STABILITY OF HOMOMORPHISMS AND (θ, ϕ) -DERIVATIONS

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ABSTRACT. In this paper, we prove the generalized Hyers–Ulam stability of homomorphisms and (θ, ϕ) -derivations on a ring $\mathcal{R}$ into a Banach $\mathcal{R}$ -bimodule $\mathcal{M}$.

1. INTRODUCTION

The stability problem of functional equations originated from a question of S.M. Ulam [37] concerning the stability of group homomorphisms : *Let $(G_1, *)$ be a group and let $(G_2, \diamond, d)$ be a metric group with the metric $d(\cdot, \cdot)$. Given $\epsilon > 0$, does there exist $\delta(\epsilon) > 0$ such that if a mapping $h : G_1 \rightarrow G_2$ satisfies the inequality*

$$d(h(x * y), h(x) \diamond h(y)) < \delta$$

for all $x, y \in G_1$, then there is a homomorphism $H : G_1 \rightarrow G_2$ with

$$d(h(x), H(x)) < \epsilon$$

for all $x \in G_1$?

In other words, we are looking for situations where homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a homomorphism near it. D.H. Hyers [12] gave a first affirmative answer to the question of Ulam for Banach spaces. Let X and Y be Banach spaces: *Assume that $f : X \rightarrow Y$ satisfies*

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon$$

for some $\varepsilon \geq 0$ and all $x, y \in X$. Then there exists a unique additive mapping $T : X \rightarrow Y$ such that

$$\|f(x) - T(x)\| \leq \varepsilon$$

for all $x \in X$.

T. Aoki [2] and Th.M. Rassias [31] provided a generalization of the Hyers' theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

2000 *Mathematics Subject Classification.* Primary: 39B72; Secondary 47H09.

Key words and phrases. Generalized metric space, fixed point, stability, Banach algebra, semiprime ring, Jordan derivation, generalized Jordan derivation.

Theorem 1.1. (Th.M. Rassias). *Let $f : E \rightarrow E'$ be a mapping from a normed vector space E into a Banach space E' subject to the inequality*

$$\|f(x+y) - f(x) - f(y)\| \leq \varepsilon(\|x\|^p + \|y\|^p) \quad (1.1)$$

for all $x, y \in E$, where ε and p are constants with $\varepsilon > 0$ and $p < 1$. Then the limit

$$L(x) = \lim_{n \rightarrow \infty} \frac{f(2^n x)}{2^n}$$

exists for all $x \in E$ and $L : E \rightarrow E'$ is the unique additive mapping which satisfies

$$\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p} \|x\|^p \quad (1.2)$$

for all $x \in E$. If $p < 0$ then inequality (1.1) holds for $x, y \neq 0$ and (1.2) for $x \neq 0$. Also, if for each $x \in E$ the mapping $t \mapsto f(tx)$ is continuous in $t \in \mathbb{R}$, then L is linear.

The inequality (1.1) has provided a lot of influence in the development of what is now known as a *generalized Hyers–Ulam stability* of functional equations. In 1994, a generalization of the Th.M. Rassias' theorem was obtained by P. Găvruta [8], who replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general control function $\varphi(x, y)$. Since then the stability problems of various functional equations and mappings and their Pexiderized versions with more general domains and ranges have been investigated by a number of authors (see [21]–[29]). We also refer the readers to the books [7], [13], [16] and [32].

Let A be a real or complex algebra. A mapping $D : A \rightarrow A$ is said to be a (*ring*) *derivation* if

$$D(a+b) = D(a) + D(b), \quad D(ab) = D(a)b + aD(b)$$

for all $a, b \in A$. If, in addition, $D(\lambda a) = \lambda D(a)$ for all $a \in A$ and all $\lambda \in \mathbb{F}$, then D is called a *linear derivation*, where $\mathbb{F}$ denotes the scalar field of A . I.M. Singer and J. Wermer [35] proved that if A is a commutative Banach algebra and $D : A \rightarrow A$ is a continuous linear derivation, then $D(A) \subseteq \text{rad}(A)$. They also conjectured that the same result holds even D is a discontinuous linear derivation. M.P. Thomas [36] proved the conjecture. As a direct consequence, we see that there are no non-zero linear derivations on a semi-simple commutative Banach algebra, which had been proved by B.E. Johnson [15]. On the other hand, it is not the case for ring derivations. O. Hatori and J. Wada [9] determined a representation of ring derivations on a semi-simple commutative Banach algebra (see also [33]) and they proved that only the zero operator is a ring derivation

on a semi-simple commutative Banach algebra with the maximal ideal space without isolated points. The stability of derivations between operator algebras was first obtained by P. Šemrl [34]. Badora [3] and T. Miura *et al.* [22] proved the generalized Hyers–Ulam stability of ring derivations on Banach algebras.

Let $\mathcal{R}$ be an associative ring, $\mathcal{N}$ be a $\mathcal{R}$ -bimodule and let θ, ϕ be automorphisms of $\mathcal{R}$. An additive mapping $D : \mathcal{R} \rightarrow \mathcal{N}$ is called a *derivation* if $D(ab) = D(a)b + aD(b)$ holds for all pairs $a, b \in \mathcal{R}$ and is called a *Jordan derivation* in case $D(a^2) = D(a)a + aD(a)$ is fulfilled for all $a \in \mathcal{R}$. Every derivation is a Jordan derivation. The converse is in general not true (see [6, 10]). The concept of generalized derivation has been introduced by Bresar [4]. Hvala [11] and Lee [18] introduced a concept of (θ, ϕ) -derivation (see also [19]). An additive mapping $F : \mathcal{R} \rightarrow \mathcal{N}$ is called a (θ, ϕ) -*derivation* in case $F(ab) = F(a)\theta(b) + \phi(a)F(b)$ holds for all pairs $a, b \in \mathcal{R}$. An additive mapping $F : \mathcal{R} \rightarrow \mathcal{N}$ is called a (θ, ϕ) -*Jordan derivation* in case $F(a^2) = F(a)\theta(a) + \phi(a)F(a)$ holds for all $a \in \mathcal{R}$. An additive mapping $F : \mathcal{R} \rightarrow \mathcal{N}$ is called a *generalized (θ, ϕ) -derivation* in case $F(ab) = F(a)\theta(b) + \phi(a)D(b)$ holds for all pairs $a, b \in \mathcal{R}$, where $D : \mathcal{R} \rightarrow \mathcal{N}$ is a (θ, ϕ) -derivation. An additive mapping $F : \mathcal{R} \rightarrow \mathcal{N}$ is called a *generalized (θ, ϕ) -Jordan derivation* in case $F(a^2) = F(a)\theta(a) + \phi(a)D(a)$ holds for all $a \in \mathcal{R}$, where $D : \mathcal{R} \rightarrow \mathcal{N}$ is a (θ, ϕ) -Jordan derivation. It is clear that every generalized (θ, ϕ) -derivation is a generalized (θ, ϕ) -Jordan derivation.

The aim of the present paper is to establish the stability problem of homomorphisms and generalized (θ, ϕ) -derivations by using the fixed point method (see [1, 5, 17, 21]).

Let E be a set. A function $d : E \times E \rightarrow [0, \infty]$ is called a *generalized metric* on E if d satisfies

- (i) $d(x, y) = 0$ if and only if $x = y$;
- (ii) $d(x, y) = d(y, x)$ for all $x, y \in E$;
- (iii) $d(x, z) \leq d(x, y) + d(y, z)$ for all $x, y, z \in E$.

We recall the following theorem by Margolis and Diaz.

Theorem 1.2. [20] *Let (E, d) be a complete generalized metric space and let $J : E \rightarrow E$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in E$, either*

$$d(J^n x, J^{n+1} x) = \infty$$

for all non-negative integers n or there exists a non-negative integer n_0 such that

- (1) $d(J^n x, J^{n+1} x) < \infty$ for all $n \geq n_0$;
- (2) the sequence $\{J^n x\}$ converges to a fixed point y^* of J ;
- (3) y^* is the unique fixed point of J in the set $Y = \{y \in E : d(J^{n_0} x, y) < \infty\}$;
- (4) $d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.

2. STABILITY OF HOMOMORPHISMS

In this section, we assume that $\mathcal{R}$ is an associative ring, $\mathcal{X}$ is a normed algebra, $\mathcal{Y}$ is a Banach algebra, and $n \geq 3$ is a fixed integer.

Lemma 2.1. *Let X and Y be linear spaces. A mapping $f : X \rightarrow Y$ (with $f(0) = 0$ if $n = 3$) satisfies*

$$\sum_{j=1}^n f\left(-x_j + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} x_i\right) = (n-2) \sum_{i=1}^n f(x_i) \quad (2.1)$$

for all $x_1, \dots, x_n \in X$, if and only if f is additive.

Proof. Let f satisfy (2.1). Letting $x_1 = \dots = x_n = 0$ in (2.1), we get $f(0) = 0$. Letting $x_2 = \dots = x_n = 0$ in (2.1), we infer that f is odd. So by letting $x_3 = \dots = x_n = 0$ in (2.1) and using the oddness of f , we get that the mapping f is additive. The converse is obvious. $\square$

Theorem 2.2. *Let $f : \mathcal{R} \rightarrow \mathcal{Y}$ be a mapping for which there exist functions $\varphi : \mathcal{R}^n \rightarrow [0, \infty)$ and $\psi : \mathcal{R}^2 \rightarrow [0, \infty)$ such that*

$$\lim_{k \rightarrow \infty} \frac{1}{r^k} \varphi(r^k a_1, \dots, r^k a_n) = 0, \quad (2.2)$$

$$\lim_{k \rightarrow \infty} \frac{1}{r^k} \psi(r^k a, b) = \lim_{k \rightarrow \infty} \frac{1}{r^k} \psi(a, r^k b) = \lim_{k \rightarrow \infty} \frac{1}{r^k} \psi(r^k a, r^k b) = 0, \quad (2.3)$$

$$\left\| \sum_{j=1}^n f\left(-a_j + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} a_i\right) - (n-2) \sum_{i=1}^n f(a_i) \right\| \leq \varphi(a_1, \dots, a_n), \quad (2.4)$$

$$\|f(ab) - f(a)f(b)\| \leq \psi(a, b) \quad (2.5)$$

for all $a, b, a_1, \dots, a_n \in \mathcal{R}$, where $r = n-2$. If there exists a constant $L < 1$ such that

$$\varphi(ra, \dots, ra) \leq rL\varphi(a, \dots, a)$$

for all $a \in \mathcal{R}$, then there exists a unique homomorphism $H : \mathcal{R} \rightarrow \mathcal{Y}$ satisfying

$$\|f(a) - H(a)\| \leq \frac{1}{n(n-2)(1-L)} \varphi(a, \dots, a), \quad (2.6)$$

$$H(a)[H(b) - f(b)] = [H(a) - f(a)]H(b) = 0 \quad (2.7)$$

for all $a, b \in \mathcal{R}$.

Proof. Letting $a_1 = \cdots = a_n = a$ in (2.4), we get

$$\|f(ra) - rf(a)\| \leq \frac{1}{n}\varphi(a, \dots, a) \quad (2.8)$$

for all $a \in \mathcal{R}$. Let $E := \{g : \mathcal{R} \rightarrow \mathcal{Y}\}$. We introduce a generalized metric on E as follows:

$$d_\varphi(g, h) := \inf\{C \in [0, \infty] : \|g(a) - h(a)\| \leq C\varphi(a, \dots, a) \text{ for all } a \in \mathcal{R}\}.$$

It is easy to show that (E, d_φ) is a generalized complete metric space [5].

Now we consider the mapping $\Lambda : E \rightarrow E$ defined by

$$(\Lambda g)(a) = \frac{1}{r}g(ra), \quad \text{for all } g \in E \text{ and } a \in \mathcal{R}.$$

Let $g, h \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d_\varphi(g, h) \leq C$.

From the definition of d_φ , we have

$$\|g(a) - h(a)\| \leq C\varphi(a, \dots, a)$$

for all $a \in \mathcal{R}$. By the assumption and last inequality, we have

$$\|(\Lambda g)(a) - (\Lambda h)(a)\| = \frac{1}{r}\|g(ra) - h(ra)\| \leq \frac{C}{r}\varphi(ra, \dots, ra) \leq CL\varphi(a, \dots, a)$$

for all $a \in \mathcal{R}$. So $d_\varphi(\Lambda g, \Lambda h) \leq Ld_\varphi(g, h)$ for any $g, h \in E$. It follows from (2.8) that $d_\varphi(\Lambda f, f) \leq \frac{1}{n(n-2)}$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^k f\}$ converges to a fixed point H of Λ , i.e.,

$$H : \mathcal{R} \rightarrow \mathcal{Y}, \quad H(a) = \lim_{k \rightarrow \infty} (\Lambda^k f)(a) = \lim_{k \rightarrow \infty} \frac{1}{r^k} f(r^k a)$$

and $H(ra) = rH(a)$ for all $a \in \mathcal{R}$. Also H is the unique fixed point of Λ in the set $E_\varphi = \{g \in E : d_\varphi(f, g) < \infty\}$ and

$$d_\varphi(H, f) \leq \frac{1}{1-L}d_\varphi(\Lambda f, f) \leq \frac{1}{n(n-2)(1-L)},$$

i.e., inequality (2.6) holds true for all $a \in \mathcal{R}$. It follows from the definition of H , (2.2) and (2.4) that

$$\sum_{j=1}^n H\left(-a_j + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} a_i\right) = (n-2) \sum_{i=1}^n H(a_i)$$

for all $a_1, \dots, a_n \in \mathcal{R}$. Since $H(0) = 0$, by Lemma 2.1 the mapping H is additive. So it follows from the definition of H , (2.3) and (2.5) that

$$\begin{aligned} \|H(ab) - H(a)H(b)\| &= \lim_{k \rightarrow \infty} \frac{1}{r^{2k}} \|f(r^{2k}ab) - f(r^k a)f(r^k b)\| \\ &\leq \lim_{k \rightarrow \infty} \frac{1}{r^{2k}} \psi(r^k a, r^k b) = 0 \end{aligned}$$

for all $a, b \in \mathcal{R}$. So H is homomorphism. Similarly, we have from (2.3) and (2.5) that

$$H(ab) = H(a)f(b), \quad H(ab) = f(a)H(b) \quad (2.9)$$

for all $a, b \in \mathcal{R}$. Since H is homomorphism, we get (2.7) from (2.9).

Finally it remains to prove the uniqueness of H . Let $H_1 : \mathcal{R} \rightarrow \mathcal{Y}$ be another homomorphism satisfying (2.6). Since $d_\varphi(f, H_1) \leq \frac{1}{n(n-2)(1-L)}$ and H_1 is additive, we get $H_1 \in E_\varphi$ and $(\Lambda H_1)(a) = \frac{1}{r}H_1(ra) = H_1(a)$ for all $a \in \mathcal{R}$, i.e., H_1 is a fixed point of Λ . Since H is the unique fixed point of Λ in E_φ , we get $H_1 = H$. $\square$

We need the following lemma in the proof of the next theorem.

Lemma 2.3. [30] *Let X and Y be linear spaces and $f : X \rightarrow Y$ be an additive mapping such that $f(\mu x) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1 := \{\mu \in \mathbb{C} : |\mu| = 1\}$. Then the mapping f is $\mathbb{C}$ -linear.*

Lemma 2.4. *Let X and Y be linear spaces. A mapping $f : X \rightarrow Y$ satisfies*

$$\sum_{j=1}^n f(-\mu x_j + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} \mu x_i) = (n-2)\mu \sum_{i=1}^n f(x_i) \quad (2.10)$$

for all $x_1, \dots, x_n \in X$ and all $\mu \in \mathbb{T}^1$, if and only if f is $\mathbb{C}$ -linear.

Proof. Let f satisfy (2.10). Letting $x_1 = \dots = x_n = 0$ in (2.10), we get $f(0) = 0$. By Lemma 2.1, the mapping f is additive. Letting $x_2 = \dots = x_n = 0$ in (2.10) and using the oddness of f , we get that $f(\mu x_1) = \mu f(x_1)$ for all $x_1 \in X$ and all $\mu \in \mathbb{T}^1$. So by Lemma 2.3, the mapping f is $\mathbb{C}$ -linear. The converse is obvious. $\square$

The following theorem is an alternative result of Theorem 2.2.

Theorem 2.5. *Let $f : \mathcal{X} \rightarrow \mathcal{Y}$ be a mapping for which there exist functions $\varphi : \mathcal{X}^n \rightarrow [0, \infty)$ and $\psi : \mathcal{X}^2 \rightarrow [0, \infty)$ such that*

$$\begin{aligned} \lim_{k \rightarrow \infty} r^k \varphi\left(\frac{1}{r^k} a_1, \dots, \frac{1}{r^k} a_n\right) &= 0, \\ \lim_{k \rightarrow \infty} r^k \psi\left(\frac{1}{r^k} a, b\right) &= \lim_{k \rightarrow \infty} r^k \psi\left(a, \frac{1}{r^k} b\right) = \lim_{k \rightarrow \infty} r^{2k} \psi\left(\frac{1}{r^k} a, \frac{1}{r^k} b\right) = 0, \\ \left\| \sum_{j=1}^n f(-\mu a_j + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} \mu a_i) - (n-2)\mu \sum_{i=1}^n f(a_i) \right\| &\leq \varphi(a_1, \dots, a_n), \\ \|f(ab) - f(a)f(b)\| &\leq \psi(a, b) \end{aligned}$$

for all $a, b, a_1, \dots, a_n \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$, where $r = n - 2$. If there exists a constant $L < 1$ such that

$$r\varphi\left(\frac{1}{r}a, \dots, \frac{1}{r}a\right) \leq L\varphi(a, \dots, a)$$

for all $a \in \mathcal{X}$, then there exists a unique homomorphism $H : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying

$$\begin{aligned} \|f(a) - H(a)\| &\leq \frac{L}{n(n-2)(1-L)}\varphi(a, \dots, a), \\ H(a)[H(b) - f(b)] &= [H(a) - f(a)]H(b) = 0 \end{aligned}$$

for all $a, b \in \mathcal{X}$.

Proof. It follows from the assumptions that $\varphi(0, \dots, 0) = 0$, and so $f(0) = 0$. Letting $\mu = 1$ and using the same method as in the proof of Theorem 2.2, we have

$$\|f(ra) - rf(a)\| \leq \frac{1}{n}\varphi(a, \dots, a) \quad (2.11)$$

for all $a \in \mathcal{R}$. Let $E := \{g : \mathcal{X} \rightarrow \mathcal{Y} \mid g(0) = 0\}$. We introduce the same definition d_φ as in the proof of Theorem 2.2 such that (E, d_φ) becomes a generalized complete metric space. Let $\Lambda : E \rightarrow E$ be the mapping defined by

$$(\Lambda g)(a) = rg\left(\frac{1}{r}a\right), \quad \text{for all } g \in E \text{ and } a \in \mathcal{X}.$$

One can show that

$$d_\varphi(\Lambda g, \Lambda h) \leq Ld_\varphi(g, h)$$

for any $g, h \in E$. It follows from the assumption and (2.11) that $d_\varphi(\Lambda f, f) \leq \frac{L}{n(n-2)}$. Due to Theorem 1.2, the sequence $\{\Lambda^k f\}$ converges to a fixed point H of Λ , i.e., $H : \mathcal{X} \rightarrow \mathcal{Y}$,

$$H(a) = \lim_{k \rightarrow \infty} (\Lambda^k f)(a) = \lim_{n \rightarrow \infty} r^k f\left(\frac{1}{r^k}a\right), \quad H(ra) = rH(a)$$

for all $a \in \mathcal{X}$. Also

$$d_\varphi(H, f) \leq \frac{1}{1-L}d_\varphi(\Lambda f, f) \leq \frac{L}{n(n-2)(1-L)}$$

i.e., the inequality

$$\|f(a) - H(a)\| \leq \frac{L}{n(n-2)(1-L)}\varphi(a, \dots, a)$$

holds true for all $a \in \mathcal{X}$.

The rest of the proof is similar to the proof of Theorem 3.1 and we omit the details. $\square$

Corollary 2.6. *Let $p, q, \delta, \varepsilon$ be non-negative real numbers with $0 < p, q < 1$. Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a mapping such that*

$$\left\| \sum_{j=1}^n f(-\mu a_j + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} \mu a_i) - (n-2)\mu \sum_{i=1}^n f(a_i) \right\| \leq \delta + \varepsilon \sum_{i=1}^n \|a_i\|^p,$$

$$\|f(ab) - f(a)f(b)\| \leq \delta + \varepsilon(\|a\|^q + \|b\|^q)$$

for all $a, b, a_1, \dots, a_n \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique homomorphism $H : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying

$$\|f(a) - H(a)\| \leq \frac{\delta}{(r+2)(r-r^p)} + \frac{\varepsilon}{r-r^p} \|a\|^p,$$

$$H(a)[H(b) - f(b)] = [H(a) - f(a)]H(b) = 0$$

for all $a, b \in \mathcal{X}$, where $r = n-2$.

Proof. The proof follows from Theorem 2.2 by taking

$$\varphi(a_1, \dots, a_n) := \delta + \varepsilon \sum_{i=1}^n \|a_i\|^p, \quad \psi(a, b) := \delta + \varepsilon(\|a\|^q + \|b\|^q)$$

for all $a, b, a_1, \dots, a_n \in \mathcal{X}$. Then we can choose $L = r^{p-1}$ and we get the desired results. $\square$

Corollary 2.7. *Let p, q, ε be non-negative real numbers with $p > 1$ and $q > 2$. Suppose that $f : \mathcal{X} \rightarrow \mathcal{Y}$ is a mapping such that*

$$\left\| \sum_{j=1}^n f(-\mu a_j + \sum_{\substack{1 \leq i \leq n \\ i \neq j}} \mu a_i) - (n-2)\mu \sum_{i=1}^n f(a_i) \right\| \leq \varepsilon \sum_{i=1}^n \|a_i\|^p,$$

$$\|f(ab) - f(a)f(b)\| \leq \varepsilon(\|a\|^q + \|b\|^q)$$

for all $a, b, a_1, \dots, a_n \in \mathcal{X}$ and all $\mu \in \mathbb{T}^1$. Then there exists a unique homomorphism $H : \mathcal{X} \rightarrow \mathcal{Y}$ satisfying

$$\|f(a) - H(a)\| \leq \frac{\varepsilon}{r^p - r} \|a\|^p$$

for all $a, b \in \mathcal{X}$, where $r = n-2$.

Proof. The proof follows from Theorem 2.5 by taking

$$\varphi(a_1, \dots, a_n) := \varepsilon \sum_{i=1}^n \|a_i\|^p, \quad \psi(a, b) := \varepsilon(\|a\|^q + \|b\|^q)$$

for all $a, b, a_1, \dots, a_n \in \mathcal{X}$. Then we can choose $L = r^{1-p}$ and we get the desired results. $\square$

3. STABILITY OF GENERALIZED (θ, ϕ) -DERIVATIONS

In this section, we assume that $\mathcal{R}$ is a 2-divisible associative ring, $\mathcal{M}$ is a Banach $\mathcal{R}$ -bimodule, and θ, ϕ are automorphisms of $\mathcal{R}$. For convenience, we use the following abbreviation for given mappings $f, g : \mathcal{R} \rightarrow \mathcal{M}$,

$$\begin{aligned} D_{f,g}^{\theta,\phi}(a, b, c, d) &:= f(ab + c + d) - f(a)\theta(b) - \phi(a)g(b) - f(c) - f(d), \\ J_{f,g}^{\theta,\phi}(a, b, c) &:= f(a^2 + b + c) - f(a)\theta(a) - \phi(a)g(a) - f(b) - f(c) \end{aligned}$$

for all $a, b, c, d \in \mathcal{R}$. Now we prove the generalized Hyers–Ulam stability of generalized (θ, ϕ) -derivations and generalized (θ, ϕ) -Jordan derivations in Banach $\mathcal{R}$ -bimodules.

Theorem 3.1. *Let $f, g : \mathcal{R} \rightarrow \mathcal{M}$ be mappings for which there exist functions $\varphi, \psi : \mathcal{R}^3 \rightarrow [0, \infty)$ such that*

$$\lim_{n \rightarrow \infty} 4^n \varphi\left(\frac{a}{2^n}, 0, 0\right) = \lim_{n \rightarrow \infty} 2^n \varphi\left(0, \frac{b}{2^n}, \frac{c}{2^n}\right) = 0, \quad (3.1)$$

$$\|J_{f,g}^{\theta,\phi}(a, b, c)\| \leq \varphi(a, b, c), \quad (3.2)$$

$$\lim_{n \rightarrow \infty} 4^n \psi\left(\frac{a}{2^n}, 0, 0\right) = \lim_{n \rightarrow \infty} 2^n \psi\left(0, \frac{b}{2^n}, \frac{c}{2^n}\right) = 0, \quad (3.3)$$

$$\|J_{g,g}^{\theta,\phi}(a, b, c)\| \leq \psi(a, b, c) \quad (3.4)$$

for all $a, b, c \in \mathcal{R}$. If there exist constants $L, K < 1$ such

$$2\varphi(0, a, a) \leq L\varphi(0, 2a, 2a), \quad 2\psi(0, a, a) \leq K\psi(0, 2a, 2a)$$

for all $a \in \mathcal{R}$, then there exist a unique (θ, ϕ) -Jordan derivation $G : \mathcal{R} \rightarrow \mathcal{M}$ and a unique generalized (θ, ϕ) -Jordan derivation $F : \mathcal{R} \rightarrow \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \leq \frac{L}{2 - 2L} \varphi(0, a, a), \quad (3.5)$$

$$\|g(a) - G(a)\| \leq \frac{K}{2 - 2K} \psi(0, a, a) \quad (3.6)$$

for all $a \in \mathcal{R}$.

Proof. It follows from (3.1) and (3.3) that $\varphi(0, 0, 0) = 0 = \psi(0, 0, 0)$ and so we get from (3.2) and (3.4) that $f(0) = g(0) = 0$. Letting $a = 0$ and $b = c$ in (3.2), we get

$$\|f(2c) - 2f(c)\| \leq \varphi(0, c, c) \quad (3.7)$$

for all $c \in \mathcal{R}$. Let $E := \{h : \mathcal{R} \rightarrow \mathcal{M} \mid h(0) = 0\}$. We introduce a generalized metric on E as follows:

$$d_\varphi(h, k) := \inf\{C \in [0, \infty] : \|h(a) - k(a)\| \leq C\varphi(0, a, a) \text{ for all } a \in \mathcal{R}\}.$$

It is easy to show that (E, d_φ) is a generalized complete metric space [5].

Now we consider the mapping $\Lambda : E \rightarrow E$ defined by

$$(\Lambda h)(a) = 2h\left(\frac{a}{2}\right), \quad \text{for all } h \in E \text{ and } a \in \mathcal{R}.$$

Let $h, k \in E$ and let $C \in [0, \infty]$ be an arbitrary constant with $d_\varphi(h, k) \leq C$. From the definition of d_φ , we have

$$\|h(a) - k(a)\| \leq C\varphi(0, a, a)$$

for all $a \in \mathcal{R}$. By the assumption and last inequality, we have

$$\|(\Lambda h)(a) - (\Lambda k)(a)\| = 2\left\|h\left(\frac{a}{2}\right) - k\left(\frac{a}{2}\right)\right\| \leq 2C\varphi\left(0, \frac{a}{2}, \frac{a}{2}\right) \leq CL\varphi(0, a, a)$$

for all $a \in \mathcal{R}$. So $d_\varphi(\Lambda h, \Lambda k) \leq Ld_\varphi(h, k)$ for any $h, k \in E$. It follows from the assumption and (3.7) that $d_\varphi(\Lambda f, f) \leq L/2$. Therefore according to Theorem 1.2, the sequence $\{\Lambda^n f\}$ converges to a fixed point F of Λ , i.e.,

$$F : \mathcal{R} \rightarrow \mathcal{M}, \quad F(a) = \lim_{n \rightarrow \infty} (\Lambda^n f)(a) = \lim_{n \rightarrow \infty} 2^n f\left(\frac{a}{2^n}\right)$$

and $F(2a) = 2F(a)$ for all $a \in \mathcal{R}$. Also F is the unique fixed point of Λ in the set $E_\varphi = \{h \in E : d_\varphi(f, h) < \infty\}$ and

$$d_\varphi(F, f) \leq \frac{1}{1-L} d_\varphi(\Lambda f, f) \leq \frac{L}{2-2L},$$

i.e., inequality (3.5) holds true for all $a \in \mathcal{R}$. Similarly, we obtain that

$$d_\psi(\Lambda h, \Lambda k) \leq Kd_\psi(h, k), \quad d_\psi(\Lambda g, g) \leq K/2.$$

for any $h, k \in E$, where

$$d_\psi(h, k) := \inf\{C \in [0, \infty] : \|h(a) - k(a)\| \leq C\psi(0, a, a) \text{ for all } a \in \mathcal{R}\}.$$

So according to Theorem 1.2, the sequence $\{\Lambda^n g\}$ converges to a fixed point G of Λ , i.e.,

$$G : \mathcal{R} \rightarrow \mathcal{M}, \quad G(a) = \lim_{n \rightarrow \infty} (\Lambda^n g)(a) = \lim_{n \rightarrow \infty} 2^n g\left(\frac{a}{2^n}\right)$$

and $G(2a) = 2G(a)$ for all $a \in \mathcal{R}$. Also G is the unique fixed point of Λ in the set $E_\psi = \{h \in E : d_\psi(g, h) < \infty\}$ and

$$d_\psi(G, g) \leq \frac{1}{1-K} d_\psi(\Lambda g, g) \leq \frac{K}{2-2K},$$

i.e., inequality (3.6) holds true for all $a \in \mathcal{R}$. It follows from the definitions of F, G , (3.1) and (3.2) that

$$\begin{aligned} \|J_{F,G}^{\theta,\phi}(a, 0, 0)\| &= \lim_{n \rightarrow \infty} 4^n \left\| J_{f,g}^{\theta,\phi}\left(\frac{a}{2^n}, 0, 0\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} 4^n \varphi\left(\frac{a}{2^n}, 0, 0\right) = 0, \\ \|J_{F,G}^{\theta,\phi}(0, b, c)\| &= \lim_{n \rightarrow \infty} 2^n \left\| J_{f,g}^{\theta,\phi}\left(0, \frac{b}{2^n}, \frac{c}{2^n}\right) \right\| \\ &\leq \lim_{n \rightarrow \infty} 2^n \varphi\left(0, \frac{b}{2^n}, \frac{c}{2^n}\right) = 0 \end{aligned}$$

for all $a, b, c \in \mathcal{R}$. Hence

$$F(a^2) = F(a)\theta(a) + \phi(a)G(a), \quad F(b+c) = F(b) + F(c) \quad (3.8)$$

for all $a, b, c \in \mathcal{R}$. Similarly, it follows from the definition of G , (3.3) and (3.4) that

$$G(a^2) = G(a)\theta(a) + \phi(a)G(a), \quad G(b+c) = G(b) + G(c) \quad (3.9)$$

for all $a, b, c \in \mathcal{R}$. Hence G is a (θ, ϕ) -Jordan derivation. So we infer from (3.8) and (3.9) that F is a generalized (θ, ϕ) -Jordan derivation.

Finally it remains to prove the uniqueness of F and G . Let $F_1, G_1 : \mathcal{R} \rightarrow \mathcal{M}$ be another additive mappings satisfying (3.5) and (3.6), respectively. Since $d_\varphi(f, F_1) \leq \frac{L}{2-2L}$, $d_\psi(g, G_1) \leq \frac{K}{2-2K}$ and F_1, G_1 are additive, we get $F_1 \in E_\varphi$, $G_1 \in E_\psi$ and $(\Lambda F_1)(a) = 2F_1(a/2) = F_1(a)$, $(\Lambda G_1)(a) = 2G_1(a/2) = G_1(a)$ for all $a \in \mathcal{R}$, i.e., F_1, G_1 are fixed points of Λ . Since F and G are the unique fixed points of Λ in E_φ and E_ψ , respectively, we get $F_1 = F$ and $G_1 = G$. $\square$

Theorem 3.2. *Let $f, g : \mathcal{R} \rightarrow \mathcal{M}$ be mappings with $f(0) = g(0) = 0$ for which there exist functions $\Phi, \Psi : \mathcal{R}^3 \rightarrow [0, \infty)$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{4^n} \Phi(2^n a, 0, 0) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \Phi(0, 2^n b, 2^n c) = 0, \quad (3.10)$$

$$\|J_{f,g}^{\theta,\phi}(a, b, c)\| \leq \Phi(a, b, c), \quad (3.11)$$

$$\lim_{n \rightarrow \infty} \frac{1}{4^n} \Psi(2^n a, 0, 0) = \lim_{n \rightarrow \infty} \frac{1}{2^n} \Psi(0, 2^n b, 2^n c) = 0, \quad (3.12)$$

$$\|J_{g,g}^{\theta,\phi}(a, b, c)\| \leq \Psi(a, b, c) \quad (3.13)$$

for all $a, b, c \in \mathcal{R}$. If there exist constants $L, K < 1$ such

$$\Phi(0, 2a, 2a) \leq 2L\Phi(0, a, a), \quad \Psi(0, 2a, 2a) \leq 2K\Psi(0, a, a)$$

for all $a \in \mathcal{R}$, then there exist a unique (θ, ϕ) -Jordan derivation $G : \mathcal{R} \rightarrow \mathcal{M}$ and a unique generalized (θ, ϕ) -Jordan derivation $F : \mathcal{R} \rightarrow \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \leq \frac{1}{2-2L}\Phi(0, a, a), \quad (3.14)$$

$$\|g(a) - G(a)\| \leq \frac{1}{2-2K}\Psi(0, a, a), \quad (3.15)$$

for all $a \in \mathcal{R}$.

Proof. Using the same method as in the proof of Theorem 3.1, we have

$$\left\| \frac{1}{2}f(2c) - f(c) \right\| \leq \frac{1}{2}\Phi(0, c, c), \quad \left\| \frac{1}{2}g(2c) - g(c) \right\| \leq \frac{1}{2}\Psi(0, c, c) \quad (3.16)$$

for all $c \in \mathcal{R}$. We introduce the same definitions for E , d_Φ and d_Ψ as in the proof of Theorem 3.1 such that (E, d_Φ) and (E, d_Ψ) become generalized complete metric spaces. Let $\Lambda : E \rightarrow E$ be the mapping defined by

$$(\Lambda h)(a) = \frac{1}{2}h(2a), \quad \text{for all } h \in E \text{ and } a \in \mathcal{R}.$$

One can show that

$$d_\Phi(\Lambda h, \Lambda k) \leq Ld_\Phi(h, k), \quad d_\Psi(\Lambda h, \Lambda k) \leq Kd_\Psi(h, k)$$

for any $h, k \in E$. It follows from (3.16) that $d_\Phi(\Lambda f, f) \leq \frac{1}{2}$ and $d_\Psi(\Lambda g, g) \leq \frac{1}{2}$. Due to Theorem 1.2, the sequences $\{\Lambda^n f\}$ and $\{\Lambda^n g\}$ converge to fixed points F and G of Λ , i.e., $F, G : \mathcal{R} \rightarrow \mathcal{M}$,

$$F(a) = \lim_{n \rightarrow \infty} (\Lambda^n f)(a) = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n a), \quad G(a) = \lim_{n \rightarrow \infty} (\Lambda^n g)(a) = \lim_{n \rightarrow \infty} \frac{1}{2^n} g(2^n a),$$

$F(2a) = 2F(a)$ and $G(2a) = 2G(a)$ for all $a \in \mathcal{R}$. Also

$$\begin{aligned} d_\Phi(F, f) &\leq \frac{1}{1-L} d_\Phi(\Lambda f, f) \leq \frac{1}{2-2L}, \\ d_\Psi(G, g) &\leq \frac{1}{1-K} d_\Psi(\Lambda g, g) \leq \frac{1}{2-2K}, \end{aligned}$$

i.e., the inequalities (3.14) and (3.15) hold true for all $a \in \mathcal{R}$.

The rest of the proof is similar to the proof of Theorem 3.1 and we omit the details. $\square$

Corollary 3.3. *Let $\varepsilon, \delta, p, q$ be non-negative real numbers with $0 < p, q < 1$ or $p, q > 2$. If $\mathcal{R}$ is a normed ring and $f, g : \mathcal{R} \rightarrow \mathcal{M}$ are mappings satisfy the inequalities*

$$\|J_{f,g}^{\theta,\phi}(a, b, c)\| \leq \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p), \quad \|J_{g,g}^{\theta,\phi}(a, b, c)\| \leq \delta(\|a\|^q + \|b\|^q + \|c\|^q)$$

for all $a, b, c \in \mathcal{R}$, then there exist a unique (θ, ϕ) -Jordan derivation $G : \mathcal{R} \rightarrow \mathcal{M}$ and a unique generalized (θ, ϕ) -Jordan derivation $F : \mathcal{R} \rightarrow \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \leq \frac{2\varepsilon}{|2 - 2^p|} \|a\|^p, \quad \|g(a) - G(a)\| \leq \frac{2\delta}{|2 - 2^q|} \|a\|^q$$

for all $a \in \mathcal{R}$.

Proof. Let

$$L := \begin{cases} 2^{p-1}, & 0 < p < 1; \\ 2^{1-p}, & p > 2. \end{cases} \quad K := \begin{cases} 2^{q-1}, & 0 < q < 1; \\ 2^{1-q}, & q > 2. \end{cases}$$

So the result follows from Theorems 3.1 and 3.2. $\square$

Corollary 3.4. *Let ε and δ be non-negative real numbers and let $f, g : \mathcal{R} \rightarrow \mathcal{M}$ be mappings satisfying $f(0) = g(0) = 0$ and the inequalities*

$$\|J_{f,g}^{\theta,\phi}(a, b, c)\| \leq \varepsilon, \quad \|J_{g,g}^{\theta,\phi}(a, b, c)\| \leq \delta$$

for all $a, b, c \in \mathcal{R}$. Then there exist a unique (θ, ϕ) -Jordan derivation $G : \mathcal{R} \rightarrow \mathcal{M}$ and a unique generalized (θ, ϕ) -Jordan derivation $F : \mathcal{R} \rightarrow \mathcal{M}$ satisfying

$$\|f(a) - F(a)\| \leq \varepsilon, \quad \|g(a) - G(a)\| \leq \delta$$

for all $a \in \mathcal{R}$.

Proof. The proof follows from Theorem 3.2 by taking

$$\Phi(a, b, c) := \varepsilon, \quad \Psi(a, b, c) := \delta$$

for all $a, b, c \in \mathcal{R}$. Then we can choose $L = K = 1/2$ and we get the desired results. $\square$

Theorem 3.5. *Let $f, g : \mathcal{R} \rightarrow \mathcal{M}$ be mappings with $f(0) = g(0) = 0$ for which there exists a function $\Phi : \mathcal{R}^4 \rightarrow [0, \infty)$ satisfying*

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{2^n} \Phi(2^n a, 2^n b, 2^n c, 2^n d) &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \Phi(2^n a, b, 0, 0) \\ &= \lim_{n \rightarrow \infty} \frac{1}{2^n} \Phi(a, 2^n b, 0, 0) = 0, \end{aligned} \tag{3.17}$$

$$\|D_{f,g}^{\theta,\phi}(a, b, c, d)\| \leq \Phi(a, b, c, d) \tag{3.18}$$

for all $a, b, c, d \in \mathcal{R}$. If $\mathcal{R}$ has the identity e , $\mathcal{M}$ is unit linked and there exists a constant $L < 1$ such

$$\Phi(0, 0, 2a, 2a) \leq 2L\Phi(0, 0, a, a)$$

for all $a \in \mathcal{R}$, then g is a (θ, ϕ) -derivation and f is a generalized (θ, ϕ) -derivation. Moreover, $f = a\theta + g$, where $a = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n e)$.

Proof. Letting $a = b = 0$ and $c = d$ in (3.18), we get

$$\|f(2c) - 2f(c)\| \leq \Phi(0, 0, c, c)$$

for all $c \in \mathcal{R}$. Using the same method as in the proof of Theorem 3.2, we infer that the limit

$$F(a) := \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n a) \quad (3.19)$$

exists for all $a \in \mathcal{R}$ and the mapping $F : \mathcal{R} \rightarrow \mathcal{M}$ is additive. Letting $c = d = 0$ and replacing a and b by $2^n e$ and $2^n b$, respectively, in (3.18), we get

$$\|f(4^n b) - f(2^n e)\theta(2^n b) - \phi(2^n e)g(2^n b)\| \leq \Phi(2^n e, 2^n b, 0, 0)$$

for all $b \in \mathcal{R}$ and all $n \in \mathbb{N}$. Since $\phi(e) = e$, we have

$$\left\| \frac{1}{4^n} f(4^n b) - \frac{1}{2^n} f(2^n e)\theta(b) - \frac{1}{2^n} g(2^n b) \right\| \leq \frac{1}{4^n} \Phi(2^n e, 2^n b, 0, 0) \quad (3.20)$$

for all $b \in \mathcal{R}$ and all $n \in \mathbb{N}$. It follows from (3.17), (3.19) and (3.20) that the limit

$$G(b) := \lim_{n \rightarrow \infty} \frac{1}{2^n} g(2^n b)$$

exists and $G(b) = F(b) - F(e)\theta(b)$ for all $b \in \mathcal{R}$. Hence G is additive. It follows from the definitions of F, G , (3.17) and (3.18) that

$$\begin{aligned} & \|F(ab) - F(a)\theta(b) - \phi(a)G(b)\| \\ &= \lim_{n \rightarrow \infty} \frac{1}{4^n} \|f(4^n ab) - f(2^n a)\theta(2^n b) - \phi(2^n a)g(2^n b)\| \\ &\leq \lim_{n \rightarrow \infty} \frac{1}{4^n} \Phi(2^n a, 2^n b, 0, 0) = 0 \end{aligned}$$

for all $a, b \in \mathcal{R}$. Therefore

$$F(ab) = F(a)\theta(b) + \phi(a)G(b) \quad (3.21)$$

for all $a, b \in \mathcal{R}$. Further, by (3.21) we have

$$\begin{aligned} G(ab) &= F(ab) - F(e)\theta(ab) \\ &= F(a)\theta(b) + \phi(a)G(b) - F(e)\theta(a)\theta(b) \\ &= [F(a) - F(e)\theta(a)]\theta(b) + \phi(a)G(b) \\ &= G(a)\theta(b) + \phi(a)G(b) \end{aligned}$$

for all $a, b \in \mathcal{R}$. Thus G is a (θ, ϕ) -derivation and (3.21) shows that F is a generalized (θ, ϕ) -derivation.

By (3.17), (3.18) and the definitions of F, G , we have

$$F(ab) - F(a)\theta(b) = \phi(a)g(b), \quad (3.22)$$

$$F(ab) - \phi(a)G(b) = f(a)\theta(b) \quad (3.23)$$

for all $a, b \in \mathcal{R}$. Since $G(e) = 0$ and $\theta(e) = \phi(e) = e$, letting $a = e$ in (3.22) and $b = e$ in (3.23), we get $g(b) = F(b) - F(e)\theta(b) = G(b)$ and $F(a) = f(a)$, respectively, for all $a, b \in \mathcal{R}$. So g is a (θ, ϕ) -derivation and f is a generalized (θ, ϕ) -derivation. Moreover, $f = F(e)\theta + g$. $\square$

Corollary 3.6. *Let ε, δ, p be non-negative real numbers with $0 < p < 1$. If $\mathcal{R}$ is a normed ring with the identity e , $\mathcal{M}$ is unit linked and $f, g : \mathcal{R} \rightarrow \mathcal{M}$ are mappings with $f(0) = g(0) = 0$ and satisfy the inequality*

$$\|D_{f,g}^{\theta,\phi}(a, b, c, d)\| \leq \delta + \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p),$$

for all $a, b, c, d \in \mathcal{R}$, then g is a (θ, ϕ) -derivation and f is a generalized (θ, ϕ) -derivation. Moreover, $f = a\theta + g$, where $a = \lim_{n \rightarrow \infty} \frac{1}{2^n} f(2^n e)$.

Theorem 3.7. *Let $f, g : \mathcal{R} \rightarrow \mathcal{M}$ be mappings for which there exist functions $\varphi, \psi : \mathcal{R}^2 \rightarrow [0, \infty)$ such that*

$$\lim_{n \rightarrow \infty} \frac{1}{n} \varphi(na, b) = \lim_{n \rightarrow \infty} \frac{1}{n} \varphi(a, nb) = 0, \quad (3.24)$$

$$\lim_{n \rightarrow \infty} \frac{1}{n} \psi(na, b) = \lim_{n \rightarrow \infty} \frac{1}{n} \psi(a, nb) = 0, \quad (3.25)$$

$$\|f(ab) - f(a)\theta(b) - \phi(a)g(b)\| \leq \varphi(a, b), \quad (3.26)$$

$$\|g(ab) - g(a)\theta(b) - \phi(a)g(b)\| \leq \psi(a, b) \quad (3.27)$$

for all $a, b, c \in \mathcal{R}$. If $\mathcal{R}$ is normed with the identity e and $\mathcal{M}$ is unit linked, then

$$g(ab) = g(a)\theta(b) + \phi(a)g(b), \quad f(ab) = f(a)\theta(b) + \phi(a)g(b) \quad (3.28)$$

for all $a, b \in \mathcal{R}$.

Proof. By (3.24) and (3.27), we get

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n} [g(nab) - g(na)\theta(b)] &= \phi(a)g(b), \\ \lim_{n \rightarrow \infty} \frac{1}{n} [g(nab) - \phi(a)g(nb)] &= g(a)\theta(b) \end{aligned} \quad (3.29)$$

for all $a, b \in \mathcal{R}$. Using the Badora's method [3] on the inequality (3.27), we have

$$\begin{aligned}
\|g(ab) - g(a)\theta(b) - \phi(a)g(b)\| &\leq \left\|g(ab) - \frac{1}{n}g(nabe) + \frac{1}{n}\phi(ab)g(ne)\right\| \\
&\quad + \left\|\frac{1}{n}g(nab) - \frac{1}{n}\phi(a)g(nb) - g(a)\theta(b)\right\| \\
&\quad + \left\|\frac{1}{n}g(nab) - \frac{1}{n}g(na)\theta(b) - \phi(a)g(b)\right\| \\
&\quad + \frac{1}{n}\|\phi(a)g(nb) - \phi(ab)g(ne) + g(na)\theta(b) - g(nab)\| \\
&\leq \left\|g(ab) - \frac{1}{n}g(nabe) + \frac{1}{n}\phi(ab)g(ne)\right\| \\
&\quad + \left\|\frac{1}{n}g(nab) - \frac{1}{n}\phi(a)g(nb) - g(a)\theta(b)\right\| \\
&\quad + \left\|\frac{1}{n}g(nab) - \frac{1}{n}g(na)\theta(b) - \phi(a)g(b)\right\| \\
&\quad + \frac{1}{n}\|\phi(a)\| \|g(nb) - g(b)\theta(ne) - \phi(b)g(ne)\| \\
&\quad + \frac{1}{n}\|g(nab) - g(na)\theta(b) - \phi(na)g(b)\| \\
&\leq \left\|g(ab) - \frac{1}{n}g(nabe) + \frac{1}{n}\phi(ab)g(ne)\right\| \\
&\quad + \left\|\frac{1}{n}g(nab) - \frac{1}{n}\phi(a)g(nb) - g(a)\theta(b)\right\| \\
&\quad + \left\|\frac{1}{n}g(nab) - \frac{1}{n}g(na)\theta(b) - \phi(a)g(b)\right\| \\
&\quad + \frac{1}{n}\|\phi(a)\| \psi(b, ne) + \frac{1}{n}\psi(na, b)
\end{aligned}$$

for all $a, b \in \mathcal{R}$. Applying (3.25) and (3.29), we observe that the right side of the last inequality tends to 0 when n tends to infinity. Therefore

$$g(ab) = g(a)\theta(b) + \phi(a)g(b) \quad (3.30)$$

for all $a, b \in \mathcal{R}$.

Similarly, by (3.24) and (3.26), we have

$$\begin{aligned}
\lim_{n \rightarrow \infty} \frac{1}{n} [f(nab) - f(na)\theta(b)] &= \phi(a)g(b), \\
\lim_{n \rightarrow \infty} \frac{1}{n} [f(nab) - \phi(a)g(nb)] &= f(a)\theta(b)
\end{aligned} \quad (3.31)$$

for all $a, b \in \mathcal{R}$. Let $a, b \in \mathcal{R}$ and $n \in \mathbb{N}$ be fixed. Since g satisfies (3.30), we have $g(nb) = g(bne) = ng(b) + \phi(b)g(ne)$. Using (3.26), we have

$$\begin{aligned}
\|f(ab) - f(a)\theta(b) - \phi(a)g(b)\| &\leq \left\| f(ab) - \frac{1}{n}f(nabe) + \frac{1}{n}\phi(ab)g(ne) \right\| \\
&\quad + \left\| \frac{1}{n}f(nab) - \frac{1}{n}\phi(a)g(nb) - f(a)\theta(b) \right\| \\
&\quad + \left\| \frac{1}{n}f(nab) - \frac{1}{n}f(na)\theta(b) - \phi(a)g(b) \right\| \\
&\quad + \frac{1}{n}\|\phi(a)g(nb) - \phi(ab)g(ne) + f(na)\theta(b) - f(nab)\| \\
&= \left\| f(ab) - \frac{1}{n}f(nabe) + \frac{1}{n}\phi(ab)g(ne) \right\| \\
&\quad + \left\| \frac{1}{n}f(nab) - \frac{1}{n}\phi(a)g(nb) - f(a)\theta(b) \right\| \\
&\quad + \left\| \frac{1}{n}f(nab) - \frac{1}{n}f(na)\theta(b) - \phi(a)g(b) \right\| \\
&\quad + \frac{1}{n}\|n\phi(a)g(b) + f(na)\theta(b) - f(nab)\| \\
&\leq \left\| f(ab) - \frac{1}{n}f(nabe) + \frac{1}{n}\phi(ab)g(ne) \right\| \\
&\quad + \left\| \frac{1}{n}f(nab) - \frac{1}{n}\phi(a)g(nb) - f(a)\theta(b) \right\| \\
&\quad + \left\| \frac{1}{n}f(nab) - \frac{1}{n}f(na)\theta(b) - \phi(a)g(b) \right\| + \frac{1}{n}\varphi(na, b).
\end{aligned}$$

Applying (3.24) and (3.31), we observe that the right side of the last inequality tends to 0 when n tends to infinity. Therefore

$$f(ab) = f(a)\theta(b) + \phi(a)g(b).$$

□

Corollary 3.8. *Let $\varepsilon, \delta, p, q$ be non-negative real numbers with $0 < p, q < 1$. If $\mathcal{R}$ is a normed ring with the identity e , $\mathcal{M}$ is unit linked and $f, g : \mathcal{R} \rightarrow \mathcal{M}$ are mappings satisfy the inequalities*

$$\begin{aligned}
\|f(ab) - f(a)\theta(b) - \phi(a)g(b)\| &\leq \delta + \varepsilon(\|a\|^p + \|b\|^q) \\
\|g(ab) - g(a)\theta(b) - \phi(a)g(b)\| &\leq \delta + \varepsilon(\|a\|^p + \|b\|^q)
\end{aligned}$$

for all $a, b \in \mathcal{R}$, then f and g satisfy (3.28) for all $a, b \in \mathcal{R}$.

ACKNOWLEDGMENT

The authors thank the referee(s) for a number of valuable suggestions on a previous version of this paper.

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