

THE FRACTIONAL K -METRIC DIMENSION OF GRAPHS

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Let G be a graph with vertex set $V(G)$. For any two distinct vertices x and y of G , let $R\{x, y\}$ denote the set of vertices z such that the distance from x to z is not equal to the distance from y to z in G . For a function g defined on $V(G)$ and for $U \subseteq V(G)$, let $g(U) = \sum_{s \in U} g(s)$. Let $\kappa(G) = \min\{|R\{x, y\}| : x \neq y \text{ and } x, y \in V(G)\}$. For any real number $k \in [1, \kappa(G)]$, a real-valued function $g : V(G) \rightarrow [0, 1]$ is a k -resolving function of G if $g(R\{x, y\}) \geq k$ for any two distinct vertices $x, y \in V(G)$. The fractional k -metric dimension, $\dim_f^k(G)$, of G is $\min\{g(V(G)) : g \text{ is a } k\text{-resolving function of } G\}$. In this paper, we initiate the study of the fractional k -metric dimension of graphs. For a connected graph G and $k \in [1, \kappa(G)]$, it's easy to see that $k \leq \dim_f^k(G) \leq \frac{k|V(G)|}{\kappa(G)}$; we characterize graphs G satisfying $\dim_f^k(G) = k$ and $\dim_f^k(G) = |V(G)|$, respectively. We show that $\dim_f^k(G) \geq k \dim_f(G)$ for any $k \in [1, \kappa(G)]$, and we give an example showing that $\dim_f^k(G) - k \dim_f(G)$ can be arbitrarily large for some $k \in (1, \kappa(G)]$; we also describe a condition for which $\dim_f^k(G) = k \dim_f(G)$ holds. We determine the fractional k -metric dimension for some classes of graphs, and conclude with two open problems, including whether $\phi(k) = \dim_f^k(G)$ is a continuous function of k on every connected graph G .

1. Introduction

Let G be a finite, simple, undirected, and connected graph with vertex set $V(G)$ and edge set $E(G)$. For $v \in V(G)$, the open neighborhood of v is $N(v) = \{u \in$

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$V(G) : uv \in E(G)\}$, and the *closed neighborhood* of v is $N[v] = N(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$, denoted by $\deg(v)$, is $|N(v)|$; a *leaf* is a vertex of degree one, and a *major vertex* is a vertex of degree at least three. The *distance* between two vertices $x, y \in V(G)$, denoted by $d(x, y)$, is the length of a shortest path between x and y in G . The *diameter*, $\text{diam}(G)$, of a graph G is $\max\{d(x, y) : x, y \in V(G)\}$. The *complement* of G , denoted by \overline{G} , is the graph whose vertex set is $V(G)$ and $xy \in E(\overline{G})$ if and only if $xy \notin E(G)$ for $x, y \in V(G)$. We denote by K_n and P_n the complete graph and the path on n vertices, respectively.

For two distinct vertices $x, y \in V(G)$, let $R\{x, y\} = \{z \in V(G) : d(x, z) \neq d(y, z)\}$. A subset $S \subseteq V(G)$ is called a *resolving set* of G if $|S \cap R\{x, y\}| \geq 1$ for any two distinct vertices x and y in G . The *metric dimension*, $\text{dim}(G)$, of G is the minimum cardinality of S over all resolving sets of G . Since metric dimension is suggestive of the dimension of a vector space in linear algebra, sometimes a minimum resolving set of G is called a *basis* of G . The concept of metric dimension was introduced independently by Slater [28], and by Harary and Melter [20]. Applications of metric dimension can be found in network discovery and verification [7], robot navigation [24], sonar [28], combinatorial optimization [27], chemistry [25], and strategies for the mastermind game [9]. It was noted in [19] that determining the metric dimension of a graph is an NP-hard problem. Metric dimension has been extensively studied. For a survey on metric dimension in graphs, see [4, 8]. The effect of the deletion of a vertex or of an edge on the metric dimension of a graph was raised as a fundamental question in graph theory in [8]; the question is essentially settled in [11].

If a minimum number of requisite robots are installed in a network to identify the exact location of an intruder in the network, one malfunctioning robot can lead to failure of detection. Thus, it is natural to build a certain level of redundancy into the detection system. As a generalization of metric dimension, k -metric dimension was introduced first by Estrada-Moreno et al. [12] and, independently, by Adar and Epstein [1] soon afterwards. Let $\kappa(G) = \min\{|R\{x, y\}| : x \neq y \text{ and } x, y \in V(G)\}$. For a positive integer $k \in \{1, 2, \dots, \kappa(G)\}$, a set $S \subseteq V(G)$ is called a k -*resolving set* of G if $|S \cap R\{x, y\}| \geq k$ for any two distinct vertices x and y in G . The k -*metric dimension*, $\text{dim}^k(G)$ ¹, of G is the minimum cardinality over all k -resolving sets of G . It was shown in [12] that k -metric dimension of a connected graph G exists for every $k \in \{1, 2, \dots, \kappa(G)\}$, and G is called $\kappa(G)$ -metric dimensional. For an application of k -metric dimension to error-correcting codes, see [5]. For other articles on the k -metric dimension of graphs, see [6, 13, 14].

The fractionalization of various graph parameters has been extensively studied (see [26]). Currie and Oellermann [10] defined fractional metric dimension as the optimal solution to a linear programming problem, by relaxing a condition of the integer programming problem for metric dimension. A formulation of fractional metric dimension as a linear programming problem can be found in [15]. Arumugam and Mathew [2] officially studied the fractional metric dimension of graphs.

¹In fact, the notation of this parameter has been $\text{dim}_k(G)$ in previous works. However, we rather prefer to use $\text{dim}^k(G)$ here, to facilitate the notation of $\text{dim}_f^k(G)$.

For a function g defined on $V(G)$ and for $U \subseteq V(G)$, let $g(U) = \sum_{s \in U} g(s)$. A real-valued function $g : V(G) \rightarrow [0, 1]$ is a *resolving function* of G if $g(R\{x, y\}) \geq 1$ for any two distinct vertices $x, y \in V(G)$. The *fractional metric dimension*, $\dim_f(G)$, of G is $\min\{g(V(G)) : g \text{ is a resolving function of } G\}$. Notice that $\dim_f(G)$ reduces to $\dim(G)$, if the codomain of resolving functions is restricted to $\{0, 1\}$. For more articles on the fractional metric dimension, as well as the closely related fractional strong metric dimension, of graphs, see [3, 16, 17, 18, 21, 22, 23, 29].

Now, we introduce fractional k -metric dimension, which can be viewed as a generalization of $\dim_f(G)$, as well as a fractionalization of $\dim^k(G)$. For any real number $k \in [1, \kappa(G)]$, a real-valued function $h : V(G) \rightarrow [0, 1]$ is a *k -resolving function* of G if $h(R\{x, y\}) \geq k$ for any distinct vertices $x, y \in V(G)$. The *fractional k -metric dimension*, $\dim_f^k(G)$, of G is $\min\{h(V(G)) : h \text{ is a } k\text{-resolving function of } G\}$. Note $\dim_f^1(G) = \dim_f(G)$ and that $\dim_f^k(G)$ reduces to $\dim^k(G)$ when the codomain of k -resolving functions is restricted to $\{0, 1\}$ and $k \in [1, \kappa(G)]$ is restricted to positive integers.

Next, we recall some results on the fractional metric dimension of graphs. One can easily see that, for any connected graph G of order at least two, $1 \leq \dim_f(G) \leq \frac{|V(G)|}{2}$ (see [2]). For the characterization of graphs G achieving the lower bound, see Theorem 3(a). Regarding the characterization of graphs G achieving the upper bound, the following result is stated in [2] and a correct proof is provided in [21].

Theorem 1. [2, 21] *Let G be a connected graph of order at least two. Then $\dim_f(G) = \frac{|V(G)|}{2}$ if and only if there exists a bijection $\alpha : V(G) \rightarrow V(G)$ such that $\alpha(v) \neq v$ and $|R\{v, \alpha(v)\}| = 2$ for all $v \in V(G)$.*

An explicit characterization of graphs G satisfying $\dim_f(G) = \frac{|V(G)|}{2}$ is given in [3]. We recall the following construction from [3]. Let $\mathcal{K} = \{K_n : n \geq 2\}$ and $\overline{\mathcal{K}} = \{\overline{K}_n : n \geq 2\}$. Let $H[\mathcal{K} \cup \overline{\mathcal{K}}]$ be the family of graphs obtained from a connected graph H by (i) replacing each vertex $u_i \in V(H)$ by a graph $H_i \in \mathcal{K} \cup \overline{\mathcal{K}}$, and (ii) each vertex in H_i is adjacent to each vertex in H_j if and only if $u_i u_j \in E(H)$.

Theorem 2. [3] *Let G be a connected graph of order at least two. Then $\dim_f(G) = \frac{|V(G)|}{2}$ if and only if $G \in H[\mathcal{K} \cup \overline{\mathcal{K}}]$ for some connected graph H .*

Now, we recall the fractional metric dimension of some classes of graphs. We begin by recalling some terminologies. Fix a graph G . A leaf u is called a *terminal vertex* of a major vertex v if $d(u, v) < d(u, w)$ for every other major vertex w . The terminal degree, $ter_G(v)$, of a major vertex v is the number of terminal vertices of v . A major vertex v is an *exterior major vertex* if it has positive terminal degree. Let $ex(G)$ denote the number of exterior major vertices of G , $ex_a(G)$ the number of exterior major vertices u with $ter_G(u) = a$, and $\sigma(G)$ the number of leaves of G .

Theorem 3.

- (a) [23] *For any graph G of order $n \geq 2$, $\dim_f(G) = 1$ if and only if $G \cong P_n$.*
- (b) [29] *For a tree T , $\dim_f(T) = \frac{1}{2}(\sigma(T) - ex_1(T))$.*

- (c) [2] For the Petersen graph \mathcal{P} , $\dim_f(\mathcal{P}) = \frac{5}{3}$.
- (d) [2] For an n -cycle C_n , $\dim_f(C_n) = \begin{cases} \frac{n}{n-2} & \text{if } n \text{ is even,} \\ \frac{n}{n-1} & \text{if } n \text{ is odd.} \end{cases}$
- (e) [2] For the wheel graph W_n of order $n \geq 5$, $\dim_f(W_n) = \begin{cases} 2 & \text{if } n = 5, \\ \frac{3}{2} & \text{if } n = 6, \\ \frac{n-1}{4} & \text{if } n \geq 7. \end{cases}$
- (f) [23] If B_m is a bouquet of m cycles with a cut-vertex (i.e., the vertex sum of m cycles at one common vertex), where $m \geq 2$, then $\dim_f(B_m) = m$.
- (g) [29] For $m \geq 2$, let $G = K_{a_1, a_2, \dots, a_m}$ be a complete m -partite graph of order $n = \sum_{i=1}^m a_i$, and let s be the number of partite sets of G consisting of exactly one element. Then
- $$\dim_f(G) = \begin{cases} \frac{n-1}{2} & \text{if } s = 1, \\ \frac{n}{2} & \text{otherwise.} \end{cases}$$
- (h) [2] For the grid graph $G = P_s \square P_t$ ($s, t \geq 2$), $\dim_f(G) = 2$.

In this paper, we initiate the study of the fractional k -metric dimension of graphs. For a connected graph G , let $\kappa(G) = \min\{|R\{x, y\}| : x \neq y \text{ and } x, y \in V(G)\}$. The paper is organized as follows. In section 2, we compare $\dim_f^k(G)$ with $\dim_f(G)$ for certain k . We prove that $\dim_f^k(G) \geq k \dim_f(G)$ for any $k \in [1, \kappa(G)]$; we describe a condition for which $\dim_f^k(G) = k \dim_f(G)$ holds for all $k \in [1, \kappa(G)]$. For $k \in [1, \kappa(G)]$, we show that $k \leq \dim_f^k(G) \leq \frac{k}{\kappa(G)} |V(G)|$, which implies $k \leq \dim_f^k(G) \leq |V(G)|$; we characterize graphs G satisfying $\dim_f^k(G) = k$ and $\dim_f^k(G) = |V(G)|$, respectively. We also show an example such that two non-isomorphic graphs H_1 and H_2 satisfy $\dim_f^k(H_1) = \dim_f^k(H_2)$ for all $k \in [1, \kappa]$, where $\kappa(H_1) = \kappa(H_2) = \kappa$. In section 3, for $k \in [1, \kappa(G)]$, we determine the fractional k -metric dimension of trees, cycles, wheel graphs, the Petersen graph, a bouquet of cycles (i.e., the vertex sum of cycles at one common vertex), complete multi-partite graphs, and grid graphs (i.e., the Cartesian product of two paths). Along the way, we give an example showing that $\dim_f^k(G) - k \dim_f(G)$ can be arbitrarily large for some $k \in (1, \kappa(G))$. We conclude with some open problems.

2. Some general results on fractional k -metric dimension

We begin with some observations. Two distinct vertices $x, y \in V(G)$ are called *twin vertices* if $N(x) - \{y\} = N(y) - \{x\}$.

Observation 4. *Let G be a connected graph and let $k \in [1, \kappa(G)]$. If two distinct vertices x and y are twin vertices in G , then $R\{x, y\} = \{x, y\}$, and thus $\kappa(G) = 2$ and $g(x) + g(y) \geq k$ for any k -resolving function g of G .*

Observation 5. *Let G be a connected graph.*

- (a) For $k \in [1, \kappa(G)]$, $\dim_f(G) \leq \dim_f^k(G)$.

(b) For $k \in \{1, 2, \dots, \kappa(G)\}$, $\dim_f^k(G) \leq \dim^k(G)$.

(c) [2, 12] For $k \in \{1, 2, \dots, \kappa(G)\}$, $\dim_f(G) \leq \dim(G) \leq \dim^k(G)$.

Observation 5 provides inequalities between any two graph parameters among $\dim_f(G)$, $\dim(G)$, $\dim_f^k(G)$ for $k \in [1, \kappa(G)]$, and $\dim^k(G)$ for $k \in \{1, 2, \dots, \kappa(G)\}$, excluding the relation between $\dim(G)$ and $\dim_f^k(G)$. So, it is natural to compare $\dim(G)$ and $\dim_f^k(G)$ for $k \in [1, \kappa(G)]$.

Remark 6. (a) The value of $\dim_f^k(G) - \dim(G)$ can be arbitrarily large, as G varies, for some $k \in (1, \kappa(G)]$. For $n \geq 4$, note that $\dim(P_n) = 1 \leq \dim_f^k(P_n)$ for $k \in [1, n-1]$ and $\dim_f^{n-1}(P_n) = n$ (see Proposition 15). Thus, $\dim_f^{n-1}(P_n) - \dim(P_n) = n-1$ can be arbitrarily large.

(b) The value of $\dim(G) - \dim_f^k(G)$ can be arbitrarily large, as G varies, for some $k \in [1, \kappa(G)]$. For $n \geq 3$, note that $\dim(K_n) = n-1$ and $\dim_f^k(K_n) = \frac{kn}{2}$ for $k \in [1, 2]$ (see Proposition 26). Now, let $k \in [1, 2)$; then $\dim(K_n) - \dim_f^k(K_n) = \frac{(2-k)n}{2} - 1$ becomes arbitrarily large as $n \rightarrow \infty$.

In light of Observation 5(b), we have the following

Theorem 7. The value of $\dim^k(G) - \dim_f^k(G)$ can be arbitrarily large, as G varies, for some $k \in \{1, 2, \dots, \kappa(G)\}$.

Proof. Let H be a connected graph with vertex set $V(H) = \{u_1, u_2, \dots, u_n\}$, where $n \geq 2$. Let G be the graph obtained from H as follows:

- (i) for each $i \in \{1, 2, \dots, n\}$, add three vertices $a_{i,1}, b_{i,1}, c_{i,1}$ and three edges $u_i a_{i,1}, u_i b_{i,1}, u_i c_{i,1}$;
- (ii) for each $i \in \{1, 2, \dots, n\}$, subdivide the edge $u_i a_{i,1}$ ($u_i b_{i,1}$ and $u_i c_{i,1}$, respectively) exactly $s-1$ times so that the edge $u_i a_{i,1}$ ($u_i b_{i,1}$ and $u_i c_{i,1}$, respectively) in (i) becomes the $u_i - a_{i,1}$ path given by $u_i, a_{i,s}, a_{i,s-1}, \dots, a_{i,1}$ (the $u_i - b_{i,1}$ path given by $u_i, b_{i,s}, b_{i,s-1}, \dots, b_{i,1}$ and the $u_i - c_{i,1}$ path given by $u_i, c_{i,s}, c_{i,s-1}, \dots, c_{i,1}$, respectively).

For each $i \in \{1, 2, \dots, n\}$, let T_i be the subtree of G consisting of the $u_i - a_{i,1}$ path, the $u_i - b_{i,1}$ path, and the $u_i - c_{i,1}$ path; further, let $P^{i,a}$ be the $a_{i,s} - a_{i,1}$ path, $P^{i,b}$ the $b_{i,s} - b_{i,1}$ path, and $P^{i,c}$ the $c_{i,s} - c_{i,1}$ path. Then, for each $i \in \{1, 2, \dots, n\}$,

$$(1) \quad \begin{cases} R\{a_{i,s}, b_{i,s}\} = V(P^{i,a}) \cup V(P^{i,b}), \\ R\{a_{i,s}, c_{i,s}\} = V(P^{i,a}) \cup V(P^{i,c}), \\ R\{b_{i,s}, c_{i,s}\} = V(P^{i,b}) \cup V(P^{i,c}). \end{cases}$$

We determine $\kappa(G)$, $\dim^k(G)$ for $k \in \{1, 2, \dots, \kappa(G)\}$, and $\dim_f^k(G)$ for $k \in [1, \kappa(G)]$.

Claim 1: $\kappa(G) = 2s$.

Proof of Claim 1. Let x and y be two distinct vertices of G . First, let $x, y \in V(T_i)$ for some $i \in \{1, 2, \dots, n\}$. If $d(u_i, x) \neq d(u_i, y)$, then $R\{x, y\} \supseteq V(T_i)$ with $|R\{x, y\}| \geq$

$|V(T_j)| = 3s+1$, where $j \in \{1, 2, \dots, n\} - \{i\}$. If $d(u_i, x) = d(u_i, y)$, say $x \in V(P^{i,a})$ and $y \in V(P^{i,b})$ without loss of generality, then $R\{x, y\} = V(P^{i,a}) \cup V(P^{i,b})$ with $|R\{x, y\}| = |V(P^{i,a})| + |V(P^{i,b})| = 2s$. Second, let $x \in V(T_i)$ and $y \in V(T_j)$ for distinct $i, j \in \{1, 2, \dots, n\}$, say x lies on the $u_i - a_{i,1}$ path and y lies on the $u_j - a_{j,1}$ path, without loss of generality. Then $|R\{x, y\}| \geq 2s$, since at most one vertex lying on a $a_{i,1} - a_{j,1}$ geodesic is at equal distance from both x and y . So, $\kappa(G) = 2s$. \square

Claim 2: For $k \in \{1, 2, \dots, 2s\}$, $\dim^k(G) = \begin{cases} \frac{3kn}{2} & \text{if } k \text{ is even,} \\ \frac{(3k+1)n}{2} & \text{if } k \text{ is odd.} \end{cases}$

Proof of Claim 2. Let $k \in \{1, 2, \dots, 2s\}$. First, we show that $\dim^k(G) \geq \frac{3kn}{2}$ for an even k , and $\dim^k(G) \geq \frac{(3k+1)n}{2}$ for an odd k . Let S be a minimum k -resolving set of G . For each $i \in \{1, 2, \dots, n\}$, (1) implies

$$(2) \quad \begin{cases} |S \cap R\{a_{i,s}, b_{i,s}\}| = |S \cap V(P^{i,a})| + |S \cap V(P^{i,b})| \geq k, \\ |S \cap R\{a_{i,s}, c_{i,s}\}| = |S \cap V(P^{i,a})| + |S \cap V(P^{i,c})| \geq k, \\ |S \cap R\{b_{i,s}, c_{i,s}\}| = |S \cap V(P^{i,b})| + |S \cap V(P^{i,c})| \geq k. \end{cases}$$

Suppose k is even. By summing over the three inequalities in (2), we obtain $|S \cap V(P^{i,a})| + |S \cap V(P^{i,b})| + |S \cap V(P^{i,c})| \geq \frac{3k}{2}$, and thus $|S \cap V(T_i)| \geq \frac{3k}{2}$ for each $i \in \{1, 2, \dots, n\}$. So, $|S \cap V(G)| \geq \sum_{i=1}^n \frac{3k}{2} = \frac{3kn}{2}$, and hence $\dim^k(G) \geq \frac{3kn}{2}$. Now, suppose k is odd. If $|S \cap V(P^{i,a})| \leq \lfloor \frac{k}{2} \rfloor$ and $|S \cap V(P^{i,b})| \leq \lfloor \frac{k}{2} \rfloor$ for some $i \in \{1, 2, \dots, n\}$, then $|S \cap R\{a_{i,s}, b_{i,s}\}| = |S \cap V(P^{i,a})| + |S \cap V(P^{i,b})| \leq \lfloor \frac{k}{2} \rfloor + \lfloor \frac{k}{2} \rfloor = k - 1$, contradicting the assumption that S is a k -resolving set of G ; thus, $|S \cap V(P^{i,t})| \leq \lfloor \frac{k}{2} \rfloor$ for at most one $t \in \{a, b, c\}$ for each $i \in \{1, 2, \dots, n\}$. Let $|S \cap V(P^{i,c})| = \alpha \leq \min\{|S \cap V(P^{i,a})|, |S \cap V(P^{i,b})|\}$ for $i \in \{1, 2, \dots, n\}$. Then $|S \cap V(P^{i,a})| \geq k - \alpha$ and $|S \cap V(P^{i,b})| \geq k - \alpha$ from (2), and thus $|S \cap V(T_i)| \geq 2k - \alpha \geq 2k - \lfloor \frac{k}{2} \rfloor = \frac{3k+1}{2}$. So, $|S \cap V(G)| \geq \sum_{i=1}^n \frac{3k+1}{2} = \frac{(3k+1)n}{2}$, and hence $\dim^k(G) \geq \frac{(3k+1)n}{2}$.

Second, we show that $\dim^k(G) \leq \frac{3kn}{2}$ for an even k , and $\dim^k(G) \leq \frac{(3k+1)n}{2}$ for an odd k . If k is even, let $W_0 = \cup_{i=1}^n (\{a_{i,1}, a_{i,2}, \dots, a_{i, \frac{k}{2}}\} \cup \{b_{i,1}, b_{i,2}, \dots, b_{i, \frac{k}{2}}\} \cup \{c_{i,1}, c_{i,2}, \dots, c_{i, \frac{k}{2}}\})$. If k is odd, let $W_1 = \cup_{i=1}^n (\{a_{i,1}, \dots, a_{i, \lceil \frac{k}{2} \rceil}\} \cup \{b_{i,1}, \dots, b_{i, \lceil \frac{k}{2} \rceil}\} \cup \{c_{i,1}, \dots, c_{i, \lfloor \frac{k}{2} \rfloor}\})$. Note that $|W_0| = \frac{3kn}{2}$, $|W_1| = \frac{(3k+1)n}{2}$, $|W_0 \cap V(P^{i,a})| = |W_0 \cap V(P^{i,b})| = |W_0 \cap V(P^{i,c})| = \frac{k}{2}$ and $|W_1 \cap V(P^{i,a})| = |W_1 \cap V(P^{i,b})| = \lceil \frac{k}{2} \rceil = 1 + |W_1 \cap V(P^{i,c})|$ for $i \in \{1, 2, \dots, n\}$. It suffices to show that W_0 (W_1 , respectively) is a k -resolving set of G when k is even (odd, respectively). Let x and y be distinct vertices of G . Suppose $x, y \in V(T_i)$ for some $i \in \{1, 2, \dots, n\}$. If $d(u_i, x) \neq d(u_i, y)$, then $|W_\ell \cap R\{x, y\}| \geq |V(T_j)| \geq \frac{3k}{2} \geq k$ for $j \neq i$ and for $\ell \in \{0, 1\}$. If $d(u_i, x) = d(u_i, y)$, say $x \in V(P^{i,a})$ and $y \in V(P^{i,c})$ (other cases can be handled similarly), then $|W_\ell \cap R\{x, y\}| = |W_\ell \cap (V(P^{i,a}) \cup V(P^{i,c}))| \geq k$ for $\ell \in \{0, 1\}$. Now, let $x \in V(T_i)$ and $y \in V(T_j)$ for distinct $i, j \in \{1, 2, \dots, n\}$; suppose $d(u_i, x) \leq d(u_j, y)$, without loss of generality. Since $R\{x, y\} \supseteq V(T_i)$, $|W_\ell \cap R\{x, y\}| \geq |W_\ell \cap V(T_i)| \geq \frac{3k}{2}$ for $\ell \in \{0, 1\}$. So, W_0 (W_1 , respectively) is a k -resolving set of G when k is even (odd, respectively). \square

Claim 3: For $k \in [1, 2s]$, $\dim_f^k(G) = \frac{3kn}{2}$.

Proof of Claim 3. Let $k \in [1, 2s]$. First, we show that $\dim_f^k(G) \geq \frac{3kn}{2}$. Let $g : V(G) \rightarrow [0, 1]$ be any k -resolving function of G . From (1), for each $i \in \{1, 2, \dots, n\}$, we have $g(R\{a_{i,s}, b_{i,s}\}) = g(V(P^{i,a})) + g(V(P^{i,b})) \geq k$, $g(R\{a_{i,s}, c_{i,s}\}) = g(V(P^{i,a})) + g(V(P^{i,c})) \geq k$, and $g(R\{b_{i,s}, c_{i,s}\}) = g(V(P^{i,b})) + g(V(P^{i,c})) \geq k$. By summing over the three inequalities, we obtain $g(V(P^{i,a})) + g(V(P^{i,b})) + g(V(P^{i,c})) \geq \frac{3k}{2}$, and thus $g(V(T_i)) \geq \frac{3k}{2}$ for each $i \in \{1, 2, \dots, n\}$. So, $g(V(G)) \geq \sum_{i=1}^n \frac{3k}{2} = \frac{3kn}{2}$, and hence $\dim_f^k(G) \geq \frac{3kn}{2}$.

Second, we show that $\dim_f^k(G) \leq \frac{3kn}{2}$. Let $h : V(G) \rightarrow [0, 1]$ be a function defined by

$$h(v) = \begin{cases} 0 & \text{if } \deg(v) \geq 3, \\ \frac{k}{2s} & \text{if } \deg(v) \leq 2. \end{cases}$$

Notice that $h(V(P^{i,a})) = h(V(P^{i,b})) = h(V(P^{i,c})) = \frac{k}{2}$ for each $i \in \{1, 2, \dots, n\}$, and $h(V(G)) = \frac{3kn}{2}$. It suffices to show that h is a k -resolving function of G . Let x and y be distinct vertices of G . Suppose $x, y \in V(T_i)$ for some $i \in \{1, 2, \dots, n\}$. If $d(u_i, x) \neq d(u_i, y)$, then $h(R\{x, y\}) \geq h(V(T_j)) \geq \frac{3k}{2}$ for $j \neq i$. If $d(u_i, x) = d(u_i, y)$, say $x \in V(P^{i,a})$ and $y \in V(P^{i,b})$ without loss of generality, then $h(R\{x, y\}) = h(V(P^{i,a})) + h(V(P^{i,b})) = k$. Now, let $x \in V(T_i)$ and $y \in V(T_j)$ for distinct $i, j \in \{1, 2, \dots, n\}$; suppose $d(u_i, x) \leq d(u_j, y)$, without loss of generality. Then $h(R\{x, y\}) \geq h(V(T_i)) \geq \frac{3k}{2}$. So, h is a k -resolving function of G . \square

By Claims 2 and 3, we see that, for each odd $k \in \{1, 2, \dots, 2s\}$, $\dim_f^k(G) - \dim_f^k(G) = \frac{(3k+1)n}{2} - \frac{3kn}{2} = \frac{n}{2}$, which can be arbitrarily large. \square

Next, we compare the fractional metric dimension and the fractional k -metric dimension of graphs.

Lemma 8. For any connected graph G and for any $k \in [1, \kappa(G)]$, $\dim_f^k(G) \geq k \dim_f(G)$.

Proof. Let $g : V(G) \rightarrow [0, 1]$ be a minimum k -resolving function of G . Then $g(R\{x, y\}) \geq k$ for any two distinct vertices $x, y \in V(G)$. Now, let $h : V(G) \rightarrow [0, 1]$ be a function defined by $h(u) = \frac{1}{k}g(u)$ for each $u \in V(G)$. Then $h(R\{x, y\}) = \frac{1}{k}g(R\{x, y\}) \geq 1$ for any distinct vertices $x, y \in V(G)$; thus, h is a resolving function of G . So, $h(V(G)) = \frac{1}{k} \dim_f^k(G) \geq \dim_f(G)$, i.e., $\dim_f^k(G) \geq k \dim_f(G)$. \square

Next, we examine the conditions for which $\dim_f^k(G) = k \dim_f(G)$ holds.

Lemma 9. Let G be a connected graph and let $k \in [1, \kappa(G)]$. If there exists a minimum resolving function $g : V(G) \rightarrow [0, 1]$ such that $g(v) \leq \frac{1}{k}$ for each $v \in V(G)$, then $\dim_f^k(G) = k \dim_f(G)$ for any $k \in [1, \kappa(G)]$.

Proof. Let $g : V(G) \rightarrow [0, 1]$ be a minimum resolving function of G satisfying $g(v) \leq \frac{1}{k}$ for each $v \in V(G)$. Let $h : V(G) \rightarrow [0, 1]$ be a function defined by $h(v) = kg(v)$ for each $v \in V(G)$. Then h is a k -resolving function of G : (i) for

each $v \in V(G)$, $0 \leq h(v) = kg(v) \leq 1$; (ii) for any two distinct $x, y \in V(G)$, $h(R\{x, y\}) = kg(R\{x, y\}) \geq k$, since $g(R\{x, y\}) \geq 1$ by the assumption that g is a resolving function of G . So, $\dim_f^k(G) \leq h(V(G)) = kg(V(G)) = k \dim_f(G)$. Since $\dim_f^k(G) \geq k \dim_f(G)$ by Lemma 8, $\dim_f^k(G) = k \dim_f(G)$. \square

Next, we obtain the lower and upper bounds of $\dim_f^k(G)$ in terms of k , $\kappa(G)$, and the order of G .

Proposition 10. *Let G be a connected graph of order n . For any $k \in [1, \kappa(G)]$, $k \leq \dim_f^k(G) \leq \frac{kn}{\kappa(G)}$, where both bounds are sharp.*

Proof. The lower bound is trivial. For the upper bound, let $g : V(G) \rightarrow [0, 1]$ be a function such that $g(v) = \frac{k}{\kappa(G)}$ for each $v \in V(G)$. Since $\kappa(G) = \min\{|R\{x, y\}| : x \neq y \text{ and } x, y \in V(G)\}$, $g(R\{u, v\}) \geq k$ for any distinct vertices $u, v \in V(G)$. So, g is a k -resolving function of G , and hence $\dim_f^k(G) \leq g(V(G)) = \sum_{i=1}^n \frac{k}{\kappa(G)} = \frac{kn}{\kappa(G)}$.

For the sharpness of the lower bound, see Proposition 12(a); for the sharpness of the upper bound, we refer to Proposition 21. \square

As an immediate consequence of Proposition 10, we have the following.

Corollary 11. *For a connected graph G of order n and for $k \in [1, \kappa(G)]$, $k \leq \dim_f^k(G) \leq n$.*

Next, we characterize graphs G achieving the lower bound and the upper bound, respectively, of Corollary 11. Let $\mathcal{R}_\kappa(G) = \bigcup_{x, y \in V(G), x \neq y, |R\{x, y\}| = \kappa} R\{x, y\}$, where $\kappa = \kappa(G)$.

Proposition 12. *For a connected graph G of order $n \geq 2$ and for $k \in [1, \kappa(G)]$,*

- (a) $\dim_f^k(G) = k$ if and only if $G \cong P_n$ and $k \in [1, 2]$,
- (b) $\dim_f^k(G) = n$ if and only if $k = \kappa(G) = \kappa$ and $V(G) = \mathcal{R}_\kappa(G)$.

Proof. **(a)** (\Leftarrow) If $G \cong P_n$ and $k \in [1, 2]$, then $\dim_f^k(G) = k$ by Proposition 15.

(\Rightarrow) Suppose that $\dim_f^k(G) = k$. Since $\dim_f^k(G) \geq k \dim_f(G) \geq k$ by Lemma 8, $\dim_f^k(G) = k$ implies $\dim_f(G) = 1$; thus $G \cong P_n$ by Theorem 3(a) and $k \in [1, 2]$ from Proposition 15.

(b) (\Leftarrow) Let $k = \kappa(G) = \kappa$ and $V(G) = \mathcal{R}_\kappa(G)$. Then, for any vertex $v \in V(G)$, there exist two distinct vertices $x, y \in V(G)$ such that $v \in R\{x, y\}$ with $|R\{x, y\}| = \kappa$. Since any κ -resolving function g of G must satisfy $g(R\{x, y\}) \geq \kappa$ and $g(v) \leq 1$ for each $v \in V(G)$, $g(u) = 1$ for each $u \in R\{x, y\}$ with $|R\{x, y\}| = \kappa$. Since $V(G) = \mathcal{R}_\kappa(G)$, $g(v) = 1$ for each $v \in V(G)$. Thus, $g(V(G)) = n$ and hence $\dim_f^k(G) = n$.

(\Rightarrow) Let $\dim_f^k(G) = n$. By Proposition 10, $k = \kappa$. Suppose that $\mathcal{R}_\kappa(G) \subsetneq V(G)$ and let $w \in V(G) - \mathcal{R}_\kappa(G)$. Then, for any vertex $w' \in V(G) - \{w\}$, $|R\{w, w'\}| \geq \kappa(G) + 1$. If $h : V(G) \rightarrow [0, 1]$ is a function defined by $h(w) = 0$ and $h(v) = 1$ for each $v \in V(G) - \{w\}$, then h is a κ -resolving function of G with $h(V(G)) = n - 1$, which contradicts the assumption that $\dim_f^k(G) = n$. \square

Remark 13. Let $G \in H[\mathcal{K} \cup \overline{\mathcal{K}}]$ for some connected graph H , as described in Theorem 2. Then $\kappa(G) = 2$ and $V(G) = \mathcal{R}_\kappa(G)$; thus $\dim_f^2(G) = |V(G)|$ by Proposition 12(b). More generally, $\dim_f^k(G) = k \dim_f(G) = \frac{k}{2}|V(G)|$ for $k \in [1, 2]$ by Lemma 9, since $g : V(G) \rightarrow [0, 1]$ defined by $g(u) = \frac{1}{2}$, for each $u \in V(G)$, forms a minimum resolving function of G .

We conclude this section with an example showing that two non-isomorphic graphs can have the same κ and identical k -fractional metric dimension for all $k \in [1, \kappa]$.

Remark 14. There exist non-isomorphic graphs H_1 and H_2 such that $\dim_f^k(H_1) = \dim_f^k(H_2)$ for all $k \in [1, \kappa]$, where $\kappa = \kappa(H_1) = \kappa(H_2)$. For example, let $H_1 \cong K_{2,2,3}$ and $H_2 \cong K_{3,4}$; then $\dim_f(H_1) = \dim_f(H_2) = \frac{7}{2}$ by Theorem 3(g). Since both H_1 and H_2 have twin vertices, $\kappa(H_1) = \kappa(H_2) = 2$. Also note that a function $g_i : V(H_i) \rightarrow [0, 1]$ defined by $g_i(u) = \frac{1}{2}$ for each $u \in V(H_i)$ forms a minimum resolving function for H_i , where $i \in \{1, 2\}$. By Lemma 9, $\dim_f^k(H_1) = \dim_f^k(H_2) = \frac{7}{2}k$ for every $k \in [1, \kappa]$, whereas $H_1 \not\cong H_2$.

3. The fractional k -metric dimension of some graphs

In this section, we determine $\dim_f^k(G)$ for $k \in [1, \kappa(G)]$ when G is a tree, a cycle, a wheel graph, the Petersen graph, a bouquet of cycles, a complete multipartite graph, or a grid graph (the Cartesian product of two paths). Along the way, we provide an example showing that $\dim_f^k(G) - k \dim_f(G)$ can be arbitrarily large for some $k \in (1, \kappa(G)]$. First, we determine $\dim_f^k(G)$ when G is a path.

Proposition 15. Let P_n be an n -path, where $n \geq 2$. Then $\dim_f^k(P_2) = k$ for $k \in [1, 2]$ and, for $n \geq 3$,

$$\dim_f^k(P_n) = \begin{cases} k & \text{if } k \in [1, 2], \\ 2 + (k - 2)\frac{n-2}{n-3} & \text{if } k \in (2, n - 1]. \end{cases}$$

Proof. Let P_n be an n -path given by u_1, u_2, \dots, u_n , where $n \geq 2$; then $\kappa(P_2) = 2$ and $\kappa(P_n) = n - 1$ for $n \geq 3$. Since a function h defined on $V(P_2)$ by $h(u_1) = h(u_2) = \frac{1}{2}$ is a minimum resolving function of P_2 , $\dim_f^k(P_2) = k \dim_f(P_2) = k$ for $k \in [1, 2]$ by Theorem 3(a) and Lemma 9. So, let $n \geq 3$ and we consider two cases.

Case 1: $k \in [1, 2]$. If $g : V(P_n) \rightarrow [0, 1]$ is a function defined by $g(u_1) = g(u_n) = \frac{1}{2}$ and $g(u_i) = 0$ for each $i \in \{2, \dots, n - 1\}$, then g is a minimum resolving

function of P_n : (i) for any two distinct vertices $x, y \in V(P_n)$, $R\{x, y\} \supseteq \{u_1, u_n\}$, and hence $g(R\{x, y\}) \geq g(u_1) + g(u_n) = \frac{1}{2} + \frac{1}{2} = 1$; (ii) $g(V(P_n)) = 1 = \dim_f(P_n)$ by Theorem 3(a). Since $g(u_i) \leq \frac{1}{2} \leq \frac{1}{k}$ for each $i \in \{1, 2, \dots, n\}$, we have $\dim_f^k(P_n) = k \dim_f(P_n) = k$ for any $k \in [1, 2]$ by Lemma 9 and Theorem 3(a).

Case 2: $k \in (2, n-1]$. Note that $n \geq 4$ in this case since $\kappa(P_3) = 2$. Let $h : V(P_n) \rightarrow [0, 1]$ be a k -resolving function of P_n . Let $h(u_1) + h(u_n) = b$; then $0 \leq b \leq 2$. Since $R\{u_i, u_{i+2}\} = V(P_n) - \{u_{i+1}\}$ for $i \in \{1, 2, \dots, n-2\}$, $h(R\{u_i, u_{i+2}\}) = h(V(P_n)) - h(u_{i+1}) \geq k$ for each $i \in \{1, 2, \dots, n-2\}$. By summing over the $(n-2)$ inequalities, we have $(n-2)h(V(P_n)) - \sum_{j=2}^{n-1} h(u_j) \geq (n-2)k$, i.e., $(n-3)h(V(P_n)) + h(u_1) + h(u_n) \geq (n-2)k$. So, $h(V(P_n)) \geq \frac{(n-2)k-b}{n-3}$ since $n > 3$; note that the minimum of $h(V(P_n))$ is $\frac{(n-2)k-2}{n-3}$ when $b = h(u_1) + h(u_n)$ takes the maximum value 2. Thus, $\dim_f^k(P_n) \geq \frac{(n-2)k-2}{n-3} = 2 + (k-2)\frac{n-2}{n-3}$.

Now, let $g : V(P_n) \rightarrow [0, 1]$ be a function defined by

$$g(u_i) = \begin{cases} 1 & \text{if } i \in \{1, n\}, \\ \frac{k-2}{n-3} & \text{otherwise.} \end{cases}$$

Then g is a k -resolving function of P_n : (i) $0 < \frac{k-2}{n-3} \leq 1$ for any $k \in (2, n-1]$; (ii) for any distinct $i, j \in \{1, 2, \dots, n\}$, $g(R\{u_i, u_j\}) \geq 2 + (n-3)\frac{k-2}{n-3} = k$ since $R\{u_i, u_j\} \supseteq \{u_1, u_n\}$ and $|R\{u_i, u_j\}| \geq n-1$. So, $\dim_f^k(P_n) \leq g(V(P_n)) = 2 + (n-2)\frac{k-2}{n-3}$.

Therefore, $\dim_f^k(P_n) = 2 + (k-2)\frac{n-2}{n-3}$ for $k \in (2, n-1]$. \square

Second, we determine $\dim_f^k(T)$ when T is a tree that is not a path. Let $M(T)$ be the set of exterior major vertices of a tree T . Let $M_1(T) = \{w \in M(T) : \text{ter}_T(w) = 1\}$, $M_2(T) = \{w \in M(T) : \text{ter}_T(w) = 2\}$, and $M_3(T) = \{w \in M(T) : \text{ter}_T(w) \geq 3\}$; note that $M(T) = M_1(T) \cup M_2(T) \cup M_3(T)$. For any vertex $v \in M(T)$, let T_v be the subtree of T induced by v and all vertices belonging to the paths joining v with its terminal vertices. Let $M^*(T) = M_2(T) \cup M_3(T)$. We recall the following result on $\kappa(T)$.

Theorem 16. [12] *For a tree T that is not a path,*

$$\kappa(T) = \min \bigcup_{w \in M^*(T)} \{d(\ell_i, \ell_j) : \ell_i \text{ and } \ell_j \text{ are two distinct terminal vertices of } w\}.$$

We begin by examining $\dim_f^k(T)$ when T is a tree with exactly one exterior major vertex.

Proposition 17. *Let T be a tree with $ex(T) = 1$. Let v be the exterior major vertex of T and let $\ell_1, \ell_2, \dots, \ell_a$ be the terminal vertices of v in T (note that $a \geq 3$). Suppose that $d(v, \ell_1) \leq d(v, \ell_2) \leq \dots \leq d(v, \ell_a)$. Then $\kappa(T) = d(v, \ell_2)$, and*

(a) *if $d(v, \ell_1) = d(v, \ell_2)$, then $\dim_f^k(T) = k \dim_f(T) = \frac{ka}{2}$ for $k \in [1, \kappa(T)]$;*

(b) if $d(v, \ell_1) < d(v, \ell_2)$, then

$$\dim_f^k(T) = \begin{cases} \frac{ka}{2} & \text{for } k \in [1, 2d(v, \ell_1)], \\ (a-1)k - (a-2)d(v, \ell_1) & \text{for } k \in (2d(v, \ell_1), \kappa(T)]. \end{cases}$$

Proof. For each $i \in \{1, 2, \dots, a\}$, let s_i be the neighbor of v lying on the $v - \ell_i$ path, and let P^i denote the $s_i - \ell_i$ path in T . By Theorem 16, $\kappa(T) = d(\ell_1, \ell_2)$. Let $k \in [1, \kappa(T)]$.

(a) Let $d(v, \ell_1) = d(v, \ell_2)$. By Lemma 8, $\dim_f^k(T) \geq k \dim_f(T)$. We will show that $\dim_f^k(T) \leq k \dim_f(T)$. Let $g : V(T) \rightarrow [0, 1]$ be a function defined by

$$(3) \quad g(u) = \begin{cases} 0 & \text{if } u = v, \\ \frac{k}{2d(v, \ell_i)} & \text{for each vertex } u \in V(P^i), \text{ where } i \in \{1, 2, \dots, a\}. \end{cases}$$

Note that, for each $i \in \{1, 2, \dots, a\}$, (i) $g(V(P^i)) = \frac{k}{2}$; (ii) $0 \leq \frac{k}{2d(v, \ell_i)} \leq 1$ since $k \leq \kappa(T) = d(\ell_1, \ell_2) = 2d(v, \ell_1) \leq 2d(v, \ell_i)$. If two distinct vertices x and y lie on the $v - \ell_i$ path for some $i \in \{1, 2, \dots, a\}$, then $g(R\{x, y\}) \geq g(V(T) - V(P^i)) \geq k$ since $a \geq 3$. For distinct $i, j \in \{1, 2, \dots, a\}$, if $x \in V(P^i)$ and $y \in V(P^j)$ with $d(v, x) \neq d(v, y)$, say $d(v, x) < d(v, y)$ without loss of generality, then $g(R\{x, y\}) \geq g(V(T) - V(P^j)) \geq k$; if $x \in V(P^i)$ and $y \in V(P^j)$ with $d(v, x) = d(v, y)$, then $g(R\{x, y\}) = g(V(P^i) \cup V(P^j)) = k$. So, g is a k -resolving function of T with $g(V(T)) = \frac{ka}{2}$. Thus $\dim_f^k(T) \leq \frac{ka}{2} = k \dim_f(T)$ by Theorem 3(b). Therefore, $\dim_f^k(T) = k \dim_f(T) = \frac{ka}{2}$ for $k \in [1, \kappa(T)]$.

(b) Let $d(v, \ell_1) < d(v, \ell_2)$, and we consider two cases.

Case 1: $k \in [1, 2d(v, \ell_1)]$. In this case, the function g in (3) is a k -resolving function of T as shown in the proof for (a); thus, $\dim_f^k(T) \leq g(V(T)) = \frac{ka}{2} = k \dim_f(T)$. Since $\dim_f^k(T) \geq k \dim_f(T)$ by Lemma 8, $\dim_f^k(T) = k \dim_f(T) = \frac{ka}{2}$.

Case 2: $k \in (2d(v, \ell_1), \kappa(T)]$. Let $h : V(T) \rightarrow [0, 1]$ be a minimum k -resolving function of T . Note that (i) $h(V(P^1)) \leq d(v, \ell_1)$ since $h(u) \leq 1$ for each $u \in V(T)$; (ii) for distinct $i, j \in \{1, 2, \dots, a\}$, $h(V(P^i)) + h(V(P^j)) \geq k$ since $R\{s_i, s_j\} = V(P^i) \cup V(P^j)$. Let $h(V(P^1)) = \beta$. From $h(V(P^1)) + h(V(P^j)) \geq k$ for each $j \in \{2, 3, \dots, a\}$, $h(V(P^j)) \geq k - \beta$. So, $h(V(T)) \geq h(V(P^1)) + \sum_{i=2}^a h(V(P^i)) \geq \beta + (a-1)(k - \beta) = (a-1)k - (a-2)\beta \geq (a-1)k - (a-2)d(v, \ell_1)$ since $\beta \leq d(v, \ell_1)$; thus $\dim_f^k(T) \geq (a-1)k - (a-2)d(v, \ell_1)$.

Next, we show that $\dim_f^k(T) \leq (a-1)k - (a-2)d(v, \ell_1)$. Let $g : V(T) \rightarrow [0, 1]$ be a function defined by

$$g(u) = \begin{cases} 0 & \text{if } u = v, \\ 1 & \text{for each vertex } u \in V(P^1), \\ \frac{k-d(v, \ell_1)}{d(v, \ell_j)} & \text{for each vertex } u \in V(P^j), \text{ where } j \in \{2, \dots, a\}. \end{cases}$$

Note that (i) $g(V(P^1)) = d(v, \ell_1)$; (ii) for $j \in \{2, 3, \dots, a\}$, $g(V(P^j)) = k - d(v, \ell_1) > k - \frac{k}{2} = \frac{k}{2}$ since $k > 2d(v, \ell_1)$; (iii) $g(V(T)) = d(v, \ell_1) + (a-1)(k - d(v, \ell_1)) = (a-1)k - (a-2)d(v, \ell_1)$. Also note that g is a k -resolving function of

T : (i) $0 \leq \frac{d(v, \ell_1)}{d(v, \ell_j)} \leq \frac{k-d(v, \ell_1)}{d(v, \ell_j)} \leq \frac{d(\ell_1, \ell_2)-d(v, \ell_1)}{d(v, \ell_j)} = \frac{d(v, \ell_2)}{d(v, \ell_j)} \leq 1$ for $j \in \{2, \dots, a\}$; (ii) if two distinct vertices x and y lie on the $v - \ell_i$ path for some $i \in \{1, 2, \dots, a\}$, then $g(R\{x, y\}) \geq g(V(T)) - g(V(P^i)) \geq \min\{k, 2(k - d(v, \ell_1))\} = k$ since $a \geq 3$ and $k - d(v, \ell_1) > \frac{k}{2}$; (iii) if $x \in V(P^i)$ and $y \in V(P^j)$ with $d(v, x) \neq d(v, y)$ for distinct $i, j \in \{1, 2, \dots, a\}$, then $g(R\{x, y\}) \geq \min\{g(V(T) - V(P^i)), g(V(T) - V(P^j))\} \geq k$, since at most one vertex in the $\ell_i - \ell_j$ path can be at equal distance from both x and y in T ; (iv) if $x \in V(P^i)$ and $y \in V(P^j)$ with $d(v, x) = d(v, y)$ for distinct $i, j \in \{1, 2, \dots, a\}$, then $g(R\{x, y\}) = g(V(P^i)) + g(V(P^j)) \geq \min\{k, 2(k - d(v, \ell_1))\} = k$. Thus, $\dim_f^k(T) \leq g(V(T)) = (a - 1)k - (a - 2)d(v, \ell_1)$.

Therefore, $\dim_f^k(T) = (a - 1)k - (a - 2)d(v, \ell_1)$. \square

Next, we determine $\dim_f^k(T)$ for a tree T with $ex(T) \geq 1$. We begin with the following lemma, which, besides being useful for Theorem 19, bears independent interest.

Lemma 18. *Let T be a tree with $ex(T) \geq 2$. For $w \in M^*(T)$, let $x \in V(T_w)$ and $y \in V(T) - V(T_w)$. Then either $R\{x, y\} \supseteq V(T_w)$ or $R\{x, y\} \supseteq V(T_{w'})$ for some $w' \in M^*(T) - \{w\}$.*

Proof. Since $ex(T) \geq 2$, $|M^*(T)| \geq 2$. Let $x \in V(T_w)$ for some $w \in M^*(T)$. First, suppose $y \in V(T_{w'})$ for some $w' \in M^*(T) - \{w\}$. Assume, for contradiction, that there exist $u \in V(T_w)$ and $v \in V(T_{w'})$ such that

$$(4) \quad d(x, u) = d(y, u) \text{ and } d(x, v) = d(y, v).$$

Put $a = d(y, v)$, $b = d(v, w')$, $c = d(w', w)$, $d = d(w, u)$, $e = d(u, x)$. Let us call a path leading from w to any of its leaves a “ w -terminal path”. We may assume that u and x lie in the same w -terminal path, since $d(x, u) = d(y, u)$ implies $d(x, w) = d(y, w)$ if u and x lie in distinct w -terminal paths. Likewise, we assume v and y lie in the same w' -terminal path. After writing the two equations (4) in terms of components a, b, c, d, e and simplifying, we obtain $b + c + d = 0$. This means that $b = c = d = 0$, since all variables denote (nonnegative) distances. In particular, the distinctness of w and w' is contradicted by $c = 0$.

Now, suppose $y \notin V(T_z)$ for any $z \in M^*(T)$; then either $y \in V(T_{y'})$ for some $y' \in M_1(T)$ or $y \notin V(T_{z'})$ for any $z' \in M(T)$. Note that there exists a vertex $w' \in M^*(T) - \{w\}$ such that either y or y' lies in the $w - w'$ path in T . Since $d(s, x) = d(s, y)$ for $s \in V(T_{w'})$ is equivalent to $d(w', x) = d(w', y)$, we may assume, for contradiction, that

$$(5) \quad d(x, w') = d(y, w') \text{ and } d(x, t) = d(y, t), \text{ where } t \in V(T_w).$$

Put $d(y, w') = a + b$ and $d(w, w') = c + b$, where $b \geq 0$ denotes the length of the path shared between the $w - w'$ path and the $y - w'$ path. Similarly, put $d(w, t) = d$ and $d(x, t) = e$. As before, we may assume that x and t lie in the same w -terminal path. After simplifying the two equations (5) in terms of a, b, c, d, e , we obtain $c = d = 0$. This implies that either $y \in V(T_w)$ (when $b > 0$) or $w = w'$ (when $b = 0$); both possibilities contradict the present assumptions. \square

Theorem 19. *Let T be a tree with $ex(T) \geq 1$. Then $\kappa(T) = \min\{\kappa(T_v) : v \in M^*(T)\}$ and, for $k \in [1, \kappa(T)]$,*

$$(6) \quad \dim_f^k(T) = k|M_2(T)| + \sum_{v \in M_3(T)} \dim_f^k(T_v).$$

Proof. If $ex(T) = 1$, then $|M_2(T)| = 0$ and $|M_3(T)| = 1$, and so (6) trivially holds; see Proposition 17 for explicit formulas for $\kappa(T)$ and $\dim_f^k(T)$. So, let $ex(T) \geq 2$; then $|M^*(T)| \geq 2$. By Theorem 16, $\kappa(T) = \min\{\kappa(T_v) : v \in M^*(T)\}$. Let $k \in [1, \kappa(T)]$; notice that $\kappa(T) \leq \kappa(T_z)$ for any $z \in M^*(T)$.

First, we show that $\dim_f^k(T) \geq k|M_2(T)| + \sum_{v \in M_3(T)} \dim_f^k(T_v)$. For $v \in M_2(T)$, let $N(v) \cap V(T_v) = \{r_1, r_2\}$. For $w \in M_3(T)$, let $\ell_1, \ell_2, \dots, \ell_\sigma$ be the terminal vertices of w with $ter_T(w) = \sigma$, and let s_i be the neighbor of w lying on the $w - \ell_i$ path, where $i \in \{1, 2, \dots, \sigma\}$; further, let P^i denote the $s_i - \ell_i$ path. Let $g : V(T) \rightarrow [0, 1]$ be a minimum k -resolving function of T . If $M_2(T) \neq \emptyset$, then, for any $v \in M_2(T)$, $R\{r_1, r_2\} = V(T_v) - \{v\}$ and $g(R\{r_1, r_2\}) = g(V(T_v) - \{v\}) \geq k$; thus $\sum_{v \in M_2(T)} g(V(T_v)) \geq \sum_{v \in M_2(T)} k = k|M_2(T)|$. If $M_3(T) \neq \emptyset$, then, for any $w \in M_3(T)$, notice $R\{s_i, s_j\} = V(P^i) \cup V(P^j)$ for any distinct $i, j \in \{1, 2, \dots, \sigma\}$. This, together with the argument used in the proof of Proposition 17, we have $\sum_{w \in M_3(T)} g(V(T_w)) \geq \sum_{w \in M_3(T)} \dim_f^k(T_w)$. Thus, we have

$$g(V(T)) \geq \sum_{v \in M_2(T)} g(V(T_v)) + \sum_{w \in M_3(T)} g(V(T_w)) \geq k|M_2(T)| + \sum_{w \in M_3(T)} \dim_f^k(T_w).$$

Next, we show that $\dim_f^k(T) \leq k|M_2(T)| + \sum_{v \in M_3(T)} \dim_f^k(T_v)$. For each $w \in M_3(T)$, let g_w be a minimum k -resolving function on $V(T_w)$. For each $w \in M_2(T)$, define a function h_w on $V(T_w)$ such that $h_w(w) = 0$ and $h_w(u) = \frac{k}{|V(T_w)|-1}$ if $u \neq w$. For $k \in [1, \kappa(T)]$, let $g : V(T) \rightarrow [0, 1]$ be the function defined by

$$g(u) = \begin{cases} g_w(u) & \text{if } u \in V(T_w) \text{ for } w \in M_3(T), \\ h_w(u) & \text{if } u \in V(T_w) \text{ for } w \in M_2(T), \\ 0 & \text{otherwise.} \end{cases}$$

Note that (i) for each $w \in M_2(T)$, $g(V(T_w)) = h_w(V(T_w) - \{w\}) = k$; (ii) for each $w \in M_3(T)$, $g(V(T_w)) = g_w(V(T_w)) = \dim_f^k(T_w) \geq k$; (iii) $g(V(T)) = k|M_2(T)| + \sum_{w \in M_3(T)} \dim_f^k(T_w)$. It suffices to show g is a k -resolving function of T . Obviously, $0 \leq g(u) \leq 1$ for each $u \in V(T)$. So, let x and y be distinct vertices of T ; we will show that $g(R\{x, y\}) \geq k$. Consider three cases: (1) there is a $w \in M^*(T)$ such that $\{x, y\} \subseteq V(T_w)$; (2) there is a $w \in M^*(T)$ such that $x \in V(T_w)$ and $y \notin V(T_w)$; (3) $\{x, y\} \subseteq V(T) - \cup_{w \in M^*(T)} V(T_w)$. In case (1), if $w \in M_3(T)$, then $g(R\{x, y\}) \geq g_w(R\{x, y\} \cap V(T_w)) \geq k$, since g_w is a k -resolving function on $V(T_w)$; if $w \in M_2(T)$ and $d(x, w) \neq d(y, w)$, then there is a $z \in M^*(T) - \{w\}$ such that $g(R\{x, y\}) \geq g(V(T_z)) \geq k$; if $w \in M_2(T)$ and $d(x, w) = d(y, w)$, then $g(R\{x, y\}) = h_w(V(T_w) - \{w\}) = k$. In case (2), by Lemma 18, either $R\{x, y\} \supseteq V(T_w)$ or $R\{x, y\} \supseteq V(T_{w'})$ for some $w' \in M^*(T) - \{w\}$; thus $g(R\{x, y\}) \geq k$. So,

we consider case (3). Note that $x \in V(T_{x'})$ for some $x' \in M_1(T)$ or $x \notin V(T_z)$ for any $z \in M(T)$; similarly, $y \in V(T_{y'})$ for some $y' \in M_1(T)$ or $y \notin V(T_{z'})$ for any $z' \in M(T)$. If $\{x, y\} \subseteq V(T_v)$ for some $v \in M_1(T)$, then $d(v, x) \neq d(v, y)$ and there exist distinct $v', v'' \in M^*(T)$ such that v lies on the $v' - v''$ path in T ; thus $g(R\{x, y\}) \geq g(V(T_{v'})) + g(V(T_{v''})) \geq 2k$. If $\{x, y\} \not\subseteq V(T_v)$ for any $v \in M(T)$, there exist distinct $w_1, w_2 \in M^*(T)$ such that both x (or x') and y (or y') lie on the $w_1 - w_2$ path in T ; then $d(w_1, x) = d(w_1, y)$ and $d(w_2, x) = d(w_2, y)$ imply either $x = y$ or $\{x, y\} \subseteq V(T_s)$ for some $s \in M_1(T)$, where both possibilities contradict the present assumptions. Thus $R\{x, y\} \supseteq V(T_{w_i})$ for at least one $i \in \{1, 2\}$, and $g(R\{x, y\}) \geq g(V(T_{w_i})) \geq k$. \square

Next, we provide an example showing that $\dim_f^k(G) - k \dim_f(G)$ can be arbitrarily large for some $k \in (1, \kappa(G)]$.

Remark 20. *The value of $\dim_f^k(G) - k \dim_f(G)$ can be arbitrarily large, as G varies, for some $k \in (1, \kappa(G)]$. Let T be a tree with $ex(T) = 1$. Let v be the exterior major vertex of T and let $\ell_1, \ell_2, \dots, \ell_\alpha$ be the terminal vertices of T such that $d(v, \ell_1) = 1 < \beta = d(v, \ell_j)$ for each $j \in \{2, 3, \dots, \alpha\}$, where $\alpha \geq 3$. By Proposition 17, $\kappa(T) = \beta + 1$ and $\dim_f^{\beta+1}(T) = (\alpha - 1)(\beta + 1) - (\alpha - 2) = (\alpha - 1)\beta + 1$. Since $\dim_f(T) = \frac{\alpha}{2}$ by Theorem 3(b), $\dim_f^{\beta+1}(T) - (\beta + 1) \dim_f(T) = (\alpha - 1)\beta + 1 - (\beta + 1)(\frac{\alpha}{2}) = (\frac{\alpha}{2} - 1)(\beta - 1)$ can be arbitrarily large, as α or β gets big enough.*

Next, we determine the fractional k -metric dimension of cycles.

Proposition 21. *Let C_n be an n -cycle, where $n \geq 3$. Then*

$$(7) \quad \dim_f^k(C_n) = k \dim_f(C_n) = \begin{cases} \frac{kn}{n-2} & \text{if } n \text{ is even and } k \in [1, n-2], \\ \frac{kn}{n-1} & \text{if } n \text{ is odd and } k \in [1, n-1]. \end{cases}$$

Proof. Note that $\kappa(C_n) = n - 2$ for an even n , and $\kappa(C_n) = n - 1$ for an odd n . Let $k \in [1, \kappa(C_n)]$. For an even $n \geq 4$, a function $g : V(C_n) \rightarrow [0, 1]$ defined by $g(u) = \frac{1}{n-2}$, for each $u \in V(C_n)$, is a minimum resolving function of C_n : (i) $0 < g(u) = \frac{1}{n-2} \leq \frac{1}{k} \leq 1$ since $n \geq 4$; (ii) for distinct $x, y \in V(C_n)$, $|R\{x, y\}| \geq n - 2$, and thus $g(R\{x, y\}) \geq (n - 2)(\frac{1}{n-2}) = 1$; (iii) $g(V(C_n)) = \frac{n}{n-2} = \dim_f(C_n)$ by Theorem 3(d). Similarly, for an odd $n \geq 3$, one can easily check that a function $h : V(C_n) \rightarrow [0, 1]$ defined by $h(u) = \frac{1}{n-1}$, for each $u \in V(C_n)$, is a minimum resolving function of C_n satisfying $h(u) \leq \frac{1}{k}$. By Lemma 9 and Theorem 3(d), (7) follows. \square

Remark 22. *Note that, for any fixed $k \in [1, \kappa(C_n)]$, $\lim_{n \rightarrow \infty} \dim_f^k(C_n) = k$ by Proposition 21 (c.f. Proposition 12(a)).*

Next, we determine the fractional k -metric dimension of wheel graphs.

Proposition 23. *For the wheel graph W_n of order $n \geq 5$,*

$$(8) \quad \dim_f^k(W_n) = k \dim_f(W_n) = \begin{cases} 2k & \text{if } n = 5 \text{ and } k \in [1, 2], \\ \frac{3k}{2} & \text{if } n = 6 \text{ and } k \in [1, 4], \\ \frac{k(n-1)}{4} & \text{if } n \geq 7 \text{ and } k \in [1, 4]. \end{cases}$$

Proof. For $n \geq 5$, the wheel graph $W_n = C_{n-1} + K_1$ is obtained from an $(n-1)$ -cycle C_{n-1} by joining an edge from each vertex of C_{n-1} to a new vertex, say v ; let the C_{n-1} be given by $u_1, u_2, \dots, u_{n-1}, u_1$. Note that $\text{diam}(W_n) = 2$ for $n \geq 5$.

Case 1: $n = 5$. Note that $\kappa(W_5) = 2$ since $R\{u_1, u_3\} = \{u_1, u_3\}$. Let $k \in [1, 2]$. Let $g : V(W_5) \rightarrow [0, 1]$ be a function defined by $g(v) = 0$ and $g(u_i) = \frac{1}{2}$ for each $i \in \{1, 2, 3, 4\}$. Then g is a minimum resolving function of W_5 : (i) $0 \leq g(x) \leq \frac{1}{k} \leq 1$ for each $x \in V(W_5)$; (ii) for distinct $i, j \in \{1, 2, 3, 4\}$, $g(R\{u_i, u_j\}) \geq g(u_i) + g(u_j) = 1$; (iii) for $i \in \{1, 2, 3, 4\}$, $g(R\{v, u_i\}) \geq g(u_i) + g(u_\ell) = 1$, where $u_\ell \in V(W_5) - N[u_i]$; (iv) $g(V(W_5)) = 2 = \dim_f(W_5)$ by Theorem 3(e). By Lemma 9 and Theorem 3(e), $\dim_f^k(W_5) = k \dim_f(W_5) = 2k$ for $k \in [1, 2]$.

Case 2: $n \geq 6$. First, we show that $\kappa(W_n) = 4$ in this case. For each $i \in \{1, 2, \dots, n-1\}$, $R\{v, u_i\} = (V(W_n) - N(u_i)) \cup \{v\}$ with $|R\{v, u_i\}| = n-2 \geq 4$. For distinct $i, j \in \{1, 2, \dots, n-1\}$, (i) if $u_i u_j \in E(W_n)$, then $R\{u_i, u_j\} = (N(u_i) \cup N(u_j)) - \{v\}$ with $|R\{u_i, u_j\}| = 4$; (ii) if $u_i u_j \notin E(W_n)$ and $|N(u_i) \cap N(u_j)| = 1$, then $R\{u_i, u_j\} = (N[u_i] \cup N[u_j]) - \{v\}$ with $|R\{u_i, u_j\}| = 6$; (iii) if $u_i u_j \notin E(W_n)$ and $|N(u_i) \cap N(u_j)| = 2$, then $R\{u_i, u_j\} = (N[u_i] \cup N[u_j]) - (N(u_i) \cap N(u_j))$ with $|R\{u_i, u_j\}| = 4$. So, $\kappa(W_n) = 4$ for $n \geq 6$.

Second, we determine $\dim_f^k(W_n)$ for $n \geq 6$. Let $k \in [1, 4]$. For $n = 6$, a function $g : V(W_6) \rightarrow [0, 1]$ defined by $g(x) = \frac{1}{4}$, for each $x \in V(W_6)$, is a minimum resolving function of W_6 : (i) $0 < g(x) \leq \frac{1}{k} \leq 1$ for each $x \in V(W_6)$; (ii) for any distinct $x, y \in V(W_6)$, $g(R\{x, y\}) \geq 4(\frac{1}{4}) = 1$; (iii) $g(V(W_6)) = \frac{3}{2} = \dim_f(W_6)$ by Theorem 3(e). For $n \geq 7$, let $h : V(W_n) \rightarrow [0, 1]$ be a function defined by $h(v) = 0$ and $h(u_i) = \frac{1}{4}$ for each $i \in \{1, 2, \dots, n-1\}$. Then h is a minimum resolving function of W_n : (i) $0 \leq h(x) \leq \frac{1}{k} \leq 1$ for each $x \in V(W_n)$; (ii) for each $i \in \{1, 2, \dots, n-1\}$, $|R\{v, u_i\}| \geq n-2 \geq 5$ since $n \geq 7$, and hence $h(R\{v, u_i\}) \geq 4(\frac{1}{4}) = 1$; (iii) for distinct $i, j \in \{1, 2, \dots, n-1\}$, $|R\{u_i, u_j\}| \geq 4$ and $v \notin R\{u_i, u_j\}$, and thus $h(R\{u_i, u_j\}) \geq 4(\frac{1}{4}) = 1$; (iv) $h(V(W_n)) = \frac{n-1}{4} = \dim_f(W_n)$ by Theorem 3(e). Therefore, by Lemma 9 and Theorem 3(e), (8) holds for $n \geq 6$ and $k \in [1, 4]$. \square

Next, we determine the fractional k -metric dimension of the Petersen graph.

Proposition 24. *For the Petersen graph \mathcal{P} , $\dim_f^k(\mathcal{P}) = k \dim_f(\mathcal{P}) = \frac{5}{3}k$ for $k \in [1, 6]$.*

Proof. Note that \mathcal{P} is 3-regular and vertex-transitive. Since $\text{diam}(\mathcal{P}) = 2$, any two distinct vertices in \mathcal{P} are either adjacent or at distance two apart.

We first show that $\kappa(\mathcal{P}) = 6$. For any distinct $x, y \in V(\mathcal{P})$, $R\{x, y\} = N[x] \cup N[y] - (N(x) \cap N(y))$ and $|R\{x, y\}| = 6$: (i) if $xy \in E(\mathcal{P})$, then $N(x) \cap N(y) = \emptyset$ and $x \in N[y]$ and $y \in N[x]$; (ii) if $xy \notin E(\mathcal{P})$, then $|N(x) \cap N(y)| = 1$. So, $\kappa(\mathcal{P}) = 6$.

Now, let $k \in [1, 6]$. Since $\dim_f^k(\mathcal{P}) \geq k \dim_f(\mathcal{P})$ by Lemma 8, it suffices to show that $\dim_f^k(\mathcal{P}) \leq k \dim_f(\mathcal{P})$. Let $g : V(\mathcal{P}) \rightarrow [0, 1]$ be a function defined by $g(v) = \frac{k}{6}$ for each $v \in V(\mathcal{P})$. Since $0 \leq g(v) \leq 1$ for each vertex $v \in V(\mathcal{P})$ and $g(R\{x, y\}) = 6(\frac{k}{6}) \geq k$ for any two distinct $x, y \in V(\mathcal{P})$, g is a k -resolving function of \mathcal{P} . So, $\dim_f^k(\mathcal{P}) \leq |V(\mathcal{P})|(\frac{k}{6}) = \frac{10k}{6} = k(\frac{5}{3}) = k \dim_f(\mathcal{P})$ by Theorem 3(c). \square

Next, we determine the fractional k -metric dimension of a bouquet of cycles.

Proposition 25. *Let B_m be a bouquet of m cycles C^1, C^2, \dots, C^m with a cut-vertex (i.e., the vertex sum of m cycles at one common vertex), where $m \geq 2$; further, let C^1 be the cycle of the minimum length among the m cycles of B_m .*

Then $\kappa(B_m) = \begin{cases} |V(C^1)| - 1 & \text{if } C^1 \text{ is an odd cycle,} \\ |V(C^1)| - 2 & \text{if } C^1 \text{ is an even cycle,} \end{cases}$ and, for $k \in [1, \kappa(B_m)]$, $\dim_f^k(B_m) = k \dim_f(B_m) = km$.

Proof. Let v be the cut-vertex of B_m . For each $i \in \{1, 2, \dots, m\}$, let C^i be given by $v, u_{i,1}, u_{i,2}, \dots, u_{i,r_i}, v$ and let $P^i = C^i - v$; further, let $P^{i,1}$ be the $u_{i,1} - u_{i, \lceil \frac{r_i}{2} \rceil}$ geodesic. Without loss of generality, let $r_1 \leq r_2 \leq \dots \leq r_m$.

Claim 1: If C^1 is an odd cycle, then $\kappa(B_m) = r_1 = |V(C^1)| - 1$; if C^1 is an even cycle, $\kappa(B_m) = r_1 - 1 = |V(C^1)| - 2$.

Proof of Claim 1. Let x and y be distinct vertices of B_m . First, let $x, y \in V(C^i)$ for some $i \in \{1, 2, \dots, m\}$. If $d(v, x) \neq d(v, y)$, then $R\{x, y\} \supseteq V(C^j)$ with $|R\{x, y\}| \geq |V(C^j)| = r_j + 1 \geq r_1 + 1$ for $j \neq i$. If $d(v, x) = d(v, y)$ and C^i is an odd cycle, then $R\{x, y\} = V(P^i)$ with $|R\{x, y\}| = r_i \geq r_1$; notice, for an odd cycle C^1 , $|R\{u_{1,1}, u_{1,r_1}\}| = r_1$. If $d(v, x) = d(v, y)$ and C^i is an even cycle, then $R\{x, y\} = V(P^i) - \{u_{i, \lceil \frac{r_i}{2} \rceil}\}$ with $|R\{x, y\}| = r_i - 1$, where $r_i - 1 \geq r_1$ if C^1 is an odd cycle, and $r_i - 1 \geq r_1 - 1$ if C^1 is an even cycle; notice, for an even cycle C^1 , $|R\{u_{1,1}, u_{1,r_1}\}| = r_1 - 1$.

Second, let $x \in V(P^i)$ and $y \in V(P^j)$ for distinct $i, j \in \{1, 2, \dots, m\}$; let $x \in V(P^{i,1})$ and $y \in V(P^{j,1})$, without loss of generality. If $d(v, x) = d(v, y)$, then $R\{x, y\} \supseteq V(P^{i,1}) \cup V(P^{j,1})$ with $|R\{x, y\}| \geq \lceil \frac{r_i}{2} \rceil + \lceil \frac{r_j}{2} \rceil \geq r_1$. So, let $d(v, x) \neq d(v, y)$, say $d(v, x) < d(v, y)$ without loss of generality; then $d(u, x) \neq d(u, y)$ for each $u \in V(P^{i,1})$, and $r_j \geq 3$. If $r_j = 3$, then $R\{x, y\} \supseteq V(P^{i,1}) \cup V(P^j)$. If $r_j \geq 4$, then at most two vertices in C^j are at equal distance from x and y ; thus, $R\{x, y\} \supseteq V(P^{i,1}) \cup (V(P^j) - \{w_1, w_2\})$ such that $w_t \in V(P^j)$ with $d(w_t, x) = d(w_t, y)$, where $t \in \{1, 2\}$. In each case, $|R\{x, y\}| \geq \lceil \frac{r_i}{2} \rceil + \lceil \frac{r_j}{2} \rceil \geq r_1$. \square

Claim 2: For $k \in [1, \kappa(B_m)]$, $\dim_f^k(B_m) = k \dim_f(B_m) = km$.

Proof of Claim 2. Let $k \in [1, \kappa(B_m)]$, and let $h : V(B_m) \rightarrow [0, 1]$ be a function defined by

$$h(u) = \begin{cases} \frac{1}{r_i} & \text{for each } u \in V(P^i) \text{ if } C^i \text{ is an odd cycle,} \\ \frac{1}{r_j-1} & \text{for each } u \in V(P^j) - \{u_{j, \lceil \frac{r_j}{2} \rceil}\} \text{ if } C^j \text{ is an even cycle,} \\ 0 & \text{otherwise.} \end{cases}$$

Note that, for each $i \in \{1, 2, \dots, m\}$, $h(V(P^i)) = 1$ if C^i is an odd cycle, and $h(V(P^i)) = h(V(P^i)) - h(u_{i, \lceil \frac{r_i}{2} \rceil}) = 1$ if C^i is an even cycle. We also note that h is a minimum resolving function of B_m : (i) $0 \leq h(u) \leq \frac{1}{k} \leq 1$ for each $u \in V(B_m)$; (ii) if $x, y \in V(C^i)$ with $d(v, x) \neq d(v, y)$, for some $i \in \{1, 2, \dots, m\}$, then $h(R\{x, y\}) \geq h(V(C^j)) \geq 1$ for $j \neq i$; (iii) if $x, y \in V(C^i)$ with $d(v, x) = d(v, y)$ and $x \neq y$, for some $i \in \{1, 2, \dots, m\}$, then $h(R\{x, y\}) = h(V(P^i)) = 1$ when

C^i is an odd cycle, and $h(R\{x, y\}) = h(V(P^i)) - h(u_{i, \lceil \frac{r_i}{2} \rceil}) = 1$ when C^i is an even cycle; (iv) if $x \in V(P^i)$ and $y \in V(P^j)$ for distinct $i, j \in \{1, 2, \dots, m\}$, then $h(R\{x, y\}) \geq \frac{1}{2}h(V(P^i)) + \frac{1}{2}h(V(P^j)) \geq 1$ using a similar argument used in the proof of Claim 1; (v) $h(V(B_m)) = m = \dim_f(B_m)$ by Theorem 3(f). So, by Lemma 9 and Theorem 3(f), $\dim_f^k(B_m) = k \dim_f(B_m) = km$ for $k \in [1, \kappa(B_m)]$. \square \square

Next, we determine the fractional k -metric dimension of complete multipartite graphs.

Proposition 26. *For $m \geq 2$, let $G = K_{a_1, a_2, \dots, a_m}$ be a complete m -partite graph of order $n = \sum_{i=1}^m a_i \geq 3$, and let s be the number of partite sets of G consisting of exactly one element. Then, for $k \in [1, 2]$,*

$$\dim_f^k(G) = k \dim_f(G) = \begin{cases} \frac{k(n-1)}{2} & \text{if } s = 1, \\ \frac{kn}{2} & \text{otherwise.} \end{cases}$$

Proof. Let $V(G)$ be partitioned into m -partite sets V_1, V_2, \dots, V_m with $|V_i| = a_i$, where $i \in \{1, 2, \dots, m\}$. Without loss of generality, let $a_1 \leq a_2 \leq \dots \leq a_m$. Note that $\kappa(G) = 2$: (i) if $a_m \geq 2$, then, for two distinct $x, y \in V_m$, $R\{x, y\} = \{x, y\}$; (ii) if $a_m = 1$, then, for $x \in V_1$ and $y \in V_2$, $R\{x, y\} = \{x, y\}$. Let $k \in [1, 2]$.

First, let $s \neq 1$. A function $g : V(G) \rightarrow [0, 1]$ defined by $g(v) = \frac{1}{2}$, for each $v \in V(G)$, is a minimum resolving function of G : (i) $0 < g(v) \leq \frac{1}{k} \leq 1$ for each $v \in V(G)$; (ii) for any distinct vertices $x, y \in V(G)$, $g(R\{x, y\}) \geq g(x) + g(y) = 1$; (iii) $g(V(G)) = \frac{n}{2} = \dim_f(G)$ by Theorem 3(g). So, $\dim_f^k(G) = k \dim_f(G) = \frac{kn}{2}$ by Lemma 9 and Theorem 3(g).

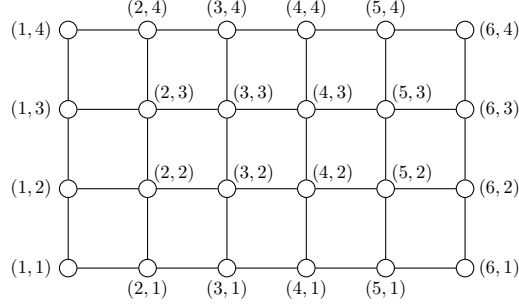
Second, let $s = 1$. Let $h : V(G) \rightarrow [0, 1]$ be a function defined by $h(u) = 0$ for $u \in V_1$ and $h(v) = \frac{1}{2}$ for each $v \in V(G) - V_1$. Then h is a minimum resolving function of G : (i) $0 \leq h(v) \leq \frac{1}{k} \leq 1$ for each $v \in V(G)$; (ii) for any two distinct vertices $x, y \in V(G) - V_1$, $h(R\{x, y\}) \geq h(x) + h(y) = 1$; (iii) for $x \in V_1$ and $y \in V_i \subseteq V(G) - V_1$, $h(R\{x, y\}) \geq h(V_i) \geq 1$, where $i \in \{2, \dots, m\}$; (iv) $h(V(G)) = \frac{n-1}{2} = \dim_f(G)$ by Theorem 3(g). So, $\dim_f^k(G) = k \dim_f(G) = \frac{k(n-1)}{2}$ for $k \in [1, 2]$ by Lemma 9 and Theorem 3(g). \square

Now, we consider the fractional k -metric dimension of grid graphs (i.e., the Cartesian product of two paths). The *Cartesian product* of two graphs G and H , denoted by $G \square H$, is the graph with the vertex set $V(G) \times V(H)$ such that (u, w) is adjacent to (u', w') if and only if either $u = u'$ and $ww' \in E(H)$, or $w = w'$ and $uu' \in E(G)$. See Figure 1 for the labeling of $P_6 \square P_4$.

We recall the following result.

Theorem 27. [5] *For $s, t \geq 2$, $\kappa(P_s \square P_t) = s + t - 2$ and $\dim^k(P_s \square P_t) = 2k$, where $k \in \{1, 2, \dots, s + t - 2\}$.*

Proposition 28. *For $k \in [1, s + t - 2]$, $\dim_f^k(P_s \square P_t) = k \dim_f(P_s \square P_t) = 2k$, where $s, t \geq 2$.*

Figure 1: Labeling of $P_6 \square P_4$.

Proof. Let $s \geq t \geq 2$, and let $G = P_s \square P_t$ and $L = \{v \in V(G) : 2 \leq \deg(v) \leq 3\}$. By Theorem 27, $\kappa(G) = s + t - 2$. Let $k \in [1, s + t - 2]$. Since $\dim_f^k(G) \geq k \dim_f(G) = 2k$ by Lemma 8 and Theorem 3(h), it suffices to show that $\dim_f^k(G) \leq 2k$. Let $g : V(G) \rightarrow [0, 1]$ be a function defined by

$$g(v) = \begin{cases} \frac{k}{s+t-2} & \text{if } v \in L, \\ 0 & \text{otherwise.} \end{cases}$$

Note that $g(V(G)) = 2k$. We will show that g is a k -resolving function of G . Clearly, $0 \leq g(v) \leq 1$ for each $v \in V(G)$. Let $x = (a, b)$ and $y = (c, d)$ be two distinct vertices of G . We consider two cases.

Case 1: $a = c$ or $b = d$. If $a = c$, then $R\{x, y\} \cap L \supseteq \cup_{i=1}^s \{(i, 1), (i, t)\}$, and thus $g(R\{x, y\}) \geq (2s) \left(\frac{k}{s+t-2}\right) \geq k$ since $s \geq t \geq 2$. So, let $b = d$; without loss of generality, let $a < c$. Let $z = (\alpha, \beta) \in L$. Note that (i) if $\alpha \leq a$, then $d(z, x) < d(z, y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{j=1}^t \{(1, j)\} \cup (\cup_{i=2}^a \{(i, 1), (i, t)\})$; (ii) if $\alpha \geq c$, then $d(z, x) > d(z, y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{j=1}^t \{(s, j)\} \cup (\cup_{i=c}^{s-1} \{(i, 1), (i, t)\})$; (iii) if $a < \alpha < c$ and $\beta = 1$, then there exists at most one such $z \in L$ satisfying $d(z, x) = d(z, y)$, since $d(z, x) = d(z, y)$ implies $\alpha - a + b - 1 = c - \alpha + b - 1$, i.e., $2\alpha = a + c$; similarly, if $a < \alpha < c$ and $\beta = t$, then there exists at most one such $z \in L$ satisfying $d(z, x) = d(z, y)$. Thus, $|R\{x, y\} \cap L| \geq 2s + 2t - 6$, and hence $g(R\{x, y\}) \geq (2s + 2t - 6) \left(\frac{k}{s+t-2}\right) = 2k \left(\frac{s+t-3}{s+t-2}\right) \geq k$, since $s + t \geq 4$.

Case 2: $a \neq c$ and $b \neq d$. Without loss of generality, let $a < c$; further, assume that $b < d$ (the case for $b > d$ can be handled similarly). Let $z' = (\alpha', \beta') \in L$. Note that (i) if $\alpha' \leq a$ and $\beta' = 1$, then $d(z', x) < d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=1}^a \{(i, 1)\}$; (ii) if $\alpha' \geq c$ and $\beta' = t$, then $d(z', x) > d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{i=c}^s \{(i, t)\}$; (iii) if $a < \alpha' < c$ (i.e., $c \neq a + 1$) and $\beta' = 1$, then there exists at most one such $z' \in L$ satisfying $d(z', x) = d(z', y)$, since $d(z', x) = d(z', y)$ implies $\alpha' - a + b - 1 = c - \alpha' + d - 1$, i.e., $2\alpha' = a - b + c + d$; similarly, if $a < \alpha' < c$ and $\beta' = t$, there exists at most one such $z' \in L$ satisfying $d(z', x) = d(z', y)$. Likewise, we note that (i) if $\alpha' = 1$ and $\beta' \leq b$, then $d(z', x) < d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{j=1}^b \{(1, j)\}$; (ii) if $\alpha' = s$ and $\beta' \geq d$, then $d(z', x) > d(z', y)$ and thus $R\{x, y\} \cap L \supseteq \cup_{j=d}^t \{(s, j)\}$; (iii) if $\alpha' = 1$ and $b < \beta' < d$ (i.e., $d \neq b + 1$), then there

exists at most one such $z' \in L$ satisfying $d(z', x) = d(z', y)$, since $d(z', x) = d(z', y)$ implies $a-1+\beta'-b = c-1+d-\beta'$, i.e., $2\beta' = -a+b+c+d$; similarly, if $\alpha' = s$ and $b < \beta' < d$, then there exists at most one such $z' \in L$ satisfying $d(z', x) = d(z', y)$. So, if $c = a+1$ or $d = b+1$, then $|R\{x, y\} \cap L| \geq s+t-2$; if $c \geq a+2$ and $d \geq b+2$, then $|R\{x, y\} \cap L| \geq a+(s-c+1)+2(c-a-2)+b+(t-d+1)+2(d-b-2) \geq s+t-2$. In each case, $g(R\{x, y\}) \geq (s+t-2)(\frac{k}{s+t-2}) = k$.

Thus, in each case, g is a k -resolving function of G , and hence $\dim_f^k(G) \leq g(V(G)) = 2k$. Therefore, $\dim_f^k(G) = k \dim_f(G) = 2k$ for $k \in [1, s+t-2]$ for $s \geq t \geq 2$. \square

4. Open Problems

We conclude this paper with two open problems.

Problem 1. Let $\phi(k) = \dim_f^k(G)$ be a function of k , for a fixed G , on domain $[1, \kappa(G)]$. Is ϕ a continuous function of k on every connected graph G ?

Problem 2. Suppose $\dim_f^k(G)$ is given by $\psi(k)$ for integral values of k . When and how can we interpolate ψ and deduce $\dim_f^k(G)$ for any real number $k \in [1, \kappa(G)]$?

For example, let $G = P_s \square P_t$, where $s, t \geq 2$. Then $\dim_f^k(G) = 2k$ for integers $k \in \{1, 2, \dots, \kappa(G)\}$ by Theorems 3(h), Lemma 8, Observation 5(b), and Theorem 27. In Proposition 28, we proved that $\dim_f^k(G) = 2k$ for any real number $k \in [1, \kappa(G)]$, by using Lemma 8 and constructing a k -resolving function g on $V(G)$ with $g(V(G)) = 2k$ for $k \in [1, \kappa(G)]$. The construction of k -resolving function for any real number $k \in [1, \kappa(G)]$ in determining $\dim_f^k(G)$ in Proposition 28 does not appear to carry to the construction of k -resolving set for any integral values $k \in \{1, 2, \dots, \kappa(G)\}$ in determining $\dim^k(G)$, and vice versa.

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