Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **x** (xxxx), xxx–xxx.
doi:10.2298/AADMxxxxxxxx

CONNECTED DOMINATION GAME

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In this paper we introduce a domination game based on the notion of connected domination. Let $G = (V, E)$ be a connected graph of order at least 2. We define a *connected domination game* on G as follows: The game is played by two players, Dominator and Staller. The players alternate taking turns choosing a vertex of G (Dominator starts). A move of a player by choosing a vertex v is legal, if (1) the vertex v dominates at least one additional vertex that was not dominated by the set of previously chosen vertices and (2) the set of all chosen vertices induces a connected subgraph of G. The game ends when none of the players has a legal move (i.e., G is dominated). The aim of Dominator is to finish as soon as possible, Staller has an opposite aim.
Let D be the set of played vertices obtained at the end of the connected domination game (D is a connected dominating set of G). The *connected game domination number* of G, denoted $\gamma_{cg}(G)$, is the minimum cardinality of D, when both players played optimally on G.
We provide an upper bound on $\gamma_{cg}(G)$ in terms of the connected domination number. We also give a tight upper bound on this parameter for the class of 2-trees. Next, we investigate the Cartesian product of a complete graph and a tree, and we give exact values of the connected game domination number for such a product, when the tree is a path or a star. We also consider some variants of the game, in particular, a Staller-start game.

1. INTRODUCTION

In this paper we define and study the new problem of connected domination game on graphs. The notion of connected domination was introduced in 1979 by Sampathkumar and Walikar in [**24**] (see also [**12**] and [**11**]). Let G be a connected graph. The set of vertices S is a *connected dominating set* of G, if S is dominating

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2010 Mathematics Subject Classification. 05C57, 05C69.
Keywords and Phrases. Connected domination, domination games on graphs, connected domination game, 2-tree, Cartesian product

and the subgraph induced by S in G is connected. The minimum cardinality of a connected dominating set of G is called the *connected domination number* of G, and denoted by $\gamma_c(G)$.

Among a variety of games on graphs, recently the games defined based on the domination notion received a lot of attention. The well known game of this type is the domination game introduced by Brešar, Klavžar and Rall in 2010 ([**1**]). One of the widely studied problems regarding this game is the 3/5 Conjecture, posed and studied in [**19**], and later in [**5, 6, 7, 13, 16**]. The other directions of investigations are, among others, graphs critical with respect to this game ([**8, 26**]), realisations of the game domination numbers ([**22**]) and computational complexity of such a game ([**20**]). Further results on this topic can be found in [**2, 3, 4, 10, 21, 23, 27**]. Recently, the studies of two new domination games were initiated, namely, on the total domination game ([**14**], see also [**15, 17**]), and the disjoint domination game ([**9**]).

In the present paper we introduce a new domination game based on the notion of connected domination. Since the connected dominating set is defined only for connected graphs, from now on we assume all considered graphs to be connected and with at least two vertices. So, let $G = (V, E)$ be such a graph. We define a *connected domination game* on G as follows: The game is played by two players, Dominator and Staller. The players alternate taking turns choosing a vertex of G (Dominator starts). A move of a player by choosing a vertex v is legal, if (1) the vertex v dominates at least one additional vertex that was not dominated by the set of previously chosen vertices and (2) the set of all chosen vertices induces a connected subgraph of G. The game ends when none of the players has a legal move (i.e., G is dominated). The aim of Dominator is to finish as soon as possible, Staller has an opposite aim.

A vertex is called *played* if it has been chosen by any player. Let D be the set of played vertices at the end of the connected domination game. Note that D is a connected dominating set of G. The *connected game domination number* of G, denoted $\gamma_{cg}(G)$, is defined as the minimum cardinality of D, when both players played optimally on G.

The following relation between the connected domination number and its game version can be observed (it is analogous to the theorem relating the game domination number and the domination number, stated by Brešar, Klavžar and Rall in 2010, see [**1**]).

Theorem 1.

$$\gamma_c(G) \leq \gamma_{cg}(G) \leq 2\gamma_c(G) - 1.$$

Proof. It is similar to the proof of Theorem 1 in [**1**], but we include it for completeness. The lower bound is obvious. For the upper bound, let S be a connected dominating set of G with $|S| = \gamma_c(G)$. Dominator has the following strategy: at each turn, he chooses one vertex from S, if such a move is legal. Obviously, if Dominator can play in this way till the game ends, then the total number of picked vertices cannot exceed $2\gamma_c(G)-1$. We show that whenever the set of played vertices is not dominating, then Dominator can play on vertices from S. On the contrary suppose that, at some turn, any move of Dominator by choosing a vertex from S is not legal. Let W be the set of already played (by both players) vertices. Since S

is a connected dominating set of G and $S \cap W \neq \emptyset$, and the game is not finished, there is a vertex in S that is adjacent to a vertex from W and has an undominated neighbour. Hence there is a legal move for Dominator on the set S, a contradiction. Thus, the game ends in at most $2\gamma_c(G) - 1$ moves, as required. □

Throughout the paper we in general follow [**25**] in definitions and notation. The degree of a vertex x in a graph G is denoted by $d(x)$. A vertex of degree 1 is called a leaf. By K_n, P_n and $K_{1,n-1}$ we denote the complete graph, the path and the star on n vertices, respectively. Given two graphs G_1 and G_2, the Cartesian product of G_1 and G_2 is the graph $G = G_1 \Box G_2$ with vertex set $V(G) = V(G_1) \times V(G_2)$ where vertex (x_1, x_2) is adjacent to vertex (y_1, y_2) if and only if (1) $x_1 = y_1$ and $x_2y_2 \in E(G_2)$, or (2) $x_2 = y_2$ and $x_1y_1 \in E(G_1)$.

The rest of the paper is organized as follows. In Section 2 we define a variant of connected domination game, called connected domination game with Chooser. This new game appears to be very useful for proving bounds and properties of the connected game domination number. In Sections 3 and 4 we use this tool to determine upper bounds on the connected game domination number of some special classes of graphs.

It is well known that for any tree T (with $n \geq 3$ vertices) it holds $\gamma_c(T) = n - |L(T)|$, where $L(T)$ is the set of all leaves of T. For the connected domination game of T, it is easy to observe that any strategy for Dominator of not starting at a leaf vertex ensures that the game ends in $n - |L(T)|$ moves. Thus $\gamma_{cg}(T) = \gamma_c(T)$. Since we know the connected game domination number of trees, it would be interesting to find the connected game domination number of a k-tree for $k \geq 2$. In Section 3 we prove that the connected game domination number of an n-vertex 2-tree is at most $\lceil 2(n-4)/3 \rceil + 1$ and the bound is tight. However, the connected domination number of an n-vertex 2-tree is at most $\lfloor (n-2)/2 \rfloor$. In Section 4 we consider Cartesian products of graphs when one factor is a complete graph and the other is a tree. In this section we also give examples of graphs for which the upper bound on the connected game domination number given in Theorem 1 is tight. Finally, in Section 5 we consider the impact of skipping moves of the player on the size of the final dominating set. Namely, we study the game in which Staller has the first move and the game in which Staller can skip some moves.

2. CONNECTED DOMINATION GAME WITH CHOOSER

In order to prove bounds for the connected game domination number, we define another game on a connected graph G, namely, a *connected domination game with Chooser*.

The general rules are the same as in the connected domination game (we have players Dominator and Staller, who alternate taking turns choosing a vertex of G according to the same rules as in the connected domination game) but there is an additional player called Chooser. The rules for Chooser are as follows:

- Chooser makes moves between the moves of other players, or as the last move of the game.

- In his move, Chooser can pick and add to the set of played vertices zero, one or more vertices.
- The move of Chooser is legal, if after his move the set of played vertices induces a connected subgraph.
- Chooser cannot start the game.

The game ends when the set of played vertices forms a connected dominating set of G. The aim of Dominator is to finish as soon as possible, Staller has an opposite aim.

The connected domination game with Chooser is still the game between Dominator and Staller, however Chooser by his moves can help either Dominator or Staller. The next lemma shows the relation between the number of moves of Chooser and the size of the dominating set obtained at the end of the game. We will use it as a very useful tool for proving our results concerning the connected domination game. In the proof of this lemma we use the so-called imagination strategy that was presented in [**1**].

Lemma 1 (Chooser Lemma)**.** *Consider the connected domination game with Chooser on a graph G. Suppose that in the game Chooser picks k vertices, and that both Dominator and Staller play optimally. Then at the end of the game the number of played vertices is at most $\gamma_{cg}(G) + k$ and at least $\gamma_{cg}(G) - k$.*

Proof. Suppose that the players play the connected domination game with Chooser on G.

First, we will show the upper bound by providing a strategy for Dominator. This strategy is as follows: he will imagine that another game is being played at the same time on G — a connected domination game — and he will play it according to an optimal strategy (hence the length of this game will not be greater than $\gamma_{cg}(G)$). So two games are played at the same time: the real game (with Chooser), and the connected domination game imagined by Dominator. In the first phase of the game (this phase of game lasts until Chooser makes his first move in the real game) Dominator will just copy each move of Staller to his imagined game, and will respond in the imagined game with an optimal move from the ordinary connected domination game. Each of these Dominator's moves is then also copied to the real game. This will continue until Chooser decides to pick a vertex in the real game (but if he does not decide to pick any vertex, then the two games are the same and thus both have $\gamma_{cg}(G)$ played vertices). After Chooser's move, Dominator plays in such a way that he maintains the following invariant: every dominated vertex in the imagined game is also dominated in the real game, and the number of played vertices in the real game is less than or equal to the number of played vertices in the imagined game plus the number of vertices played by Chooser in the real game. We are going to prove that Dominator can always play to satisfy the above defined invariant.

Observe that only the following two situations, if occur, may cause problems for Dominator: (1) Staller chooses his move in the real game, but in the imagined game the copy of this move is not legal; (2) Dominator chooses a vertex to pick according to an optimal strategy in the imagined game, but in the real game such a move is not legal.

First, we consider the case when it is Staller's turn. If the move of Staller in the real game is legal in the imagined game, then Dominator copies this move in the imagined game. It is easy to see that after such a move the invariant is maintained. So suppose that the move of Staller in the real game is not legal in the imagined game (so situation (1) occurs). Since before the move of Staller the invariant was maintained, it follows that the vertex v picked by Staller is not adjacent to any vertex picked in the imagined game. Let D_R (D_I, respectively) denote the set of played vertices at this stage of game in the real game (in the imagined game, respectively). So, $D_I \cup \{v\}$ does not induce a connected subgraph in G. Observe that $|D_R \cap D_I| \geq 1$ since at least the first move is the same in the real and the imagined game (because Chooser cannot start). Let P be the shortest path from v to D_I such that all internal vertices of P are in D_R. The fact that $|D_R \cap D_I| \geq 1$ implies that such a path exists. Dominator chooses the internal vertex w of P which is adjacent to a vertex from D_I, and he picks this vertex in the imagined game instead of v. We claim that choosing w is legal in the imagined game. Since w is adjacent to at least one vertex from D_I (namely, the end vertex of P), the subgraph induced in G by $D_I \cup \{w\}$ is connected. Let w' be the other neighbour of w on the path P, i.e. $w' \notin D_I$. Observe that w' is not dominated by any vertex of D_I, since otherwise P would not be the shortest path. Thus, w dominates a new vertex. Furthermore, since $w \in D_R$, the invariant is maintained.

Now, we consider the case when it is the turn of Dominator. If the move of Dominator in the imagined game made according to an optimal strategy is legal also in the real game, then Dominator copies this move in the real game. It is easy to see that after such a move the invariant is maintained. Suppose now that the move of Dominator made in the imagined game is not legal in the real game (so situation (2) occurs). Since before Dominator's move the invariant was maintained, it follows that the vertex v chosen by Dominator in the imagined game dominates no new vertex in the real game. In such a case Dominator chooses any legal vertex in the real game (such a vertex exists, otherwise the real game would be finished). It is an easy observation that after such a move of Dominator the invariant still holds.

By maintaining the invariant Dominator ensures that the real game ends no later than the imagined game, and hence the total number of played vertices in the real game is at most $\gamma_{cg}(G) + k$, what completes the proof of the upper bound.

Next, we will prove the lower bound, by presenting the strategy for Staller.

Now, Staller will play two games on G — the real game and the imagined game. In the imagined game Staller plays optimally according to the strategy for the connected domination game, so Staller in the imagined game always answer on moves of Dominator optimally. In the first phase of the game Staller will just copy each move of Dominator to his imagined game, and respond in the imagined game with an optimal move from the connected domination game. Then Staller also copies this move to the real game. This phase of game lasts until Chooser makes his first move in the real game. If Chooser does not decide to pick any vertex, then the two games are the same and thus both have $\gamma_{cg}(G)$ played vertices.

After Chooser's move Staller plays in such a way that until the set of played vertices in the real game does not form a connected dominating set, he maintains the following invariant: every vertex dominated in the imagined game is also dominated in the real game and the number of played vertices in the real game is equal to the number of played vertices in the imagined game plus the number of vertices picked by Chooser in the real game. We argue that Staller can maintain such an invariant.

Again, the following problems may occur: (1) Dominator chooses his move in the real game, but in the imagined game the copy of this move is not legal, or (2) Staller chooses a vertex to pick according to an optimal strategy in the imagined game, but in the real game such a move is not legal.

Suppose first that in the real game Dominator picks a vertex v that is not legal in the imagined game. Thus, the vertex v picked by Dominator is not adjacent to any vertex picked in the imagined game. Let D_R (D_I, respectively) denote the set of played vertices at this stage of the game in the real game (in the imagined game, respectively). So, $D_I \cup \{v\}$ does not induce a connected graph. Similarly as in the first part of the proof we show that there is a vertex w that can be chosen in the imagined game instead of v. Since at least the first move is the same at the real game and the imagined game, we have $|D_R \cap D_I| \geq 1$. Let P be the shortest path from v to D_I such that all internal vertices of P are in D_R. The fact that $|D_R \cap D_I| \geq 1$ implies that such a path exists. Staller chooses the internal vertex w of P that is adjacent to a vertex of D_I, and he picks this vertex in the imagined game instead of v. Similar arguments as above shows that choosing w is legal in the imagined game.

Consider now the situation when the optimal move of Staller made in the imagined game is not legal in the real game. Let v be the vertex picked by Staller in the imagined game. Since before the move of Staller the invariant was maintained, it follows that v is already played in the real game or v does not dominate new vertex. In such a case Staller chooses any legal vertex in the real game. It is straightforward that such a move does not violate the invariant.

Finally, suppose that the real game is finished. So, the number of vertices played in the real game is equal to the number of vertices played in the imagined game plus the number of vertices picked by Chooser in the real game. Let D_R^f (D_I^f, respectively) denote the set of played vertices at this stage of game in the real game (in the imagined game, respectively), so $|D_R^f| = |D_I^f| + k$ and D_R is a connected dominating set of G. We have to prove that $|D_R^f| \geq \gamma_{cg}(G) - k$. Staller continues the imagined game, carrying out both his and Dominator moves. Let $S = D_R^f \setminus D_I^f$. The strategy of Staller is as follows: Staller, when making a move as Dominator, chooses such vertices from S that are legal for the game, i.e., each vertex must dominate a new vertex and be adjacent to at least one of played vertices. Observe that such a vertex does not exist only if the set of played vertices in the imagined game form a connected dominating set, but in such a case the imagined game is finished. Next, Staller makes his move according to his optimal strategy.

First, we consider the case that when the real game ends, the next turn in the imagined game is Staller's turn. Staller makes his move according to his optimal strategy. Next he chooses (playing as Dominator) a vertex from S that is legal for the game. If such a vertex does not exist, the imagined game is over. In this case the number of played vertices in the imagined game is $|D_I^f| + 1$. Since the game is over, we have $|D_I^f| + 1 \geq \gamma_{cg}(G)$. But $|D_I^f| = |D_R^f| - k$, hence we get $|D_R^f| \geq \gamma_{cg}(G) + k - 1$. Thus, suppose that this is not the case and there is a legal vertex in S, say s_1, that can be chosen. Staller continues the game in the same way, i.e, playing as Dominator he chooses one by one legal vertices from S and answers optimally. Finally, after at most $|S|$ moves of Dominator and $|S|$ moves of Staller the game ends. Thus, $|D_I^f| + 2|S| = |D_I^f| + 2k \geq \gamma_{cg}(G)$ and hence $|D_R^f| - k + 2k \geq \gamma_{cg}(G)$.

Next, suppose that it is a Dominator's move. Staller chooses (as a move of Dominator) a vertex from S that is legal for the game. If such a vertex does not exist, the imagined game is over. Then $|D_I^f| \geq \gamma_{cg}(G)$, implying $|D_R^f| - k \geq \gamma_{cg}(G)$. If there is a legal vertex in S, Staller continues the game in the same manner as above. Finally, after at most $|S|$ moves of Dominator and $|S| - 1$ moves of Staller, the game ends. Thus, $|D_I^f| + 2|S| - 1 = |D_I^f| + 2k - 1 \geq \gamma_{cg}(G)$ and hence $|D_R^f| - k + 2k - 1 \geq \gamma_{cg}(G)$. □

It is straightforward that for any integer $k \geq 0$ there are graphs for which the upper bound given in Lemma 1 is attained (of course, for a certain sequence of Chooser's moves). The next proposition shows that the lower bound is also tight.

Proposition 1. *Let GameCh(k) stands for the connected domination game with Chooser in which Dominator and Staller play optimally and Chooser picks k vertices. For every $k \geq 1$ there exists a graph G such that*

$$\min\{|D| : D \text{ is the set of vertices played in GameCh(k) on } G\} = \gamma_{cg}(G) - k.$$

Proof. We define graphs $H_1, H_2, \ldots$ as follows: Let H_1 be the graph presented in Figure 1 (H_1 is obtained from two copies of H_0 in such a way that we identify the vertex u from one copy with u' from the other, similarly v and v', and then we remove one parallel edge). For $i \geq 2$, H_i is obtained from H_{i-1} by adding a copy of H_0 and identifying the vertex u from this copy with one of degree-2 vertices of H_{i-1}, say u', being adjacent to a center of a fan; then we identify v from H_0 with a degree-3 neighbour of u' in H_{i-1}, and remove one parallel edge (see Figure 1).

Observe that $\gamma_c(H_1) = 5$ (for instance, red vertices, as shown in Figure 1, form a minimum connected dominating set of the graph). But $\gamma_{cg}(H_1) = 6$, since Staller can always force one additional vertex during the game (one such play is presented in Figure 1; vertices x_i, $i = 1, \ldots, 6$, are consecutively chosen by players, in this order). Consider now the connected domination game with one move of Chooser played on H_1. Observe that an optimal strategy of Dominator is to start at x_1 (or x_5). So suppose Dominator starts at x_1. Then Staller has only four legal moves: x_2 (or x_2'), x_3 (or x_4). If Staller picks x_3 or x_4, then suppose Chooser picks x_5. Dominator chooses x_6, and Staller ends at x_2 (or x_2') with five moves. For otherwise, if Staller picks x_2 or x_2', then suppose that Chooser picks x_3. Dominator chooses x_5, and each legal move of Staller ends the game with five moves. Thus, for H_1, there is a move of Chooser such that the number of played vertices is $\gamma_{cg}(H_1) - 1$.

Similar reasoning can be applied to prove that for any $k \geq 2$, it holds $\min\{|D| : D$ is the set of vertices played in GameCh(k) on H_k $\} = \gamma_{cg}(H_k) - k$, i.e., there is a choice of k moves of Chooser such that, if Dominator and Staller play optimally, the game ends in $\gamma_{cg}(H_k) - k$ moves. □

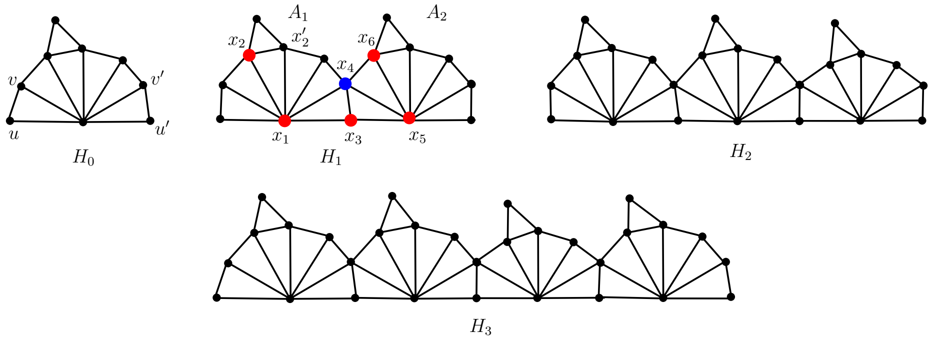


Figure 1: Graphs H_0, H_1, H_2 and H_3

3. CONNECTED DOMINATION GAME ON 2-TREES

As we observed earlier, the connected domination game on trees is trivial. So, we would like to consider next some tree-like structures. The first family of graphs that we deal with are 2-trees.

Let us recall that a k-tree can be recursively defined as follows: The complete graph on k vertices is a k-tree. If G is a k-tree on n vertices, then the graph obtained by adding a new vertex adjacent to the vertices of a k-clique of G is also a k-tree. The classes of k-trees and, in particular, of 2-trees are widely studied.

First we present a simple upper bound for the connected domination number of a 2-tree (although there are some polynomial-time algorithms for computing the connected domination number in the class of 2-trees, we were not able to find an upper bound for the connected domination number in terms of the order of a 2-tree).

Theorem 2. *If G is a 2-tree of order $n \geq 4$, then*

$$\gamma_c(G) \leq \left\lfloor \frac{n-2}{2} \right\rfloor.$$

Proof. We use induction on the number of vertices in a 2-tree. It is easy to see that the bound holds for 2-trees of order 4 and 5. So assume that G is a 2-tree of order n, $n > 5$. Since G is a 2-tree, it has a vertex of degree 2, say v. Let u, u' be the neighbours of v in G. Obviously, uu' is an edge in G.

Suppose first that n is even. Consider the graph $G' = G - v$. Clearly, G' is a 2-tree (of odd order equaling $n-1$). From the induction hypothesis, $\gamma_c(G') \leq (n-4)/2 = (n-2)/2 - 1$. Let D' be a connected dominating set of G' with size $\gamma_c(G')$. It is obvious that $D = D' \cup \{u\}$ is a connected dominating set of G, $|D| \leq |D'| + 1 \leq (n-2)/2$. Hence, $\gamma_c(G) \leq \lfloor (n-2)/2 \rfloor$, as desired.

Assume now that n is odd. We need to distinguish cases.

Case 1. Suppose that there is a degree-2 vertex v' in G, $v' \neq v$, such that v' is adjacent to u and u'. Let $G' = G - v$. G' is a 2-tree of even order $n-1$, thus it has a connected dominating set, say D', of size at most $(n-3)/2$. Since v' must have a neighbour in D', we can take D' as the minimum connected dominating set for the graph G. Thus the bound holds.

Case 2. Suppose that there is a degree-2 vertex v' in G, $v' \neq v$, such that v' is adjacent to one of the neighbours of v, w.l.o.g., let v' and u be neighbours. Let $G' = G - \{v, v'\}$. G' is a 2-tree of odd order equal to $n-2$. Thus let D' be the connected dominating set of G' of size $\gamma_c(G')$. The vertex u must have a neighbour in D' (or u itself is in D'). Hence $D = D \cup \{u\}$ is a connected dominating set of G with the desired number of vertices.

Now suppose that in G there is no degree-2 vertex satisfying either Case 1 or Case 2. Thus, in G there must be a degree-2 vertex, say x, adjacent to a degree-3 vertex, say y. Denote the other neighbour of x by z. Let $G' = G - \{x, y\}$. Observe that there is a connected dominating set D' of G' such that the size of D' equals $\gamma_c(G')$ and $z \in D'$ or z has a neighbour in D'. In either case, the set $D = D \cup \{z\}$ is a connected dominating set of G with the desired number of vertices. □

The bound presented in Theorem 2 is tight. To see this, consider some special 2-trees, called 2-*paths*, which can be defined inductively as follows: The unique 2-tree on 4 vertices (a diamond) is a 2-path. If G is a 2-path and x and y are adjacent vertices of G such that $d(x) = 2$ and $d(y) = 3$, and we add a new vertex of degree 2 joined with x and y, then the obtained graph is also a 2-path. It is easy to observe that if G is a 2-path (of order n, $n \geq 4$), then $\gamma_c(G) = \lfloor (n-2)/2 \rfloor$. (Small 2-paths and their minimal connected dominating sets realising the connected domination number are presented in Figure 2.)

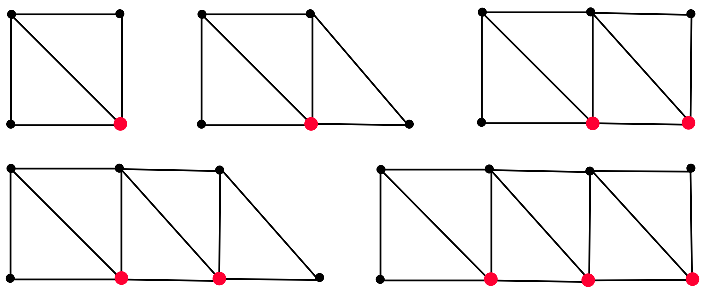


Figure 2: 2-paths

An interesting and useful characterization of 2-trees is through the notion of perfect elimination ordering (an analogous version of this property holds also for k-trees for any $k \geq 2$, but in the present paper we need it only for 2-trees, as stated below).

Let $(v_1, \ldots, v_n)$ be an ordering of the vertices of a graph G. If the vertex v_i precedes the vertex v_j in the ordering $(v_1, \ldots, v_n)$, then we write $v_i < v_j$. Let $N_p(v_i) = N(v_i) \cap \{v_1, \ldots, v_{i-1}\}$ and $N_s(v_i) = N(v_i) \cap \{v_{i+1}, \ldots, v_n\}$ for $i \in \{1, \ldots, n\}$. The ordering $(v_1, \ldots, v_n)$ is a *perfect* 2-*elimination ordering* for G if $v_1 v_2 \in E(G)$ and vertices of $N_p(v_i)$ induce K_2 for each $i \in \{3, \ldots, n\}$.

The next lemma is well known and follows directly from the definition, nevertheless, it will be very useful in our proofs.

Lemma 2. *A graph G is 2-tree if and only if there exists a perfect 2-elimination ordering for G.*

Let G be a 2-tree and $(v_1, \ldots, v_n)$ be a perfect 2-elimination ordering for G. Let v_i $(i \geq 2)$ be a vertex of G and S be the minimum sub-ordering of $(v_1, \ldots, v_n)$ such that $v_1, v_2, v_i \in S$ and the vertices of S induce a 2-tree. The *level* of v_i is the number of vertices in S, denoted by $l(v_i)$. We assume that the level of v_1 equals one.

Remark 1. Note that it may happen that $v_i < v_j$ and $l(v_i) > l(v_j)$.

Lemma 3. *If G is 2-tree of order at least 6, then G contains at least one of the configurations (three vertices x, y, z with given degrees and neighbourhoods) presented in Figure 3.*

Proof. The proof will be by contradiction. Let G be a 2-tree with at least 6 vertices and suppose that none of the configurations (1) to (14) occurs in G. Let $\mathbf{v}$ be a perfect 2-elimination ordering for G. Let x be a vertex of highest level in $\mathbf{v}$. Let $N_p(x) = \{x', x''\}$, $x'' < x'$ and $N_p(x') = \{x'', w\}$. Observe that either $x'' < w$ or $w < x''$.

First assume that there are at least three common successors x, y, z of x' and x''. Since x is the vertex of highest level, it follows that y, z are adjacent to both x' and x'' and $d(y) = d(z) = 2$, so we have configuration (1), a contradiction. Thus, we may assume that there are at most two common successors of x', x''. Suppose first that there are exactly two vertices x, y in $N_s(x') \cap N_s(x'')$. If there is a vertex z in $N_s(x') \cap N_s(w)$, then $d(z) = 2$ (since x is the vertex of highest level) and this implies configuration (2); otherwise we have configuration (3); in both cases we get a contradiction. Thus, in the rest part of the proof we assume that there is exactly one vertex, namely x, in $N_s(x') \cap N_s(x'')$. If there is a vertex in $N_s(x') \cap N_s(w)$, then we get configuration (4). Observe that since we assume that none of the configurations (1), (2), (3) is present, the vertices x' and w can have at most one common neighbour.

Now we use the fact that there is none of the configurations (1), (2), (3), (4) in G. Thus the vertices x', x'' have exactly one common successor, namely x (i.e. $N_s(x') \cap N_s(x'') = \{x\}$), and there is no successor of both x', w that is adjacent to both x', w (i.e. $N_s(x') \cap N_s(w) = \emptyset$). Consider the edge $x''w$. Suppose that there is a vertex, different from x', adjacent to this edge and such that it succeeds both x'' and w. Using the fact that x is the vertex of highest level and since configurations (1),(2),(3),(4) are forbidden, we obtain one of the configurations (5), or (6), or (7), that again leads to a contradiction.

We are now at position that we can assume that there is no vertex adjacent to both x'' and w that succeeds both x'' and w. From the above discussion we get the following claim (since otherwise one of the configurations (1)–(7) would appear).

Claim 1.

1. There is exactly one vertex in $N_s(x') \cap N_s(x'')$ (namely, x);
2. $N_s(x') \cap N_s(w) = \emptyset$;
3. there is exactly one vertex in $N_s(w) \cap N_s(x'')$ (namely, x').

Assume for a moment that $x'' < w$. Let $N_p(w) = \{x'', u'\}$. Suppose that there is a vertex v in $N_s(w) \cap N_s(u')$. Our assumption that configurations (1)–(7) are forbidden implies that there is at most one vertex that succeeds v and it has

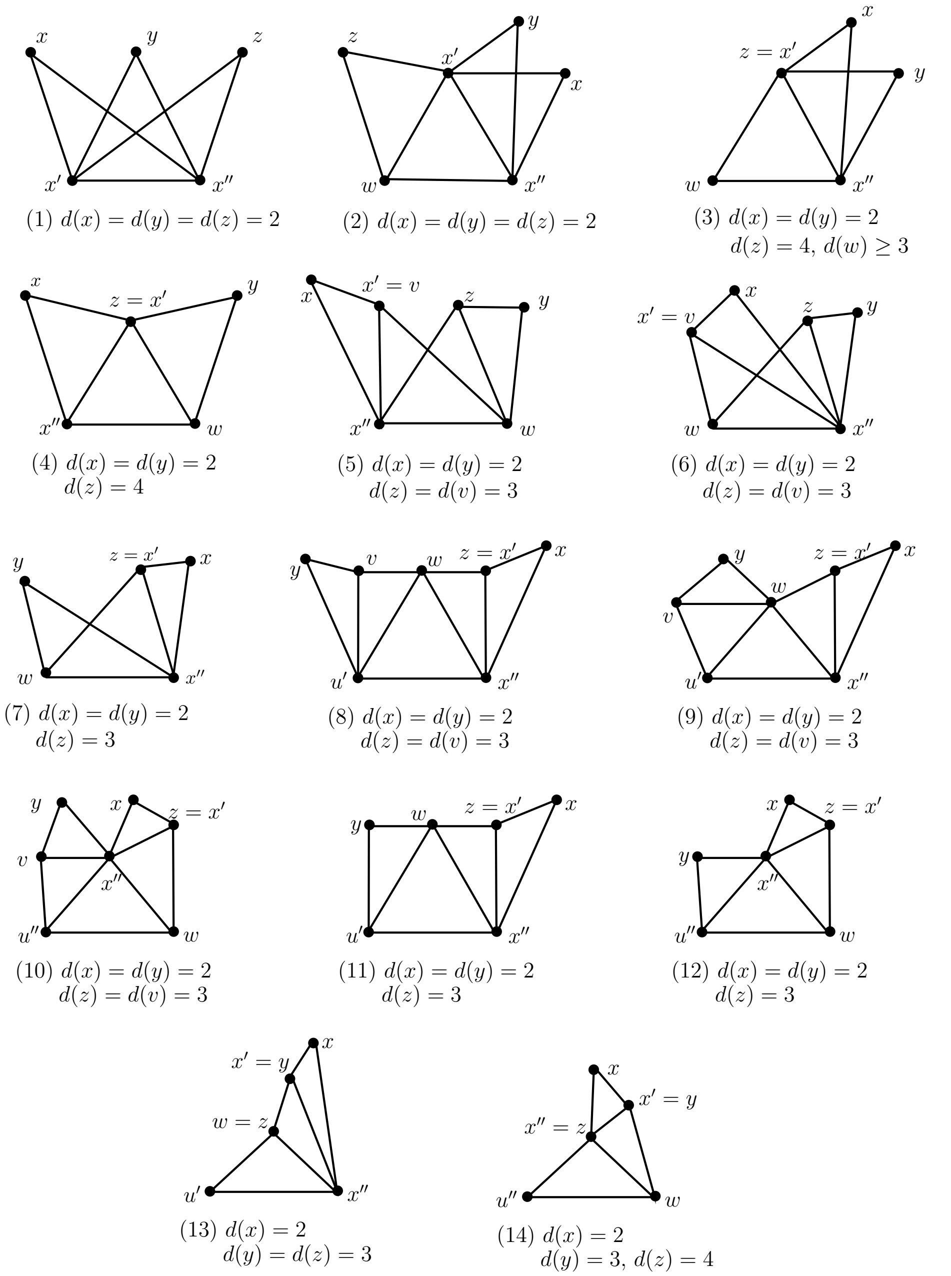


Figure 3: Configurations

degree 2. If such a vertex exists we obtain either configuration (8) or (9), otherwise configuration (11).

Suppose now that $w < x''$. Let $N_p(x'') = \{w, u''\}$. Assume that there is a vertex v in $N_s(x'') \cap N_s(u'')$. If there is a vertex that succeeds v, then we obtain configuration (10) or again configuration (9). Otherwise, we obtain configuration (12).

Now, since we forbid configurations (1)–(12), we can state the next claim.

Claim 2. 1. There is exactly one vertex in $N_s(x') \cap N_s(x'')$ (namely, x);

2. $N_s(x') \cap N_s(w) = \emptyset$;

3. there is exactly one vertex in $N_s(w) \cap N_s(x'')$ (namely, x');

4. (if $x'' < w$ then $N_s(u') \cap N_s(w) = \emptyset$) and (if $w < x''$, then $N_s(u'') \cap N_s(x'') = \emptyset$), where u' and u'' are defined as above.

Thus, for $x'' < w$ we obtain configuration (13) in G, and for $w < x''$ we have configuration (14); this final contradiction ends the proof. □

Our next result is an upper bound for the connected game domination number of a 2-tree. In the proof of this theorem we use the structural lemma given above, the Chooser Lemma (Lemma 1), and the following general result due to Kanté *et al.* [**18**], which establishes the connection between connected dominating sets and minimal cut-sets in a graph.

Theorem 3 ([**18**])**.** *In every connected graph $G = (V, E)$, a set $D \subseteq V$ is a connected dominating set if and only if $D \cap S \neq \emptyset$ for every minimal cut-set S in G.*

Theorem 4. *If G is a 2-tree of order $n \geq 4$, then $\gamma_{cg}(G) \leq \left\lceil \frac{2(n-4)}{3} \right\rceil + 1$.*

Proof. The proof is by induction on the number of vertices of the graph. It is easy to check that the assertion holds for 2-trees with 4, 5 or 6 vertices. Assume that $n \geq 7$, and that the theorem is true for all graphs with at most $n-1$ vertices. Let G be a 2-tree on n vertices. Lemma 3 implies that at least one of the configurations (1)–(14) is contained in G; we call this configuration F. Thus, G contains three vertices x, y, z with prescribed degrees and neighborhoods (we assume that the vertices of the configuration F are labelled as presented in Figure 3). Let $G' = G \setminus \{x, y, z\}$. Using Lemma 3, it is easy to observe that G' is a 2-tree. We show that $\gamma_{cg}(G) \leq \gamma_{cg}(G') + 2$, so the induction hypothesis implies the inequality in the theorem. The strategy of Dominator is that he will play two games at the same time: the imagined game — the connected domination game with Chooser on G', and the real game — the connected domination game on G. So two games are played at the same time: the real game on G, and the game with Chooser on G', imagined by Dominator. At the beginning of the game Dominator chooses a vertex in G' according to his optimal strategy and copy this move to G; then he will just copy each move of Staller to his imagined game, and respond in the imagined game with an optimal move from the connected domination game on G'. Each of these moves of Dominator is then also copied to the real game. However, the move of Staller may not be legal in the imagined game, since Staller chooses his moves with respect to the real game. The move of Staller may be not legal in the following two situations: either if Staller picks one of the vertices $\{x, y, z\}$ or the only new vertex dominated by the vertex picked by Staller is one of $\{x, y, z\}$.

First assume that each move of Staller in the real game is also legal in the imagined game. Thus, Dominator just copies the move of Staller and picks a vertex according to the optimal strategy for the game on G', and copies this move to G. When the game on G' is finished, some of the vertices of G still may not be dominated. More precisely, some vertices from $\{x, y, z\}$ may not be dominated. The game is continued on G; we show that after at most two more moves all vertices of G are also dominated.

To this end, suppose first that the next move is the move of Staller. If F is configuration (1), then vertices x', x'' could not be in the dominating set, since otherwise the game on G would end. Thus, Staller must pick either x' or x'' and the real game ends on G. Suppose that F is configuration (2). Observe that in G' the vertices w, x'' form a minimal cut-set. Thus, by Theorem 3, one of vertices w, x'' must be in the obtained dominating set of G'. Hence, the next move of Staller finishes the real game. Assume that F is configuration (3). If neither w nor x'' is in the dominating set, then Staller must pick one of these vertices. If he picks x'', then the real game ends, otherwise in the next move Dominator picks x'', and again the game on G ends. Observe that x'' cannot be in the dominating set of G', since otherwise the game on G would end. If w is in the dominating set obtained in the imagined game, then Staller must pick either x' or x'', and all vertices of G are dominated. Suppose that F is configuration (4). If one of w, x'' is in the dominating set, then Staller in the next move finishes the real game. Otherwise, Staller must pick either w or x''. Then, Dominator in the next move dominates all vertices of G. Suppose that F is either configuration (5) or (6). In both cases, vertices w, x'' form a minimal cut-set of G', so by Theorem 3 one of vertices w, x'' must be in the dominating set of G'. If x'' is in the dominating set of G', then this must be configuration (5), since otherwise the game on G would be finished as well. However, any legal move of Staller ends the game on G. So suppose now that w is in the dominating set of G'. Then if this is configuration (5), then Staller ends the game in one move, otherwise, if this is configuration (6), then either Staller ends the game or he chooses x' or z, but then Dominator ends the game in his next move. If F is configuration (7), then Staller makes the last move of the game on G when w is already in the dominating set of G', otherwise Staller must pick either w or x''. In the former case Dominator picks x'' and dominates all vertices of G. In the latter case after the move of Staller the game ends. Now suppose that F is configuration (8) or (9). The vertices u', x'' form a minimal cut-set of G' and also the vertices u', w form a minimal cut-set of G' (we may assume $n \geq 8$ here, since for otherwise, it is easily seen $\gamma_{cg}(G) \leq 3$). Thus, by Theorem 3 we conclude that either u' or both w, x'' are in the dominating set of G'. If both w, x'' are in the dominating set of G', then the next move of Staller finishes the game. If u' is in the dominating set of G', then Dominator can finish the game regardless of which vertex has been picked by Staller. If F is configuration (10), the very similar arguments show that after at most two moves the game is over, in this case sets $\{u'', x''\}$ and $\{u'', w\}$ form the cut-sets of G' (again, we may assume $n \geq 8$, since otherwise it is straightforward that $\gamma_{cg}(G) = 1$). If F is configuration (11) or (12), then $\{u', x''\}$ or $\{u'', w\}$ is a cut-set of G'. So, again argument that at least one vertex from a cut-set must be in a connected dominating set of G' implies that after at most two moves the game is finished. For the case when F is configuration (13) or (14), Dominator picks x'', whenever the game is not over after the move of Staller, and hence after at most two moves all vertices of G are dominated.

Suppose that the next move is of Dominator. We can see that if F is one of the configurations (1)–(7), (10)–(14), then Dominator can pick a vertex of G' (such a vertex is dominated in G' what implies that we obtain a connected dominating set) in such a way that after the move of Dominator at most one vertex of G is not dominated, so the game on G finishes after at most two more moves than the game on G'. Suppose that F is configuration (8) or (9). As above, we may assume $n \geq 8$. The vertices u', x'' form a minimal cut-set of G' and also the vertices u', w form a minimal cut-set of G'. Thus, by Theorem 3 either u' or both w, x'' are in the dominating set of G'. If both w, x'' are in the dominating set of G', then Dominator picks v. If u' is in the dominating set of G', then Dominator picks x''.

To finish the proof it remains to show that even if Staller makes a move that is not legal in the imagined game, Dominator can continue the real and imagined games in such a way that the real game on G finishes after at most two more moves than the imagined game on G'. Below we describe the strategy for Dominator. Several cases have to be distinguished.

Case 1. Assume that Staller picks a vertex v of G such that the only new vertex dominated by v is one of x, y, z and after Staller's move the vertices x, y, z are dominated. (Note that if x, y and z are dominated, then all their neighbours are also dominated, so none of x, y, z can be chosen in next moves.) Dominator copies the move of Staller to the imagined game as the move of Chooser, next picks any legal vertex of G and copies this move to G', again, as the move of Chooser. Observe that these last two moves were legal moves of Chooser in the imagined game on G'. Dominator then continues the imagined game on G' according to the optimal strategy for the game in which Chooser picks two vertices. Since vertices x, y, z are already dominated, it follows that when the imagined game on G' is finished, the real game on G also is finished. Lemma 1 implies that the number of moves is at most $\gamma_{cd}(G') + 2$.

Case 2. Suppose now that Staller picks a vertex v of G such that the only new vertex dominated by v is one of x, y, z and after Staller's move there is a vertex in x, y, z that is not dominated. In such a case Dominator copies this move to the imagined game as the move of Chooser and picks a vertex that is legal for Chooser in G' such that after such a move in G all vertices x, y, z are dominated in G. Similar discussion as in the case when Staller makes only legal moves shows that Dominator can find such a vertex. Dominator treats this move as a move of Chooser on G'. Next, Dominator copies the second move to G and continues the imagined game on G' according to the optimal strategy for the game in which Chooser picks two vertices. Both games — the real game and the imagined game — ends at the same time and by Lemma 1, the number vertices in the obtained connected dominating set is at most $\gamma_{cd}(G') + 2$.

Case 3. Suppose that Staller picks a vertex that is not in G' (i.e., Staller picks one of x, y, z) and all vertices of x, y, z are dominated. Let s be the vertex picked by Staller. Since this move must be legal in the real game on G (s must have an undominated neighbour), it follows that $s = z$ and F is one of configurations

(3)–(12) or $s = z$ or $s = y$ and F is the configuration (14). Observe that new vertex dominated by s is one of x, y, z, what implies that the set of dominated vertices in the imagined game on G' and in the real game on G differ only on vertices of x, y, z. Dominator picks any legal vertex in G (such a move is legal for Chooser in the game on G'). Next, Dominator copies this move to G' (as a move of Chooser) and continues the imagined game on G' according to the optimal strategy for the game in which Chooser picks one vertex. By Lemma 4 when the imagined game on G' is over, then the number of picked vertices is at most $\gamma_{cg}(G') + 1$. Furthermore, the real game is also over and the set of picked vertices in G contains one more vertex, namely, the vertex s.

Case 4. Suppose that Staller picks a vertex that is not in G' (i.e., Staller picks one of x, y, z) and there is a vertex in x, y, z that is undominated. Let s be the vertex picked by Staller. Since this move must be legal, we see that $s = z$ and the new vertex dominated by s is one of x, y, z. Moreover, F is one of configurations (5)–(13). Dominator picks a vertex of G' such that after such a move in G all vertices of x, y, z are dominated in G. Next Dominator copies this move on G' (as Chooser's move) and continues the imagined game on G' according to the optimal strategy for the game in which Chooser picks one vertex. Thus, by Lemma 1 when the games are finished the number of vertices in the obtained connected dominating set of G is at most $\gamma_{cg}(G') + 2$.

Therefore, $\gamma_{cg}(G) \leq \gamma_{cg}(G') + 2$, as required. This concludes the proof. □

To see that the bound given in Theorem 4 is tight, we again consider 2-paths. As we observed earlier, if G is a 2-path (of order $n \geq 5$), then $\gamma_c(G) = \left\lfloor \frac{n-2}{2} \right\rfloor$. In case of the connected domination game on 2-paths, we obtain the following formulas, which in particular imply that the bound given in Theorem 4 is tight.

Proposition 2. *If G is a 2-path of order $n \geq 5$, then*

$$\gamma_{cg}(G) = \begin{cases} \left\lceil \frac{2(n-4)}{3} \right\rceil + 1, & if\ n \equiv 1 (\mathrm{mod}\ 3), \\ \left\lceil \frac{2(n-4)}{3} \right\rceil, & otherwise. \end{cases}$$

Proof. To prove the equality, we will separately show the two inequalities.

For the lower bound, it is enough to observe that Staller can follow the strategy: choose the vertex that increases the number of dominated vertices only by one. He can maintain this during the game, and since the moves of Dominator (except possibly the first move) does not increase the number of dominated vertices by more than two, the desired bound follows by a simple calculation.

If $n \equiv 1 (\mathrm{mod}\ 3)$, then the upper bound follows from Theorem 4. Otherwise, let Dominator start at a degree-4 vertex adjacent to a degree-2 vertex (this move dominates five vertices). Next, Dominator can play in such a way that by each pair of consecutive moves of Staller and Dominator the number of dominated vertices increases by at least three. Again, the bound follows by a simple calculation. □

In order to obtain bounds for the connected domination and connected game domination number we also take into account also some other structural properties of 2-trees. Namely, we observe that degree-2 vertices in a 2-tree play a role somehow analogous to leaves in a tree. If G is any connected graph, with at least 3 vertices, then obviously a leaf cannot be included in a minimal connected dominating set. In case of trees, we have $\gamma_c(T) = n - |L(T)|$, where $L(T)$ is the set of all leaves in a tree T. We can observe that for a 2-tree T a similar result holds. Namely, let $L_1(T)$ be the set of degree-2 vertices in T, and let $L_2(T)$ be the set of degree-2 vertices in a graph $T - L_1(T)$. Using this notation we give another simple upper bound for the connected domination number of 2-trees.

Observation 1. *For any 2-tree T of order $n \geq 5$ which is not a 2-star (here a 2-star is a 2-tree such that $L_1 \cup L_2 = V$), it holds*

$$\gamma_c(T) \leq n - (|L_1(T)| + |L_2(T)|).$$

This bound is also tight — it is quite easy to construct 2-trees for which we have equality in the above formula.

Observe next that for any graph G, and for any vertex $x \in V(G)$ with the property $G[N(x)]$ is a clique in G, it holds that x can be chosen only in the first move of the game. Thus the game on a 2-tree T (of order $n \geq 4$) can always end in at most $n - |L_1(T)|$ moves. Similar result holds for k-trees.

Observation 2. *Let T be a k-tree of order $n \geq k + 2$, $k \geq 2$. Let L be the set of all vertices of degree k in T. It holds*

$$\gamma_{cg}(T) \leq n - |L|.$$

4. CONNECTED DOMINATION GAME ON CARTESIAN PRODUCTS

In this section we consider the connected domination game played on certain Cartesian product graphs. In our considerations, we take a complete graph as one factor, and a tree as the other. These graphs show that the upper bound for the connected game domination number given in Theorem 1 is tight. However, we start with the following more general result.

Theorem 5. *Let G be a connected graph with m vertices, $m \geq 4$, and let $\gamma_c(G) = \gamma$. Then for any $n \geq 2$ it holds*

$$\gamma_{cg}(K_n \square G) \leq \min\{2m - 1, 2n\gamma - 1\}.$$

Proof. Let X be any connected dominating set of G with γ vertices. Observe at the beginning that each of the following two sets is a connected dominating set of $K_n \square G$: D_1 containing all vertices corresponding to one, fixed, vertex of K_n, and D_2 containing all vertices corresponding in the product to vertices of the set X. Take D to be the smallest (with respect to the order) set among D_1, D_2. An approach similar to that in the proof of Theorem 1 gives the required bound for $\gamma_{cg}(K_n \square G)$. □

From now on, we consider Cartesian product with a complete graph as one factor, and a tree as the other. Since in a tree (of order at least three) there is only one minimal connected dominating set — the set of all non-leaves — the following is an immediate corollary of the previous theorem.

Corollary 1. *Let T be a tree of order $m \geq 4$ with l leaves, and let $n \geq 2$ be an integer. Then*

$$\gamma_{cg}(K_n \Box T) \leq \min\{2m-1, 2n(m-l)-1\}.$$

Bounds given in Theorem 5 are tight, what follows from the next result, which deals with Cartesian products when one of the factors is a star. In fact, in such a case we obtain exact formulas for the connected game domination number. Moreover, parts (1) and (2) of Theorem 6 show that the general upper bound given in Theorem 1 is also tight.

Theorem 6. *Let $m \geq 4$, $n \geq 2$. Let $G = K_n \Box K_{1,m-1}$.*

(1) *If $m > n$, then $\gamma_{cg}(G) = 2n - 1 = 2\gamma_c(G) - 1$.*

(2) *If $m < n$, then $\gamma_{cg}(G) = 2m - 1 = 2\gamma_c(G) - 1$.*

(3) *If $m = n$, then $\gamma_{cg}(G) = 2n - 3$.*

Proof. Observe first that both the following sets are connected dominating sets of G: the set of all m vertices corresponding to one, fixed, vertex of K_n and the set of all n vertices corresponding to a center of the star. Further, the one which is smaller, call it D, realises the connected domination number. Hence D has $\gamma_c(G)$ vertices.

From Theorem 1 we get that if $m > n$, then $\gamma_{cg}(G) \leq 2n - 1$, and if $m < n$, then $\gamma_{cg}(G) \leq 2m - 1$. To get the equalities, we propose a certain strategy for Staller.

(1) Let $m > n$. Hence D has n vertices corresponding to the center of $K_{1,m-1}$. Roughly speaking, Staller can play in such a way that whenever Dominator chooses a vertex from D, then Staller answers with a move not in D. More precisely, Staller plays to hold the following invariant: (*) $\min\{c, a\}$ can drop maximally by one between two consecutive moves of Staller, where c is the number of not played vertices in D and a is the number of not played lines (a *line* here is understood as a set of vertices of one copy of K_n, and a line is *played* when at least one its vertex is chosen). A *level* is a set of vertices corresponding to one copy of the star.

At the beginning of the game, $\min\{c, a\} = n$. The game ends when $\min\{c, a\} = 0$. Clearly, if (*) is true during the game, then at least $2n - 1$ vertices have to be chosen. We will prove that as long as the game does not end or the number of chosen vertices is less than $2n - 1$, then Staller can play to hold (*). If the first move of Dominator was to choose a vertex x from the set D, then Staller plays at any vertex in the same level as x. If Dominator's first move was at a vertex y not from D, then Staller chooses a vertex from D in the same level as y. In either case, after such a move of Staller we have $\min\{c, a\} = \min\{n - 1, m - 2\} = n - 1$ and (*) holds.

The strategy for Staller for next moves is similar. If Dominator plays at a vertex from D, then Staller answers in the same level, but in a new line. Suppose Dominator plays at a vertex not from D. Then Staller chooses a vertex from D in the same level, if this is a legal move; otherwise, a vertex from D in the next level. Observe that in all cases, (*) still holds after Staller's move.

(2) The case when $m < n$ can be done similarly, as above.

(3) Suppose $m = n$. Observe that $\gamma_{cg}(K_n \Box K_{1,m-1}) \geq 2n-3$, since otherwise Dominator would have a strategy for finishing a game on $K_n \Box K_{1,m}$ with at most $2n-2$ moves. It remains to prove that $\gamma_{cg}(K_n \Box K_{1,m-1}) \leq 2n-3$. Let D be the set of vertices corresponding to the center of star. Dominator plays according to the following strategy: the first move of Dominator is at any vertex, say x, from D. Next, if the first move of Staller is in the same copy of the star, then Dominator chooses the next vertex in this copy. We denote the set of all vertices in this copy by Y. Otherwise, if Staller's first move is to choose another vertex from D, then Dominator continues by choosing any vertex from D. Dominator next played in this way: if the first moves where on D then he chooses vertices from D; if the game was started at Y, then he chooses vertices from Y. Observe that in either case, the game ends after at most $2n-6$ next moves. Hence, the total number of moves does not exceed $2n-6+3 = 2n-3$. □

Next, we consider products with paths, and give exact formulas for the connected game domination number of such graphs.

Theorem 7. *Let* $m \geq 4$. *Then*

(1) $\gamma_{cg}(K_2 \Box P_m) = 2m-4$;

(2) $\gamma_{cg}(K_3 \Box P_m) = 2m-3$;

(3) $\gamma_{cg}(K_n \Box P_m) = 2m-1$, *for* $n \geq 4$.

For short paths, it holds

- $\gamma_{cg}(K_2 \Box P_2) = 2$, $\gamma_{cg}(K_n \Box P_2) = 3$, *for* $n \geq 3$.
- $\gamma_{cg}(K_2 \Box P_3) = \gamma_{cg}(K_3 \Box P_3) = 3$, $\gamma_{cg}(K_n \Box P_3) = 5$, *for* $n \geq 4$.

Proof. Suppose $m \geq 4$.

(1) Let $G = K_2 \Box P_m$. We denote the vertices of the first (second, respectively) P_m-level of G by $x_1, \ldots, x_m$ ($y_1, \ldots, y_m$, respectively). For $1 \leq i \leq m$, the vertices x_i, y_i form a line number i in G. A line is *played*, if at least one vertex from it is already chosen, otherwise a line is *unplayed*. Two lines $i, i+1$ are said to be *adjacent*.

To prove that $\gamma_{cg}(K_2 \Box P_m) \leq 2m-4$, we give the strategy for Dominator. Dominator will play according to the following rules: (R1) he chooses a vertex neither from line 1 nor m; (R2) he chooses a vertex v of a played line only if v is either in line 2 or line $m-1$ and all lines $2, \ldots m-1$ are already played; otherwise, he always chooses a vertex of an unplayed line (different from 1 and m). First we observe that Dominator can always apply this strategy. Indeed, if during the game there is an unplayed line among $2, \ldots, m-1$, he chooses an unplayed line adjacent to a played one. Since the set of played vertices must induce a connected graph, there is at least one vertex that is legal in this line. If all lines in $2, \ldots, m-1$ are played and some vertices are still not dominated (the undominated vertices are in the first or the last line), then Dominator chooses an unplayed vertex of the line 2 or line $m-1$. At the end of the game at least four vertices remain unplayed, namely, two vertices on each of the ends of the graph G. More precisely, on one end it remains unplayed either x_1, y_1, or x_1, x_2 or y_1, y_2, while on the other end it remains unplayed either x_m, y_m, or x_m, x_{m-1} or y_m, y_{m-1}. Hence, $\gamma_{cg}(G) \leq 2m-4$.

Now, we prove that $\gamma_{cg}(G) \geq 2m-4$. To this end, we present a strategy for Staller. The strategy for Staller is as follows: he chooses a vertex from a played line. Staller can always apply this strategy, since each time he forces Dominator to play a vertex of an unplayed line. At the end of the game either four or two vertices remain unplayed (two vertices remain unplayed when Dominator starts the game with a vertex of the first or last line).

(2) Let $G = K_3 \Box P_m$. As above, we denote the vertices of the first (second and third, respectively) P_m-level of G by $x_1, \ldots, x_m$ ($y_1, \ldots, y_m$ and $z_1, \ldots, z_m$, respectively). For $1 \leq i \leq m$, the vertices x_i, y_i and z_i form a line number i in G.

First, we will prove that $\gamma_{cg}(K_3 \Box P_m) \leq 2m-3$. We use induction on m. Take $K_3 \Box P_4$. Dominator starts at x_2. Then, if Staller picks the vertex in line 2, say y_2, then Dominator answers at z_2. For otherwise, if Staller takes x_3 or x_1, then Dominator answers by choosing a vertex in a new line. In either case, the game will end after at most two more moves. Hence, $\gamma_{cg}(K_3 \Box P_4) \leq 5$.

So suppose now that $m \geq 5$ and consider $G = K_3 \Box P_m$. Let $G' = G - \{x_1, y_1, z_1\}$. From the induction hypothesis, $\gamma_{cg}(G') \leq 2m-5$. Dominator has the following strategy: he will be playing two games — the real one on G, and the imagined game with Chooser on G'. In the first phase of the game, Dominator will copy each Staller's move from the real game to the imagined game, and then answer in the imagined game with an optimal move, which will be then copied to the real game. This phase of game lasts until the first illegal move. Observe that it cannot be a move of Dominator copied from the imagined game to the real one, hence it must be the move of Staller, which is legal in the real game, but is not legal in the imagined game. This means that either (a) Staller chooses a vertex from line 2, say x_2, and the only vertex that is "privately" dominated by x_2 is x_1 or (b) Staller chooses a vertex from the first line, say x_1.

Suppose that we have the situation (a). From the fact that only x_1 is newly dominated by x_2, it follows that all vertices from line 3 are chosen. If in line 1 there is a vertex that is not dominated, then Dominator chooses in the real game x_1. In the imagined game Dominator copies Staller's move and sees it as a move of Chooser. Observe that since now all moves of Staller in the real game are legal in the imagined game and if all vertices will be dominated in the imagined game then also all vertices will be dominated in the real game. By Lemma 1 at the end of the imagined game the number of played vertices is at most $\gamma_{cg}(G')+1$. Hence the real game ends after at most $\gamma_{cg}(G')+2$ moves. If in line 1 all vertices are dominated, then Dominator chooses in the real game any legal vertex. In the imagined game Dominator copies both moves (of Staller and of Dominator) and sees them as moves of Chooser. Again after such moves all moves of Staller in the real game are legal in the imagined game and if all vertices will be dominated in the imagined game then also all vertices will be dominated in the real game. Since Chooser made two moves in the imagined game, the imagined game ends after at most $\gamma_{cg}(G')+2$ moves, so also the real game ends after at most $\gamma_{cg}(G')+2$ moves.

So let us assume that the situation (b) occurs. Thus, the vertex x_2 was chosen (in both games). Now, the vertices of line 1 and line 2 are dominated. Dominator chooses any legal vertex in the real game and copies this move to the imagined game as the move of Chooser. At the remaining part of the game all moves of Staller in the real game are legal in the imagined game. When the imagined game is finished the number of played vertices is at most $\gamma_{cg}(G')+1$, since Chooser picked one vertex. Thus, at the end of real game the number of played vertices is at most $\gamma_{cg}(G')+2$.

To finish the proof for the upper bound, we need to consider the situation, when all moves of Dominator and Staller were legal, and the imagined game ends. Observe that we only need to consider the case, when the next move it is the Dominator's move. Suppose first that there is vertex from line 2, say x_2, that is already chosen. Dominator picks x_1 and ends the real game with at most $\gamma_{cg}(G')+1$ moves. So let none of the vertices from line 2 be chosen. Since the imagined game ends, all vertices from line 3 must be chosen. Dominator picks a vertex from line 2. Regardless of the move of Staller, Dominator can end the real game in his next turn. So the real game ends at most 3 moves after the imagined. But observe that $\gamma_{cg}(G')$ is odd, hence if the imagined game ends on the move of Staller, it means it ends with at most $\gamma_{cg}(G')-1$ moves. Thus the real game ends with at most $\gamma_{cg}(G')+2$ moves, as required.

Now, we are going to show that $\gamma_{cg}(K_3\Box P_m)\geq 2m-3$.

As previously, let A denote the set of all played lines (at some stage of the game). The strategy for Staller is as follows: (R1) if Staller can choose a vertex from a line from the set A, then he picks it; (R2) otherwise, Staller makes any legal move.

We will prove that during the game, if neither line 1 nor m is played, then Staller can hold the following invariant:

(*) after the move of Staller, $|W|\geq 2|A|$, where W is the set of all vertices that are chosen so far.

We use induction on the number of moves. Clearly, after first two moves, Staller's strategy guarantee that (*) holds. So suppose that after some moves, it is Staller's turn. Let W' be the set of vertices chosen before the last move of Dominator, A' — the set of lines played before the last move of Dominator. Consider first the situation, when Staller plays according to Rule R1. Hence, $|W|=|W'|+2$, $|A|\leq|A'|+1$. Using the induction hypothesis we get $|W|\geq 2|A|$. So suppose now that Staller is forced to play according to Rule R2. It follows that all vertices from the first and the last line in A' are chosen before Staller's move. Thus, in one of these lines (which in fact can be the same line, if this will be the forth move of the game) the vertex which was chosen last was chosen by Dominator in his last move. So, $|W|=|W'|+2$ and $|A|=|A'|+1$. Thus, from the induction hypothesis, we get $|W|\geq 2|A|$.

It may happen, that neither line 1 nor m is played, but the game ends. Thus, all vertices from lines 2 and $m-2$ are chosen. Consider first the case, when the game ends after the move of Dominator. Clearly, this move have to be the move of choosing the third, not played yet, vertex from line 2 or $m-1$. Before this move, (*) was satisfied. Thus after this move the number of played lines is $m-2$, the number of chosen vertices is at least $2(m-2)+1=2m-3$, as required. So suppose that the final move was done by Staller. Observe that before his move and Dominator's last move, we have the same number of played lines, as now. Thus using (*) we get that the number of vertices chosen at the end of the game is at least $2(m-2)+2=2m-2$.

Suppose now, that at some turn of the game line 1 or m is played for the first time, w.l.o.g., we may assume that one of the players picks a vertex x_1 from line 1. Consider two cases.

Case 1. Suppose it was Staller who chose x_1. But from Staller's strategy it follows that all vertices from line 2 were chosen before the move of Staller. Thus such a move is not legal.

Case 2. It was a move of Dominator. Let s be the largest index of played lines.

Case 2.1. Suppose there is a vertex in line s that is not chosen. Staller picks this vertex. After this move, the condition from (*) still holds. Suppose that the game is continued and observe that Staller's strategy guarantees that (*) is true, until one of the players played at line m or the game ends. Suppose first that there is a move on line m. Obviously, the game ends and this is Dominator's move. Before this move (*) was satisfied, hence at the end of game we have at least $2(m-1)+1=2m-1$ played vertices. Consider now the case when the game ends without a move at line m. This means that all vertices from line $m-1$ are chosen. It is easy to observe that regardless who made the last move, the number of played vertices is at least $2(m-1)=2m-2$.

Case 2.2. Assume that all vertices from line s are already chosen. Then if the game is not finished yet, Staller plays according to Rule R2, i.e., he plays at line $s+1$. Hence after his move we get that the following condition is satisfied (**) the number of played vertices is at least $2|A|-1$. The game continues as previously. Observe that after each Staller's move, (**) is satisfied. An analysis similar, to that of Case 2.1, but with condition (*) replaced by (**), leads to a conclusion, that also in this case the number of played vertices at end of the game is at least $2m-3$.

(3) Let $G=K_n\Box P_m$, where $m,n\geq 4$. The fact that $\gamma_{cg}(K_n\Box P_m)\leq 2m-1$ follows from Theorem 1. We will present a strategy for Staller that guarantees at least $2m-1$ moves. We denote the vertices of the i-th P_m-level of G by $x_{i,1},x_{i,2},\ldots,x_{i,m}$. For $1\leq j\leq m$, the vertices $x_{1,j},x_{2,j},\ldots,x_{n,j}$ form a line in G.

As previously, let A denote the set of all played lines. The strategy for Staller is as follows: (R1) if Staller can choose a vertex from a line from the set A, then he picks it; (R2) otherwise, Staller makes any legal move.

We will prove that during the game, if neither line 1 nor m is played, then Staller can hold the following invariant:

(*) after the move of Staller, $|W|\geq 2|A|$, where W is the set of all vertices that are chosen so far.

We use induction on the number of moves and assume that neither line 1 nor m is played. Clearly, after first two moves, Staller's strategy guarantees that (*) holds. So suppose that after some moves, it is Staller's turn. Let W' be the set of vertices chosen before the last move of Dominator, A' — the set of lines played before the last move of Dominator. Consider first the situation, when Staller plays according to Rule R1. Hence, $|W|=|W'|+2$, $|A|\leq|A'|+1$. Using the induction hypothesis we get $|W|\geq 2|A|$. So suppose now that Staller is forced to play according to Rule R2. It follows that all vertices from the first and the last line in

A' are chosen before Staller's move. Thus, in one of these lines (which in fact can be the same line at the beginning of the game) the vertex which was chosen last was chosen by Dominator in his last move. So, $|W| = |W'| + 2$ and $|A| = |A'| + 1$. Thus, from the induction hypothesis, we get $|W| \geq 2|A|$.

Consider now the following cases.

Case 1. Neither line 1 nor m is played, but the game ends. Thus, all vertices from lines 2 and $m-2$ are chosen. Since $n \geq 4$, it follows that in line 2 at least 4 vertices are chosen, and the same in line $m-2$. Suppose first that the last move in the game was done by Dominator. Let W' be the set of vertices chosen before the last three moves, A' — the set of lines played then. Using (*) we get $|W'| \geq 2|A'|$. Observe that $|A'| = m-2$. Thus, $|W| = |W'|+3 \geq 2(m-2)+3 = 2m-1$. Consider now the case when the last move was done by Staller. Let W' be the set of vertices chosen before the last four moves, A' — the set of lines played then. Using (*) we get $|W'| \geq 2|A'|$. If $n \geq 5$, then $|A'| = m-2$. Thus, $|W| = |W'|+4 \geq 2(m-2)+4 = 2m$. If $n = 4$, but $|A'| = m-2$, then again $|W| \geq 2m$. So suppose $n = 4$ and $|A'| = m-3$. Then let W'' be the set of vertices chosen before the last six moves, A'' — the set of lines played then. Again, $|W''| \geq 2|A''|$. Moreover, $|A| = |A''| - 1 = m-3$, thus $|W| = |W''| + 6 \geq 2(m-3) + 6 = 2m$.

Thus from now on we may assume, that at some turn of the game line 1 or m is played for the first time, w.l.o.g., we may assume that one of the players picks a vertex $x_{1,1}$ from line 1. Consider two cases.

Case 2. Suppose it was Staller who chose $x_{1,1}$. But from Staller's strategy it follows that all vertices from line 2 were chosen before the move of Staller. Thus such a move is not legal.

Case 3. It was a move of Dominator. Let s be the largest index of played lines.

Case 3.1. Suppose there is a vertex in line s that is not chosen. Staller picks this vertex. After this move, the condition from (*) still holds. Suppose that the game is continued and observe that Staller's strategy guarantees that (*) is true, until one of the players played at line m or the game ends. Suppose first that there is a move on line m. Obviously, the game ends and this is Dominator's move. Before this move (*) was satisfied, hence at the end of game we have at least $2(m-1)+1 = 2m-1$ played vertices. Consider now the case when the game ends without a move at line m. This means that all vertices from line $m-1$ are chosen. A similar consideration as in Case 1 gives the desired bound for $|W|$.

Case 3.2. Assume that all vertices from line s are already chosen. Then if the game is not finished yet, Staller plays according to Rule R2, i.e., he plays at line $s+1$. Hence after his move we get that the number of played vertices is at least $2|A|-1$. But after Staller's move, we have an even number of picked vertices. Thus $|W| \geq 2|A|$ also in this case and (*) still holds. The game continues as previously. Observe that after each next Staller's move, (*) is satisfied. Using similar arguments as in Case 3.1, we get that the game ends with at least $2m-1$ chosen vertices. $\square$

5. STALLER-START AND STALLER-PASS GAMES

An interesting task is to consider some variants of the connected domination game, namely, the game in which Staller has the first move and the game in which Staller can skip some moves.

At the beginning of this section we consider the connected domination game with the first move of Staller. Such a game is called a *Staller-start game.* The connected game domination number in such a game is denoted by $\gamma'_{cg}(G)$. Games of this type are extensively studied for the domination game. In particular, the connection between the number of played vertices in a domination game and in a Staller-start domination game was established. Namely, it was proved that these two numbers can differ by at most one, see [**1, 19**].

First, we observe that similarly as the upper bound in Lemma 1 one can prove the following lemma:

Lemma 4. *Consider the connected domination game with Chooser on G when Staller starts the game. Suppose that in the game Chooser picks k vertices and that Dominator and Staller plays optimally. Then at the end of the game the number of picked vertices is at most $\gamma'_{cg}(G) + k$.*

It is easy to see that the two parameters, $\gamma_{cg}(G)$ and $\gamma'_{cg}(G)$, may differ. It is easy to observe that if Staller starts the game, then the number of played vertices can be larger than in the Dominator-start game. For example, consider a tree T with n vertices, $n \geq 3$, and l leaves. We have $\gamma_{cg}(T) = n - l$, but $\gamma'_{cg}(T) = n - l + 1$. Below we present examples of graphs for which the difference between the connected game domination number and the connected game domination number in Staller-start game equals 1, 2, 3, and 4, respectively. We have no examples of graphs with bigger difference, but we also cannot see a reason, why such graphs would not exist. Hence, we pose the problem

Problem 1. *Find the maximum k for which there exists a graph G satisfying*

$$\gamma'_{cg}(G) = \gamma_{cg}(G) + k.$$

The graphs for which the difference between the connected game domination number and the connected game domination number in Staller-start game is 1, 2, 3, and 4, respectively, are presented in Figure 4. (G_1 is a called a *fan.*) Observe that for each of them the connected game domination number equals the connected domination number. To see this, consider the strategy for Dominator to start at vertex labelled by x, and then always choose a vertex in such a way that the set of all already played vertices can be extended to a minimum connected dominating set. Observe that Staller cannot make any move that violates this condition. Thus we get $\gamma_{cg}(G_i) = \gamma_c(G_i)$, for $i = 1, 2, 3, 4$. On the other hand, if we play in the Staller-start game, and Staller starts at vertex labelled by y, then it can be proved (by the case by case analysis) that $\gamma'_{cg}(G_1) = \gamma_{cg}(G_1) + 1 = 2$, $\gamma'_{cg}(G_2) = \gamma_{cg}(G_2) + 2 = 4$, $\gamma'_{cg}(G_3) = \gamma_{cg}(G_3) + 3 = 8$, and $\gamma'_{cg}(G_4) = \gamma_{cg}(G_4) + 4 = 16$.

It is an easy observation that the fans in graphs $G_1, \ldots, G_4$ can be extended (in the sense, that each of them can have more triangles). So in fact the above example shows that for each $k \in \{1, \ldots, 4\}$ there is an infinite family of graphs G

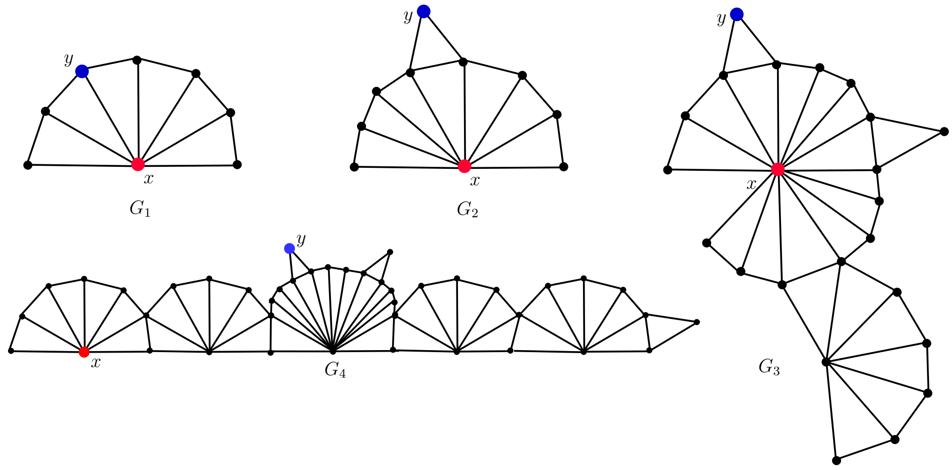


Figure 4: Graphs G_1, G_2, G_3 and G_4

such that $\gamma'_{cg}(G) = \gamma_{cg}(G) + k$. For $k \in \{1, 2, 3\}$ we have also constructed some other (more complicated) examples of such graphs.

On the other hand, if Staller starts the game, then the number of moves can be at most one less than in the ordinary game, what shows the next theorem.

Theorem 8. *For any graph G,*

$$\gamma'_{cg}(G) \geq \gamma_{cg}(G) - 1.$$

Proof. We will present the strategy for Staller, who will be playing two games on G: the real Staller-start game, and the imagined Dominator-start game with Chooser. First move of Staller in the real game is any legal move, say at vertex s, and it is not copied to the imagined game. Then, Staller will copy the first move of Dominator in the real game to the imagined game. After the first move of Dominator in the imagined game, Staller picks a vertex s in the imagined game as the move of Chooser. Now, in both games the set of played vertices is the same and it is Staller's tour. Staller will continue the imagined game according to the optimal strategy for the connected domination game with Chooser and copy each move in the imagined game to the real game, and copy each move of Dominator in the real game to the imagined game. Thus the real and the imagined game end at the same time and with the same number of moves (including one move of Chooser). From Lemma 1 it follows that the number of moves in the imagined game is at least $\gamma_{cg}(G) - 1$, hence so it is in the real one. □

To see that he bound given in the previous theorem is tight, consider for example the graph $K_n \Box P_3$, $n \geq 4$. We have $\gamma_{cg}(K_n \Box P_3) = 5$, but in the Staller-start game, no matter from which vertex Staller starts, Dominator can end in at most four moves.

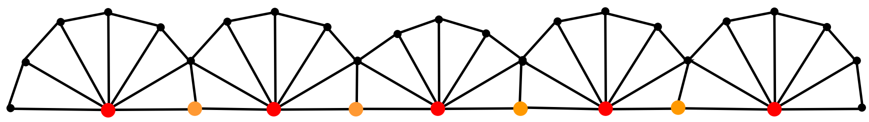
The *Staller-pass game* is the variant of the domination game in which, at each turn, Staller may pass instead of playing a vertex. Denote the size of the final dominating set in the Staller-pass game on G, under optimal play, by $\hat{\gamma}_{cg}(G)$ if Dominator moves first and by $\hat{\gamma}'_{cg}(G)$ if Staller moves first. (Turns on which Staller passes do not count toward the size of the final dominating set.)

Since Staller has additional options in the Staller-pass game and Dominator does not, $\hat{\gamma}_{cg}(G) \geq \gamma_{cg}(G)$ and $\hat{\gamma}'_{cg}(G) \geq \gamma'_{cg}(G)$ for any graph G. Moreover, since Staller may pass initially, it always holds $\hat{\gamma}_{cg}(G) \geq \hat{\gamma}'_{cg}(G)$. This property contrasts with the usual game, where sometimes $\gamma_{cg}(G) > \gamma'_{cg}(G)$.

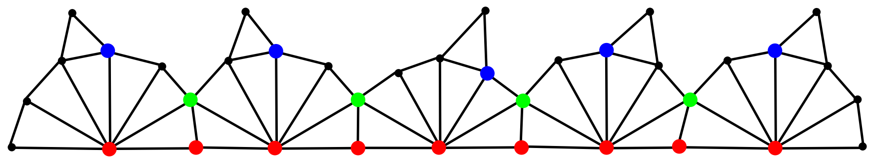
Let us denote by $\hat{\gamma}^k_{cg}(G)$ the size of the final dominating set in the Staller-pass game on G when Staller may pass at most k moves.

Lemma 5. *Consider the connected domination Staller-pass game on G when Staller may pass at most k moves. Then $\hat{\gamma}^k_{cg}(G) \leq \gamma_{cg}(G) + k$.*

Proof. Dominator will play two games on G — the real game and the imagined game. In the real game Staller may pass, in the imagined game Dominator will play according to an optimal strategy for the connected domination game with Chooser. In the first part of the game Dominator will just copy each move of Staller to his imagined game, and respond in the imagined game with an optimal move from the ordinary domination game. This will continue until Staller decides to pass in the real game. Then Dominator picks any legal vertex in the real game, next copies this move in the imagined game and in the imagined game sees this move as the vertex picked by Chooser. The game continues in the same way, when Staller passes, Dominator chooses any legal vertex and sees it as the move of Chooser. So, by Lemma 1 we have that at the end of the game in the dominating set there are at most $\gamma_{cg}(G) + k$ vertices. $\square$



$\gamma_c(F_5) = 9 = \gamma_{cg}(F_5)$



$\gamma_{cg}(F'_5) = 18 = \gamma_{cg}(F_5) + 9$

Figure 5: Graphs F_5 and F'_5

To see that the above bound is tight, consider the following family of graphs: Let F_1 be a fan with $n \geq 7$ vertices. We construct F_i from F_{i-1} in such a way that we take a copy of a fan F_1 and identify a degree-2 vertex from this copy with a degree-2 vertex from F_{i-1}, and the same we do for its degree-3 neighbours. For example, a graph F_5 is presented in Figure 5. Observe that for any $i \geq 1$,

$\gamma_c(F_i) = 2i-1 = \gamma_{cg}(F_i)$ (see Figure 5, top, the red and orange vertices form a minimum connected dominating set). Consider now the game on F_i, when Staller can pass at most $k = i-1$ moves, $i \geq 2$. Let the strategy for Staller be as follows: if Dominator chooses a vertex from the center of a fan (vertices coloured red in Figure 5, top), then Staller passes. Observe that such a game is equivalent to a connected domination game on a graph F_i', obtained from F_i by adding i vertices of degree 2 and joining each of them with two degree-3 vertices of a different fan (as shown in Figure 5, bottom). The blue vertices are the vertices that are chosen by Staller in F_i' in the turns in a game on F_i when he passes his moves. Next, observe that $\gamma_{cg}(F_i') = 4i-1$. It follows that $\hat{\gamma}_{cg}^{i-1}(F_i) = 4i-2-i = 3i-2 = \gamma_{cg}(F_i)+i-1$.

REFERENCES

1. B. Brešar, S. Klavžar and D. Rall, Domination game and an imagination strategy, SIAM J. Discrete Math. 24 (2010) 979–991.
2. B. Brešar, S. Klavžar and D. Rall, Domination game played on trees and spanning subgraphs, Discrete Math. 313 (2013) 915–923.
3. B. Brešar, P. Dorbec, S. Klavžar and G. Košmrlj, Domination game: Effect of edge- and vertex-removal, Discrete Math. 330 (2014) 1–10.
4. B. Brešar, P. Dorbec, S. Klavžar and G. Košmrlj, How long can one bluff in the domination game?, Discuss. Math. Graph Theory 37 (2017) 337–352.
5. B. Brešar, S. Klavžar, G. Košmrlj and D.F. Rall, Domination game: extremal families of graphs for the 3/5-conjectures, Discrete Appl. Math. 161 (2013) 1308–1316.
6. Cs. Bujtás, On the game domination number of graphs with given minimum degree, Electron. J. Combin. 22(3) (2015) #P3.29.
7. Cs. Bujtás, Domination game on forests, Discrete Math. 338 (2015) 2220–2228.
8. Cs. Bujtás, S. Klavžar and G. Košmrlj, Domination game critical graphs, Discuss. Math. Graph Theory 35 (2015) 781–796.
9. Cs. Bujtás and Zs. Tuza, The Disjoint Domination Game, Discrete Math. 339 (2016) 1985–1992.
10. P. Dorbec, G. Košmrlj and G. Renault, The domination game played on unions of graphs, Discrete Math. 338 (2015) 71–79.
11. T.W. Haynes, S.T. Hedetniemi and P.J. Slater, Fundamentals of Domination in Graphs (Marcel Dekker, Inc., New York, 1998).
12. S.T. Hedetniemi and R.C. Laskar, Connected domination in graphs, in: B. Bollobás, ed., Graph Theory and Combinatorics (Academic Press, London, 1984).
13. M.A. Henning, W.B. Kinnersley, Domination Game: A proof of the 3/5-Conjecture for Graphs with Minimum Degree at Least Two, SIAM J. Discrete Math. 30 (2016) 20–35.
14. M.A. Henning, S. Klavžar and D.F. Rall, Total Version of the Domination Game, Graphs Combin. 31 (2015) 1453–1462.
15. M.A. Henning, S. Klavžar and D.F. Rall, Game total domination critical graphs, Discrete Appl. Math. 250 (2018) 28–37.
16. M.A. Henning, C. Löwenstein, Domination Game: Extremal Families for the 3/5-Conjecture for Forests, Discuss. Math. Graph Theory 37 (2017) 369–381.
17. M.A. Henning and D.F. Rall, Trees with equal total domination and game total domination numbers, Discrete Appl. Math. 226 (2018) 58–70.

18. M.M. Kanté, V. Limouzy, A. Mary and L. Nourine, On the enumeration of minimal dominating sets and related notions, SIAM J. Discrete Math. 28 (2014) 1916–1929.
19. W.B. Kinnersley, D.B. West, R. Zamani, Extremal problems for game domination number, SIAM J. Discrete Math. 27 (2013) 2090–2107.
20. S. Klavžar, G. Košmrlj and S. Schmidt, On the computational complexity of the domination game, Iran. J. Math. Sci. Inform. 10(2) (2015) 115–122.
21. S. Klavžar, G. Košmrlj and S. Schmidt, On graphs with small game domination number, Appl. Anal. Discrete Math. 10 (2016) 30–45.
22. G. Košmrlj, Realizations of the game domination number, J. Comb. Optim. 28 (2014) 447–461.
23. G. Košmrlj, Domination game on paths and cycles, Ars Math. Contemp. 13 (2017) 125–136.
24. E. Sampathkumar and H.B. Walikar, The connected domination number of a graph, J. Math. Phys. Sci. 13 (1979) 607–613.
25. D. West, Introduction to Graph Theory, 2nd ed. (Upper Saddle River, Prentice Hall, 2001).
26. K. Xu and X. Li, On domination game stable graphs and domination game edge-critical graphs, Discrete Appl. Math. 250 (2018) 47–56.
27. K. Xu, X. Li and S. Klavžar, On graphs with largest possible game domination number, Discrete Math. 341 (2018) 1768–1777.

(Received 26.11.2017)
(Revised 31.10.2018)

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