \frac{1}{2}(|S_4| - |N(v) \cap S_3|) + \frac{|S_2| - 1}{2\delta} + 1$$

$$\leq |S_1| + \frac{1}{2}|S_3| + \frac{|S_2| - 1}{2\delta}$$

$$\leq |S_1| + \frac{1}{2}|S_3| + \frac{|S_2|}{2\delta}$$

$$= \left(\frac{\delta(\Delta + \delta) + \delta - 1}{\delta(\Delta + \delta)}\right) im(G) + \left(\frac{\Delta + 1}{\Delta + \delta}\right) \left(\frac{n}{2}\right)$$

It is a contradiction. Hence $|N(v) \cap S_3| \leq 1$. Since $N(S_3) = S_2$, it follows that $|N(v) \cap S_3| = 1$.

Since $d(u) = \delta$ for each $u \in S_3$, by Claim 1 we have $|S_2| = \delta|S_3|$. Thus $\gamma_t(G) = |S_1| + |S_3|$. 


Claim 2: For each vertex $v \in S_1$, $|PN(v, S_1)| \geq 1$, where $PN(v, S_1) = (N(v) \cap S_2) - [N(S_1 - \{v\}) \cap S_2]$.

Otherwise, if there exists a vertex $v \in S_1$ such that $|PN(v, S_1)| = 0$. Let $vu \in M$ for some $u \in V$, $w \in N(u) \cap S_2$, and a minimum subset $A \subseteq N(S_3)$ containing $w$ that dominates $S_3$. By Claim 1, we have $|A| = |S_3|$. Then $(S_1 - \{v\}) \cup A$ is a total dominating set of $G$, which contradicts $\gamma_t(G) = |S_1| + |S_3|$. So $|PN(v, S_1)| \geq 1$ for $v \in S_1$.

Moreover, it follows Claims 1 and 2 that $\delta = \Delta = 2$ and hence $G \cong C_5$.

Conversely, it is obvious that if $G$ is isomorphic to either $P_2$ or $C_5$, then the equality holds. □

An edge incident with a leaf is called a leaf edge. A pendant triangle in a graph $G$ is a triangle where two vertices of it are of degree 2 and the third vertex is of degree greater than 2.

Lemmma 2.3 ([1]). Let $G$ be a connected graph. Then $\alpha'(G) = \text{im}(G)$ if and only if $G$ is a star, $C_3$, or the graph obtained from a connected bipartite graph with bipartite vertex sets $X$ and $Y$ by attaching at least one leaf edge to each vertex of $X$, and possibly some pendant triangles to some vertices of $Y$.

By Theorem 2.2 and Lemma 2.3, we have the following corollary.

Corollary 2.4. For any connected graph $G$,

$$\gamma_t(G) - \alpha'(G) \leq \left(\frac{\Delta + 1}{\Delta + \delta}\right) \left(\frac{n_1}{2}\right) + \left(\frac{\delta - 1}{\delta(\Delta + \delta)}\right) \text{im}(G).$$

Furthermore equality holds if and only if $G$ is isomorphic to either $P_2$ or $C_3$.

Remark: Suppose $G$ is a connected graph with minimum degree $\delta$. It is conjectured that the bound in Theorem 2.2 can be improved for large enough $\delta$.

3. Graphs with total domination numbers at most their matching numbers

3.1 Characterization of trees with total domination numbers at most their matching numbers
A total dominating set of a graph $G$ of cardinality $\gamma_t(G)$ is called a $\gamma_t(G)$-set while a tree $T$ with total domination number $\gamma_t(T)$ and matching number $\alpha'(T)$ is called a $(\gamma_t(T), \alpha'(T))$-tree. Before presenting our results, we make a couple of straightforward observations.

**Observation 1:** If $v$ is a support of a graph $G$, then $v$ is in every $\gamma_t(G)$-set.

**Observation 2:** For any connected graph $G$ with diameter at least three, there exists a $\gamma_t(G)$-set that contains no leaves of $G$.

Let $T'$ be a $(\gamma_t(T'), \alpha'(T'))$-tree with $|V(T')| \geq 2$. For any $v \in V(T')$, let $T'_v = T'' - \{v\}$. Define graphs $T(i, 1)$ as in Figure 2 for $i = 2, 3, 4$, and $S(j, k)$ for $j = 3, 4, 8$ and $k \geq 1$ as in Figures 3. Let $S(5, k)$ denote the disjoint union of $k$ copies of $T(3, 1)$ and $S(6, k) = S(5, k)$. Let $S(7, k)$ denote the disjoint union of $k$ copies of $P_3$. The vertex $v_1$ shown in the Figure 3 is called the central vertex of each of the graphs.

Cockayne, Henning and Mynhardt [3] characterized the set of vertices of a tree that are contained in all, or in no, respectively, minimum total dominating sets of the tree as $D(T') = \{u \in V(T') \mid$ there exists a $\gamma_t(T')$-set containing $u\}$ and $D'(T') = V(T') - D(T')$.

**Lemma 3.1.** Let a tree $T$ be obtained from a tree $T'$ by joining a vertex $v$ of $T'$ to a leaf of $P_4$ (with an edge). Then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\gamma_t(T') \leq \alpha'(T')$.

**Proof.** Suppose that $T$ is obtained from $T'$ by joining $v$ to a leaf $x$ of path $P = xyzw$. Let $S$ be a $\gamma_t(T')$-set of $T'$. Then it is obvious that $S \cup \{y, z\}$ is a total dominating set of $T$. So $\gamma_t(T) \leq \gamma_t(T') + 2$. 

![Figure 2](image)

![Figure 3](image)
Let $D$ be a $\gamma_T(T)$-set of $T$ containing no leaves. Then $y, z \in D$. If $x \notin D$, then $D - \{y, z\}$ is a total dominating set of $T'$ and $\gamma_T(T') \leq \gamma_T(T) - 2$. Suppose that $x \in D$. If $v \in D$, then $v \notin D$ for any vertex $u \in N_{T'}(v)$; otherwise, $D - \{x\}$ is a total dominating set of $T'$, which is a contradiction. So $(D - \{x, y, z\}) \cup \{u\}$ is a total dominating set of $T'$. Hence, $\gamma_T(T') \leq \gamma_T(T) - 2$. If $v \notin D$, and there exists a vertex $u \in N_{T'}(v)$ such that $u \notin D$. Then $(D - \{x, y, z\}) \cup \{u\}$ is a total dominating set of $T'$. Hence, $\gamma_T(T') \leq \gamma_T(T) - 2$. Thus, we have $\gamma_T(T) = \gamma_T(T') + 2$.

Let $M'$ be a maximum matching of $T'$. It is obvious that $M' \cup \{xy, zw\}$ is a matching of $T$. So, $\alpha'(T) \geq \alpha'(T') + 2$. Let $M$ be a maximum matching of $T$ saturating the largest number of vertices of $P$. Then $zw, xy \in M$. So $M - \{xy, zw\}$ is a matching of $T'$. Hence $\alpha'(T') \geq \alpha'(T) - 2$. Therefore $\alpha'(T) = \alpha'(T') + 2$. Hence we have $\gamma_T(T) \leq \alpha'(T)$ if and only if $\gamma_T(T') \leq \alpha'(T')$.

From Lemma 3.2 to Lemma 3.8 below, we suppose $v$ is a vertex of a tree $T'$. By the similar way of the previous proof, we have the following results.

**Lemma 3.2.** Suppose $T_v'$ contains a component $P_1$. Let $T$ be a tree obtained from $T'$ by joining $v$ to the vertex of $P_1$ or the leaf $v_1$ of $T(2, 1)$ described in Figure 2. Then $\gamma_T(T) \leq \alpha'(T)$ if and only if $\gamma_T(T') \leq \alpha'(T')$.

**Lemma 3.3.** Suppose that $T_v'$ contains a component $P_2$. Let $T$ be a tree obtained from $T'$ by joining $v$ to a leaf of $P_2$ or a support of $P_4$. Then $\gamma_T(T) \leq \alpha'(T)$ if and only if $\gamma_T(T') \leq \alpha'(T')$.

**Lemma 3.4.** Suppose that $T_v'$ contains a component $P_3$ such that $v$ is adjacent to a leaf of $P_3$. Let $T$ be obtained from $T'$ by joining $v$ to a support of $P_4$ or the leaf $v_1$ of $T(2, 1)$. Then $\gamma_T(T) \leq \alpha'(T)$ if and only if $\gamma_T(T') \leq \alpha'(T')$.

**Lemma 3.5.** Suppose that $T_v'$ contains a component $P_4$ such that $v$ is adjacent to a support of $P_4$. Let $T$ be obtained from $T'$ by joining $v$ to a support of another $P_4$. Then $\gamma_T(T) \leq \alpha'(T)$ if and only if $\gamma_T(T') \leq \alpha'(T')$.

**Lemma 3.6.** Suppose that $T_v'$ contains a component $T(2, 1)$ such that $v$ is adjacent to the leaf $v_1$ of $T(2, 1)$. Let $T$ be obtained from $T'$ by joining $v$ to the leaf $v_1$ of another $T(2, 1)$. Then $\gamma_T(T) \leq \alpha'(T)$ if and only if $\gamma_T(T') \leq \alpha'(T')$.

**Lemma 3.7.** Suppose that $T_v'$ contains a component $P_5$ such that $v$ is adjacent to a support of $P_5$. Let $T$ be obtained from $T'$ by joining $v$ to a support of $P_4$. Then $\gamma_T(T) \leq \alpha'(T)$ if and only if $\gamma_T(T') \leq \alpha'(T')$.

**Lemma 3.8.** Suppose that $T_v'$ contain components $P_1$ and $P_3$ such that $v$ is adjacent to a leaf of $P_3$. Let $T$ be obtained from $T'$ by joining $v$ to a vertex of $P_2$. Then $\gamma_T(T) \leq \alpha'(T)$ if and only if $\gamma_T(T') \leq \alpha'(T')$.

Let $v$ be a vertex of $T$, and $\zeta$ be a family of trees such that each tree $T \in \zeta$ has the following properties:

1. Let $C(T) = \{u \in V(T) \mid d(u) \geq 3\}$. For any $u \in C(T)$, $T_u$ does not contain a component $P_1$ ($t \geq 4$) with a leaf adjacent to $u$. 

(2) If one of the components of \( T_v \) is \( P_1 \), then other components of \( T_v \) are neither \( P_1 \) nor \( T(2,1) \) with the leaf \( v_1 \) adjacent to \( v \).
(3) If one of the components of \( T_v \) is \( P_2 \), then other components of \( T_v \) are neither \( P_2 \) nor \( P_4 \) with a support adjacent to \( v \).
(4) If \( P_3 \) with a leaf adjacent to \( v \) is a component of \( T_v \), then no other components of \( T_v \) are \( P_4 \) with a support adjacent to \( v \), or \( T(2,1) \) with the leaf \( v_1 \) adjacent to \( v \).
(5) At most one component of \( T_v \) is \( P_4 \) with a support adjacent to \( v \).
(6) At most one component of \( T_v \) is \( T(2,1) \) with the leaf \( v_1 \) of \( T(2,1) \) is adjacent to \( v \).
(7) If one of the components of \( T_v \) is \( P_5 \) with a support adjacent to \( v \), then no other components of \( T_v \) are \( P_4 \) with a support adjacent to \( v \).
(8) If \( P_1 \), and \( P_3 \) with a leaf adjacent to \( v \) are components of \( T_v \), then \( P_2 \) is not a component of \( T_v \).

For any tree \( T \), by Lemmas 3.1 to 3.8, either \( T \in \zeta \) or \( T \) can be transformed into some tree \( T' \in \zeta \) such that \( \gamma(T) \leq \alpha'(T) \) if and only if \( \gamma(T') \leq \alpha'(T') \).
In this situation, we say that \( T \) is an extension of \( T' \). Thus, in order to give a characterization of the trees with \( \gamma(T) \leq \alpha'(T) \), we define the following operations on trees.

Suppose that a tree \( T \) is obtained from another tree \( T' \) by the following operations.

**Operation 1:** \( T \) is obtained from tree \( T' \in \zeta \) as an extension.

**Operation 2:** Suppose that \( T_v' \) contains a component \( P_1 = \{ w \} \) and \( P_2 = xy \) such that \( ex \in E(T') \), where \( v \in V(T') \). Join \( x \) to a leaf of another \( P_2 \) or join \( w \) to a support of \( P_4 \).

**Operation 3:** Join the central vertex \( v_1 \) of \( S(3, k) \) to a vertex of \( T' \) for some \( k \).

**Operation 4:** Join the central vertex \( v_1 \) of \( S(4, k) \) to a vertex of \( T' \) for some \( k \).

**Operation 5:** For each \( u \in D(T') \), attach \( S(5, k) \) by joining each vertex \( v_1 \) to \( u \) for some \( k \).

**Operation 6:** For each \( u \in D'(T') \), attach \( S(6, k) \) by joining each vertex \( v_1 \) to \( u \) for some \( k \).

**Operation 7:** Suppose that \( T_v' \) contains components \( P_1 = \{ w \} \) and \( P_2 = xy \) such that \( ex \in E(T') \). Delete the component \( P_1 \) or \( P_2 \) and attach \( S(7, k) \) by joining a leaf of each \( P_3 \) to \( v \), for some integer \( k \).

**Operation 8:** For each \( v \in V(T') \), attach \( S(8, k) \) by joining vertex \( v_1 \) to \( v \) for some \( k \).

As the following lemmas can be obtained by similar arguments of the proof of Lemma 3.1, the proofs of the lemmas are omitted.

**Lemma 3.9.** Suppose that \( T \) is obtained from \( T' \) by using operation 2. Then \( \gamma(T) - \alpha'(T) = \gamma(T') - \alpha'(T') \).

**Lemma 3.10.** Suppose that \( T \) is obtained from \( T' \) by using operation 3. Then \( \gamma(T) - \alpha'(T) = \gamma(T') - \alpha'(T') - 1 \).
LEMMA 3.11. Suppose that $T$ is obtained from $T'$ by using operation 4. Then
\[ \gamma_c(T) - \alpha'(T) = \gamma_c(T') - \alpha'(T') + k - 1. \]

LEMMA 3.12. Suppose that $T$ is obtained from $T'$ by using operation 5. Then
\[ \gamma_c(T) - \alpha'(T) = \gamma_c(T') - \alpha'(T') - k. \]

LEMMA 3.13. Suppose that $T$ is obtained from $T'$ by using operation 6. Then
\[ \gamma_c(T) - \alpha'(T) = \gamma_c(T') - \alpha'(T') - k + 1. \]

LEMMA 3.14. Suppose that $T$ is obtained from $T'$ by using operation 7. Then
\[ \gamma_c(T) - \alpha'(T) = \gamma_c(T') - \alpha'(T') + k. \]

LEMMA 3.15. Suppose that $T$ is obtained from $T'$ by using operation 8. Then
\[ \gamma_c(T) - \alpha'(T) = \gamma_c(T') - \alpha'(T') + k - 1. \]

Let $c(i)$ denote the number of operations $i$ required to construct the tree $T$ from $P_2$ or $P_3$ for $i = 1, 2, \ldots, 8$. For each operation $i$, assume that $S(i, k_{ij})$ is attached for some integer $k_{ij}$, where $j = 1, 2, \ldots, c(i)$ and $i = 3, 4, 5, 6, 7, 8$.

THEOREM 3.16. Suppose $T$ is a tree of order $n$ for $n \geq 3$. Then $T$ can be obtained from a path $P_1$ by a finite sequence of operations $i$ for $i = 1, 2, \ldots, 8$, where $l = 2$ or 4. Furthermore, $\gamma_c(T) - \alpha'(T) = \gamma_c(P_l) - \alpha'(P_l) - c(3) + \sum_{j=1}^{c(4)}(k_{ij} - 1) - \sum_{j=1}^{c(5)}k_{ij} - \sum_{j=1}^{c(6)}(k_{ij} - 1) + \sum_{j=1}^{c(7)}k_{ij} + \sum_{j=1}^{c(8)}(k_{ij} - 1)$.

PROOF. We proceed by induction on the order $n$ of $T$. If $\text{diam}(T) = 2$, then $T$ is a star. So $T$ is obtained from $P_2$ by using operation 1. If $\text{diam}(T) = 3$, then $T$ is a double star. So $T$ is obtained from $P_4$ by using operation 1. If $T$ is isomorphic to $P_5, T(3, 1)$ or $T(4, 1)$, then it is obvious that the result holds. Assume that every tree $T'$ of order $5 \leq n' < n$ can be obtained from $P_2$ or $P_4$ by a finite sequence of operations $i$ for $i = 1, 2, \ldots, 8$.

Let $T$ be a tree of order $n$ such that $T$ is not isomorphic to $T(3, 1)$ and $T(4, 1)$. Assume that the longest path $P$ of $T$ is $u_1u_2\ldots u_t$. Without loss of generality, we can assume that $t \geq 5$. If $T \notin \zeta$, then $T$ can be obtained from some tree $T' \in \zeta$ by using operation 1. Without loss of generality, we can assume that $T \in \zeta$. Then $d(u_1) = 1, d(u_2) = 2$ and $2 \leq d(u_3) \leq 3$, and we may have the following cases.

**Case 1:** $d(u_3) = 3$. Then $u_3$ is a support of $T$. Let $N(u_3) - \{u_2, u_4\} = \{u'_4\}$. If $d(u_4) \geq 3$, then $d(u_4) = 3$ and $u_4$ is a support of $T$. Let $T' = T - \{u_1, u_2\}$. So $T$ is obtained from $T'$ by using operation 2. Without loss of generality, we can assume that $d(u_4) = 2$. Since $T$ is not isomorphic to $T(3, 1)$, it follows that $d(u_5) \geq 2$.

**Case 1.1:** $d(u_5) \geq 3$.

Let $T_{u_5}, T_{u_2}, \ldots, T_{u_5d(u_5)}$ denote components of $T_{u_5} = T - \{u_5\}$ such that $u_1 \in T_{u_51}$ and $u_4 \in T_{u_5d(u_5)}$. Since $T \in \zeta$, it follows that $T_{u_5i}$ is isomorphic to $P_2, P_3$ or $P_5$ for $i = 2, \ldots, d(u_5) - 1$, where one support of each $P_2$ and $P_3$ is adjacent to $u_5$.

If there exists $i$ such that $T_{u_5i}$ is isomorphic to $P_2$ for $i = 2, \ldots, d(u_5) - 1$, then let $T' = T - \{u_1, u_2, u_3, u'_4\}$. So $T$ is obtained from $T'$ by using operation 2.

Now assume $T_{u_5i}$ is not isomorphic to $P_2$ for any $i = 2, \ldots, d(u_5) - 1$. 

If there exists \( i \) such that \( T_{u_{ij}} \) is isomorphic to \( P_5 \) for \( i = 2, \ldots, d(u_5) - 1 \), then \( T_{u_{ij}} \) is \( P_5 \) for \( i = 2, \ldots, d(u_5) - 1 \). Let \( T' = T_{u_{ij}(u_5)} \). It follows that \( T \) is obtained from \( T' \) by using operation 4.

If \( T_{u_{ij}} \) is neither isomorphic to \( P_2 \) nor \( P_5 \) for any \( i = 2, \ldots, d(u_5) - 1 \), then \( T_{u_{ij}} \) is \( P_4 \) for \( i = 2, \ldots, d(u_5) - 1 \). Let \( T' = T_{u_{ij}(u_5)} \). It follows that \( T \) is obtained from \( T' \) by using operation 3.

**Case 1.2:** \( d(u_5) = 2 \). Since \( T \) is not isomorphic to \( T(4, 1) \), it follows that \( d(u_6) \geq 2 \). Assume that \( T_{u_6} \) has \( k \) components \( T(3, 1) \). Let \( T_{u_1}, T_{u_2}, \ldots, T_{u_k} \) denote components \( T(3, 1) \) of \( T_{u_6} \). If \( k = d(u_6) \), let \( T' = T - T_{u_1} - T_{u_2} - \cdots - T_{u_{k(k-1)}} \); otherwise, \( T' = T - T_{u_1} - T_{u_2} - \cdots - T_{u_k} \). Thus, \( T \) is obtained from \( T' \) by operation 5 or 6.

**Case 2:** \( d(u_4) = 2 \). Suppose that \( d(u_4) = 2 \). Since \( T \) is not isomorphic to \( P_5 \), it follows that \( d(u_5) \geq 2 \). Let \( T' = T - \{u_1, u_2, u_3, u_4\} \). So \( T \) is obtained from \( T' \) by using operation 1. Without loss of generality, assume that \( d(u_4) \geq 3 \).

Let \( T_{u_1}, T_{u_2}, \ldots, T_{u_d(u_4)} \) denote the components of \( T_{u_4} = T - \{u_4\} \) such that \( u_1 \in T_{u_4} \) and \( u_1 \in T_{u_d(u_4)} \). Then \( T_{u_4} \) is isomorphic to \( P_1, P_2 \) or \( P_3 \) for \( i = 2, \ldots, d(u_4) - 1 \), where one leaf of each \( P_i \) is adjacent to \( u_4 \).

If there exists \( i \) such that \( T_{u_{ij}} \) is isomorphic to \( P_1 \) for \( i = 2, \ldots, d(u_4) - 1 \), say \( T_{u_{4d(u_4)-1}} = P_1 \), then \( T_{u_{ij}} = P_3 \) for \( i = 1, 2, \ldots, d(u_4) - 2 \). Let \( T' = T - T_{u_1} - T_{u_2} - \cdots - T_{u_{d(u_4)-2}} - \{u_1, u_2\} \). So \( T \) is obtained from \( T' \) by using operation 7.

If \( T_{u_{ij}} \) is not isomorphic to \( P_1 \) for any \( i = 2, \ldots, d(u_4) - 1 \), then \( T_{u_{ij}} \) is isomorphic to \( P_2 \) for \( i = 2, \ldots, d(u_4) - 1 \), say \( T_{u_{4d(u_4)-1}} = P_2 \), then \( T_{u_{ij}} = P_3 \) for \( i = 1, 2, \ldots, d(u_4) - 2 \). Let \( T' = T - T_{u_1} - T_{u_2} - \cdots - T_{u_{d(u_4)-2}} - \{u_1, u_2\} \). So \( T \) is obtained from \( T' \) by using operation 7.

If \( T_{u_{ij}} \) is neither isomorphic to \( P_1 \) nor \( P_2 \) for any \( i = 2, \ldots, d(u_4) - 1 \), then \( T_{u_{ij}} = P_3 \) for \( i = 1, 2, \ldots, d(u_4) - 1 \). Let \( T' = T_{u_{4d(u_4)}} \). It follows that \( T \) is obtained from \( T' \) by using operation 8.

By Cases 1 and 2, any tree \( T \) can be obtained from \( T' \) by using operation \( i \) for \( i = 1, 2, \ldots, 8 \). Since \( |V(T')| < n \), \( T' \) can be obtained from a path \( P_2 \) or \( P_4 \) by a finite sequence of operations \( i \) for \( i = 1, 2, \ldots, 8 \). It follows that \( T \) can be obtained from \( P_2 \) or \( P_4 \) by a finite sequence of operations \( i \) for \( i = 1, 2, \ldots, 8 \).

Let \( c(i) \) denote the number of operations \( i \) required to construct the tree \( T' \) from \( P_3, l = 2, 4 \), for \( i = 1, 2, \ldots, 8 \). For each operation \( i \), \( S(i, k_{ij}) \) is attached, where \( j = 1, 2, \ldots, c(i) \) and \( i = 3, 4, 5, 6, 7, 8 \). So, \( \gamma_i(T') - \alpha'(T') = \gamma_i(P_l) - \alpha'(P_l) - c'(3) + \sum_{j=1}^{c(i)}(k'_{ij} - 1) - \sum_{j=1}^{c(i)}(k'_{ij} - 1) = 1 + \sum_{j=1}^{c(i)}k'_{ij} - \sum_{j=1}^{c(i)}k'_{kj} - 1). \)

Let \( c(i) \) denote the number of operations \( i \) required to construct the tree \( T \) from \( P_3, l = 2, 4 \), for \( i = 1, 2, \ldots, 8 \). For each operation \( i \), \( S(i, k_{ij}) \) is attached, where \( j = 1, 2, \ldots, c(i) \) and \( i = 3, 4, 5, 6, 7, 8 \). Since \( T \) is obtained from \( T' \) by using some operation \( h \), it follows that \( c(h) = c'(h) + 1 \), and \( c(j) = c'(j) \) for \( j \neq h \). Furthermore, \( k'_{ij} = k_{ij} \) for \( j = 1, \ldots, c'(h) \), and \( k_{ij} = k'_{ij} \) for \( j = 1, \ldots, c(t) \) and \( t \neq h \). By Lemmas 3.9-3.15, it follows that \( \gamma_i(T) - \alpha'(T) = \gamma_i(P_l) - \alpha'(P_l) - c(3) + \sum_{j=1}^{c(i)}(k_{ij} - 1) - \sum_{j=1}^{c(i)}(k_{ij} - 1) = \sum_{j=1}^{c(i)}k_{ij} + \sum_{j=1}^{c(i)}(k'_{ij} - 1). \)
Corollary 3.17. If a tree $T$ can be obtained from path $P_2$ by a finite sequence of operations $i$ for $i = 1, 2, \ldots, 8$, then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\sum_{j=1}^{c(4)}(k_{4j} - 1) - \sum_{j=1}^{c(5)}k_{5j} - \sum_{j=1}^{c(6)}(k_{6j} - 1) + \sum_{j=1}^{c(7)}k_{7j} + \sum_{j=1}^{c(8)}(k_{8j} - 1) \leq c(3) - 1$.

Corollary 3.18. If a tree $T$ can be obtained from path $P_4$ by a finite sequence of operations $i$ for $i = 1, 2, \ldots, 8$, then $\gamma_t(T) \leq \alpha'(T)$ if and only if $\sum_{j=1}^{c(4)}(k_{4j} - 1) - \sum_{j=1}^{c(5)}k_{5j} - \sum_{j=1}^{c(6)}(k_{6j} - 1) + \sum_{j=1}^{c(7)}k_{7j} + \sum_{j=1}^{c(8)}(k_{8j} - 1) \leq c(3)$.

3.2 A family of graphs with total domination numbers at most their matching numbers

A family of graphs with total domination number at most their matching number will be given in the following. Let $\eta = \{T \mid \gamma_t(T) \leq \alpha'(T)\}$. Define a family of graphs $\varrho$. A graph $G \in \varrho$ if and only if $G$ contains a spanning tree $T \in \eta$.

Lemma 3.19. Let $G$ be a connected graph. If $G$ contains a spanning tree $T \in \eta$, then $\gamma_t(G) \leq \alpha'(G)$.

Proof. We will prove by induction on the number of edges of $G$. If $|E(G)| = n - 1$, then $G$ is a tree and $G \in \eta$. So $\gamma_t(G) \leq \alpha'(G)$. Suppose that the property is true for all graph with the number of edges less than $k$. Let $G$ be a connected graph with $k$ edges and $k > n - 1$. Suppose that $T \in \eta$ is a spanning tree of $G$. Let $e \in E(G) - E(T)$ and $G' = G - e$. Then $T$ is also a spanning tree of $G'$. By the induction hypothesis, we have $\gamma_t(G') \leq \alpha'(G')$. It is obvious that $\gamma_t(G) \leq \gamma_t(G')$ and $\alpha'(G') \leq \alpha'(G)$. Hence $\gamma_t(G) \leq \alpha'(G)$. \qed

By Lemma 3.19 we have the following result.

Theorem 3.20. For any graph $G \in \varrho$, $\gamma_t(G) \leq \alpha'(G)$.

Acknowledgement: Partially supported by GRF, Research Grant Council of Hong Kong; and Faculty Research Grant, Hong Kong Baptist University.

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