Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **x** (xxxx), xxx–xxx.
https://doi.org/10.2298/AADM170716024G

Maximally and super connected multisplit graphs and digraphs

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Fault tolerance is especially important for interconnection networks, since the growing size of networks increases their vulnerability to component failures. A classical measure for the fault tolerance of a network in the case of vertex failures is its connectivities. A network based on a graph $G = (X_1 \cup X_2 \cup \cdots \cup X_k, I, E)$ is called a k-multisplit network, if its vertex set V can be partitioned into $k+1$ stable sets I, X_1, $X_2, \cdots, X_k$ such that $X_1 \cup X_2 \cup \cdots \cup X_k$ induces a complete k-partite graph and I is an independent set.

In this note, we first show that: for any non-complete connected k-multisplit graph $G = (X_1 \cup X_2 \cup \cdots \cup X_k, I, E)$ with $k \geq 3$ and $|X_1| \leq |X_2| \leq \cdots \leq |X_k|$, each of the following holds

(1) If $|X_1 \cup X_2 \cup \cdots \cup X_{k-1}| \geq \delta$, then $\kappa(G) = \delta(G)$.

(2) If $|X_1 \cup X_2 \cup \cdots \cup X_{k-1}| < \delta$, then $\kappa(G) \geq |X_1 \cup X_2 \cup \cdots \cup X_{k-1}|$.

(3) $\lambda(G) = \delta(G)$.

(4) If $|X_1 \cup X_2 \cup \cdots \cup X_{k-1}| > \delta$ with respect to $|X_1| \geq 2$ and $\delta \geq 2$, then G is super-κ.

(5) G is super-λ.

In addition, we present sufficient conditions for digraphs to be maximally edge-connected and super-edge connected in terms of the zeroth-order general Randić index of digraphs.

1. INTRODUCTION

Interconnection networks is an important research area for parallel and distributed computer systems. Network reliability is one of the most significant factors

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2010 Mathematics Subject Classification. 05C12, 05C05.
Keywords and Phrases. Maximally edge-connected, super-edge-connected, k-multisplit graphs, digraphs, the zeroth-order general Randić index.

in designing the topology of an interconnection network. Fault tolerance is an important property of network performance. A classic measure of the fault tolerance of a network is the connectivity and edge-connectivity of the corresponding graph. In general, the larger connectivity/edge-connectivity is, the more reliable the network is.

A network can be conveniently modelled as a graph or digraph. In the note, we mainly explore sufficient conditions for k-multisplit graphs and digraphs to be super-edge-connected and maximally edge-connected, respectively. We begin with some notations and concepts for graphs and digraphs.

1.1 Notations for graphs

Let $G = (V, E)$ be a connected graph. For any vertex $x \in V$, we use $N(x)$ to denote the set of neighbors of x in G and the number $|N(x)|$ is said to be the **degree** of x. The minimum value of these numbers is the **minimum degree** of G, denoted by $\delta = \delta(G)$. Let $E(x : G)$ denote the set of edges incident with x in G. For $X, Y \subseteq V$, we use $[X, Y]$ to denote the set of edges with one end in X and the other in Y. The **connectivity**, denoted by $\kappa = \kappa(G)$, of a connected graph G is the least positive integer k such that there is a vertex subset $S \subset V$ with $|S| = k$ and $G - S$ is disconnected or reduces to the trivial graph K_1. We call that a connected graph G with at least two vertices have **edge-connectivity** $\lambda = \lambda(G)$ if the removal of some set of λ edges disconnects G, but there is no smaller set of edges whose removal disconnects G.

In 1932, Whitney presented a basic inequality on connectivity, edge-connectivity and minimum degree of a graph.

Theorem 1.1. *([**15**]) Let G be a connected graph, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$.*

It was shown in [**3**] that this result is best possible in the sense that if a, b and c are positive integers such that $a \leq b \leq c$, then there must exist a graph G with $\kappa(G) = a$, $\lambda(G) = b$ and $\delta(G) = c$.

Because of Theorem 1.1, it is reasonable to say that a graph G is **maximally connected** if $\kappa(G) = \delta(G)$ and write max-κ for simplicity; a graph G is **maximally-edge connected** if $\lambda(G) = \delta(G)$ and write max-λ for short. Furthermore, a graph G is **super-connected** or simply super-κ if every minimum vertex cut is the set of the neighbors of a vertex of G, that is every minimum vertex cut isolates a vertex. Similarly, a graph G is called **super-edge-connected** or super-λ if every minimum edge-cut is trivial; that is, if every minimum edge-cut consists of edges incident with a vertex of minimum degree. Thus every super connected (resp. super-edge-connected) graph is maximally connected (resp. maximally edge-connected). We encourage the interested reader to consult [**8, 12**] for more information on connectivities of graphs.

A graph $G = (V, E)$ is called a **split graph** if its vertex set V can be partitioned into a clique C and an independent set I, more details refer to [**4**]. Usually, the split graph G is denoted by $G = (C, I, E)$. In what follows, we always assume that $N(I) = C$. Otherwise, if $N(I) \neq C$, then by choosing a vertex $v \in C \setminus N(I)$, and replacing C by $C \setminus \{v\}$ and I by $I \cup \{v\}$, there would produce a new split graph $G = (C \setminus \{v\}, I \cup \{v\}, E)$ such that $N(I \cup \{v\}) = C \setminus \{v\}$. It immediately follows that $\delta(G) \leq |C|$. Analogously, a graph $G = (Y \cup Z, I, E)$ is called a **bisplit graph** if its vertex set V can be partitioned into three stable sets I, X_1 and X_2 such that $X_1 \cup X_2$ induces a complete bipartite graph and I is an independent set, where $|X_2| \geq |X_1|$. We encourage the interested reader to consult [**2**] for more details. It is similar to split graphs, we always assume that $N(I) = X_1 \cup X_2$ for any bisplit graph $G = (X_1 \cup X_2, I, E)$.

Inspired by the definitions of split graphs and bisplit graphs, a natural generalization could be presented as follows: a graph $G = (X_1 \cup X_2 \cup \cdots \cup X_k, I, E)$ is called a k**-multisplit graph**, if its vertex set V can be partitioned into $k+1$ stable sets I, X_1, $X_2, \cdots, X_k$ such that $X_1 \cup X_2 \cup \cdots \cup X_k$ induces a complete k-partite graph and I is an independent set, where $|X_1| \leq |X_2| \leq \cdots \leq |X_k|$. As usual, in what follows we always assume that $N(I) = X_1 \cup X_2 \cup \cdots \cup X_k$.

1.2 Notations for digraphs

Let D be a finite and simple digraph with vertex set $V(D)$. For any vertex v of a digraph D, we denote the set of **out-neighbors** and **in-neighbors** of v be $N^+(v) = N_D^+(v)$ and $N^-(v) = N_D^-(v)$, respectively. For a vertex $v \in V(D)$, the degree of v, denoted by $d(v)$, is defined as the minimum value of its out-degree $d^+(v) = |N^+(v)|$ and its in-degree $d^-(v) = |N^-(v)|$. The **minimum out-degree** and **minimum in-degree** of a digraph D are denoted by $\delta^+(D)$ and $\delta^-(D)$. In addition, let $\delta = \delta(D) = \min\{\delta^+(D), \delta^-(D)\}$ be the **minimum degree** of D.

If X and Y are two subsets of $V(D)$, then we denote by (X, Y) the set of arcs with tail in X and head in Y. A digraph is **strongly connected** if for every pair u, v of distinct vertices there exists a directed path from u to v, D is k**-edge-connected** if for any set S of at most $k-1$ arcs the subdigraph $D - S$ is strongly connected. The **edge-connectivity** $\lambda = \lambda(D)$ of a digraph D is defined as the largest value k such that D is k-edge-connected. The inequality $\lambda(D) \leq \delta(D)$ is immediate. We call a digraph D **maximally edge-connected**, if $\lambda(D) = \delta(D)$. A digraph is called **super-edge-connected** or **super-**λ, if every minimum edge cut is trivial, that means, every minimum edge cut consists of edges adjacent to or from a vertex of minimum degree. Clearly, if D is super-λ, then $\lambda(D) = \delta(D)$.

1.3 Motivation

Split graphs have many interesting properties such as simple induced subgraph characterization, linear time recognition as well as polynomial time algorithms for some NP-hard problems. A series of excellent contributions have been obtained on this topic up to now. For example, Woeginger [**14**] and Zhang et al. [**16**] studied the toughness and the scattering number of split graphs in terms of algorithms, respectively. In 2008, Metsidik and Vumar [**10**] considered the connectivity and edge toughness of bisplit graphs. In 2010, Guo and co-workers investigated super connectivity of Kronecker product of graphs, see more information from [**5**].

Three years later, Guo et al. explored the super connectivity of Kronecker product of complete graph and split graphs, see [**6**].

More recently, the first author of the presented paper [**7**] showed that split graphs and bisplit graphs are super-connected and super-edge-connected. Motivated by these results, we now turn our attention to the connectivity and super connectivity of k-multisplit graphs with $k \geq 3$. For convenience, in the following discussions, we always assume that $I \neq \emptyset$.

In this note, we first prove that: Let $G = (X_1 \cup X_2 \cdots \cup X_k, I, E)$ be a non-complete connected k-multisplit graph with $k \geq 3$ and $|X_1| \leq |X_2| \leq \cdots \leq |X_k|$, each of the following holds

(1) If $|X_1 \cup X_2 \cdots \cup X_{k-1}| \geq \delta(G)$, then G is maximally connected.

(2) If $|X_1 \cup X_2 \cdots \cup X_{k-1}| < \delta(G)$, then $\kappa(G) \geq |X_1 \cup X_2 \cdots \cup X_{k-1}|$.

(3) G is maximally edge-connected.

(4) If $|X_1 \cup X_2 \cdots \cup X_{k-1}| > \delta(G)$ with respect to $|X_1| \geq 2$ and $\delta(G) \geq 2$, then G is super-connected.

(5) G is super-edge connected.

Sufficient conditions for graphs to be maximally edge-connected or super-edge-connected were given by several authors, see for example some recent papers [**11**], [**12**] and the survey paper by Hellwig and Volkmann [**8**]. To our knowledge, there are no results on this topic for digraphs. In Section 3, we shall explore sufficient conditions for digraphs to be maximally edge-connected in terms of the zeroth-order general Randić index of digraphs.

All (di)-graphs considered in this paper are simple and finite. For notation and terminology not defined here, we refer to Bondy [**1**].

2. MAXIMALLY EDGE-CONNECTED AND SUPER-EDGE-CONNECTED SPLIT GRAPHS

Problems concerning on relations between the (edge)-connectivity and minimum degree attracted mathematicians and theoretical computer scientists for a long time. We begin with some results for (bi)-split graphs which will be used in later proofs.

Lemma 2.2. *([**17**]) Let $G = (C, I, E)$ be a non-complete connected split graph, then $\kappa(G) = \lambda(G) = \delta(G)$.*

Lemma 2.3. *([**10**]) Let $G = (X_1 \cup X_2, I, E)$ be a non-complete connected bisplit graph, each of the following holds:*

(1) If $|X_1| \geq \delta(G)$, then $\kappa(G) = \delta(G)$.
(2) If $|X_1| < \delta(G)$, then $\kappa(G) \geq |X_1|$.

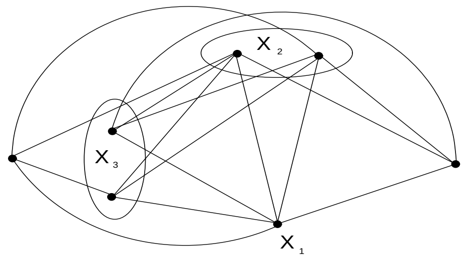

Fig.1

Illustration of Remark 1.

(3) $\lambda(G) = \delta(G)$.

In what follows, we shall focus on investigating sufficient conditions for k-multisplit graphs to be maximally connected. We begin with the following result.

Theorem 2.4. *Let $G = (X_1 \cup X_2 \cup \cdots \cup X_k, I, E)$ be a non-complete multipartite connected k-multisplit graph, each of the following holds:*

(1) If $|X_1 \cup X_2 \cup \cdots \cup X_{k-1}| \geq \delta(G)$, then $\kappa(G) = \delta(G)$ and each minimum vertex cut $S \subseteq X_1 \cup X_2 \cup \cdots \cup X_k$.

(2) If $|X_1 \cup X_2 \cup \cdots \cup X_{k-1}| < \delta(G)$, then $\kappa(G) \geq |X_1 \cup X_2 \cup \cdots \cup X_{k-1}|$.

Proof. Suppose that $X = X_1 \cup X_2 \cup \cdots \cup X_k$ and $Y = X_1 \cup X_2 \cup \cdots \cup X_{k-1}$. Let S be a minimum vertex cut with $|S| \leq \delta$.

Case 1. $G - S$ does not contain the isolated vertex of I. Then $S \cap I = \emptyset$.

Otherwise, $|S \cap X| = |S| - |S \cap I| \leq \delta - 1$. Hence $G[X] - S$ is connected. Note that $N(I) = X$. We obtain $G - S$ is connected, a contradiction.

So $S \cap I = \emptyset$ and $S \subseteq X$. Furthermore, $|S| \geq \delta$. Then $\kappa(G) = \delta(G)$.

Case 2. $G - S$ contains the isolated vertex v of I. Then $N(v) \subseteq S$. We know $\delta \leq d(v) \leq |S| \leq \delta$. That is $\delta(G) = d(v) = |S| = \kappa(G)$ and $S \subseteq X$.

Let $|Y| < \delta$. Hence $G - S$ is connected for any S with $|S| < |Y|$. That is $\kappa \geq |Y|$. □

Remark 1. It follows from Fig.1 that $X_1 \cup X_2 \cup X_3$ induces a $K_{1,2,2}$ and $|I| = 2$. But $\delta = 4$, and $X_1 \cup X_2$ is a vertex cut. Hence, Theorem 2.4 is the best possible.

Remark 2. In Fig.2, $X_1 \cup X_2 \cup X_3$ induces a $K_{1,2,2}$ and $|I| = 2$. But $\delta = 3$ and $X_1 \cup X_2$ is a vertex cut, and G is not super-connected.

Theorem 2.5. *Let $G = (X_1 \cup X_2 \cup \cdots \cup X_k, I, E)$ be a non-complete multipartite connected k-multisplit graph with $|X_1| \geq 2$, $\delta \geq 2$. If $|X_1 \cup X_2 \cup \cdots \cup X_{k-1}| > \delta$, then G is super-connected.*

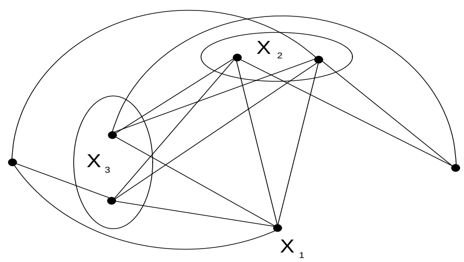

Fig.2

Illustration of Remark 2.

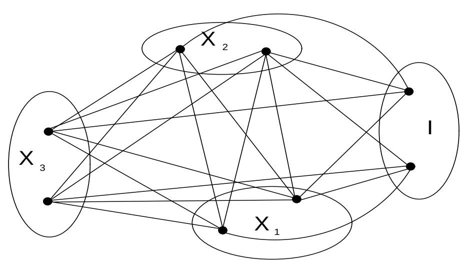

Fig.3

Illustration of Remark 3.

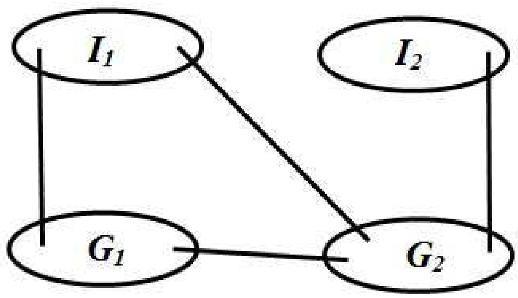

Fig.4 Illustration of Case 1.

Proof. By contradiction. Let $X = X_1 \cup X_2 \cup \cdots \cup X_k$, and S be a minimum vertex cut of G with $|S| = \kappa = \delta$. Then $G - S$ is not connected and has no isolated vertices. Note that $|X_1 \cup X_2 \cup \cdots \cup X_{k-1}| > \delta$, it follows from Theorem 2.4 that $S \subseteq X$. Hence $G[X - S]$ is connected. Consequently, each vertex of I is either connected to $G[X - S]$ or not connected to $G[X - S]$. This implies that $G - S$ is connected or has isolated vertices, a contradiction. □

Remark 3. In Fig.3, $X_1 \cup X_2 \cup X_3$ induces a $K_{2,2,2}$ and $|I| = 2$. But $\delta = 4$ and $X_1 \cup X_2$ is a vertex cut, and G is not super-connected. If $k \geq 3$, $\delta = 1$, then G is super-connected. Hence, Theorem 2.5 is the best possible.

We will give a useful lemma for the edge cut of k-multisplit graphs.

Lemma 2.6. *Let $G = (X_1 \cup X_2 \cup \cdots \cup X_k, I, E)$ be a non-complete multipartite connected k-split graph and $X = X_1 \cup X_2 \cup \cdots \cup X_k$. And F is any minimum edge cut of G such that $F \cap [X, I] \neq \varnothing, F \cap E(G[X]) \neq \varnothing$. Set G_1 is the connected component of $G[X] - F$. Then*

(1) If $|G_1| = 1$, then G is super edge connected.

(2) $F|_{G[X]}$ is the minimum edge cut of $G[X]$.

Proof. (1) If $|G_1| = 1$ for any F, then it is obviously right.

(2) Suppose F is the minimum edge cut of G such that $F \cap [X, I] \neq \varnothing, F \cap E(G[X]) \neq \varnothing$. And assume that $G[X] - F$ is not connected. Assume that $F|_{G[X]}$ is not the minimum edge cut of $G[X]$ and each component of $G[X] - F$ has at least two vertices. Set $G_2 = G[X] - F - G_1$, and $I_1 = N_I(G_1), I_2 = I - I_1$.

Notice that $|G_1| \geq 2, |G_2| \geq 2$. It is depicted in Fig.4. Clearly, $[G_1, G_2]$ must be in F.

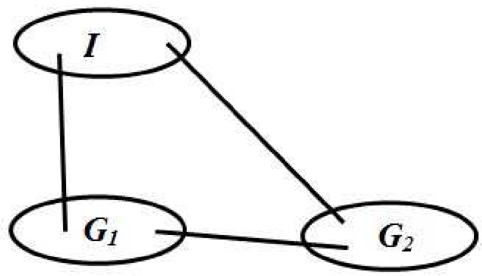

Fig.5 Illustration of Case 2.

Case 1. $I_2 \neq \emptyset$. We can see Fig.4.

Since $N(I) = X$. In G, each vertex of G_1 has a neighbor of I_1, each vertex of I_1 has a neighbor of G_1, and each vertex of I_2 has a neighbor of G_2.

Note that $G[X]$ is a complete k-partite graph. If there exist two vertices $u, v \in G_1$ such that u and v are not in the same X_i, then for any vertex $y \in G_2$, we have $uy \in E$ or $vy \in E$. Hence $|[G_1, G_2]| \geq |G_2|$.

Therefore, if all of vertices of G_1 are in the same X_i, then G_1 is an independent set, a contradiction.

For any $u \in I_2, d(u) \leq |G_2|$. Since $F \cap [X, I] \neq \varnothing$, $|F| \geq 1 + |G_2|$. Then $E(u : G)$ is another edge cut of G smaller than F, a contradiction.

Case 2. $I_2 = \emptyset$. We can see Fig.5.

There is not a vertex $u \in I$ such that $N(u) \subseteq G_1$ or $N(u) \subseteq G_2$. Otherwise we can find another edge cut smaller than F. Hence for any vertex $u \in I$, we have $N(u) \cap G_1 \neq \emptyset$ and $N(u) \neq \emptyset$.

Notice that $|[G_1, G_2]| \geq |G_2|$, $|[G_1, I]| \geq |G_1|$, and $|[G_2, I]| \geq |G_2|$. We obtain $|F| \geq |G_1| + |G_2| = |X|$.

For any $u \in I, d(u) \leq |X|$. If for any $u \in I, d(u) = |X|$, then G is a complete $k+1$-partite graph, a contradiction. Hence there is a vertex $v \in I$ such that $d(v) < |X|$. Then $E(v : G)$ is another edge cut smaller than F, a contradiction.

□

Theorem 2.7. *Let $G = (X_1 \cup X_2 \cup \cdots \cup X_k, I, E)$ be a non-complete connected multipartite k-multisplit graph. Then G is super-edge connected.*

Proof. For convenience, let $X = X_1 \cup X_2 \cup \cdots \cup X_k$. If $\delta = 1$, then the minimum degree vertices are in I. Hence, G is super-edge connected.

In what follows, we assume that $\delta \geq 2$. Let F be a minimum edge cut of G

with $|F| = \lambda \leq \delta$. Note that a complete k-partite graph is super-edge connected for any integer k not less than three.

For simplicity, we distinguish the following three cases:

Case 1. $F \subseteq G[X]$.

If $G[X] - F$ is connected, then $G - F$ is connected, a contradiction. Hence $G[X] - F$ is not connected. It follows that there exists a vertex $u \in X$ such that $E(u : G[X]) = F$. Note that the initial condition $N(I) = X$, it yields that there must exists a vertex $v \in I$ which is adjacent to u. Hence, v must be connected to other vertices of $X \setminus \{u\}$ since $\delta \geq 2$. Therefore $G - F$ is connected, again a contradiction. Hence G is super-edge connected.

Case 2. $F \subseteq [X, I]$.

It follows that $G[X]$ is connected. Since $G - F$ is not connected, there must exist a vertex $u \in I$ such that $E(u : G[X]) = F$. Hence G is super-edge connected.

Case 3. $F \cap [X, I] \neq \emptyset$ and $F \cap G[X] \neq \emptyset$.

If $G[X] - F$ is connected, then we know each vertex of I is connected to $G[X] - F$ in $G - F$. So $G - F$ is connected, a contradiction.

Hence assume that $G[X] - F$ is not connected. Because F is the minimum edge cut of G, $F \cap E(G[X])$ is the minimum edge cut of $G[X]$ by Lemma 2.6. Let $F_1 = F \cap [X, I]$ and $F_2 = F \cap E(G[X])$. Then $G[X] - F_2$ has only two components G_1 and an isolated vertex u.

If u has some neighbors $u_1, u_2, \cdots, u_m (m \geq 1)$ of I in $G - F$, then $G[\{u, u_1, u_2, \cdots, u_m\}] \cong K_{1,m}$ and $|[\{u, u_1, u_2, \cdots, u_m\}, X]| \geq (m+1)\delta - (m+1) > \delta$, a contradiction. Then u has no neighbors in I. That is G is super-edge connected.

This completes the proof of Theorem 2.7. □

From Theorem 2.7, we directly obtain the following result.

Corollary 2.8. *Let $G = (X_1 \cup X_2 \cup \cdots \cup X_k, I, E)$ be a non-complete multipartite connected k-multisplit graph of order n. Then $\lambda(G) = \delta(G)$.*

3. MAXIMALLY EDGE-CONNECTED AND SUPER-EDGE-CONNECTED DIGRAPHS

In this section, we will explore sufficient conditions for digraphs to be maximally edge-connected and super-edge connected in terms of a new topological index.

Let D be a digraph with minimum degree $\delta \geq 1$. If α is a real number, the **zeroth-order general Randić index** of D was proposed by Volkmann in [**13**]:

$$R^0_\alpha(D) = \sum_{x \in V(D)} \big[d(x)\big]^\alpha.$$

We begin with several basic lemmas, which we use in the proofs of our main results.

Lemma 3.9. *([**9**]) If $x - 2 \geq y \geq 1$ and $\alpha \in (-\infty, 0) \cup (1, +\infty)$ is a real number, then*

$$(x-1)^\alpha + (y+1)^\alpha < x^\alpha + y^\alpha.$$

Lemma 3.10. *([**12**]) Let $\alpha > 1$ be a real number, and let $a_1, a_2, \cdots, a_p$ and A be positive reals such that $\sum_{i=1}^{p} a_i \leq A$. If in addition, $a_1, a_2, \cdots, a_p, A$ are positive integers, and a, b are integers with $A = ap + b$ and $0 \leq b < p$, then*

$$\sum_{i=1}^{p} a_i^\alpha \geq (p-b)a^\alpha + b(a+1)^\alpha.$$

The **associated digraph** $D(G)$ of a graph G is obtained by replacing each edge of G by a pair of mutually opposite oriented arcs.

The following observation is simple but useful.

Observation 3.11. *Let G be a graph and $D(G)$ its associated digraph, then $\lambda(G) = \lambda(D(G))$.*

Theorem 3.12. *Let D be a strongly connected digraph of order $n \geq 3$, minimum degree δ and edge-connectivity λ, and let $\alpha > 1$ be a real number. If*

$$R_\alpha^0(D) < n\left(\frac{n-2}{2}\right)^\alpha + (\delta - 1)\left[2\left(\frac{n}{2}\right)^\alpha - \delta^\alpha - (n-\delta-2)^\alpha\right],$$

then D is maximally-edge connected.

Proof. If $\delta = 1$, then $\lambda = \delta$ in every case. In what follows, we always assume that $\delta \geq 2$. Suppose to the contrary that $\lambda \leq \delta - 1$, then there exist two disjoint sets $X, Y \subset V(D)$ such that $X \cup Y = V(D)$ and $|(X,Y)| = \lambda$. We first show that $\delta + 1 \leq |X|, |Y| \leq n - \delta - 1$. Suppose that X contains at most δ vertices. Since every vertex in X has at most $|X| - 1$ out-neighbors in X and there are at most λ arcs from X to Y, we obtain the contradiction

$$\delta|X| \leq \sum_{x \in X} d^+(x) \leq |X|(|X|-1) + \lambda \leq \delta(|X|-1) + \delta - 1 = \delta|X| - 1.$$

Therefore $|X| \geq \delta + 1$. By the similar approach, we can show that $|Y| \geq \delta + 1$.

It follows from the above

$$\sum_{x \in X} d(x) \leq \sum_{x \in X} d^-(x) \leq |X|(|X|-1) + \delta - 1.$$

Applying Lemma 3.10, we deduce that

$$\begin{aligned}\sum_{x\in X}\big[d(x)\big]^{\alpha} \geq&(|X|-(\delta-1))(|X|-1)^{\alpha}+(\delta-1)|X|^{\alpha}\\ =&(|X|-1)^{\alpha}+(|X|-1)^{\alpha+1}+(\delta-1)\big[|X|^{\alpha}-(|X|-1)^{\alpha}\big].\end{aligned}$$

Similarly, we have

$$\sum_{y\in Y}\big[d(y)\big]^{\alpha} \geq (|Y|-1)^{\alpha}+(|Y|-1)^{\alpha+1}+(\delta-1)\big[|Y|^{\alpha}-(|Y|-1)^{\alpha}\big].$$

Combing the previous two inequalities, together with the initial condition $|X|+|Y|=n$, we get

$$\begin{aligned}R^0_{\alpha}(D)=&\sum_{x\in X}\big[d(x)\big]^{\alpha}+\sum_{y\in Y}\big[d(y)\big]^{\alpha}\\ \geq&(|X|-1)^{\alpha+1}+(|X|-1)^{\alpha}+(|Y|-1)^{\alpha+1}+(|Y|-1)^{\alpha}\\ &+(\delta-1)\big[|X|^{\alpha}-(|X|-1)^{\alpha}+|Y|^{\alpha}-(|Y|-1)^{\alpha}\big].\\ =&(|X|-1)^{\alpha+1}+(n-|X|-1)^{\alpha+1}+(|X|-1)^{\alpha}+(n-|X|-1)^{\alpha}\\ &+(\delta-1)\big[|X|^{\alpha}+(n-|X|)^{\alpha}-((|X|-1)^{\alpha}+(n-X-1)^{\alpha})\big].\end{aligned}$$

To accomplish the proof, it is sufficient to find the sharp lower bound of the following function

$$\begin{aligned}Q_1(X,Y)=&(|X|-1)^{\alpha+1}+(n-|X|-1)^{\alpha+1}+(|X|-1)^{\alpha}+(n-|X|-1)^{\alpha}\\ &+(\delta-1)\big[|X|^{\alpha}+(n-|X|)^{\alpha}-((|X|-1)^{\alpha}+(n-X-1)^{\alpha})\big].\end{aligned}$$

Note that the functions $H_1(t)=(t-1)^{\alpha+1}+(n-t-1)^{\alpha+1}$, $H_2(t)=(t-1)^{\alpha}+(n-t-1)^{\alpha}$ and $H_3(t)=t^{\alpha}+(n-t)^{\alpha}$ have their minimum for $t=n/2$, and the function $H_2(t)$ has its maximum for $H_2(t)=\delta^{\alpha}+(n-\delta-2)^{\alpha}$ when $\delta+1\leq t\leq n-\delta-1$ and $\alpha>1$, which immediately yields that

$$\begin{aligned}R^0_{\alpha}(D)\geq&\left(\frac{n}{2}-1\right)^{\alpha+1}+\left(\frac{n}{2}-1\right)^{\alpha}+\left(\frac{n}{2}-1\right)^{\alpha+1}+\left(\frac{n}{2}-1\right)^{\alpha}\\ &+(\delta-1)\left[\left(\frac{n}{2}\right)^{\alpha}+\left(\frac{n}{2}\right)^{\alpha}-\delta^{\alpha}-(n-\delta-2)^{\alpha}\right]\\ =&n\left(\frac{n-2}{2}\right)^{\alpha}+(\delta-1)\left[2\left(\frac{n}{2}\right)^{\alpha}-\delta^{\alpha}-(n-\delta-2)^{\alpha}\right],\end{aligned}$$

which contradicts the hypothesis. Therefore $\lambda=\delta$. □

We write $D(K_n)$ for the complete digraph of order n. The following example will show that the condition in Theorem 3.12 is best possible.

Example 3.13. *Let n and δ be integers such that $n = 2\delta + 2 \geq 6$. Furthermore, let $H_1 = D(K_{\delta+1})$ with vertex set $V(H_1) = \{x_1^1, x_2^1, \cdots, x_{\delta+1}^1\}$, and let $H_2 = D(K_{\delta+1})$ with vertex set $V(H_2) = \{x_1^2, x_2^2, \cdots, x_{\delta+1}^2\}$. Define the digraph H by the union of H_1 and H_2 together with the $2\delta - 2$ arcs $x_1^1x_1^2, x_2^1x_2^2, \cdots, x_{\delta-1}^1x_{\delta-1}^2$ as well as $x_1^2x_1^1, x_2^2x_2^1, \cdots, x_{\delta-1}^2x_{\delta-1}^1$. Then $n(H) = n$, $\delta(H) = \delta$ and*

$$R_\alpha^0(H) = 4\delta^\alpha + (2\delta - 2)(\delta + 1)^\alpha.$$

Therefore

$$R_\alpha^0(H) = n\left(\frac{n-2}{2}\right)^\alpha + (\delta - 1)\left[2\left(\frac{n}{2}\right)^\alpha - \delta^\alpha - (n - \delta - 2)^\alpha\right]$$

and so equality in the inequality of Theorem 3.12. However, $\lambda(H) = \delta(H) - 1$.

Corollary 3.14. *Let G be a connected graph of order $n \geq 3$, minimum degree δ and edge-connectivity λ, and let $\alpha > 1$ be a real number. If*

$$R_\alpha^0(G) < n\left(\frac{n-2}{2}\right)^\alpha + (\delta - 1)\left[2\left(\frac{n}{2}\right)^\alpha - \delta^\alpha - (n - \delta - 2)^\alpha\right],$$

then D is maximally-edge connected.

Proof. Since $N_G(v) = N_{D(G)}^+(v) = N_{D(G)}^-(v)$, for each vertex $v \in V(G) = V(D(G))$, we observe that $n(G) = n(D(G))$, $\delta(G) = \delta(D(G))$ and $R_\alpha^0(G) = R_\alpha^0(D(G))$. Thus Theorem 3.12 and Observation 3.11 imply the desired result. □

Theorem 3.15. *Let D be a strongly connected digraph of order n, minimum degree $\delta \geq 2$ and edge-connectivity λ, and let $\alpha > 1$ be a real number. If*

$$R_\alpha^0(D) < n\left(\frac{n-2}{2}\right)^\alpha + \delta\left[2\left(\frac{n}{2}\right)^\alpha - (\delta - 1)^\alpha - (n - \delta - 1)^\alpha\right],$$

then D is super-edge connected.

Proof. Suppose to the contrary that D is not super-λ, then there exist two disjoint sets $X, Y \subset V(D)$ such that $X \cup Y = V(D)$, $|X|, |Y| \geq 2$ and $|(X, Y)| = \lambda$. We first show that $\delta \leq |X|, |Y| \leq n - \delta$. Suppose that X contains at most $\delta - 1$ vertices when $\delta \geq 3$. Since every vertex in X has at most $|X| - 1$ out-neighbors in X and there are at most λ arcs from X to Y, we obtain the contradiction

$$\delta|X| \leq \sum_{x \in X} d^+(x) \leq |X|(|X| - 1) + \lambda \leq (\delta - 1)(|X| - 1) + \delta = \delta|X| - |X| + 1.$$

Therefore $|X| \geq \delta$. Similar analysis shows that $|Y| \geq \delta$.

It immediately follows that

$$\sum_{x\in X} d(x) \le \sum_{x\in X} d^-(x) \le |X|(|X|-1)+\delta.$$

Applying Lemma 3.10 in the case $|X|\ge \delta+1$, we deduce that

$$\begin{aligned}\sum_{x\in X}\big[d(x)\big]^\alpha \ge&(|X|-\delta)(|X|-1)^\alpha+\delta|X|^\alpha\\ =&(|X|-1)^\alpha+(|X|-1)^{\alpha+1}+\delta[|X|^\alpha-(|X|-1)^\alpha].\end{aligned}$$

If $|X|=\delta$, then

$$\sum_{x\in X} d(x) \le \sum_{x\in X} d^-(x) \le |X|(|X|-1)+\delta=|X|^2.$$

Applying again Lemma 3.10, we see that

$$\sum_{x\in X}\big[d(x)\big]^\alpha \ge \delta^{\alpha+1}=(|X|-1)^\alpha+(|X|-1)^{\alpha+1}+\delta[|X|^\alpha-(|X|-1)^\alpha].$$

Similar calculations shows that

$$\sum_{y\in Y}\big[d(y)\big]^\alpha \ge (|Y|-1)^\alpha+(|Y|-1)^{\alpha+1}+\delta\big[|Y|^\alpha-(|Y|-1)^\alpha\big].$$

It follows from the previous two inequalities, in conjunction with the initial condition $|X|+|Y|=n$, that

$$\begin{aligned}R_\alpha^0(D)=&\sum_{x\in X}\big[d(x)\big]^\alpha+\sum_{y\in Y}\big[d(y)\big]^\alpha\\ \ge&(|X|-1)^{\alpha+1}+(|X|-1)^\alpha+(|Y|-1)^{\alpha+1}+(|Y|-1)^\alpha\\ &+\delta\big[|X|^\alpha-(|X|-1)^\alpha+|Y|^\alpha-(|Y|-1)^\alpha\big].\\ =&(|X|-1)^{\alpha+1}+(n-|X|-1)^{\alpha+1}+(|X|-1)^\alpha+(n-|X|-1)^\alpha\\ &+\delta\big[|X|^\alpha+(n-|X|)^\alpha-((|X|-1)^\alpha+(n-X-1)^\alpha)\big].\end{aligned}$$

To accomplish the proof, it is sufficient to find the minimum of the function

$$\begin{aligned}Q_2(X,Y)=&(|X|-1)^{\alpha+1}+(n-|X|-1)^{\alpha+1}+(|X|-1)^\alpha+(n-|X|-1)^\alpha\\ &+\delta\big[|X|^\alpha+(n-|X|)^\alpha-((|X|-1)^\alpha+(n-X-1)^\alpha)\big].\end{aligned}$$

Note that the functions $H_1(t)=(t-1)^{\alpha+1}+(n-t-1)^{\alpha+1}$, $H_2(t)=(t-1)^\alpha+(n-t-1)^\alpha$ and $H_3(t)=t^\alpha+(n-t)^\alpha$ have their minimum for $t=n/2$, and the function $H_2(t)$ has its maximum $H_2(t)=(\delta-1)^\alpha+(n-\delta-1)^\alpha$ when $\delta\le t\le n-\delta$ and $\alpha>1$, it yields that

$$\begin{aligned}R_\alpha^0(D)\ge&\left(\frac{n}{2}-1\right)^{\alpha+1}+\left(\frac{n}{2}-1\right)^{\alpha}+\left(\frac{n}{2}-1\right)^{\alpha+1}+\left(\frac{n}{2}-1\right)^{\alpha}\\ &+\delta\left[\left(\frac{n}{2}\right)^\alpha+\left(\frac{n}{2}\right)^\alpha-(\delta-1)^\alpha-(n-\delta-1)^\alpha\right]\\ =&n\left(\frac{n-2}{2}\right)^\alpha+\delta\left[2\left(\frac{n}{2}\right)^\alpha-(\delta-1)^\alpha-(n-\delta-1)^\alpha\right]\end{aligned}$$

again a contradiction. Therefore D is super-edge connected. □

The next example shows that the condition in Theorem 3.15 is best possible.

Example 3.16. *Let n and δ be integers such that $n = 2\delta \geq 6$. Furthermore, let $H_1 = D(K_\delta)$ with vertex set $V(H_1) = \{x_1^1, x_2^1, \cdots, x_\delta^1\}$, and let $H_2 = D(K_\delta)$ with vertex set $V(H_2) = \{x_1^2, x_2^2, \cdots, x_\delta^2\}$. Define the digraph H by the union of H_1 and H_2 together with the 2δ arcs $x_1^1x_1^2, x_2^1x_2^2, \cdots, x_\delta^1x_\delta^2$ as well as $x_1^2x_1^1, x_2^2x_2^1, \cdots, x_\delta^2x_\delta^1$. Then $n(H) = n$, $\delta(H) = \delta$ and*

$$R_\alpha^0(H) = 2\delta^{\alpha+1}.$$

Therefore

$$R_\alpha^0(H) = n\left(\frac{n-2}{2}\right)^\alpha + \delta\left[2\left(\frac{n}{2}\right)^\alpha - (\delta-1)^\alpha - (n-\delta-1)^\alpha\right]$$

and so equality in the inequality of Theorem 3.15. However, H is not super-edge connected.

The following is an immediate consequence of Theorem 3.15.

Corollary 3.17. *Let G be a connected graph of order n, minimum degree $\delta \geq 2$ and edge-connectivity λ, and let $\alpha > 1$ be a real number. If*

$$R_\alpha^0(G) < n\left(\frac{n-2}{2}\right)^\alpha + \delta\left[2\left(\frac{n}{2}\right)^\alpha - (\delta-1)^\alpha - (n-\delta-1)^\alpha\right],$$

then G is super-edge connected.

Acknowledgments The authors would like to thank the anonymous referees for their valuable comments and constructive suggestions, which resulted in an improved version of this manuscript. We also thank NSFC for supporting this project under the Grant No.11301440.

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(Received 16.07.2017)
(Revised 14.11.2018)

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