

COVERING COMPLETE GRAPHS BY MONOCHROMATICALLY BOUNDED SETS

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Given a k -colouring of the edges of the complete graph K_n , are there $k - 1$ monochromatic components that cover its vertices? This important special case of the well-known Lovász-Ryser conjecture is still open. In this paper we consider a strengthening of this question, where we insist that the covering sets are not merely connected but have bounded diameter. In particular, we prove that for any colouring of $E(K_n)$ with four colours, there is a choice of sets A_1, A_2, A_3 that cover all vertices, and colours c_1, c_2, c_3 , such that for each $i = 1, 2, 3$ the monochromatic subgraph induced by the set A_i and the colour c_i has diameter at most 80.

1. INTRODUCTION

Given a graph G , whose edges are coloured with a colouring $\chi: E(G) \rightarrow C$ (where adjacent edges are allowed to use the same colour), given a set of vertices A , and a colour $c \in C$, we write $G[A, c]$ for the subgraph induced by A and the colour c , namely the graph on the vertex set A and the edges $\{xy: x, y \in A, \chi(xy) = c\}$. In particular, when $A = V(G)$, we write $G[c]$ instead of $G[V(G), c]$. Finally, we also use the usual notion of the induced subgraph $G[A]$ which is the graph on the vertex set A with edges $\{xy: x, y \in A, xy \in E(G)\}$. We usually write $[n] = \{1, 2, \dots, n\}$ for the vertex set of K_n .

Our starting point is the following conjecture of Gyárfás.

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Conjecture 1. ([2], [4]) *Let k be fixed. Let G be a k -edge-coloured K_n . Then we can find sets A_1, \dots, A_{k-1} whose union is $[n]$, and colours c_1, \dots, c_{k-1} such that $G[A_i, c_i]$ is connected for each $i \in [k-1]$.*

This is an important special case of the well-known Lovász-Ryser conjecture, which we now state.

Conjecture 2. (Lovász-Ryser conjecture. [6], [9]) *Let G be a graph whose maximum independent sets have size $\alpha(G)$. Then, whenever $E(G)$ is k -coloured, we can cover G by at most $(k-1)\alpha(G)$ monochromatic components.*

Conjectures 1 and 2 have attracted a great deal of attention. When it comes to the Lovász-Ryser conjecture, we should note the result of Aharoni ([1]), who proved the case of $k = 3$. For $k \geq 4$, the conjecture is still open. The special case of complete graphs was proved by Gyárfás ([3]) for $k \leq 4$, and by Tuza ([10]) for $k = 5$. For $k > 5$, the conjecture is open.

Let us also mention some results similar in the spirit to Conjecture 6. In [8], inspired by questions of Gyárfás ([2]), Ruszinkó showed that every k -colouring of the edges of K_n has a monochromatic component of order at least $n/(k-1)$ and of diameter at most 5. This was improved by Letzter ([5]), who showed that in fact there are monochromatic triple stars of order at least $n/(k-1)$. This bound is known to be tight in some cases, as shown by Gyárfás in [3]. In fact, Gyárfás proves that if $(k-1)^2|n$ and there is an affine plane of order $k-1$, then one may find a k -edge-colouring of K_n where no monochromatic component is larger than $n/(k-1)$. For more results and questions along these lines, we refer the reader to surveys of Gyárfás ([2], [4]).

In a completely different direction, relating to contraction mappings on metric spaces, the following theorem is proved in [7]. (We mention in passing that the current paper is self-contained, and in particular no knowledge of [7] is assumed.)

Theorem 3. *There is an absolute constant $C > 0$ such that the following holds. If $0 < \lambda < C$, and if $\{f, g, h\}$ are commuting continuous maps on a complete metric space (X, d) with the property that for any two distinct points $x, y \in X$ we have $\min\{d(f(x), f(y)), d(g(x), g(y)), d(h(x), h(y))\} \leq \lambda d(x, y)$, then the maps f, g, h have a common fixed point. In fact, we may take $C = 10^{-18}$.*

One of the ingredients in the proof of Theorem 3 was the following simple lemma.

Lemma 4 ([7], Lemma 5.5). *Suppose that G is a K_n with a 3-edge-colouring. Then we may find colours c_1, c_2 , (not necessarily distinct), and sets V_1, V_2 such that $V_1 \cup V_2 = [n]$ and $G[V_1, c_1], G[V_2, c_2]$ are each connected and of diameter at most 8.*

Proof sketch. Let x be a vertex and let χ be the given colouring. Define $A_i = \{a: \chi(ax) = i\}$. Thus, A_1, A_2, A_3 are non-empty, as otherwise we are done, by

taking $A_i \cup \{x\}$ for the non-empty A_i . We now try to ‘absorb’ vertices from A_i inside A_j . To this end, we define $B_{i,j} = \{a \in A_i : (\forall u \in A_j) \chi(au) \neq j\}$, so that $\text{diam}_j \left(\{x\} \cup A_j \cup (A_i \setminus B_{i,j}) \right) \leq 4$. Hence, when $\{i, j, k\} = [3]$ and $B_{i,j}$ and $B_{i,k}$ are disjoint, then we are done. Thus, we may assume that all $B_{i,j}$ are non-empty.

All edges between $B_{i,j}$ and $B_{j,i}$ are of colour k , hence $\text{diam}_k G[B_{i,j} \cup B_{j,i}] \leq 2$. Pick any $v \in B_{3,1} \cap B_{3,2}$. If $\chi(vu) = 3$ for some $u \in B_{1,2}$, then

$$\text{diam}_3 G[B_{1,2} \cup B_{2,1} \cup A_3 \cup \{x\}] \leq 5, \text{diam}_1 G[A_1 \cup (A_2 \setminus B_{2,1}) \cup \{x\}] \leq 4$$

completing the proof. Similarly, we deduce that when $w \in B_{2,1}$, $\chi(vw) \neq 3$. Therefore, when $v \in B_{3,1} \cap B_{3,2}$, then $\chi(B_{1,2}, v) = 2$ and $\chi(B_{2,1}, v) = 1$.

If $\chi(uv) = 2$ for some $u \in A_1 \setminus B_{1,2}$, then take $A_1 \cup A_2 \cup \{x, v\}$ with colour 2, and $A_3 \cup \{x\}$ with colour 3 to prove the lemma, and argue analogously when $\chi(wv) = 1$ for $w \in A_2 \setminus B_{2,1}$. Hence, we may assume that $\chi(A_1 \setminus B_{1,2}, v) = 3$ and $\chi(A_2 \setminus B_{2,1}, v) = 3$. But now it follows that

$$\text{diam}_3 G[B_{1,2} \cup B_{2,1}] \leq 2, \text{diam}_3 G[[n] \setminus (B_{1,2} \cup B_{2,1})] \leq 4$$

completing the proof. \square

In the case of two colours, we also note a very slight strengthening of the classical observation due to Erdős and Rado which says that for every graph G either G or G^c is connected.

Lemma 5. *Suppose that G is K_n with a 2-edge-colouring. Then we may find a colour c such that $G[c]$ is connected and of diameter at most 3.*

Proof sketch. After a short inspection, it is clear that the claim holds for $n \leq 4$. For $n \geq 5$, for each pair of vertices x, y , let $C(x, y)$ be the set of colours $c \in [2]$ such that there is a set of at most four vertices V such that $x, y \in V$ and $G[V]$ is c -connected and of diameter at most 3. If $1 \in C(x, y)$ for all pairs, we are done, and similarly if $2 \in C(x, y)$ for all pairs, we are also done. Otherwise, there are u, v, w, z such that $1 \notin C(u, v), 2 \notin C(w, z)$. Note that some of these vertices might coincide, but $V = \{u, v, w, z\}$ has size at least 3. Consider the subgraph $G[V]$ which is monochromatically connected and of diameter at most 3, to obtain a contradiction. \square

In [7], a common generalization of these statements and a strengthening of Conjecture 1 was conjectured.

Conjecture 6. *For every k , there is an absolute constant C_k such that the following holds. Given any colouring of the edges of K_n in k colours, we can find sets A_1, A_2, \dots, A_{k-1} whose union is $[n]$, and colours c_1, c_2, \dots, c_{k-1} such that $K_n[A_i, c_i]$ is connected and of diameter at most C_k , for each $i \in [k-1]$.*

The main result of this paper is

Theorem 7. *Conjecture 6 holds for four colours, and one may take $C_4 = 80$.*

1.1. An outline of the proof

We begin the proof by establishing the weaker Conjecture 1 for the case of four colours. Although this was proved by Gyárfás in [3], the reasons for giving a proof here are twofold. Firstly, we actually give a different reformulation of Conjecture 1 that has a more geometric flavour. The proof given here and the reformulation we consider emphasize the importance of the graph G_k , defined as a direct product of k copies of K_n , to Conjecture 1. Secondly, by giving this proof we make the paper self-contained.

We also need some auxiliary results about colourings with two or three colours, like Lemmas 5 and 4 mentioned above. In particular, we generalize the case of two colours to complete multipartite graphs. Another auxiliary result we use is the fact that G_k essentially cannot have large very sparse graphs.

In the remainder of the outline, let G stand for 4-edge-colouring of K_n .

The main tool in our proof is the notion of (c_3, c_4) -layer mappings, where c_3, c_4 are two colours. For a non-empty $P \subset \mathbb{N}_0^2$ (where $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$), this is a mapping $L: P \rightarrow \mathcal{P}(n)$, (where $[n]$ is the vertex set of our graph G), with the property that

1. sets $L(A)$ partition $[n]$ as A ranges over P ,
2. and for $A, B \in P$ with $|A_1 - B_1|, |A_2 - B_2| \geq 2$, we have all edges between $L(A)$ and $L(B)$ in G coloured using only c_3, c_4 .

This is a generalization of the idea that if we fix a vertex x_0 and we assign $A^{(x)} = (d_{c_1}(x_0, x), d_{c_2}(x_0, x)) \in \mathbb{N}_0^2$ to each vertex x , where d_{c_1}, d_{c_2} are distances in colours c_1, c_2 (which are the remaining two colours), then if $A^{(x)}, A^{(y)}$ satisfy $|A_1^{(x)} - A_1^{(y)}|, |A_2^{(x)} - A_2^{(y)}| \geq 2$, the edge xy cannot be coloured by c_1 or c_2 .

Given a subset P' of the domain P , we say that it is k -distant if for all distinct $A, B \in P'$ we have $|A_1 - B_1|, |A_2 - B_2| \geq k$. Once we have all this terminology set up, we begin building up structure in our graph, essentially as follows:

Step 1. We prove that if a (c_3, c_4) -layer mapping has a 3-distant set of size at least 4, then Theorem 7 holds.

Step 2. We continue the analysis of distant sets, and prove essentially that if a (c_3, c_4) -layer mapping has a 6-distant set of size at least 3, then Theorem 7 holds.

Step 3. We prove Theorem 7 when every colour induces a connected subgraph.

Step 4. We prove Theorem 7 when any two monochromatic components of different colours intersect.

Step 5. We put everything together to finish the proof.

Organization of the paper. In the next subsection, we briefly discuss a reformulation of Conjecture 1. In Section 2, we collect some auxiliary results, including results on 2-colourings of edges of complete multipartite graphs and the results on sparse subgraphs of G_k and independent sets in G_3 . In Section 3, we prove Conjecture 1 for four colours, reproving a result of Gyárfás. The proof of Theorem 7 is given in Section 4, with subsections splitting the proof into the steps described above. Finally, we end the paper with some concluding remarks in Section 5.

1.2. Another version of Conjecture 1

Let l be an integer, define the graph G_l with vertex set \mathbb{N}_0^l and put an edge between any two sequences that differ at every coordinate. Equivalently, G_l is the direct product of l copies of $K_{\mathbb{N}_0}$ (the complete graph on the vertex set \mathbb{N}_0). We formulate the following conjecture.

Conjecture 8. *Given a finite set of vertices $X \subset \mathbb{N}_0^l$, we can find l sets $X_1, \dots, X_l \subseteq X$ that cover X and each X_i is either contained in a hyperplane of the form $\{x_i = c\}$ or $G_l[X_i]$ is connected.*

This conjecture is actually equivalent to Conjecture 1.

Proposition 9. *Conjectures 1 and 8 are equivalent for $k = l + 1$.*

Proof. Conjecture 1 implies Conjecture 8. Let $X \subset \mathbb{N}_0^l$ be a finite set. Let $n = |X|$ and define an $(l + 1)$ -colouring $\chi: E(K_n) \rightarrow [l + 1]$ by setting $\chi(xy) = i$, where i is the smallest coordinate index such that $x_i = y_i$, otherwise, when x and y differ in all coordinates, set $\chi(xy) = l + 1$. If Conjecture 1 holds, we may find sets A_1, A_2, \dots, A_l that cover $[n]$, and colours c_1, c_2, \dots, c_l such that $K_n[A_i, c_i]$ are all connected. Fix now any i , and let $B \subset X$ be the set of vertices corresponding to A_i . If $c_i \leq l$, then for any $x, y \in B$, there is a sequence of vertices $z_1, z_2, \dots, z_m \in B$ such that $x_i = (z_1)_i = (z_2)_i = \dots = (z_m)_i = y_i$, so $x_i = y_i$. Hence, B is subset of the plane $\{x_i = v\}$ for some value v . Otherwise, if $c_i = l + 1$, that means that the edges of $K_n[A_i, c_i]$ correspond to edges of $G_l[B]$, so $G_l[B]$ is connected, as desired.

Conjecture 8 implies Conjecture 1. Let $\chi: E(K_n) \rightarrow [k]$ be any k -colouring of the edges of K_n . For every colour c , look at components $C_1^{(c)}, \dots, C_{n_c}^{(c)}$ of $K_n[c]$. For each choice of x_1, x_2, \dots, x_{k-1} with $x_c \in [n_c]$ for $c \in [k - 1]$, we define $C_x = C_{x_1, x_2, \dots, x_{k-1}} = \bigcap_{c \in [k-1]} C_{x_c}^{(c)}$, which is the intersection of monochromatic components, one for each colour except k . Let $X \subset \mathbb{N}^{k-1}$ be the set of all $(k - 1)$ -tuples x for which C_x is non-empty. If Conjecture 8 holds, then we can

find A_1, A_2, \dots, A_{k-1} that cover X such that each A_i is either contained in a hyperplane, or induces a connected subgraph of G_{k-1} . If $A_i \subset \{x_c = v\}$, then the corresponding intersections C_x for $x \in A_i$ are all subset of $C_v^{(c)}$. On the other hand, if $G_{k-1}[A_i]$ is connected, then taking any adjacent $x, y \in G_{k-1}[A_i]$, we have that $x_c \neq y_c$ for all $c \in [k-1]$. Hence all the edges of between C_x and C_y are coloured by k . Hence, all the sets C_x for $x \in A_i$ are subset of the same component of $K_n[k]$. This completes the proof of the proposition. \square

2. AUXILIARY RESULTS

As suggested by its title, this section is devoted to deriving some auxiliary results. Firstly we extend Lemma 5 to complete multipartite graphs. The case of bipartite graphs is slightly different from the general case of more than two parts, and is stated separately. We also introduce additional notation. Given a colour c and vertices x, y we write $d_c(x, y)$ for the distance between x and y in $G[c]$. If they are not in the same c -component, we write $d_c(x, y) = \infty$. In particular, $d_c(x, y) < \infty$ means that x, y are in the same component of $G[c]$. Further, we write $B_c(x, r)$ for the c -ball of radius r around x , defined as $B_c(x, r) = \{y: d_c(x, y) \leq r\}$, where c is a colour, x is a vertex, and r is a nonnegative integer. For any graph G , throughout the paper, the *diameter* of G , written $\text{diam } G$, is the supremum of all finite distances between two vertices of G . Note that this is a slightly unusual definition, since more traditionally infinite diameter implies that the graph is disconnected. However, in our case $\text{diam } G = \infty$ only happens when G has arbitrarily long induced paths (as we focus on the finite graphs in this paper, this will not occur). In other words, the diameter of a graph G defined here is the supremum of diameters of all components of G . For a colour c and a set of vertices A , the c -diameter of A , written $\text{diam}_c A$, is the diameter of $G[A, c]$. We use the standard notation for complete multipartite graphs, so K_{n_1, n_2, \dots, n_r} stands for the graph with r vertex classes, which are independent sets, of sizes n_1, n_2, \dots, n_r , and all edges between different classes are present in the graph.

Lemma 10. *Suppose that the edges of $G = K_{n_1, n_2}$ are coloured in two colours. Then, one of the following holds:*

1. *either there are two vertices u, v in the same vertex class, such that all edges from u have the first colour, and all edges from v have the second colour, or*
2. *there is a colour c , such that $G[c]$ is connected and of diameter at most 7, or*
3. *there are partitions $[n_1] = A_1 \cup B_1$ and $[n_2] = A_2 \cup B_2$ such that all edges in $(A_1 \times A_2) \cup (B_1 \times B_2)$ are of one colour, and all the edges in $(A_1 \times B_2) \cup (B_1 \times A_2)$ are of the other colour.*

Proof. Let $\chi: E \rightarrow [2]$ be the given colouring. Suppose that the first possibility in the conclusion does not hold. Note that if u and v are vertices such that all edges incident to u are coloured by 1 and all edges incident to v are coloured by 2, then u

and v are forced to be in the same vertex class, which we forbade. Hence, without loss of generality, every vertex has at least one edge coloured by 1. Furthermore, if all edges of some vertex w are coloured by 1, then the second possibility in the conclusion holds. Thus, we may in fact assume that every vertex is incident to edges of both colours.

We start by observing the following. If there are two vertices v_1, v_2 such that for colour 1 the inequality $6 \leq d_1(v_1, v_2) < \infty$ holds, then for every vertex u such that $\chi(uv_1) = 1$, we must also have $d_2(u, v_1) \leq 3$. Indeed, let $(v_1 = w_0, w_1, w_2, \dots, w_r = v_2)$ be a minimal 1-path from v_1 to v_2 . Hence $r \geq 6$, the vertices w_i with an index of the same parity belong to the same vertex class of $G = K_{n_1, n_2}$ and the edges $v_1w_3 = w_0w_3, w_3w_6, w_6u \in E(G)$ are all of colour 2 (otherwise, we get a contradiction to the fact that $d_1(w_i, v_2) = r - i$), implying that $d_2(v_1, u) \leq 3$.

Now, suppose that a 1-component has diameter at least 7. Let $x_1, x_2 \in 1$ be such that $d_1(x_1, x_2) = 7$. The observation above tells us that if a vertex y is adjacent (in G) to x_1 , and $d_2(x_1, y) > 1$, then $\chi(x_1, y) = 1$, so $d_2(x_1, y) \leq 3$. Hence, every vertex y adjacent (in G) to x_1 , satisfies $d_2(x_1, y) \leq 3$. Similarly, any vertex y adjacent to x_2 satisfies $d_2(x_2, y) \leq 3$. But, x_1, x_2 are in different vertex classes (as their 1-distance is odd), so their neighbourhoods cover the whole vertex set, and x_1x_2 is an edge of colour 2 as well, from which we conclude that $G[2]$ is connected and of diameter at most 7. Thus, if any monochromatic component has diameter at least 7, the lemma follows, so assume that this does not occur.

Now we need to understand the monochromatic components. From the work above, it suffices to find monochromatic components of the desired structure, the diameter is automatically bounded by 6. Suppose that there are at least three 1-components, $X_1 \cup X_2, Y_1 \cup Y_2, Z_1 \cup Z_2$ with X_1, Y_1, Z_1 subsets of one class of K_{n_1, n_2} and X_2, Y_2, Z_2 subsets of the other. Recall that each vertex has a 1-coloured edge, so the sets X_1, \dots, Z_2 are non-empty. Let $u, v \in X_1 \cup Y_1 \cup Z_1$ be arbitrary vertices. Then we can find $w \in X_2 \cup Y_2 \cup Z_2$ in different 1-component from u, v . Hence, $\chi(uw) = \chi(vw) = 2$, so $d_2(u, v) \leq 2$. Therefore, both vertex classes of G are 2-connected and consequently the whole graph is 2-connected.

Finally, assume that each colour has exactly two monochromatic components. Let $[n_1] = A_1 \cup B_1, [n_2] = A_2 \cup B_2$ be such that $A_1 \cup A_2, B_1 \cup B_2$ are the 1-components. Hence, $A_1 \cap B_1 = A_2 \cap B_2 = \emptyset$, and all edges in $A_1 \times B_2$ and $B_1 \times A_2$ are of colour 2. Thus, sets $A_1 \cup B_2$ and $B_1 \cup A_2$ are 2-connected and cover the vertices of G , so they must be the two 2-components. Thus, all edges in $A_1 \times A_2$ and $B_1 \times B_2$ must be coloured by 1, proving the lemma. \square

Lemma 11. *Let $r \geq 3$, and suppose that $G = K_{n_1, n_2, \dots, n_r}$ is a complete r -partite graph. Suppose that the edges of G are 2-coloured. Then, one of the following holds.*

1. *Either there are two vertices u and v in the same vertex class such that all edges incident to u are of one colour, and all edges incident to v are of the other colour, or*
2. *there is a colour c such that $G[c]$ is connected and of diameter at most 9.*

Proof. Let $\chi: E(G) \rightarrow [2]$ be the given colouring. We begin the proof by making the following observations.

Claim. For any two vertices u, v , we have $d_c(u, v) \leq 3$ for some colour or u, v lie in the same vertex class and all edges from u are coloured in one colour, and all edges from v are coloured in the other colour.

Proof of the claim. Suppose that $d_c(u, v) \leq 3$ does not hold for either colour. Clearly, u and v lie in the same vertex class, and for any vertex w in a different vertex class we have $\chi(uw) \neq \chi(vw)$. Suppose that u has edges of both colours, let w and z be vertices satisfying $\chi(uw) = 1, \chi(uz) = 2$. Without loss of generality, w and z are in different vertex classes (otherwise, pick arbitrary vertex t in a vertex class different from those of u and w , and change w to t if $\chi(ut) = 1$, otherwise change z to t). Hence, $\chi(vw) = 2, \chi(vz) = 1$. If $\chi(wz) = 1$, then the path (u, w, z, v) shows $d_1(u, v) \leq 3$, and if $\chi(wz) = 2$, then the path (u, z, w, v) shows $d_2(u, v) \leq 3$, both of which result in a contradiction, proving the claim. \square

Hence, we are either done, or we may assume that for any two vertices x, y in the given graph we have $d_c(x, y) \leq 3$ for a suitable colour. Define a new 2-edge-colouring χ' of the complete graph on vertices $V(G)$, by $\chi'(x, y) = c$ for some c such that $d_c(x, y) \leq 3$. By Lemma 5, we may find c such that all pairs of vertices are on distance at most 3 in colour c under the edge-colouring χ' . Returning to the original colouring χ , we get that $G[c]$ is connected and of diameter at most 9, as desired. \square

2.1. Induced subgraphs of G_l

Recall that G_l is the graph on \mathbb{N}_0^l , with edges between pairs of points all of whose coordinates differ. In this subsection we prove a few properties of such graphs, particularly focusing on G_3 . We begin with a general statement, which will be reproved for specific cases with stronger conclusions.

Lemma 12. *If S is a set of vertices in G_l and the maximum degree of $G_l[S]$ is at most d , then the number of non-isolated vertices of $G_l[S]$ is at most $O_{l,d}(1)$.*

Proof. By Ramsey's theorem we have an N such that whenever $E(K_N)$ is coloured using $2^l - 1$ colours, there is a monochromatic K_{l+1} . Let S' be the set of non-isolated vertices in S . We show that $|S'| < (d^2 + d + 1)N$. Suppose not. Since the maximum degree is at most d , we have a subset $S'' \subset S'$ of size $|S''| \geq N$ such that sets $\{s\} \cup N(s)$ are disjoint for all $s \in S''$ (simply pick a maximal such subset, their second neighbourhoods must cover the whole S'). In particular, S'' is an independent set in G_l , so for every pair of vertices $x, y \in S''$, the set $I(x, y) = \{i \in [l]: x_i = y_i\}$ is non-empty. Thus, $I: E(K_{S''}) \rightarrow \mathcal{P}(l) \setminus \{\emptyset\}$ is a $(2^l - 1)$ -colouring of the edges of a complete graph $K_{S''}$ on the vertex set S'' . Due to the choice of N , there is a monochromatic clique on subset $T \subset S''$ of size at least $l + 1$, whose edges are coloured by some set $I_0 \neq \emptyset$. Hence, $x_i \neq t_i$ for all $i \in [l]$ and for distinct $t', t'' \in T$ we have $t'_i = t''_i$ if and only if $i \in I_0$. Take a vertex $t \in T$, and since t

is not isolated and the neighbourhoods of vertices in S'' are disjoint, we can find $x \in S'$ such that tx is an edge, but $t'x$ is not for other $t' \in T$. Thus, $x_i \neq t'_i$ for all $t' \in T$ and $i \in I_0$. But, xt' is not an edge for $t' \in T \setminus \{t\}$, so we always have $i \in [l] \setminus I_0$ such that $x_i = t'_i$. But, for each $i \in I_0$, the values of t'_i are distinct for each $t' \in T$. Hence, for each i , there is at most one vertex $t' \in T \setminus \{t\}$ such that $x_i = t'_i$. Therefore $|T|-1 \leq |[l] \setminus I_0| \leq l-1$, so $|T| \leq l$, which is a contradiction. \square

We may somewhat improve on the bound in the proof of the lemma above by observing that for colour I_0 we only need a clique of size $l - |I_0| + 2$. Thus, instead of the Ramsey number

$$R(\underbrace{l+1, l+1, \dots, l+1}_{2^{l-1}}),$$

we could use

$$R(l+2 - |I_1|, l+2 - |I_2|, \dots, l+2 - |I_{2^{l-1}}|),$$

where I_i are the non-empty sets of $[l]$. But, even for paths in G_3 , which we shall use later, taking $l=3, d=2$, we get the final bound of $7R(2, 3, 3, 3, 4, 4, 4)$, where 7 comes from $d^2 + d + 1$ factor we lose when moving from S' to S'' . We now improve this bound.

Lemma 13. *If S is a set of vertices of G_3 such that $G_3[S]$ is a path, then $|S| \leq 30$.*

Proof. Let $S = \{s_1, s_2, \dots, s_r\}$ be such that (s_1, s_2, \dots, s_r) is an induced path in G_3 , so the only edges are $s_i s_{i+1}$.

Case 1. For all $i \in \{4, 5, \dots, 10\}$, s_i coincides with one of s_1 or s_2 in at least two coordinates.

Since $s_1 s_2$ is an edge, s_1 and s_2 have all three coordinates different. Thus, for $i \in \{4, 5, \dots, 10\}$, we have $(s_i)_c \in \{(s_1)_c, (s_2)_c\}$ for all coordinates c . Hence, there are only at most 6 possible choices of s_i (as $s_i \neq s_1, s_2$), so $r \leq 9$.

Case 2. There is $i_0 \in \{4, 5, \dots, 10\}$ with at most one common coordinate with each of s_1, s_2 . Since $s_1 s_{i_0}, s_2 s_{i_0}$ are not edges, w.l.o.g. we have $s_1 = (x_1, x_2, x_3), s_2 = (y_1, y_2, y_3), s_{i_0} = (x_1, y_2, z_3)$, where $x_i \neq y_i, z_3 \notin \{x_3, y_3\}$. Consider any point s_j , for $j \geq i_0 + 2$. It is not adjacent to any of s_1, s_2, s_{i_0} . If $(s_j)_1 = x_1$ and $(s_j)_2 \neq y_2$, then $(s_j)_3 = y_3$. Similarly, if $(s_j)_1 \neq x_1$ and $(s_j)_2 = y_2$, then $(s_j)_3 = x_3$. Also, if $(s_j)_1 \neq x_1, (s_j)_2 \neq y_2$, then $s_j = (y_1, x_2, z_3)$. Hence, for $j \geq i_0 + 2$, the point s_j is on one of the lines

$$(x_1, y_2, \cdot), (x_1, \cdot, y_3), (\cdot, y_2, x_3) \text{ or it is the point } (y_1, x_2, z_3),$$

where (a, b, \cdot) stands for the line $\{(a, b, z) : z \text{ arbitrary}\}$, etc. Note that a point on (x_1, y_2, \cdot) is not adjacent to any point on (\cdot, y_2, x_3) , and the same holds for lines (x_1, y_2, \cdot) and (x_1, \cdot, y_3) . Hence, along our path, a point on the line (x_1, \cdot, y_3) is followed either by a point on (\cdot, y_2, x_3) or the point (y_1, x_2, z_3) (the latter may happen only once). In any case, if $|S| \geq 30$, then among $s_{i_0+2}, s_{i_0+3}, \dots, s_{i_0+20}$, we must get a contiguous sequence $s_j, s_{j+1}, \dots, s_{j+7}$ of points

$$s_j, s_{j+2}, s_{j+4}, s_{j+6} \in (x_1, \cdot, y_3), s_{j+1}, s_{j+3}, s_{j+5}, s_{j+7} \in (\cdot, y_2, x_3).$$

Finally, we look at $A = s_j, B = s_{j+2}, C = s_{j+5}, D = s_{j+7}$. These four points form an independent set, but $A \neq B$ gives $A_2 \neq B_2$, so one of $A_2 \neq y_2, B_2 \neq y_2$ holds, and similarly, one of $C_1 \neq x_1, D_1 \neq x_1$ holds as well. Choosing a point among A, B and a point among C, D for which equality does not hold gives an edge, which is impossible. \square

Finally, we study independent sets in G_3 . Note that Lemma 12 in this case does not tell us anything about the structure of such sets. When we refer to lines or planes, we always think of very specific cases, namely the lines are the sets of the form $\{x: x_i = a, x_j = b\}$ and the planes are $\{x: x_i = a\}$. Similarly, collinearity and coplanarity of points have stronger meaning, and imply that points lie on a common line or plane defined as above.

Lemma 14. *Let S be a set of vertices in G_3 . If every two points of S are collinear, then S is a subset of a line. If every three points of S are coplanar, then S is a subset of a plane.*

Proof. We first deal with the collinear case. Take any pair of points, $x, y \in S$, w.l.o.g. they coincide in the first two coordinates. Take third point $z \in S$. If z does not share the values of the first two coordinates with x and y , then we must have $x_3 = z_3 = y_3$, which is impossible. As z was arbitrary, we are done.

Suppose now that we have all triples coplanar. W.l.o.g. we have a noncollinear pair x, y , which only coincide in the first coordinate. Then all other points may only be in the plane $\{p: p_1 = x_1\}$. \square

Lemma 15. *(Structure of the independent sets of size 4.) Given an independent set I of G_3 of size 4 (at least) one of the following alternatives holds.*

(S1) I is coplanar, or

(S2) $I = \{(a, b, c), (a', b', c), (a', b, c'), (a, b', c')\}$, where $a \neq a'; b \neq b'$ and $c \neq c'$, or

(S3) up to permutation of coordinates $I = \{(a, b, c), (a, b, c'), (a, b', x), (a', b, x)\}$, where $a \neq a'; b \neq b'$ and $c \neq c'$.

Proof. Suppose that $I = \{A, B, C, D\}$ is not a subset of any plane. We distinguish between two cases.

Case 1. There are no collinear pairs in I .

Let $A = (a, b, c)$. But AB is not an edge and not collinear so A and B differ in precisely two coordinates. Thus, w.l.o.g. $B = (a', b', c)$ where $a \neq a'$ and $b \neq b'$. If C_3 also equals c , then we must have $C_3 = (a'', b'', c)$ with a'' different from a, a' and b'' from b, b' . However, looking at D , we cannot have $D_3 = c$ as otherwise $I \subset \{x_3 = c\}$, so D must differ at all three coordinates from one of the points A, B, C , making them joined by an edge, which is impossible. Thus $C_3 = c'$, with $c' \neq c$. Since AC and BC are not edges, $C \in \{(a, b', c'), (a', b, c')\}$. The same

argument works for D , so $D_3 = c'' \neq c$, and $D \in \{(a, b', c''), (a', b, c'')\}$. However, if $c' \neq c''$, then C, D are either collinear or adjacent in G_3 , which are both impossible. Hence $c'' = c'$, and $\{C, D\} = \{(a, b', c'), (a', b, c')\}$, as desired.

Case 2. W.l.o.g. A and B are collinear.

Let $A = (a, b, c), B = (a, b, c')$ with $c \neq c'$. Since $\{x_1 = a\}$ does not contain the whole set I , we have w.l.o.g. $C_1 = a' \neq a$. If $C_2 \neq b$, then AC or BC is an edge, which is impossible. Therefore, $C_2 = b$. Hence $D_2 = b' \neq b$, and by a similar argument $D_1 = a$. Finally CD is not an edge, so their third coordinate must be the same, proving the lemma. \square

Lemma 16. *(Structure of the independent sets of size 5.) Given an independent set I of G_3 of size 5 (at least) one of the following alternatives holds*

1. I is coplanar, or
2. I is a subset of a union of three lines, all sharing the same point.

Proof. List the vertices of I as x_1, x_2, x_3, x_4, x_5 . By Lemma 14, we may w.l.o.g. assume that x_1, x_2, x_3 are not coplanar. By the previous lemma, $\{x_1, x_2, x_3, x_i\}$ for $i = 4, 5$ may have structure **S2** or **S3**. But if both structures are **S2**, then we must have that in both quadruples, at each coordinate, each value appears precisely two times. This implies $x_4 = x_5$. Hence, w.l.o.g. $\{x_1, x_2, x_3, x_4\}$ has structure **S3**. Therefore, assume w.l.o.g. that

$$x_1 = (1, 0, 0), x_2 = (0, 1, 0), x_3 = (0, 0, 1), x_4 = (0, 0, c')$$

for some $c' \neq 1$ (which corresponds to the choice $a = 0, a' = 1, b = 0, b' = 1, x = 0, c = 1$ in the previous Lemma, switching the roles of c and c' if necessary). Looking at $\{x_1, x_2, x_3, x_5\}$, if it had **S2** for its structure, we would get $x_5 = (1, 1, 1)$, which is adjacent to x_4 , and thus impossible. Hence $\{x_1, x_2, x_3, x_5\}$ also has structure **S3**. Permutting the coordinates only permutes x_1, x_2, x_3 , and does not change the number of zeros in x_5 . Thus, w.l.o.g.

$$\begin{aligned} \{(1, 0, 0), (0, 1, 0), (0, 0, 1), x_5\} &= \{x_1, x_2, x_3, x_5\} \\ &= \{(d, e, f), (d, e, f'), (d', e, y), (d, e', y)\}, \end{aligned}$$

for some $d \neq d', e \neq e', f \neq f'$. But in the first coordinate, only zero can appear three times, so $d = 0$. Similarly, $e = 0$, so $x_5 \in (0, 0, \cdot)$, after a permutation of coordinates. Thus x_5 has at least two zeros, so our independent set I is a subset of the union of lines passing through the point $(0, 0, 0)$, as required. \square

Corollary 17. *(Structure of the independent sets of size at least 5.) Given an independent set I of G_3 of size at least 5 (at least) one of the following alternatives holds*

1. I is coplanar, or

2. I is a subset of a union of three lines, all sharing the same point.

Proof. Let $I = \{x_1, \dots, x_r\}$ and $r \geq 5$. Suppose that I is not coplanar, since we are done otherwise. By Lemma 14, w.l.o.g. $\{x_1, x_2, x_3\}$ is not a coplanar set. Take any distinct $i, j \geq 4$, and apply Lemma 16 to $\{x_1, x_2, x_3, x_i, x_j\}$. Thus, these five points lie on a union of three lines, sharing the same point p_{ij} . But, this point is the same for all choices of i and j , and so are the three lines, which completes the proof. \square

3. CONJECTURE 1 FOR FOUR COLOURS

In this short section we reprove the result of Gyárfás.

Theorem 18. (*Gyárfás*) *Conjecture 1 for four colours and Conjecture 8 for G_3 are true.*

Proof. By the equivalence of conjectures, it suffices to prove Conjecture 8 for G_3 . Let X be the given finite set of vertices in G_3 . Assume that $G_3[X]$ has at least four components, otherwise we are done immediately. By a *representatives set* we mean any set of vertices that contains at most one vertex from each component of X . A *complete representatives set* is a representatives set that intersects every component of X .

Observation 19. *If there are three colinear points, each in a different component, then X can be covered by two planes. In other words, if two planes do not suffice, then among every three points in different components, there is a non-colinear pair.*

Proof. W.l.o.g. these are points $(0, 0, 1), (0, 0, 2), (0, 0, 3)$. Then, unless $X \subset \{x_1 = 0\} \cup \{x_2 = 0\}$, we have a point of the form (a, b, c) with a, b both non-zero, so it is a neighbour of at least two of the points we started with, contradicting the fact that they belong to different components. For the second part, recall that if every pair in a triple is colinear, then the whole triple lies on a line. \square

By the observation above, every representatives set of size at least 3 has a noncollinear pair. Suppose firstly that every complete representatives set is a subset of a plane. Pick a complete representatives set $\{x_1, x_2, \dots, x_r\}$, with $x_i \in C_i$, where C_i are the components. W.l.o.g. x_1, x_2 is a noncollinear pair, therefore, it determines a plane π , forcing components C_3, C_4, \dots, C_r to be entirely contained in this plane. Hence, we may cover the whole set X by components C_1 and C_2 , and the plane π . Therefore, by Lemma 14, we may assume that we have a representatives set of size 3 which does not lie in any plane.

Case 1. X has more than four components.

Let $\{x_1, x_2, x_3\}$ be a representatives set, $x_i \in C_i$, which is not coplanar. Then, for any choice of y_4, \dots, y_r , such that $\{x_1, x_2, x_3, y_4, \dots, y_r\}$ is a complete representatives set, by Corollary 17 we have three lines that meet in a single point that contain all these points. Observe that this structure is determined entirely by

x_1, x_2, x_3 . Indeed, since these three points are not coplanar, they cannot coincide in any coordinate. However, since there are at least five components, $\{x_1, x_2, x_3\}$ extends to an independent set of size 5, which must be a subset of three lines sharing a point p . But we can identify p , since p_i must be the value that occurs precisely two times among $(x_1)_i, (x_2)_i, (x_3)_i$, and hence the lines are $l_1 = px_1, l_2 = px_2, l_3 = px_3$.¹ Thus, the union of the lines l_1, l_2, l_3 contains the components C_4, \dots, C_r and the vertices x_1, x_2, x_3 . By Observation 19, each l_i has representatives from at most two components. Hence, we may not have the common point of the three lines p present in X , as otherwise some line l_i would have three components meeting it. W.l.o.g. l_2, l_3 intersect two components, and l_1 may intersect one or two. Then, picking any $y \in l_2, z \in l_3$ such that $\{x_1, x_2, x_3, y, z\}$ is a representatives set, using the argument above applied to $\{x_1, y, z\}$ instead of $\{x_1, x_2, x_3\}$, we deduce that $C_2 \subset l_2, C_3 \subset l_3$. Therefore, we can cover all vertices by C_1 and lines l_1, l_2 and l_3 . These three lines lie inside the planes l_1l_2 and l_1l_3 ,² which completes the proof in this case.

Case 2. X has precisely four components and there exists a coplanar complete representatives set.

Let $\{x_1, x_2, x_3, x_4\}$ be a complete representatives set, with $x_i \in C_i$. W.l.o.g. we have $x_i = (a_i, b_i, 0)$. By Observation 19, we do not have a collinear triple among these 4 points, so each of the sequences $(a_i)_{i=1}^4$ and $(b_i)_{i=1}^4$ has the property that a value may appear at most twice in the sequence.

Suppose for a moment that each of these two sequences has at most one value that appears twice. Write v for the value that appears two times in (a_i) , if it exists, and let u be the corresponding value for (b_i) . If we take a point $y \in X$ outside the plane $(\cdot, \cdot, 0)$, then the number of appearances of y_1 in (a_i) and y_2 in (b_i) combined is at least three. So, either y_1 is the unique doubly-appearing value u for a_i or $y_2 = v$, so the three planes $(u, \cdot, \cdot), (\cdot, v, \cdot)$ and $(\cdot, \cdot, 0)$ cover X .

Now, assume that w.l.o.g. $\{a_i\}$ has two doubly-appearing values, i.e. $a_1 = a_2 = u \neq a_3 = a_4 = v$. If y is outside the plane $(\cdot, \cdot, 0)$, then if $y_1 \neq u$, one of the pairs x_1y, x_2y must be an edge, so x_3y and x_4y are not edges, so we must have $y_1 = v$. Similarly, if y is outside the plane $(\cdot, \cdot, 0)$ and $y_1 \neq v$, then $y_1 = u$. Hence, for all points $y \in X$, we have $y_1 \in \{u, v\}$ or $y_3 = 0$, and three planes cover once again.

Case 3. X has precisely four components, but no complete representatives set is coplanar.

Thus, by Lemma 15, every complete representatives set has either **S2** or **S3** as its structure. Observe that if **S2** is always the structure, then all the components are singletons, because any three points in the structure **S2** determine the fourth point, and we are done by taking a plane to cover two vertices. So, there is a representatives set with structure **S3**. Take such a representatives set x_1, x_2, a, b , w.l.o.g. $x_1 = (0, 0, 1), x_2 = (0, 0, 2)$. Take any y that shares the component with a ,

¹For points x, y , we write xy for the line determined by points x and y , i.e. if $x_i = y_i, x_j = y_j, x_k \neq y_k$, this is $xy = \{z: z_i = x_i, z_j = x_j\}$.

²If lines $l_1 = \{t: t_i = a_i, t_j = a_j\}$ and $l_2 = \{t: t_i = a_i, t_k = a_k\}$ are given and $\{i, j, k\} = [3]$, we write l_1l_2 for the plane $\{t: t_i = a_i\}$.

and any z that shares the component with b . Then, $\{x_1, x_2, y, z\}$ is also a complete representatives set, so it is not coplanar. But, as x_1, x_2 are collinear, it may not have structure **S2**, so the structure must be **S3**, which forces $y_3 = z_3$. Hence, we can cover X by components of x and y and the plane (\cdot, \cdot, a_3) . This completes the proof. \square

Note that the theorem is sharp – we can take $X = \{0, 2e_1, 2e_2, 2e_3, e_1 + e_2, e_1 + e_3, e_2 + e_3\}$, where $e_1 = (1, 0, 0), e_2 = (0, 1, 0), e_3 = (0, 0, 1)$.

4. CONJECTURE 6 FOR FOUR COLOURS

Recall, by a *diameter* of a colour c , written $\text{diam}_c(G)$, we mean the maximum distance between vertices sharing the same component of $G[c]$. In the remaining part of the paper, for a given 4-colouring $\chi: E(K_n) \rightarrow [4]$, we say that χ *satisfies Conjecture 6 with (constant) K* if there are sets A_1, A_2, A_3 whose union is $[n]$ and colours c_1, c_2, c_3 such that each $K_n[A_i, c_i]$ is connected and of diameter at most K . Thus, our goal can be phrased as: there is an absolute constant K such that every 4-colouring χ of $E(K_n)$ satisfies Conjecture 6 with K .

We begin the proof of the main result by observing that essentially we may assume that at least two colours have arbitrarily large diameters. We argue by modifying the colouring slightly.

Lemma 20. *Suppose χ is a 4-colouring of $E(K_n)$ such that three colours have diameters bounded by N_1 . Then χ satisfies Conjecture 6 with $\max\{N_1, 30\}$.*

Proof. Write $G = K_n$, and observe that if a point does not receive all four colours at its edges, we are immediately done. Let χ be the given colouring of the edges, and suppose that colours 1, 2 and 3 have diameter bounded by N_1 . We begin by modifying the colouring slightly. Let xy be any edge coloured by colour 4. If x and y share the same component in $G[c]$ for some $c \in \{1, 2, 3\}$, change the colour of xy to the colour c (if there is more than one choice, pick any). Note that such a modification does not change the monochromatic components, except possibly shrinking the components for the colour 4. Let χ' stand for the modified colouring. Observe that the diameter of colour 4 in χ' is also bounded. Indeed, begin by listing all the components for colours $i \in \{1, 2, 3\}$ as $C_1^{(i)}, C_2^{(i)}, C_3^{(i)}, \dots$. For $x \in \mathbb{N}^3$, consider the sets $C_x = C_{x_1, x_2, x_3} = C_{x_1}^{(1)} \cap C_{x_2}^{(2)} \cap C_{x_3}^{(3)}$. Let X be the set of all x such that $C_x \neq \emptyset$. If $G^{(\chi')}[4]$ (where the superscript indicates the relevant colouring) has an induced path (v_1, v_2, \dots, v_r) , then if $x_i \in \mathbb{N}^3$ is defined to be such that $v_i \in C_{x_i}$, in fact (x_1, x_2, \dots, x_r) becomes an induced path in G_3 . But Lemma 13 implies that $r \leq 30$. Hence, the 4-diameter in the colouring χ' is at most 30.

Applying Theorem 18 for the colouring χ' , gives three monochromatic components that cover the vertex set, let these be $G^{(\chi')}[A_1, c_1], G^{(\chi')}[A_2, c_2], G^{(\chi')}[A_3, c_3]$, where the superscript indicates the relevant colouring. Using the same sets and colours, but returning to the original colouring, we have that $G^{(\chi)}[A_1, c_1], G^{(\chi)}[A_2, c_2], G^{(\chi)}[A_3, c_3]$ are all still connected, as 1, 2 and 3-components are the same in χ and χ' , while

there can only be more 4-coloured edges in the colouring χ . Also, 1, 2 and 3-diameters are bounded by N_1 , and 4-diameters of sets may only decrease when returning to colouring χ' , so the lemma follows. \square

Let us recall some additional notions. Let $P \subset \mathbb{N}_0^2$ be a set, and let $L: P \rightarrow \mathcal{P}(n) \setminus \{\emptyset\}$ be a function with the property that $\{L(A): A \in P\}$ forms a partition of $[n]$ and there are two colours c_3, c_4 ³ such that whenever $A, B \in P$ and $|A_1 - B_1|, |A_2 - B_2| \geq 2$, then all edges between the sets $L(A)$ and $L(B)$ are coloured with c_3 and c_4 only. We call L the (c_3, c_4) -layer mapping and we refer to P as the layer index set. Further, we call a subset $S \subset P$ a k -distant set if for every two distinct points $A, B \in S$ we have $|A_1 - B_1|, |A_2 - B_2| \geq k$.

Let us briefly motivate this notion. Suppose that $K_n[c_1]$ and $K_n[c_2]$ are both connected. Fix a vertex x_0 and let $P = \{(d_{c_1}(x_0, v), d_{c_2}(x_0, v)): v \in [n]\} \subset \mathbb{N}_0^2$. Let $L(A) := \{v \in [n]: (d_{c_1}(x_0, v), d_{c_2}(x_0, v)) = A\}$ for all $A \in P$ (this also motivates the choice of the letter L , we think of $L(A)$ as a layer). Then, if $x \in L(A), y \in L(B)$ for $A, B \in P$ with $|A_1 - B_1| \geq 2, |A_2 - B_2| \geq 2$, by triangle inequality, we cannot have $d_{c_1}(x, y) \leq 1$ nor $d_{c_2}(x, y) \leq 1$, so xy takes either the colour c_3 or the colour c_4 . As we shall see, we may have more freedom in the definition of P and L if there is more than one component in a single colour.

We now explore these notions in some detail, before using them to obtain some structural results on the 4-colourings that possibly do not satisfy Conjecture 6.

Lemma 21. *Let χ be a 4-edge-colouring, L a (c_3, c_4) -layer mapping with layer index set P , and suppose that $\{A, B, C\} \subset P$ is a 3-distant set. Write $G = K_n$. Then at least one of the following holds.*

1. The colouring χ satisfies Conjecture 6 with (constant) 22, or
2. for some colour $c \in \{c_3, c_4\}$ we have $G[L(A) \cup L(B) \cup L(C), c]$ connected and of diameter at most 9.

Proof of Lemma 21. Observe that all edges between $L(A)$ and $L(B)$, between $L(A)$ and $L(C)$, and between $L(B)$ and $L(C)$, are of colours c_3 and c_4 . This is a complete tripartite graph with classes $L(A), L(B)$ and $L(C)$ and we may apply Lemma 11. If we get the second outcome of that lemma, then w.l.o.g. $L(A) \cup L(B) \cup L(C)$ is c_3 -connected and of c_3 -diameter at most 9, proving the claim. Otherwise, w.l.o.g. we obtain vertices $u, v \in L(A)$ such that $\chi(uz) = c_3, \chi(vz) = c_4$ for all $z \in L(B) \cup L(C)$. Let A_1 be the set of all $a \in L(A)$ such that all edges from a are of colour c_3 , and let A_2 be the set of all $a \in L(A)$ which have at least one c_4 -coloured edge. Thus, $L(A) = A_1 \cup A_2$ is a partition, and $u \in A_1, v \in A_2$. Define the following three sets of vertices

$$U = \{w: d_{c_3}(w, L(B)) \leq 10\}, W = \{w: d_{c_4}(w, L(B)) \leq 10\} \text{ and } T = V(G) \setminus (U \cup W).$$

³This choice of indices was made on purpose – we shall first use colours c_1, c_2 to define P and L , and the remaining colours will be c_3 and c_4 .

Clearly, U and W monochromatically connected and of diameter at most 22 for colours c_3 and c_4 respectively. Moreover, by the properties of u and v , we also have $A_1 \cup L(B) \cup L(C) \subset U$ and $A_2 \cup L(B) \cup L(C) \subset W$.

Consider any other point $D \in P$. By pigeonhole principle, D is 2-distant from at least one of the points A, B, C . Suppose that D is 2-distant from $E \in \{B, C\}$. Then, all edges from $L(D)$ to $L(E)$ are of colours c_3 and c_4 , so for each $z \in L(D)$, we have $d_{c_3}(z, u) \leq 2$ or $d_{c_4}(z, v) \leq 2$, so $L(D) \subset U \cup W$. Therefore, if $z \in T$, then $z \in L(D)$ for a point D which is 2-distant from A . Then all edges between $L(A)$ and $L(D)$ are coloured by c_3 and c_4 and we may apply Lemma 10. Since $z \notin U \cup W$, we have $d_{c_3}(z, B) > 10, d_{c_4}(z, B) > 10$, so $\chi(za) = c_4$ for all $a \in A_1$, and $\chi(za) = c_3$ for all $a \in A_2$. Thus, $T \cup A_1$ is monochromatically connected and of diameter at most 2, so the claim follows after taking $U, V, T \cup A_1$. \square

Lemma 22. *Let χ be a 4-edge-colouring, L a (c_3, c_4) -layer mapping with layer index set P , and suppose that $\{A, B, C\} \subset P$ is a 3-distant set. Write $G = K_n$. Suppose that $G[L(A) \cup L(B) \cup L(C), c_3]$ is contained in a subgraph $H_3 \subset G[c_3]$ which is connected and of c_3 -diameter at most N_3 . Suppose additionally that $G[L(A) \cup L(B), c_4]$ is contained in a subgraph $H_4 \subset G[c_4]$ that is connected and of diameter at most N_4 . Then the given colouring satisfies Conjecture 6 with $\max\{N_3, N_4\} + 2$.*

The following observation is very useful when we work with 3-distant sets.

Observation 23. *Suppose that $A, B, C, D \in P$ and that A, B, C are 3-distant. Then D is 2-distant from at least one of the points A, B, C .*

Proof. Suppose to the contrary that D is not 2-distant from either of the points A, B, C . By pigeonhole principle we may find an index i and two points A', B' among A, B, C such that $|D_i - A'_i|, |D_i - B'_i| \leq 1$, so $|A'_i - B'_i| \leq 2$, which is a contradiction. \square

Proof of Lemma 22. Pick any $D \in P$. Note that since A, B, C are 3-distant, by Observation 23 D is 2-distant from a point E among A, B, C .

Let U be the set of all vertices x in G such that $d_{c_3}(x, L(A) \cup L(B) \cup L(C)) \leq 1$. Hence, $G[U, c_3] \cup H_3$ is connected and of diameter at most $N_3 + 2$. Consider any vertex $v \notin U$. Let $D \in P$ be such that $v \in L(D)$. If $E \in \{A, B, C\}$ is 2-distant from D , then all edges from v to $L(E)$ must have colour c_4 . Define $W = \{w \notin U : d_{c_4}(w, L(C)) \leq 1\}$ and $Z = \{z \notin U : d_{c_4}(z, L(A) \cup L(B)) \leq 1\}$, so if $v \notin U$, by the argument above, we have that $v \in W$ or $v \in Z$. Moreover, if $E = C$, all edges between v and $L(C)$ must have colour c_4 , so $\text{diam}_{c_4}(L(C) \cup W) \leq 2$. Hence, we may take

$$\begin{aligned} &G[L(A) \cup L(B) \cup L(C) \cup U, c_3] \cup H_3, \\ &H_4 \cup G[Z, c_4], \text{ and} \\ &G[L(C) \cup W, c_4], \end{aligned}$$

to prove the lemma (after omitting the corresponding subgraphs if W or Z are empty). \square

Lemma 24. *Let G be K_n with a 4-edge-colouring χ and suppose that L is a (c_3, c_4) -layer mapping for some colours $c_3, c_4 \in [4]$ with a 3-distant set of size at least 4. Then χ satisfies Conjecture 6 with constant 29.*

Proof. Suppose that some $A, B, C, D \in P$ are 3-distant. All edges between $L(A) \cup L(B) \cup L(C) \cup L(D)$ are of colours c_3 and c_4 only, so by Lemma 11 either $G[L(A) \cup L(B) \cup L(C) \cup L(D), c]$ is connected for some $c \in \{c_3, c_4\}$ and of diameter at most 9, or w.l.o.g. there are vertices $u, v \in A$, such that $\chi(uz) = c_3$ and $\chi(vz) = c_4$ for all $z \in L(B) \cup L(C) \cup L(D)$. However, in the latter case, Lemma 21 tells us that either χ satisfies Conjecture 6 with constant 22, or w.l.o.g. $G[L(A) \cup L(B) \cup L(C), c_3]$ is connected and of diameter at most 9. Thus, in either case we conclude that w.l.o.g. $G[L(A) \cup L(B) \cup L(C) \cup L(D), c_3]$ is connected and of diameter at most 11.

Pick any $E \in P$. Applying Observation 23 twice, we see that E is 2-distant from at least two points $A'(E), B'(E)$ among A, B, C, D . Hence, $A'(E), B'(E), E$ is a 2-distant set, so edges between $L(A'(E)), L(B'(E))$ and $L(E)$ are of colours c_3 and c_4 only. By Lemma 21, unless we are done, for some colour $c(E) \in \{c_3, c_4\}$ we have $G[L(A'(E)) \cup L(B'(E)) \cup L(E), c(E)]$ connected and of diameter at most 9. However, if $c(E) = c_4$, then by Lemma 22 we get that χ satisfies Conjecture 6 with constant 13, since $L(A'(E)) \cup L(B'(E))$ is also contained in c_3 -connected subgraph of diameter at most 11. Hence, $c(E) = c_3$ for all E , and the whole graph is c_3 -connected and of c_3 -diameter at most 29. \square

Lemma 25. *Suppose that χ is a 4-colouring of $E(K_n)$ and that L is a (c_3, c_4) -layer mapping for some colours $c_3, c_4 \in [4]$ with a 7-distant set $\{A, B, C\}$. Suppose additionally that there are $X, Y \in P$ such that $|X_1 - A_1|, |X_1 - B_1|, |X_1 - C_1| \geq 5$ and $|Y_2 - A_2|, |Y_2 - B_2|, |Y_2 - C_2| \geq 5$. Then χ satisfies Conjecture 6 with constant 31.*

Proof. Pick any other $D \in P$. If D is 3-distant from each of A, B, C , we obtain a 3-distant set of size 4, so by Lemma 24 we are done. Hence, for every $D \in P$ we have $E \in \{A, B, C\}$ such that $|E_i - D_i| \leq 2$ for some i .

Since $\{A, B, C\}$ is a 7-distant set, by Lemma 21, we have w.l.o.g. $G[L(A) \cup L(B) \cup L(C), c_3]$ connected and of diameter at most 9. We now derive some properties of $L(D)$ for points $D \in P$ be such that $|D_i - A_i|, |D_i - B_i|, |D_i - C_i| \geq 3$ for some $i \in \{1, 2\}$. (Note that such points exist by assumptions.)

Let D be such a point and let j be such that $\{i, j\} = \{1, 2\}$. Since the set $\{A, B, C\}$ is 7-distant, there are distinct $E_1, E_2 \in \{A, B, C\}$ such that $|D_j - (E_1)_j|, |D_j - (E_2)_j| \geq 3$. Thus, $\{D, E_1, E_2\}$ is also a 3-distant set. Applying Lemma 21 to $\{D, E_1, E_2\}$ implies that $G[L(D) \cup L(E_1) \cup L(E_2), c]$ is connected and of diameter at most 9, for some $c \in \{c_3, c_4\}$, unless we are already done. However, if $c = c_4$, $G[L(E_1) \cup L(E_2), c_4]$ is contained in a subgraph of $G[c_4]$ that is connected and of diameter at most 9, so Lemma 22 applies and the claim follows. Hence, we must have $G[L(D) \cup L(E_1) \cup L(E_2), c_3]$ connected and of diameter at most 9. In particular, whenever $D \in P$ satisfies $|D_i - A_i|, |D_i - B_i|, |D_i - C_i| \geq 3$

for some $i \in \{1, 2\}$, then every point in $L(D)$ is at c_3 -distance at most 9 from $L(A) \cup L(B) \cup L(C)$.

Let X, Y be the given points that obey $|X_1 - A_1|, |X_1 - B_1|, |X_1 - C_1| \geq 5$ and $|Y_2 - A_2|, |Y_2 - B_2|, |Y_2 - C_2| \geq 5$. W.l.o.g. $|X_2 - A_2| \leq 2$. If $|Y_1 - A_1| \leq 2$, then X, Y, B, C form a 3-distant set of size 4, and once again the claim follows from Lemma 24. Hence, w.l.o.g. $|Y_1 - B_1| \leq 2$. By the work above, we also have that every point in $L(X) \cup L(Y)$ is at c_3 -distance at most 9 from $L(A) \cup L(B) \cup L(C)$. Note also that X, Y are 3-distant.

It remains to analyse $D \in P$ such that for both $i = 1, 2$ there is an $E \in \{A, B, C\}$ such that $|E_i - D_i| \leq 2$. We show that in most cases, depending on the choice of E , we have that $L(D)$ is, in fact, at bounded c_3 -distance from $L(A) \cup L(B) \cup L(C)$. If we have an $E \in \{A, B, C\}$ such that both $|E_1 - D_1| \leq 2$ and $|E_2 - D_2| \leq 2$ hold, then taking E', E'' such that $\{E, E', E''\} = \{A, B, C\}$, we have $\{D, E', E''\}$ 3-distant, so Lemma 21 once again implies that every vertex in $L(D)$ is at c_3 -distance at most 9 from $L(A) \cup L(B) \cup L(C)$ (or we are done by Lemma 22). We distinguish the following cases.

- If $|D_1 - A_1| \leq 2, |D_2 - B_2| \leq 2$, then D is 3-distant from X, Y . Let us check this. By triangle inequality, we obtain $|X_1 - D_1|, |Y_1 - D_1|, |D_2 - X_2|, |Y_2 - D_2| \geq 3$. We also know that $L(X) \cup L(Y) \cup L(C)$ is contained in the subgraph $G[L(A) \cup L(B) \cup L(C) \cup L(X) \cup L(Y)] \subset G[c_3]$ that is connected and of diameter at most 27, so every vertex in $L(D)$ is at c_3 -distance at most 10 from $L(A) \cup L(B) \cup L(C)$, unless there is a vertex $v \in L(D)$ such that all edges from v to $L(X) \cup L(Y)$ are of colour c_4 . However, in such a case, $L(X) \cup L(Y)$ is also contained in c_4 -connected subgraph of diameter 2 (with vertex set $\{v\} \cup L(X) \cup L(Y)$) and we are done by Lemma 22.
- If $|D_1 - C_1| \leq 2, |D_2 - B_2| \leq 2$, note that $\{D, X, Y\}$ is 3-distant and then the same argument we had in the case above proves that $L(D)$ is on c_3 -distance at most 10 from $L(A) \cup L(B) \cup L(C)$.
- If $|D_1 - A_1| \leq 2, |D_2 - C_2| \leq 2$, note that $\{D, X, Y\}$ is 3-distant and then the same argument we had in the case above proves that $L(D)$ is on c_3 -distance at most 10 from $L(A) \cup L(B) \cup L(C)$.

Finally, we define $P_1, P_2, P_3 \subset P$ as

$$\begin{aligned} P_1 &= \{D \in P: |D_1 - B_1|, |D_2 - A_2| \leq 2\} \\ P_2 &= \{D \in P: |D_1 - C_1|, |D_2 - A_2| \leq 2\} \\ P_3 &= \{D \in P: |D_1 - B_1|, |D_2 - C_2| \leq 2\} \end{aligned}$$

which are disjoint and if $D \in P \setminus (P_1 \cup P_2 \cup P_3)$ we know that $L(D)$ is on c_3 -distance at most 10 from $L(A) \cup L(B) \cup L(C)$. Let also $L_i = \cup_{D \in P_i} L(D)$. Hence, since for $D \in P_1$ we have $|D_1 - C_1|, |D_2 - C_2| \geq 2$, all edges between $L(D)$ and

$L(C)$ are coloured using c_3 and c_4 , thus all edges between L_1 and $L(C)$ take these two colours. Similarly, all edges between L_2 and Y , and all edges between L_3 and X are taking only the colours c_3 and c_4 . Observe that if $D \in P_2, D' \in P_3$ then $|D_1 - D'_1|, |D_2 - D'_2| \geq 3$. Thus all edges between L_2 and L_3 are only of colours c_3 and c_4 .

Set $M = \{v \in [n]: d_{c_3}(v, L(A) \cup L(B) \cup L(C)) \leq 11\}$, and note that $G[M, c_3]$ is connected and of diameter at most 31. We observe the following.

- If $x \in L_1 \setminus M$, then all edges from x to $L(C)$ take only colour c_4 . Thus, if $c \in L(C)$ is arbitrary, we have $G[\{c\} \cup (L_1 \setminus M), c_4]$ connected and of diameter 2.
- Similarly, if $x \in L_2 \setminus M$, then all edges from x to $L(Y)$ take only colour c_4 . Thus, if $y \in L(Y)$ is arbitrary, we have $G[\{y\} \cup (L_2 \setminus M), c_4]$ connected and of diameter 2.
- Similarly, if $x \in L_3 \setminus M$, then all edges from x to $L(A)$ take only colour c_4 . Thus, if $a \in L(A)$ is arbitrary, we have $G[\{a\} \cup (L_3 \setminus M), c_4]$ connected and of diameter 2.

From this we conclude that unless all $L_1 \setminus M, L_2 \setminus M, L_3 \setminus M$ are non-empty, we are done, since we can take $G[M, c_3]$ as one of the graphs, and two additional c_4 monochromatic graphs as above. Set $U_2 = L_2 \setminus M, U_3 = L_3 \setminus M$. Let $y \in L(Y), a \in L(A)$ be arbitrary. As noted above, all edges $\{y\} \times U_2$ and $\{a\} \times U_3$ are of colour c_4 . Hence, if there is an edge $u_2 u_3$ of colour c_4 with $u_2 \in U_2, u_3 \in U_3$, we have $G[\{a, y\} \cup U_2 \cup U_3, c_4]$ connected and of diameter at most 5. Taking this graph with $G[M, c_3]$ and $G[\{c\} \cup (L_1 \setminus M), c_4]$, proves the claim.

Finally, the remaining possibility is that all edges between U_2 and U_3 are of colour c_3 . In this case, $G[U_2 \cup U_3, c_3]$ is connected and of diameter at most 2. Take also $G[M, c_3]$ and $G[\{c\} \cup (L_1 \setminus M), c_4]$ to finish the proof. \square

Let us now briefly discuss a way of defining (c_3, c_4) -layer mappings. Pick two colours $c_1, c_2 \in [4]$, and take c_3, c_4 to be the remaining two colours. List all vertices as v_1, v_2, \dots, v_n . To each vertex, we shall assign two nonnegative integers, $D_1(v_i)$ and $D_2(v_i)$, initially marked as undefined. We apply the following procedure.

- Step 1** Pick the smallest index i such that $D_1(v_i)$ or $D_2(v_i)$ is undefined. If there is no such i , terminate the procedure.
- Step 2** For $j = 1, 2$, if $D_j(v_i)$ is undefined, pick an arbitrary value for it.
- Step 3** For $j = 1, 2$, if $D_j(v_i)$ was undefined before the second step, for all vertices u in the same c_j -component of v_i set $D_j(u) := d_{c_j}(v_i, u) + D_j(v_i)$. Return to Step 1.

Upon the completion of the procedure, set $P = \{(D_1(v), D_2(v)): v \in [n]\}$ and $L: P \rightarrow \mathcal{P}(n)$ as $L(x, y) := \{v \in [n]: (D_1(v), D_2(v)) = (x, y)\}$.

Claim. The mapping L above is well-defined and is a (c_3, c_4) -layer mapping.

Proof. Observe that each time we pick v_i whose value(s) are to be defined, we end up defining D_1 on one c_1 -component or D_2 on one c_2 -component or both. Hence, for every vertex v , the values $D_1(v), D_2(v)$ change precisely once from undefined to a nonnegative integer value. Hence, $(D_1(v), D_2(v))$ are well-defined and take values in \mathbb{N}_0^2 , so P and L are well-defined and $L(A)$ forms a partition of $[n]$ as A ranges over P . Finally, consider an edge xy coloured by c_1 . Let $D_1(x)$ be defined with v_i chosen in Step 2 (possibly $x = v_i$). Since xy is of colour c_1 , these are in the same c_1 -component, and hence $D_1(x) = d_{c_1}(v_i, x) + D_1(v_i)$ and $D_1(y) = d_{c_1}(v_i, y) + D_1(v_i)$. Therefore,

$$\begin{aligned} |D_1(x) - D_1(y)| &= |(d_{c_1}(v_i, x) + D_1(v_i)) - (d_{c_1}(v_i, y) + D_1(v_i))| \\ &= |d_{c_1}(v_i, x) - d_{c_1}(v_i, y)| \leq d_{c_1}(x, y) = 1 \end{aligned}$$

hence, if $\chi(xy) = c_1$, then $|D_1(x) - D_1(y)| \leq 1$. Similarly, we get the corresponding statement for the colour c_2 . It follows that if $A, B \in P$ are such that $|A_1 - B_1|, |A_2 - B_2| \geq 2$, then if $x \in L(A), y \in L(B)$, we have $(D_1(x), D_2(x)) = A, (D_1(y), D_2(y)) = B$, so xy is coloured by c_3 or c_4 , as desired. \square

4.1. Monochromatically connected case

Proposition 26. *Suppose that χ is a 4-colouring of $E(K_n)$ such that every colour induces a connected subgraph of K_n . Then χ satisfies Conjecture 6 with constant 80.*

Proof. Suppose to the contrary, in particular every colour has diameter greater than 80. Our main goal in the proof is to find a pair of vertices x, y with a control over their 1-distance and 2-distance. We need both distances sufficiently large so that we can make a use of distant sets in (3, 4)-layer mappings, and also bounded by a constant so that if a vertex is on small 1-distance from x , it is also on small 1-distance from y and vice-versa.

More precisely,

Lemma 27. *Suppose that there are vertices x', y' such that $d_1(x', y'), d_2(x', y') \geq 7$. Then χ satisfies Conjecture 6 with constant $2 \max\{d_1(x', y'), d_2(x', y')\} + 12$.*

Proof. Pick any point $z \neq x', y'$. Apply the procedure for defining (3, 4)-layer mapping starting from x' . Note that the (3, 4)-layer is uniquely defined as all colours are connected. If we obtain a 7-distant set of size at least 3, we obtain a contradiction to Lemma 25, which maybe used due to the diameter assumption. Hence, the distances corresponding to x', y', z cannot give such a set, so we must have one of

$$d_1(x', z) \leq 6 \text{ or } |d_1(x', y') - d_1(x', z)| \leq 6 \text{ or } d_2(x', z) \leq 6 \text{ or } |d_2(x', z) - d_2(x', y')| \leq 6.$$

In particular, we must have $d_1(x', z) \leq d_1(x', y') + 6$ or $d_2(x', z) \leq d_2(x', y') + 6$. Thus, monochromatic balls $B_1(x, d_1(x', y') + 6)$ and $B_2(x, d_2(x', y') + 6)$ cover all the vertices. \square

Claim. There are x, y such that $d_1(x, y) \in \{21, 22\}$ and $d_2(x, y) \geq 14$.

Proof of the claim. Suppose contrary, for every x, y such that $d_1(x, y) \in \{21, 22\}$, we must have $d_2(x, y) \leq 13$. Pick any $y_1, y_2 \in [n]$ such that $\chi(y_1 y_2) = 1$. We claim that $d_2(y_1, y_2) \leq 26$. Suppose on the contrary that $d_2(y_1, y_2) > 26$. Since the 1-diameter is greater than 22, we can find $x \in [n]$ such that $d_1(x, y_1) = 22$, which further implies that $d_2(x, y_1) \leq 13$. Take a minimal 1-path ($x = u_0, \dots, u_{22} = y_1$). If $d_1(x, y_2) \in \{21, 22\}$, then we would have $d_2(x, y_2) \leq 13$, which, together with $d_2(x, y_1) \leq 13$, would result in a contradiction. Therefore, we must have $d_1(x, y_2) = 23$. A similar argument applied to u_1 instead of x gives $d_1(u_1, y_1) = 21$, $d_1(u_1, y_2) = 20$. But the triangle inequality implies that $|d_1(x, y_2) - d_1(u_1, y_2)| \leq 1$, which is a contradiction.

Hence, taking any $x \in [n]$ the balls

$$B_2(x, 26), B_3(x, 1), B_4(x, 1)$$

cover the vertex set. However, these have diameter at most 52, which is a contradiction. \square

Take x, y given by the claim above. Since the subgraph $G[2]$ is connected, there is a minimal 2-path ($x = u_0, u_1, \dots, u_r, u_{r+1} = y$) between x and y , with $r \geq 14$. Look at the vertices $z_1 = u_7, z_2 = u_{14}, \dots, z_k = u_{7k}$ with k such that $7 \leq r - 7k \leq 13$.

Consider x, y, z_i for some $1 \leq i \leq k$ and check whether we can define a (3, 4)-layer mapping so that these three points become a 7-distant set. Apply the procedure for defining (3, 4)-layers mapping, starting from x , i.e. we want to see whether $(0, 0), (d_1(x, y), d_2(x, y))$ and $(d_1(x, z_i), d_2(x, z_i))$ are 7-distant. If they are 7-distant, Lemma 25 gives us a contradiction. Since

$$\begin{aligned} d_1(x, y) &\geq 21, d_2(x, y) \geq 14 \\ 7 \leq d_2(x, z_i) &= 7i \leq 7k < d_2(x, y) - 6 \end{aligned}$$

we must have either $d_1(x, z_i) \leq 6$ or $|d_1(x, z_i) - d_1(x, y)| \leq 6$ (implying $d_1(x, z_i) \in \{15, \dots, 28\}$). Similarly, if we start from y instead of x in our procedure, we see that either $d_1(y, z_i) \leq 6$ or $|d_1(y, z_i) - d_1(x, y)| \leq 6$ (implying $d_1(y, z_i) \in \{15, \dots, 28\}$) must hold.

Observe that for the vertex z_1 we must have $d_1(x, z_1) \leq 6$. Otherwise, we would have $15 \leq d_1(x, z_1) \leq 28$ and $d_2(x, z_1) = 7$, resulting in a contradiction by Lemma 27 (applied to the pair x, z_1). For every z_i we must have either the first inequality ($d_1(x, z_i) \leq 6$) or the second ($15 \leq d_1(x, z_i) \leq 28$), and we have that the first vertex among these, namely z_1 , satisfies the first inequality. Suppose that there

was an index i such that z_{i+1} obeys the second inequality, and pick the smallest such i . Then, by the triangle inequality, we would have

$$9 \leq d_1(z_{i+1}, x) - d_1(x, z_i) \leq d_1(z_i, z_{i+1}) \leq d_1(z_{i+1}, x) + d_1(x, z_i) \leq 34$$

and $d_2(z_i, z_{i+1}) = 7$, so Lemma 27 applies now to the pair z_i, z_{i+1} and gives a contradiction. Hence, for all $i \leq k$ we must have the first inequality for z_i . But then z_k and y satisfy the conditions of Lemma 27, giving the final contradiction, since $7 \leq d_2(y, z_k) < 14$ and

$$15 \leq d_1(y, x) - d_1(x, z_k) \leq d_1(y, z_k) \leq d_1(y, x) + d_1(x, z_k) \leq 28.$$

This completes the proof. \square

4.2. Intersecting monochromatic components

Proposition 28. *Suppose that $\chi: E(K_n) \rightarrow [4]$ be a 4-colouring with the property that, whenever C and C' are monochromatic components of different colours, and one of them has diameter at least 80 (in the relevant colour), then C and C' intersect. Then χ satisfies Conjecture 6 with constant 80.*

Proof. Suppose to the contrary, we have a colouring χ that satisfies the assumptions but for which the conclusion fails. By Lemma 20, we have that at least two colours have monochromatic diameters greater than 80. Further, by Proposition 26 we have a colour with at least two components. W.l.o.g. there is a 1-component C_1 of 1-diameter at least 80, and colour 2 has at least two components.

Next, we find a pair of vertices x, y with the property that $7 \leq d_1(x, y) \leq 9$ and x, y are in different 2-components. We do this as follows. Since C_1 intersects all 2-components, we have vertices $x_1, x_2 \in C_1$ which are in different 2-components. Looking at any 1-path from x_1 to x_2 , we can find two consecutive vertices z_1, z_2 which are in different 2-components. Pick an arbitrary vertex y with $d_1(z_1, y) = 8$. Then one of the pairs z_1, y or z_2, y satisfies the desired properties.

Pick any vertex z outside $B_1(x, 16)$. We now apply our procedure for defining (3, 4)-layers mapping with vertices listed as x, y, z, \dots . Note that may assume that $|D_1(x) - D_1(y)|, |D_1(x) - D_1(z)|, |D_1(y) - D_1(z)| \geq 7$ (recall the D_1, D_2 notation from the procedure), since $d_1(x, y) \in \{7, 8, 9\}$, and z is either in a different 1-component from x and y , or at distance at least 7 from both. Hence, we get a 7-distant set, unless $d_2(x, z) \leq 6$ or $d_2(y, z) \leq 6$. Therefore, if there are no 7-distant sets, $B_1(x, 16), B_2(x, 6)$ and $B_2(y, 6)$ cover the vertex set and we get a contradiction. On the other hand, if there z such that $\{D(x), D(y), D(z)\}$ is a 7-distant set, we may apply Lemma 25 to get obtain another contradiction, since its assumptions can be satisfied, as we shall now see. Indeed, since the 1-diameter of C_1 is at least 80, we may find $X \in P$ such that $|X_1 - D_1(x)|, |X_1 - D_1(y)|, |X_1 - D_1(z)| \geq 5$. On the other hand, for the same condition for colour 2, it is similarly satisfied if there is a 2-component of 2-diameter at least 28, or if there are at least four 2-components. Hence, if the condition fails, taking the 2-components results in contradiction. \square

4.3. Final steps

In the final part of the proof, we show how to reduce the general case to the case of intersecting monochromatic components.

Theorem 29. *Conjecture 6 holds for 4 colours and we may take 80 for the diameter bounds.*

Proof. Let χ be the given 4-colouring of $E(K_n)$. Our goal is to apply Proposition 28. We start with an observation.

Observation 30. *Suppose that C is a c -component, that is disjoint from a c' -component C' with $c' \neq c$. Then for every pair of vertices $x, y \in C$ we have $d_c(x, y) \leq 6$ or $d_{c'}(x, y) \leq 6$ or the colouring satisfies Conjecture 6 with the constant 80.*

Proof of the Observation 30. Pick $x, y \in C$ with $d_c(x, y) \geq 7$ and take arbitrary $z \in C'$. Apply our procedure for generating c_3, c_4 -layers mapping to the list x, y, z, \dots , with c_3, c_4 chosen to be the two colours different from c, c' . Since z is in different c - and c' -components from x, y , these three vertices result in a 7-distant set, unless $d_{c'}(x, y) \leq 6$, as desired. \square

Corollary 31. *Suppose that we have a c -component C , that is disjoint from a c' -component C' with $c' \neq c$ and has c -diameter at least 30. Then the colouring χ satisfies Conjecture 6 with the constant 80.*

Proof. By the Observation 30 we are either done, or any two vertices $x, y \in C$ with $d_c(x, y) > 6$ satisfy $d_{c'}(x, y) \leq 6$. Furthermore, given any two vertices $x, y \in C$, since the c -diameter of C is at least 30, we can find $z \in C$ such that $d_c(x, z), d_c(y, z) \geq 7$, so by triangle inequality $d_{c'}(x, y) \leq 12$ holds for all $x, y \in C$. Now, take an arbitrary vertex $v \in C$, let c'', c''' be the two remaining colours, and consider the sets

$$B_{c'}(v, 12), B_{c''}(v, 1), B_{c'''}(v, 1).$$

Given any $u \in [n]$, if vu is coloured by any of c', c'' or c''' , it is already in the sets above. On the other hand, if uv is of colour c , then $v \in C$ so $d_{c'}(u, v) \leq 12$, thus $u \in B^{(c')}(v, 12)$. Thus, these sets cover the vertex sets and have monochromatic diameters at most 24, so we are done. \square

Finally, we are in the position to apply Proposition 28 which finishes the proof of the theorem. \square

5. CONCLUDING REMARKS

Apart from the main conjectures 1 (and its equivalent 8) and 6, here we pose further questions. Recall the section 2 that contains the auxiliary results. There we first discussed Lemmas 10 and 11, which were variants of the main conjectures

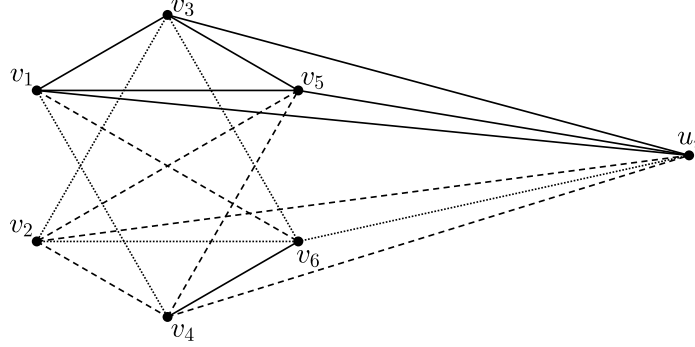


Figure 1: An example of a 3-colouring of K_n with a matching of size 3 removed that cannot be covered by two monochromatic components.

with different underlying graph instead of K_n . Recall that Lovasz-Ryser conjecture is also about different underlying graphs. Another natural question would be the following.

Question 32. *Let G be a graph, and let k be fixed. Suppose that $\chi: E(G) \rightarrow [k]$ is a k -colouring of the edges of G . For which G is it possible to find $k - 1$ monochromatically connected sets that cover the vertices of G ? What bounds on their diameter can we take?*

Notice that if there is an independent set $\{v_1, \dots, v_k\}$, then we can colour all edges from v_i by colour i , and we cannot cover the vertices with fewer than k monochromatic components. On the other hand, in the case of complete multipartite graphs and two colours, this was the only way to avoid having the graph monochromatically connected. Observe that for the case of three colours, we may have other graphs and colourings where two monochromatic components do not suffice. Consider the following example.

Pick $n + 6$ vertices labelled as v_1, v_2, \dots, v_6 and u_1, u_2, \dots, u_n . Define the graph G to be the complete graph on these vertices with three edges v_1v_2, v_3v_4 and v_5v_6 removed. Define the colouring $\chi: E(G) \rightarrow [3]$ as follows.

- Edges of colour 1 are $v_1v_3, v_3v_5, v_1v_5, v_4v_6$ and v_1u_i, v_3u_i, v_5u_i for all i .
- Edges of colour 2 are $v_2v_4, v_2v_5, v_4v_5, v_1v_6$ and v_2u_i, v_4u_i for all i .
- Edges of colour 3 are $v_2v_3, v_2v_6, v_3v_6, v_1v_4$ and v_6u_i for all i .
- Edges of the form u_iu_j are coloured arbitrarily.

It is easy to check that this colouring has no covering of vertices by two monochromatic components. Is this essentially the only way the conjecture might fail for such a graph?

Question 33. Let $G = K_n \setminus \{e_1, e_2, e_3\}$ be the complete graph with a matching of size three omitted. Suppose that $\chi: E(G) \rightarrow [3]$ is a 3-colouring of the edges such that no two monochromatic components cover G . Is such a colouring isomorphic to an example similar to the one above? What about K_{2n} with a perfect matching removed?

Finally, recall that the one of the main contributions in the final bound in Theorem 7 came from Lemma 13 and that in general the Ramsey approach of Lemma 12 would give much worse value. It would be interesting to study the right bounds for this problem as well.

Question 34. For fixed l , what is the maximum size of a set of vertices S of G_l such that $G_l[S]$ is a path? What about other families of graphs of bounded degree? In particular, for fixed l and d , what is the maximum size of a set of vertices S of G_l such that $G_l[S]$ is a connected graph of degrees bounded by d ?

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