STABILITY OF HOMOMORPHISMS AND \((\theta, \phi)\)-DERIVATIONS

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Abstract. In this paper, we prove the generalized Hyers–Ulam stability of homomorphisms and \((\theta, \phi)\)-derivations on a ring \(R\) into a Banach \(R\)-bimodule \(M\).

1. Introduction

The stability problem of functional equations originated from a question of S.M. Ulam [37] concerning the stability of group homomorphisms: Let \((G_1, \ast)\) be a group and let \((G_2, \odot, d)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \(\epsilon > 0\), does there exist \(\delta(\epsilon) > 0\) such that if a mapping \(h : G_1 \to G_2\) satisfies the inequality
\[ d(h(x \ast y), h(x) \odot h(y)) < \delta \]
for all \(x, y \in G_1\), then there is a homomorphism \(H : G_1 \to G_2\) with
\[ d(h(x), H(x)) < \epsilon \]
for all \(x \in G_1\)?

In other words, we are looking for situations where homomorphisms are stable, i.e., if a mapping is almost a homomorphism, then there exists a homomorphism near it. D.H. Hyers [12] gave a first affirmative answer to the question of Ulam for Banach spaces. Let \(X\) and \(Y\) be Banach spaces: Assume that \(f : X \to Y\) satisfies
\[ \|f(x + y) - f(x) - f(y)\| \leq \epsilon \]
for some \(\epsilon \geq 0\) and all \(x, y \in X\). Then there exists a unique additive mapping \(T : X \to Y\) such that
\[ \|f(x) - T(x)\| \leq \epsilon \]
for all \(x \in X\).

T. Aoki [2] and Th.M. Rassias [31] provided a generalization of the Hyers’ theorem for additive and linear mappings, respectively, by allowing the Cauchy difference to be unbounded.

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Theorem 1.1. (Th.M. Rassias). Let \( f : E \to E' \) be a mapping from a normed vector space \( E \) into a Banach space \( E' \) subject to the inequality
\[
\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p)
\] (1.1)
for all \( x, y \in E \), where \( \varepsilon \) and \( p \) are constants with \( \varepsilon > 0 \) and \( p < 1 \). Then the limit
\[
L(x) = \lim_{n \to \infty} \frac{f(2^n x)}{2^n}
\]
exists for all \( x \in E \) and \( L : E \to E' \) is the unique additive mapping which satisfies
\[
\|f(x) - L(x)\| \leq \frac{2\varepsilon}{2 - 2^p}\|x\|^p
\] (1.2)
for all \( x \in E \). If \( p < 0 \) then inequality (1.1) holds for \( x, y \neq 0 \) and (1.2) for \( x \neq 0 \). Also, if for each \( x \in E \) the mapping \( t \mapsto f(tx) \) is continuous in \( t \in \mathbb{R} \), then \( L \) is linear.

The inequality (1.1) has provided a lot of influence in the development of what is now known as a generalized Hyers–Ulam stability of functional equations. In 1994, a generalization of the Th.M. Rassias' theorem was obtained by P. Găvruta [8], who replaced the bound \( \varepsilon (\|x\|^p + \|y\|^p) \) by a general control function \( \varphi(x, y) \). Since then the stability problems of various functional equations and mappings and their Pexiderized versions with more general domains and ranges have been investigated by a number of authors (see [21]–[29]). We also refer the readers to the books [7], [13], [16] and [32].

Let \( A \) be a real or complex algebra. A mapping \( D : A \to A \) is said to be a (ring) derivation if
\[
D(a + b) = D(a) + D(b), \quad D(ab) = D(a)b + aD(b)
\]
for all \( a, b \in A \). If, in addition, \( D(\lambda a) = \lambda D(a) \) for all \( a \in A \) and all \( \lambda \in \mathbb{F} \), then \( D \) is called a linear derivation, where \( \mathbb{F} \) denotes the scalar field of \( A \). I.M. Singer and J. Wermer [35] proved that if \( A \) is a commutative Banach algebra and \( D : A \to A \) is a continuous linear derivation, then \( D(A) \subseteq \text{rad}(A) \). They also conjectured that the same result holds even \( D \) is a discontinuous linear derivation. M.P. Thomas [36] proved the conjecture. As a direct consequence, we see that there are no non-zero linear derivations on a semi-simple commutative Banach algebra, which had been proved by B.E. Johnson [15]. On the other hand, it is not the case for ring derivations. O. Hatori and J. Wada [9] determined a representation of ring derivations on a semi-simple commutative Banach algebra (see also [33]) and they proved that only the zero operator is a ring derivation.
on a semi-simple commutative Banach algebra with the maximal ideal space without isolated points. The stability of derivations between operator algebras was first obtained by P. Šemrl [34]. Badora [3] and T. Miura et al. [22] proved the generalized Hyers–Ulam stability of ring derivations on Banach algebras.

Let \( \mathcal{R} \) be an associative ring, \( \mathcal{N} \) be a \( \mathcal{R} \)-bimodule and let \( \theta, \phi \) be automorphisms of \( \mathcal{R} \). An additive mapping \( D : \mathcal{R} \to \mathcal{N} \) is called a derivation if \( D(ab) = D(a)b + aD(b) \) holds for all pairs \( a, b \in \mathcal{R} \) and is called a Jordan derivation in case \( D(a^2) = D(a)a + aD(a) \) is fulfilled for all \( a \in \mathcal{R} \). Every derivation is a Jordan derivation. The converse is in general not true (see [6, 10]). The concept of generalized derivation has been introduced by Bresar [4]. Hvala [11] and Lee [18] introduced a concept of \((\theta, \phi)\)-derivation (see also [19]). An additive mapping \( F : \mathcal{R} \to \mathcal{N} \) is called a \((\theta, \phi)\)-derivation in case \( F(ab) = F(a)\theta(b) + \phi(a)F(b) \) holds for all pairs \( a, b \in \mathcal{R} \). An additive mapping \( F : \mathcal{R} \to \mathcal{N} \) is called a \((\theta, \phi)\)-Jordan derivation in case \( F(a^2) = F(a)\theta(a) + \phi(a)F(a) \) holds for all \( a \in \mathcal{R} \). An additive mapping \( F : \mathcal{R} \to \mathcal{N} \) is called a generalized \((\theta, \phi)\)-derivation in case \( F(ab) = F(a)\theta(b) + \phi(a)F(b) \) holds for all pairs \( a, b \in \mathcal{R} \), where \( D : \mathcal{R} \to \mathcal{N} \) is a \((\theta, \phi)\)-derivation. An additive mapping \( F : \mathcal{R} \to \mathcal{N} \) is called a generalized \((\theta, \phi)\)-Jordan derivation in case \( F(a^2) = F(a)\theta(a) + \phi(a)D(a) \) holds for all \( a \in \mathcal{R} \), where \( D : \mathcal{R} \to \mathcal{N} \) is a \((\theta, \phi)\)-derivation. It is clear that every generalized \((\theta, \phi)\)-derivation is a generalized \((\theta, \phi)\)-Jordan derivation.

The aim of the present paper is to establish the stability problem of homomorphisms and generalized \((\theta, \phi)\)-derivations by using the fixed point method (see [1, 5, 17, 21]).

Let \( E \) be a set. A function \( d : E \times E \to [0, \infty] \) is called a generalized metric on \( E \) if \( d \) satisfies

1. \( d(x, y) = 0 \) if and only if \( x = y \);
2. \( d(x, y) = d(y, x) \) for all \( x, y \in E \);
3. \( d(x, z) \leq d(x, y) + d(y, z) \) for all \( x, y, z \in E \).

We recall the following theorem by Margolis and Diaz.

**Theorem 1.2.** [20] Let \( (E, d) \) be a complete generalized metric space and let \( J : E \to E \) be a strictly contractive mapping with Lipschitz constant \( L < 1 \). Then for each given element \( x \in E \), either

\[ d(J^n x, J^{n+1} x) = \infty \]

for all non-negative integers \( n \) or there exists a non-negative integer \( n_0 \) such that
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(1) \( d(J^n x, J^{n+1} x) < \infty \) for all \( n \geq n_0 \);
(2) the sequence \( \{J^n x\} \) converges to a fixed point \( y^* \) of \( J \);
(3) \( y^* \) is the unique fixed point of \( J \) in the set \( Y = \{ y \in E : d(J^{n_0} x, y) < \infty \} \);
(4) \( d(y, y^*) \leq \frac{1}{1-L} d(y, Jy) \) for all \( y \in Y \).

2. Stability of homomorphisms

In this section, we assume that \( \mathcal{R} \) is an associative ring, \( \mathcal{X} \) is a normed algebra, \( \mathcal{Y} \) is a Banach algebra, and \( n \geq 3 \) is a fixed integer.

Lemma 2.1. Let \( X \) and \( Y \) be linear spaces. A mapping \( f : X \rightarrow Y \) (with \( f(0) = 0 \) if \( n = 3 \)) satisfies
\[
\sum_{j=1}^{n} f(-x_j + \sum_{1 \leq i \leq n, i \neq j} x_i) = (n-2) \sum_{i=1}^{n} f(x_i) \tag{2.1}
\]
for all \( x_1, \ldots, x_n \in X \), if and only if \( f \) is additive.

Proof. Let \( f \) satisfy (2.1). Letting \( x_1 = \cdots = x_n = 0 \) in (2.1), we get \( f(0) = 0 \). Letting \( x_2 = \cdots = x_n = 0 \) in (2.1), we infer that \( f \) is odd. So by letting \( x_3 = \cdots = x_n = 0 \) in (2.1) and using the oddness of \( f \), we get that the mapping \( f \) is additive. The converse is obvious. \( \square \)

Theorem 2.2. Let \( f : \mathcal{R} \rightarrow \mathcal{Y} \) be a mapping for which there exist functions \( \varphi : \mathcal{R}^n \rightarrow [0, \infty) \) and \( \psi : \mathcal{R}^2 \rightarrow [0, \infty) \) such that
\[
\lim_{k \to \infty} \frac{1}{r^k} \varphi(r^k a_1, \cdots, r^k a_n) = 0, \tag{2.2}
\]
\[
\lim_{k \to \infty} \frac{1}{r^k} \psi(r^k a, b) = \lim_{k \to \infty} \frac{1}{r^k} \psi(a, r^k b) = \lim_{k \to \infty} \frac{1}{r^k} \psi(r^k a, r^k b) = 0, \tag{2.3}
\]
\[
\left\| \sum_{j=1}^{n} f(-a_j + \sum_{1 \leq i \leq n, i \neq j} a_i) - (n - 2) \sum_{i=1}^{n} f(a_i) \right\| \leq \varphi(a_1, \cdots, a_n), \tag{2.4}
\]
\[
\| f(ab) - f(a)f(b) \| \leq \psi(a, b) \tag{2.5}
\]
for all \( a, b, a_1, \cdots, a_n \in \mathcal{R} \), where \( r = n - 2 \). If there exists a constant \( L < 1 \) such that
\[
\varphi(ra, \cdots, ra) \leq rL \varphi(a, \cdots, a)
\]
for all \( a \in \mathcal{R} \), then there exists a unique homomorphism \( H : \mathcal{R} \rightarrow \mathcal{Y} \) satisfying
\[
\| f(a) - H(a) \| \leq \frac{1}{n(n-2)(1-L)} \varphi(a, \cdots, a), \tag{2.6}
\]
\[
H(a)[H(b) - f(b)] = [H(a) - f(a)]H(b) = 0 \tag{2.7}
\]
for all \( a, b \in \mathcal{R} \).
Proof. Letting \( a_1 = \cdots = a_n = a \) in (2.4), we get
\[
\| f(ra) - rf(a) \| \leq \frac{1}{n} \varphi(a, \cdots, a) \tag{2.8}
\]
for all \( a \in \mathcal{R} \). Let \( E := \{ g : \mathcal{R} \to \mathcal{Y} \} \). We introduce a generalized metric on \( E \) as follows:
\[
d_\varphi(g, h) := \inf \{ C \in [0, \infty] : \| g(a) - h(a) \| \leq C \varphi(a, \cdots, a) \text{ for all } a \in \mathcal{R} \}. \]
It is easy to show that \((E, d_\varphi)\) is a generalized complete metric space \([5]\).

Now we consider the mapping \( \Lambda : E \to E \) defined by
\[
(\Lambda g)(a) = \frac{1}{r} g(ra), \quad \text{for all } g \in E \text{ and } a \in \mathcal{R}.
\]
Let \( g, h \in E \) and let \( C \in [0, \infty] \) be an arbitrary constant with \( d_\varphi(g, h) \leq C \). From the definition of \( d_\varphi \), we have
\[
\| g(a) - h(a) \| \leq C \varphi(a, \cdots, a) \text{ for all } a \in \mathcal{R}.
\]
By the assumption and last inequality, we have
\[
\| (\Lambda g)(a) - (\Lambda h)(a) \| = \frac{1}{r} \| g(ra) - h(ra) \| \leq \frac{C}{r} \varphi(ra, \cdots, ra) \leq C L \varphi(a, \cdots, a)
\]
for all \( a \in \mathcal{R} \). So \( d_\varphi(\Lambda g, \Lambda h) \leq L d_\varphi(g, h) \) for any \( g, h \in E \). It follows from (2.8) that \( d_\varphi(\Lambda f, f) \leq \frac{1}{n(n-2)} \). Therefore according to Theorem 1.2, the sequence \( \{ \Lambda^k f \} \) converges to a fixed point \( H \) of \( \Lambda \), i.e.,
\[
H : \mathcal{R} \to \mathcal{Y}, \quad H(a) = \lim_{k \to \infty} (\Lambda^k f)(a) = \lim_{k \to \infty} \frac{1}{r^k} f(r^k a)
\]
and \( H(ra) = rH(a) \) for all \( a \in \mathcal{R} \). Also \( H \) is the unique fixed point of \( \Lambda \) in the set \( E_\varphi = \{ g \in E : d_\varphi(f, g) < \infty \} \) and
\[
d_\varphi(H, f) \leq \frac{1}{1 - L} d_\varphi(\Lambda f, f) \leq \frac{1}{n(n-2)(1 - L)},
\]
i.e., inequality (2.6) holds true for all \( a \in \mathcal{R} \). It follows from the definition of \( H \), (2.2) and (2.4) that
\[
\sum_{j=1}^{n} H(-a_j + \sum_{1 \leq i \leq n \atop i \neq j} a_i) = (n - 2) \sum_{i=1}^{n} H(a_i)
\]
for all \( a_1, \cdots, a_n \in \mathcal{R} \). Since \( H(0) = 0 \), by Lemma 2.1 the mapping \( H \) is additive. So it follows from the definition of \( H \), (2.3) and (2.5) that
\[
\| H(ab) - H(a)H(b) \| = \lim_{k \to \infty} \frac{1}{r^{2k}} \| f(r^{2k}ab) - f(r^k a)f(r^k b) \|
\]
\[
\leq \lim_{k \to \infty} \frac{1}{r^{2k}} \psi(r^k a, r^k b) = 0
\]
for all $a, b \in \mathcal{R}$. So $H$ is homomorphism. Similarly, we have from (2.3) and (2.5) that

$$H(ab) = H(a)f(b), \quad H(ab) = f(a)H(b)$$

for all $a, b \in \mathcal{R}$. Since $H$ is homomorphism, we get (2.7) from (2.9).

Finally it remains to prove the uniqueness of $H$. Let $H_1 : \mathcal{R} \to \mathcal{Y}$ be another homomorphism satisfying (2.6). Since $d_{\varphi}(f, H_1) \leq \frac{1}{n(n-2)(1-L)}$ and $H_1$ is additive, we get $H_1 \in E_\varphi$ and $(\Lambda H_1)(a) = \frac{1}{r}H_1(ra) = H_1(a)$ for all $a \in \mathcal{R}$, i.e., $H_1$ is a fixed point of $\Lambda$. Since $H$ is the unique fixed point of $\Lambda$ in $E_\varphi$, we get $H_1 = H$.  

We need the following lemma in the proof of the next theorem.

**Lemma 2.3.** [30] Let $X$ and $Y$ be linear spaces and $f : X \to Y$ be an additive mapping such that $f(mx) = \mu f(x)$ for all $x \in X$ and all $\mu \in \mathbb{T}^1 := \{ \mu \in \mathbb{C} : |\mu| = 1 \}$. Then the mapping $f$ is $\mathbb{C}$-linear.

**Lemma 2.4.** Let $X$ and $Y$ be linear spaces. A mapping $f : X \to Y$ satisfies

$$\sum_{j=1}^{n} f(-\mu x_j + \sum_{1 \leq i \leq n \atop i \neq j} \mu x_i) = (n-2)\mu \sum_{i=1}^{n} f(x_i)$$

(2.10)

for all $x_1, \ldots, x_n \in X$ and all $\mu \in \mathbb{T}^1$, if and only if $f$ is $\mathbb{C}$-linear.

**Proof.** Let $f$ satisfy (2.10). Letting $x_1 = \cdots = x_n = 0$ in (2.10), we get $f(0) = 0$. By Lemma 2.1, the mapping $f$ is additive. Letting $x_2 = \cdots = x_n = 0$ in (2.10) and using the oddness of $f$, we get that $f(\mu x_1) = \mu f(x_1)$ for all $x_1 \in X$ and all $\mu \in \mathbb{T}^1$. So by Lemma 2.3, the mapping $f$ is $\mathbb{C}$-linear. The converse is obvious.  

The following theorem is an alternative result of Theorem 2.2.

**Theorem 2.5.** Let $f : \mathcal{X} \to \mathcal{Y}$ be a mapping for which there exist functions $\varphi : \mathcal{X}^n \to [0, \infty)$ and $\psi : \mathcal{X}^2 \to [0, \infty)$ such that

$$\lim_{k \to \infty} r^k \varphi\left(\frac{1}{r^k} a_1, \ldots, \frac{1}{r^k} a_n\right) = 0,$$

$$\lim_{k \to \infty} r^k \psi\left(\frac{1}{r^k} a, b\right) = \lim_{k \to \infty} r^k \psi\left(a, \frac{1}{r^k} b\right) = \lim_{k \to \infty} r^k \psi\left(\frac{1}{r^k} a, \frac{1}{r^k} b\right) = 0,$$

$$\left\| \sum_{j=1}^{n} f(-\mu a_j + \sum_{1 \leq i \leq n \atop i \neq j} \mu a_i) - (n-2)\mu \sum_{i=1}^{n} f(a_i) \right\| \leq \varphi(a_1, \cdots, a_n),$$

$$\|f(ab) - f(a)f(b)\| \leq \psi(a, b)$$
for all \(a, b, a_1, \cdots, a_n \in \mathcal{X}\) and all \(\mu \in \mathbb{T}^1\), where \(r = n - 2\). If there exists a constant \(L < 1\) such that
\[
r \varphi \left( \frac{1}{r} a, \cdots, \frac{1}{r} a \right) \leq L \varphi (a, \cdots, a)
\]
for all \(a \in \mathcal{X}\), then there exists a unique homomorphism \(H : \mathcal{X} \rightarrow \mathcal{Y}\) satisfying
\[
\| f(a) - H(a) \| \leq \frac{L}{n(n - 2)(1 - L)} \varphi (a, \cdots, a),
\]
\[
H(a)[H(b) - f(b)] = [H(a) - f(a)]H(b) = 0
\]
for all \(a, b \in \mathcal{X}\).

Proof. It follows from the assumptions that \(\varphi (0, \cdots, 0) = 0\), and so \(f(0) = 0\). Letting \(\mu = 1\) and using the same method as in the proof of Theorem 2.2, we have
\[
\| f(ra) - rf(a) \| \leq \frac{1}{n} \varphi (a, \cdots, a) \tag{2.11}
\]
for all \(a \in \mathcal{R}\). Let \(E := \{ g : \mathcal{X} \rightarrow \mathcal{Y} \mid g(0) = 0 \}\). We introduce the same definition \(d_\varphi\) as in the proof of Theorem 2.2 such that \((E, d_\varphi)\) becomes a generalized complete metric space. Let \(\Lambda : E \rightarrow E\) be the mapping defined by
\[
(\Lambda g)(a) = rg \left( \frac{1}{r} a \right), \quad \text{for all } g \in E \text{ and } a \in \mathcal{X}.
\]
One can show that
\[
d_\varphi (\Lambda g, \Lambda h) \leq L d_\varphi (g, h)
\]
for any \(g, h \in E\). It follows from the assumption and (2.11) that \(d_\varphi (\Lambda f, f) \leq \frac{L}{n(n - 2)}\). Due to Theorem 1.2, the sequence \(\{\Lambda^k f\}\) converges to a fixed point \(H\) of \(\Lambda\), i.e., \(H : \mathcal{X} \rightarrow \mathcal{Y}\),
\[
H(a) = \lim_{k \rightarrow \infty} (\Lambda^k f)(a) = \lim_{n \rightarrow \infty} r^k f \left( \frac{1}{r^k} a \right), \quad H(ra) = rH(a)
\]
for all \(a \in \mathcal{X}\). Also
\[
d_\varphi (H, f) \leq \frac{1}{1 - L} d_\varphi (\Lambda f, f) \leq \frac{L}{n(n - 2)(1 - L)}
\]
i.e., the inequality
\[
\| f(a) - H(a) \| \leq \frac{L}{n(n - 2)(1 - L)} \varphi (a, \cdots, a)
\]
holds true for all \(a \in \mathcal{X}\).

The rest of the proof is similar to the proof of Theorem 3.1 and we omit the details. \(\square\)
Corollary 2.6. Let $p, q, \delta, \varepsilon$ be non-negative real numbers with $0 < p, q < 1$. Suppose that $f : X \to Y$ is a mapping such that
\[
\left\| \sum_{j=1}^{n} f\left(-\mu a_j + \sum_{1 \leq i \leq n, i \neq j} \mu a_i\right) - (n-2)\mu \sum_{i=1}^{n} f(a_i) \right\| \leq \delta + \varepsilon \sum_{i=1}^{n} \|a_i\|^p,
\]
\[
\|f(ab) - f(a)f(b)\| \leq \delta + \varepsilon (\|a\|^q + \|b\|^q)
\]
for all $a, b, a_1, \cdots, a_n \in X$ and all $\mu \in T^1$. Then there exists a unique homomorphism $H : X \to Y$ satisfying
\[
\|f(a) - H(a)\| \leq \frac{\delta}{(r+2)(r-r^p)} + \frac{\varepsilon}{r-r^p} \|a\|^p,
\]
\[
H(a)[H(b) - f(b)] = [H(a) - f(a)]H(b) = 0
\]
for all $a, b \in X$, where $r = n - 2$.

Proof. The proof follows from Theorem 2.2 by taking
\[
\varphi(a_1, \cdots, a_n) := \delta + \varepsilon \sum_{i=1}^{n} \|a_i\|^p, \quad \psi(a, b) := \delta + \varepsilon (\|a\|^q + \|b\|^q)
\]
for all $a, b, a_1, \cdots, a_n \in X$. Then we can choose $L = r^{p-1}$ and we get the desired results. \hfill \Box

Corollary 2.7. Let $p, q, \varepsilon$ be non-negative real numbers with $p > 1$ and $q > 2$. Suppose that $f : X \to Y$ is a mapping such that
\[
\left\| \sum_{j=1}^{n} f\left(-\mu a_j + \sum_{1 \leq i \leq n, i \neq j} \mu a_i\right) - (n-2)\mu \sum_{i=1}^{n} f(a_i) \right\| \leq \varepsilon \sum_{i=1}^{n} \|a_i\|^p,
\]
\[
\|f(ab) - f(a)f(b)\| \leq \varepsilon (\|a\|^q + \|b\|^q)
\]
for all $a, b, a_1, \cdots, a_n \in X$ and all $\mu \in T^1$. Then there exists a unique homomorphism $H : X \to Y$ satisfying
\[
\|f(a) - H(a)\| \leq \frac{\varepsilon}{r^{p-1}} \|a\|^p
\]
for all $a, b \in X$, where $r = n - 2$.

Proof. The proof follows from Theorem 2.5 by taking
\[
\varphi(a_1, \cdots, a_n) := \varepsilon \sum_{i=1}^{n} \|a_i\|^p, \quad \psi(a, b) := \varepsilon (\|a\|^q + \|b\|^q)
\]
for all $a, b, a_1, \cdots, a_n \in X$. Then we can choose $L = r^{1-p}$ and we get the desired results. \hfill \Box
3. Stability of Generalized \((\theta, \phi)\)-Derivations

In this section, we assume that \(R\) is a 2-divisible associative ring, \(M\) is a Banach \(R\)-bimodule, and \(\theta, \phi\) are automorphisms of \(R\). For convenience, we use the following abbreviation for given mappings \(f, g : R \to M\),

\[
D_{f,g}^{\theta,\phi}(a,b,c,d) := f(ab + c + d) - f(a)\theta(b) - \phi(a)g(b) - f(c) - f(d),
\]

\[
J_{f,g}^{\theta,\phi}(a,b,c) := f(a^2 + b + c) - f(a)\theta(a) - \phi(a)g(a) - f(b) - f(c)
\]

for all \(a, b, c, d \in R\). Now we prove the generalized Hyers–Ulam stability of generalized \((\theta, \phi)\)-derivations and generalized \((\theta, \phi)\)-Jordan derivations in Banach \(R\)-bimodules.

**Theorem 3.1.** Let \(f, g : R \to M\) be mappings for which there exist functions \(\varphi, \psi : R^3 \to [0, \infty)\) such that

\[
\lim_{n \to \infty} 4^n \varphi\left(\frac{a}{2^n}, 0, 0\right) = \lim_{n \to \infty} 2^n \varphi\left(0, \frac{b}{2^n}, \frac{c}{2^n}\right) = 0,
\]

\[
\|J_{f,g}^{\theta,\phi}(a,b,c)\| \leq \varphi(a, b, c),
\]

\[
\lim_{n \to \infty} 4^n \psi\left(\frac{a}{2^n}, 0, 0\right) = \lim_{n \to \infty} 2^n \psi\left(0, \frac{b}{2^n}, \frac{c}{2^n}\right) = 0,
\]

\[
\|J_{g,g}^{\theta,\phi}(a,b,c)\| \leq \psi(a, b, c)
\]

for all \(a, b, c \in R\). If there exist constants \(L, K < 1\) such

\[
2\varphi(0,a,a) \leq L\varphi(0,2a,2a), \quad 2\psi(0,a,a) \leq K\psi(0,2a,2a)
\]

for all \(a \in R\), then there exist a unique \((\theta, \phi)\)-Jordan derivation \(G : R \to M\) and a unique generalized \((\theta, \phi)\)-Jordan derivation \(F : R \to M\) satisfying

\[
\|f(a) - F(a)\| \leq \frac{L}{2 - 2L}\varphi(0, a, a),
\]

\[
\|g(a) - G(a)\| \leq \frac{K}{2 - 2K}\psi(0, a, a)
\]

for all \(a \in R\).

**Proof.** It follows from (3.1) and (3.3) that \(\varphi(0,0,0) = 0 = \psi(0,0,0)\) and so we get from (3.2) and (3.4) that \(f(0) = g(0) = 0\). Letting \(a = 0\) and \(b = c\) in (3.2), we get

\[
\|f(2c) - 2f(c)\| \leq \varphi(0, c, c)
\]

for all \(c \in R\). Let \(E := \{ h : R \to M \mid h(0) = 0 \}\). We introduce a generalized metric on \(E\) as follows:

\[
d_{\varphi}(h, k) := \inf\{ C \in [0, \infty] : \|h(a) - k(a)\| \leq C\varphi(0, a, a) \text{ for all } a \in R \}.
\]
It is easy to show that \((E, d_\varphi)\) is a generalized complete metric space [5].

Now we consider the mapping \(\Lambda : E \to E\) defined by

\[
(\Lambda h)(a) = 2h\left(\frac{a}{2}\right), \quad \text{for all } h \in E \text{ and } a \in \mathcal{R}.
\]

Let \(h, k \in E\) and let \(C \in [0, \infty]\) be an arbitrary constant with \(d_\varphi(h, k) \leq C\). From the definition of \(d_\varphi\), we have

\[
\|h(a) - k(a)\| \leq C\varphi(0, a, a)
\]

for all \(a \in \mathcal{R}\). By the assumption and last inequality, we have

\[
\|(\Lambda h)(a) - (\Lambda k)(a)\| = 2\left\|h\left(\frac{a}{2}\right) - k\left(\frac{a}{2}\right)\right\| \leq 2C\varphi(0, \frac{a}{2}, \frac{a}{2}) \leq C\varphi(0, a, a)
\]

for all \(a \in \mathcal{R}\). So \(d_\varphi(\Lambda h, \Lambda k) \leq Ld_\varphi(h, k)\) for any \(h, k \in E\). It follows from the assumption and (3.7) that \(d_\varphi(\Lambda f, f) \leq L/2\). Therefore according to Theorem 1.2, the sequence \(\{\Lambda^n f\}\) converges to a fixed point \(F\) of \(\Lambda\), i.e.,

\[
F : \mathcal{R} \to \mathcal{M}, \quad F(a) = \lim_{n \to \infty} (\Lambda^n f)(a) = \lim_{n \to \infty} 2^n f\left(\frac{a}{2^n}\right)
\]

and \(F(2a) = 2F(a)\) for all \(a \in \mathcal{R}\). Also \(F\) is the unique fixed point of \(\Lambda\) in the set \(E_\varphi = \{h \in E : d_\varphi(f, h) < \infty\}\) and

\[
d_\varphi(F, f) \leq \frac{1}{1 - L}d_\varphi(\Lambda f, f) \leq \frac{L}{2 - 2L},
\]

i.e., inequality (3.5) holds true for all \(a \in \mathcal{R}\). Similarly, we obtain that

\[
d_\varphi(\Lambda h, \Lambda k) \leq Kd_\varphi(h, k), \quad d_\varphi(\Lambda g, g) \leq K/2.
\]

for any \(h, k \in E\), where

\[
d_\varphi(h, k) := \inf\{ C \in [0, \infty] : \|h(a) - k(a)\| \leq C\varphi(0, a, a) \text{ for all } a \in \mathcal{R} \}.
\]

So according to Theorem 1.2, the sequence \(\{\Lambda^n g\}\) converges to a fixed point \(G\) of \(\Lambda\), i.e.,

\[
G : \mathcal{R} \to \mathcal{M}, \quad G(a) = \lim_{n \to \infty} (\Lambda^n g)(a) = \lim_{n \to \infty} 2^n g\left(\frac{a}{2^n}\right)
\]

and \(G(2a) = 2G(a)\) for all \(a \in \mathcal{R}\). Also \(G\) is the unique fixed point of \(\Lambda\) in the set \(E_\psi = \{h \in E : d_\psi(g, h) < \infty\}\) and

\[
d_\psi(G, g) \leq \frac{1}{1 - K}d_\psi(\Lambda g, g) \leq \frac{K}{2 - 2K},
\]
i.e., inequality (3.6) holds true for all \( a \in \mathcal{R} \). It follows from the definitions of \( F, G \), (3.1) and (3.2) that

\[
\|J_{f,G}^a(0,0,0)\| = \lim_{n \to \infty} 4^n \|J_{f,g}^{a}(a/2^n,0,0)\| \\
\leq \lim_{n \to \infty} 4^n \varphi\left(\frac{a}{2^n}, 0, 0\right) = 0,
\]

\[
\|J_{f,G}^a(0,b,c)\| = \lim_{n \to \infty} 2^n \|J_{f,g}^{a}(0,b/2^n, c/2^n)\| \\
\leq \lim_{n \to \infty} 2^n \varphi\left(0, \frac{b}{2^n}, \frac{c}{2^n}\right) = 0
\]

for all \( a, b, c \in \mathcal{R} \). Hence

\[
F(a^2) = F(a)\theta(a) + \phi(a)G(a), \quad F(b + c) = F(b) + F(c) \tag{3.8}
\]

for all \( a, b, c \in \mathcal{R} \). Similarly, it follows from the definition of \( G \), (3.3) and (3.4) that

\[
G(a^2) = G(a)\theta(a) + \phi(a)G(a), \quad G(b + c) = G(b) + G(c) \tag{3.9}
\]

for all \( a, b, c \in \mathcal{R} \). Hence \( G \) is a \((\theta, \phi)\)-Jordan derivation. So we infer from (3.8) and (3.9) that \( F \) is a generalized \((\theta, \phi)\)-Jordan derivation.

Finally it remains to prove the uniqueness of \( F \) and \( G \). Let \( F_1, G_1 : \mathcal{R} \to \mathcal{M} \) be another additive mappings satisfying (3.5) and (3.6), respectively. Since \( d_\varphi(f, F_1) \leq \frac{L}{2-2R}, \ d_\psi(g, G_1) \leq \frac{K}{2-2R} \) and \( F_1, G_1 \) are additive, we get \( F_1 \in E_\varphi \), \( G_1 \in E_\psi \) and \((\Lambda F_1)(a) = 2F_1(a/2) = F_1(a), \ (\Lambda G_1)(a) = 2G_1(a/2) = G_1(a)\) for all \( a \in \mathcal{R} \), i.e., \( F_1, G_1 \) are fixed points of \( \Lambda \). Since \( F \) and \( G \) are the unique fixed points of \( \Lambda \) in \( E_\varphi \) and \( E_\psi \), respectively, we get \( F_1 = F \) and \( G_1 = G \). \( \square \)

**Theorem 3.2.** Let \( f, g : \mathcal{R} \to \mathcal{M} \) be mappings with \( f(0) = g(0) = 0 \) for which there exist functions \( \Phi, \Psi : \mathcal{R}^3 \to [0, \infty) \) such that

\[
\lim_{n \to \infty} \frac{1}{4^n} \Phi(2^n a, 0, 0) = \lim_{n \to \infty} \frac{1}{2^n} \Phi(0, 2^n b, 2^n c) = 0, \tag{3.10}
\]

\[
\|J_{f,g}^{a}(a, b, c)\| \leq \Phi(a, b, c), \tag{3.11}
\]

\[
\lim_{n \to \infty} \frac{1}{4^n} \Psi(2^n a, 0, 0) = \lim_{n \to \infty} \frac{1}{2^n} \Psi(0, 2^n b, 2^n c) = 0, \tag{3.12}
\]

\[
\|J_{g,g}^{a}(a, b, c)\| \leq \Psi(a, b, c) \tag{3.13}
\]

for all \( a, b, c \in \mathcal{R} \). If there exist constants \( L, K < 1 \) such

\[
\Phi(0, 2a, 2a) \leq 2L\Phi(0, a, a), \quad \Psi(0, 2a, 2a) \leq 2K\Psi(0, a, a)
\]
for all \( a \in \mathcal{R} \), then there exist a unique \((\theta, \phi)\)-Jordan derivation \( G : \mathcal{R} \to \mathcal{M} \) and a unique generalized \((\theta, \phi)\)-Jordan derivation \( F : \mathcal{R} \to \mathcal{M} \) satisfying

\[
\| f(a) - F(a) \| \leq \frac{1}{2 - 2L} \Phi(0, a, a),
\]

\[
\| g(a) - G(a) \| \leq \frac{1}{2 - 2K} \Psi(0, a, a),
\]

for all \( a \in \mathcal{R} \).

**Proof.** Using the same method as in the proof of Theorem 3.1, we have

\[
\left\| \frac{1}{2} f(2c) - f(c) \right\| \leq \frac{1}{2} \Phi(0, c, c), \quad \left\| \frac{1}{2} g(2c) - g(c) \right\| \leq \frac{1}{2} \Psi(0, c, c)
\]

for all \( c \in \mathcal{R} \). We introduce the same definitions for \( \Phi, \) and \( \Psi \) as in the proof of Theorem 3.1 such that \( (E, \Phi) \) and \( (E, \Psi) \) become generalized complete metric spaces. Let \( \Lambda : E \to E \) be the mapping defined by

\[
(\Lambda h)(a) = \frac{1}{2} h(2a), \quad \text{for all } h \in E \text{ and } a \in \mathcal{R}.
\]

One can show that

\[
d_\Phi(\Lambda h, \Lambda k) \leq Ld_\Phi(h, k), \quad d_\Psi(\Lambda h, \Lambda k) \leq Kd_\Psi(h, k)
\]

for any \( h, k \in E \). It follows from (3.16) that \( d_\Phi(\Lambda f, f) \leq \frac{1}{2} \) and \( d_\Psi(\Lambda g, g) \leq \frac{1}{2} \).

Due to Theorem 1.2, the sequences \( \{\Lambda^n f\} \) and \( \{\Lambda^n g\} \) converge to fixed points \( F \) and \( G \) of \( \Lambda \), i.e., \( F, G : \mathcal{R} \to \mathcal{M} \),

\[
F(a) = \lim_{n \to \infty} (\Lambda^n f)(a) = \lim_{n \to \infty} \frac{1}{2^n} f(2^n a), \quad G(a) = \lim_{n \to \infty} (\Lambda^n g)(a) = \lim_{n \to \infty} \frac{1}{2^n} g(2^n a),
\]

\( F(2a) = 2F(a) \) and \( G(2a) = 2G(a) \) for all \( a \in \mathcal{R} \). Also

\[
d_\Phi(F, f) \leq \frac{1}{1 - L} d_\Phi(\Lambda f, f) \leq \frac{1}{2 - 2L},
\]

\[
d_\Psi(G, g) \leq \frac{1}{1 - K} d_\Psi(\Lambda g, g) \leq \frac{1}{2 - 2K},
\]

i.e., the inequalities (3.14) and (3.15) hold true for all \( a \in \mathcal{R} \).

The rest of the proof is similar to the proof of Theorem 3.1 and we omit the details. \( \square \)

**Corollary 3.3.** Let \( \varepsilon, \delta, p, q \) be non-negative real numbers with \( 0 < p, q < 1 \) or \( p, q > 2 \). If \( \mathcal{R} \) is a normed ring and \( f, g : \mathcal{R} \to \mathcal{M} \) are mappings satisfy the inequalities

\[
\| J_{f, g}^\theta, \phi (a, b, c) \| \leq \varepsilon (\| a \|^p + \| b \|^p + \| c \|^p), \quad \| J_{f, g}^\theta, \phi (a, b, c) \| \leq \delta (\| a \|^q + \| b \|^q + \| c \|^q)
\]
for all $a, b, c \in \mathcal{R}$, then there exist a unique $(\theta, \phi)$-Jordan derivation $G : \mathcal{R} \to \mathcal{M}$ and a unique generalized $(\theta, \phi)$-Jordan derivation $F : \mathcal{R} \to \mathcal{M}$ satisfying

$$
\| f(a) - F(a) \| \leq \frac{2\varepsilon}{|2 - 2^p|} \| a \|^p, \quad \| g(a) - G(a) \| \leq \frac{2\delta}{|2 - 2^q|} \| a \|^q
$$

for all $a \in \mathcal{R}$.

**Proof.** Let

$$
L := \begin{cases} 
2^{p-1}, & 0 < p < 1; \\
2^{1-p}, & p > 2.
\end{cases} \quad K := \begin{cases} 
2^{q-1}, & 0 < q < 1; \\
2^{1-q}, & q > 2.
\end{cases}
$$

So the result follows from Theorems 3.1 and 3.2. □

**Corollary 3.4.** Let $\varepsilon$ and $\delta$ be non-negative real numbers and let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings satisfying $f(0) = g(0) = 0$ and the inequalities

$$
\| J^{\theta, \varphi}_{f,g}(a,b,c) \| \leq \varepsilon, \quad \| J^{\theta, \varphi}_{g,g}(a,b,c) \| \leq \delta
$$

for all $a, b, c \in \mathcal{R}$. Then there exist a unique $(\theta, \phi)$-Jordan derivation $G : \mathcal{R} \to \mathcal{M}$ and a unique generalized $(\theta, \phi)$-Jordan derivation $F : \mathcal{R} \to \mathcal{M}$ satisfying

$$
\| f(a) - F(a) \| \leq \varepsilon, \quad \| g(a) - G(a) \| \leq \delta
$$

for all $a \in \mathcal{R}$.

**Proof.** The proof follows from Theorem 3.2 by taking

$$
\Phi(a,b,c) := \varepsilon, \quad \Psi(a,b,c) := \delta
$$

for all $a, b, c \in \mathcal{R}$. Then we can choose $L = K = 1/2$ and we get the desired results. □

**Theorem 3.5.** Let $f, g : \mathcal{R} \to \mathcal{M}$ be mappings with $f(0) = g(0) = 0$ for which there exists a function $\Phi : \mathcal{R}^4 \to [0, \infty)$ satisfying

$$
\lim_{n \to \infty} \frac{1}{2^n} \Phi(2^n a, 2^n b, 2^n c, 2^n d) = \lim_{n \to \infty} \frac{1}{2^n} \Phi(2^n a, b, 0, 0) = \lim_{n \to \infty} \frac{1}{2^n} \Phi(a, 2^n b, 0, 0) = 0, \quad (3.17)
$$

$$
\| D^{\theta, \varphi}_{f,g}(a,b,c,d) \| \leq \Phi(a,b,c,d) \quad (3.18)
$$

for all $a, b, c, d \in \mathcal{R}$. If $\mathcal{R}$ has the identity $e$, $\mathcal{M}$ is unit linked and there exists a constant $L < 1$ such

$$
\Phi(0,0,2a,2a) \leq 2L \Phi(0,0,a,a)
$$

for all $a \in \mathcal{R}$, then $g$ is a $(\theta, \phi)$-derivation and $f$ is a generalized $(\theta, \phi)$-derivation. Moreover, $f = a\theta + g$, where $a = \lim_{n \to \infty} \frac{1}{2^n} f(2^n e)$.
Proof. Letting $a = b = 0$ and $c = d$ in (3.18), we get
\[
\|f(2c) - 2f(c)\| \leq \Phi(0, 0, c, c)
\]
for all $c \in \mathcal{R}$. Using the same method as in the proof of Theorem 3.2, we infer that the limit
\[
F(a) := \lim_{n \to \infty} \frac{1}{2^n} f(2^n a)
\]
exists for all $a \in \mathcal{R}$ and the mapping $F : \mathcal{R} \to \mathcal{M}$ is additive. Letting $c = d = 0$ and replacing $a$ and $b$ by $2^ne$ and $2^nb$, respectively, in (3.18), we get
\[
\|f(2^n b) - f(2^ne)\theta(2^n b) - \phi(2^ne)g(2^n b)\| \leq \Phi(2^ne, 2^nb, 0, 0)
\]
for all $b \in \mathcal{R}$ and all $n \in \mathbb{N}$. Since $\phi(e) = e$, we have
\[
\|\frac{1}{4^n} f(2^n b) - \frac{1}{2^n} f(2^ne)\theta(b) - \frac{1}{2^n} g(2^n b)\| \leq \frac{1}{4^n} \Phi(2^ne, 2^nb, 0, 0)
\]
for all $b \in \mathcal{R}$ and all $n \in \mathbb{N}$. It follows from (3.17), (3.19) and (3.20) that the limit
\[
G(b) := \lim_{n \to \infty} \frac{1}{2^n} g(2^n b)
\]
exists and $G(b) = F(b) - F(e)\theta(b)$ for all $b \in \mathcal{R}$. Hence $G$ is additive. It follows from the definitions of $F, G$, (3.17) and (3.18) that
\[
\|F(ab) - F(a)\theta(b) - \phi(a)G(b)\|
= \lim_{n \to \infty} \frac{1}{4^n} \|f(4^n ab) - f(2^na)\theta(2^n b) - \phi(2^na)g(2^n b)\|
\leq \lim_{n \to \infty} \frac{1}{4^n} \Phi(2^na, 2^nb, 0, 0) = 0
\]
for all $a, b \in \mathcal{R}$. Therefore
\[
F(ab) = F(a)\theta(b) + \phi(a)G(b)
\]
for all $a, b \in \mathcal{R}$. Further, by (3.21) we have
\[
G(ab) = F(ab) - F(e)\theta(ab)
= F(a)\theta(b) + \phi(a)G(b) - F(e)\theta(a)\theta(b)
= [F(a) - F(e)\theta(a)]\theta(b) + \phi(a)G(b)
= G(a)\theta(b) + \phi(a)G(b)
\]
for all $a, b \in \mathcal{R}$. Thus $G$ is a $(\theta, \phi)$-derivation and (3.21) shows that $F$ is a generalized $(\theta, \phi)$-derivation.

By (3.17), (3.18) and the definitions of $F, G$, we have
\[
F(ab) - F(a)\theta(b) = \phi(a)g(b),
\]
\[
F(ab) - \phi(a)G(b) = f(a)\theta(b)
\]
for all \( a, b \in \mathcal{R} \). Since \( G(e) = 0 \) and \( \theta(e) = \phi(e) = e \), letting \( a = e \) in (3.22) and \( b = e \) in (3.23), we get \( g(b) = F(b) - F(e)\theta(b) = G(b) \) and \( F(a) = f(a) \), respectively, for all \( a, b \in \mathcal{R} \). So \( g \) is a \((\theta, \phi)\)-derivation and \( f \) is a generalized \((\theta, \phi)\)-derivation. Moreover, \( f = F(e)\theta + g \).

**Corollary 3.6.** Let \( \varepsilon, \delta, p \) be non-negative real numbers with \( 0 < p < 1 \). If \( \mathcal{R} \) is a normed ring with the identity \( e \), \( \mathcal{M} \) is unit linked and \( f, g : \mathcal{R} \to \mathcal{M} \) are mappings with \( f(0) = g(0) = 0 \) and satisfy the inequality

\[
\|D_{f,g}(a,b,c,d)\| \leq \delta + \varepsilon(\|a\|^p + \|b\|^p + \|c\|^p + \|d\|^p),
\]

for all \( a, b, c, d \in \mathcal{R} \), then \( g \) is a \((\theta, \phi)\)-derivation and \( f \) is a generalized \((\theta, \phi)\)-derivation. Moreover, \( f = a\theta + g \), where \( a = \lim_{n \to \infty} \frac{1}{2^n} f(2^n e) \).

**Theorem 3.7.** Let \( f, g : \mathcal{R} \to \mathcal{M} \) be mappings for which there exist functions \( \varphi, \psi : \mathcal{R}^2 \to [0, \infty) \) such that

\[
\lim_{n \to \infty} \frac{1}{n} \varphi(na, b) = \lim_{n \to \infty} \frac{1}{n} \varphi(a, nb) = 0,
\]

\[
\lim_{n \to \infty} \frac{1}{n} \psi(na, b) = \lim_{n \to \infty} \frac{1}{n} \psi(a, nb) = 0,
\]

\[
\|f(ab) - f(a)\theta(b) - \phi(a)g(b)\| \leq \varphi(a, b),
\]

\[
\|g(ab) - g(a)\theta(b) - \phi(a)g(b)\| \leq \psi(a, b)
\]

for all \( a, b, c \in \mathcal{R} \). If \( \mathcal{R} \) is normed with the identity \( e \) and \( \mathcal{M} \) is unit linked, then

\[
g(ab) = g(a)\theta(b) + \phi(a)g(b), \quad f(ab) = f(a)\theta(b) + \phi(a)g(b)
\]

for all \( a, b \in \mathcal{R} \).

**Proof.** By (3.24) and (3.27), we get

\[
\lim_{n \to \infty} \frac{1}{n} [g(nab) - g(na)\theta(b)] = \phi(a)g(b),
\]

\[
\lim_{n \to \infty} \frac{1}{n} [g(nab) - \phi(a)g(nb)] = g(a)\theta(b)
\]
for all \( a, b \in \mathcal{R} \). Using the Badora’s method \([3]\) on the inequality (3.27), we have

\[
\|g(ab) - g(a)\theta(b) - \phi(a)g(b)\| \leq \|g(ab) - \frac{1}{n}g(nab) + \frac{1}{n}\phi(ab)g(ne)\|
\]

\[
+ \|\frac{1}{n}g(nab) - \frac{1}{n}\phi(a)g(nb) - g(a)\theta(b)\|
\]

\[
+ \|\frac{1}{n}g(nab) - \frac{1}{n}g(na)\theta(b) - \phi(a)g(b)\|
\]

\[
+ \frac{1}{n}\|\phi(a)g(nb) - \phi(ab)g(ne) + g(na)\theta(b) - g(nab)\|
\]

\[
\|g(ab) - \frac{1}{n}g(nab) + \frac{1}{n}\phi(ab)g(ne)\|
\]

\[
+ \|\frac{1}{n}g(nab) - \frac{1}{n}\phi(a)g(nb) - g(a)\theta(b)\|
\]

\[
+ \|\frac{1}{n}g(nab) - \frac{1}{n}g(na)\theta(b) - \phi(a)g(b)\|
\]

\[
+ \frac{1}{n}\|\phi(a)\| \|g(nb) - g(b)\theta(ne) - \phi(b)g(ne)\|
\]

\[
+ \frac{1}{n}\|g(nab) - g(na)\theta(b) - \phi(na)g(b)\|
\]

\[
\leq \|g(ab) - \frac{1}{n}g(nab) + \frac{1}{n}\phi(ab)g(ne)\|
\]

\[
+ \|\frac{1}{n}g(nab) - \frac{1}{n}\phi(a)g(nb) - g(a)\theta(b)\|
\]

\[
+ \|\frac{1}{n}g(nab) - \frac{1}{n}g(na)\theta(b) - \phi(a)g(b)\|
\]

\[
+ \frac{1}{n}\|\phi(a)\| \|\psi(b, ne) + \frac{1}{n}\psi(na, b)\|
\]

for all \( a, b \in \mathcal{R} \). Applying (3.25) and (3.29), we observe that the right side of the last inequality tends to 0 when \( n \) tends to infinity. Therefore

\[
g(ab) = g(a)\theta(b) + \phi(a)g(b) \quad (3.30)
\]

for all \( a, b \in \mathcal{R} \).

Similarly, by (3.24) and (3.26), we have

\[
\lim_{n \to \infty} \frac{1}{n}[f(nab) - f(na)\theta(b)] = \phi(a)g(b),
\]

\[
\lim_{n \to \infty} \frac{1}{n}[f(nab) - \phi(a)g(nb)] = f(a)\theta(b) \quad (3.31)
\]
for all \(a, b \in \mathcal{R}\). Let \(a, b \in \mathcal{R}\) and \(n \in \mathbb{N}\) be fixed. Since \(g\) satisfies (3.30), we have \(g(nb) = g(bne) = n\phi(b)g(ne)\). Using (3.26), we have

\[
\|f(ab) - f(a)\theta(b) - \phi(a)g(b)\| \leq \|f(ab) - \frac{1}{n}f(nab) + \frac{1}{n}\phi(ab)g(ne)\| \\
+ \|\frac{1}{n}f(nab) - \frac{1}{n}\phi(a)g(nb) - f(a)\theta(b)\| \\
+ \|\frac{1}{n}f(nab) - \frac{1}{n}f(na)\theta(b) - \phi(a)g(b)\| \\
+ \frac{1}{n}\|\phi(a)g(nb) - \phi(ab)g(ne) + f(na)\theta(b) - f(nab)\|
\]

\[
= \|f(ab) - \frac{1}{n}f(nab) + \frac{1}{n}\phi(ab)g(ne)\| \\
+ \|\frac{1}{n}f(nab) - \frac{1}{n}\phi(a)g(nb) - f(a)\theta(b)\| \\
+ \|\frac{1}{n}f(nab) - \frac{1}{n}f(na)\theta(b) - \phi(a)g(b)\| \\
+ \frac{1}{n}\|n\phi(a)g(b) + f(na)\theta(b) - f(nab)\|
\]

\[
\leq \|f(ab) - \frac{1}{n}f(nab) + \frac{1}{n}\phi(ab)g(ne)\| \\
+ \|\frac{1}{n}f(nab) - \frac{1}{n}\phi(a)g(nb) - f(a)\theta(b)\| \\
+ \|\frac{1}{n}f(nab) - \frac{1}{n}f(na)\theta(b) - \phi(a)g(b)\| + \frac{1}{n}\phi(na,b).
\]

Applying (3.24) and (3.31), we observe that the right side of the last inequality tends to 0 when \(n\) tends to infinity. Therefore

\[f(ab) = f(a)\theta(b) + \phi(a)g(b).\]

\[\Box\]

**Corollary 3.8.** Let \(\varepsilon, \delta, p, q\) be non-negative real numbers with \(0 < p, q < 1\). If \(\mathcal{R}\) is a normed ring with the identity \(e\), \(\mathcal{M}\) is unit linked and \(f, g : \mathcal{R} \to \mathcal{M}\) are mappings satisfy the inequalities

\[
\|f(ab) - f(a)\theta(b) - \phi(a)g(b)\| \leq \delta + \varepsilon(\|a\|^p + \|b\|^q)
\]

\[
\|g(ab) - g(a)\theta(b) - \phi(a)g(b)\| \leq \delta + \varepsilon(\|a\|^p + \|b\|^q)
\]

for all \(a, b \in \mathcal{R}\), then \(f\) and \(g\) satisfy (3.28) for all \(a, b \in \mathcal{R}\).

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