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ON AN OPEN PROBLEM OF SIMSEK CONCERNING THE COMPUTATION OF A FAMILY OF SPECIAL NUMBERS

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Simsek [Y. Simsek, New families of special numbers for computing negative order Euler numbers and related numbers and polynomials, Appl. Anal. Discrete Math. 12(2018), 1-35.] conjectured that

$$B(d,k) = (k^d + x_1 k^{d-1} + x_2 k^{d-2} + \cdots + x_{d-1}k)2^{k-d},$$

where $x_1, x_2, \ldots, x_{d-1}, d$ are positive integers, and proposed the following open problem: (I) How can we compute the coefficients x_1, x_2, …, x_{d-1}? (II) Is it possible to find the function $f_d(x) = \sum_{k=1}^{\infty} B(d,k)x^k$? By using the familiar Stirling numbers of the first and second kind, we solve this problem. We further obtain a general result on the generalized numbers of $B(d,k)$.

1. INTRODUCTION

Simsek [**12**] constructed a new family of special numbers denote by $y_1(n,k;\lambda)$:

$$F_{y_1}(t,k;\lambda) = \frac{1}{k!}(\lambda e^t + 1)^k = \sum_{n=0}^{\infty} y_1(n,k;\lambda)\frac{t^n}{n!}. \tag{1}$$

The numbers $y_1(n,k;\lambda)$ are related to the many well-known numbers including the Bernoulli numbers, the Fibonacci numbers, the Lucas numbers, the Stirling

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numbers of the second kind and the central numbers. By using (1), an explicit expression of $y_1(n,k;\lambda)$ is given by

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{k}\binom{k}{j} j^n \lambda^j. \tag{2}$$

In particular, the first few values of the numbers $y_1(n,k;\lambda)$ are listed by the table in [**12**]. Substituting $\lambda = 1$ into (2) yields

$$B(n,k) = k! y_1(n,k;1), \tag{3}$$

where $B(n,k)$ is defined by the Golombek's identity [**6**]:

$$B(n,k) = \frac{d^n}{dt^n}(e^t+1)^k|_{t=0}.$$

The numbers $B(n,k)$ are related to the following numbers $a_k 2^k$. The first few cases are

$$\begin{aligned} B(0,k) &= 2^k, \\ B(1,k) &= k \cdot 2^{k-1}, \\ B(2,k) &= k(k+1)2^{k-2}, \end{aligned}$$

see for example [**12**]. Simsek [**12**] obtained the following recurrence

$$B(d,k) = \frac{2^{k-d}}{m_0}\binom{k}{d} - \sum_{\nu=1}^{d}\frac{m_\nu}{m_0}B(d-\nu,k),$$

where d is a positive integer and $m_0, m_1, \ldots, m_d \in \mathbb{Q}$. Simsek [**12**] further conjectured that

$$B(d,k) = (k^d + x_1 k^{d-1} + x_2 k^{d-2} + \cdots + x_{d-1}k)2^{k-d}, \tag{4}$$

where $x_1, x_2, \ldots, x_{d-1}, d$ are positive integers, and proposed the following open problem:

(I) How can we compute the coefficients x_1, x_2, $\ldots$, x_{d-1}?

(II) We assume that for $|x| < r$

$$\sum_{k=1}^{\infty} B(d,k)x^k = f_d(x). \tag{5}$$

Is it possible to find $f_d(x)$ function? For the open problem see also [**13**].

The aim of this paper is to give an answer to this open problem, see Corollaries 1, 3, 4 and 5 in Section 3, respectively. We further generalize the open problem to the numbers $Y(d,k;\lambda) = k! y_1(d,k;\lambda)$.

2. PRELIMINARIES

2.1. Faà di Bruno's formula and the Stirling numbers

Let $h = f \circ g$ be a composite function. There are several ways to represent the n-th derivative of the composite function h. One of the best known ways is given by Faà di Bruno's formula [**5, 8, 15**]

$$h^{(n)}(t) = \sum_{\pi(n)} \frac{n!}{k_1!k_2!\cdots k_n!} f^{(k)}(g(t)) \left(\frac{g'(t)}{1!}\right)^{k_1} \left(\frac{g''(t)}{2!}\right)^{k_2} \cdots \left(\frac{g^{(n)}(t)}{n!}\right)^{k_n}, \tag{6}$$

where the sum runs over all partitions $\pi(n)$ of the integer n, k_i denotes the number of parts of size i, and $k = k_1 + k_2 + \cdots + k_n$ denotes the number of parts of the considered partition. This well-known formula may be also described in terms of the Bell polynomials:

$$h^{(n)}(t) = \sum_{k=1}^{n} f^{(k)}(g(t)) B_{n,k}\left(g'(t), g''(t), \ldots, g^{(n-k+1)}(t)\right), \tag{7}$$

where the Bell polynomial $B_{n,k}(x_1, x_2, \ldots, x_{n-k+1})$ is given by an explicit expression

$$\begin{aligned} &B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}) \\ &= \sum_{\substack{c_1+2c_2+\cdots=n \\ c_1+c_2+\cdots=k}} \frac{n!}{c_1!c_2!\cdots c_{n-k+1}!} \left(\frac{x_1}{1!}\right)^{c_1} \left(\frac{x_2}{2!}\right)^{c_2} \cdots \left(\frac{x_{n-k+1}}{(n-k+1)!}\right)^{c_{n-k+1}}. \end{aligned} \tag{8}$$

The first few cases of the Bell polynomial are

$$\begin{aligned} &B_{1,1}(x_1) = x_1, B_{2,1}(x_1, x_2) = x_2, B_{2,2}(x_1) = x_1^2, \\ &B_{3,1}(x_1, x_2, x_3) = x_3, B_{3,2}(x_1, x_2) = 3x_1x_2, B_{3,3}(x_1) = x_1^3. \end{aligned}$$

There are many interesting and important identities for the Bell polynomials. One of the useful identities is

$$B_{n,k}(abx_1, ab^2x_2, \ldots, ab^{n-k+1}x_{n-k+1}) = a^k b^n B_{n,k}(x_1, x_2, \ldots, x_{n-k+1}), \tag{9}$$

where a and b are any complex numbers. See Chapter 3 in [**5**].

The Stirling number of the second kind denoted by $S(n,k)$ counts the number of partitionks of a set of n elements into k indistinguishable boxes in which no box is empty, and the Stirling number of the first kind denoted by $s(n,k)$ counts the number of arrangements of n objects into k nonempty circular permutations. It is worth noting that

$$S(n,k) = B_{n,k}(\underbrace{1, 1, \ldots, 1}_{n-k+1}), \tag{10}$$

$$s(n,k) = B_{n,k}(0!, -1!, 2!, \ldots, (-1)^{n-k}(n-k)!). \tag{11}$$

In particular, $S(0,0) = s(0,0) = 1$, $S(n,0) = s(n,0) = 0$ for $n \geq 1$ and $S(0,k) = s(0,k) = 0$ for $k \geq 1$. The Stirling numbers of the first and second kind have the "horizontal" generating function:

$$(x)_n = \sum_{k=0}^{n} s(n,k)x^k, \tag{12}$$

$$x^n = \sum_{k=0}^{n} S(n,k)(x)_k, \tag{13}$$

where the falling factorial $(x)_i = x(x-1)\cdots(x-i+1)$. They also have the "vertical" generating function:

$$\frac{(e^t-1)^k}{k!} = \sum_{n\geq k} S(n,k)\frac{t^n}{n!}, \tag{14}$$

$$\frac{(\log(1+t))^k}{k!} = \sum_{n\geq k} s(n,k)\frac{t^n}{n!}. \tag{15}$$

For more details see Chapter 5 in [**5**].

2.2. An explicit expression of $y_1(n,k;\lambda)$

By using Faà di Bruno's formula we establish a new explicit expression for $y_1(n,k;\lambda)$ which is different from (2).

Proposition 1. *If n is a nonnegative integer, then*

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=0}^{n}\binom{k}{j} j! S(n,j)\lambda^j(\lambda+1)^{k-j}. \tag{16}$$

Proof. By (1) we have

$$y_1(n,k;\lambda) = \frac{1}{k!}\frac{d^n}{dt^n}(\lambda e^t+1)^k|_{t=0}.$$

Thus, (16) is obviously valid for $n = 0$. According to (7), we obtain for $n \geq 1$,

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=1}^{n}(k)_j(\lambda+1)^{k-j}B_{n,j}(\lambda,\lambda,\cdots,\lambda).$$

Combining with (9) and (10) yields

$$y_1(n,k;\lambda) = \frac{1}{k!}\sum_{j=1}^{n}\binom{k}{j} j!\lambda^j(\lambda+1)^{k-j}S(n,j),$$

which is equivalent to (16) because $S(n,0) = 0$ for $n \geq 1$. □

Remark 1. *Here we apply the well-known Faà di Bruno's formula to obtain the explicit expression of* $y_1(n,k;\lambda)$ *which is helpful in solving the open problem in next section. In fact, by the work of Boyadzhiev [**3**], we also arrive at* (16). *But our method is very different from that of Boyadzhiev.*

Remark 2. *If we let* $\lambda = 1$*, then* (16) *reduces to*

$$B(n,k) = k!y(n,k;1) = \sum_{j=0}^{n} \binom{k}{j} j!S(n,j)2^{k-j}, \tag{17}$$

*see [**14**, Identity 12].*

3. MAIN RESULTS

In this section we will answer the open problem proposed by Simsek in the first section.

3.1. Answer to the open problem

Let $Y(d,k;\lambda) = k!y_1(d,k;\lambda)$. Obviously, by (16) we arrive at

$$Y(d,k;\lambda) = \sum_{j=0}^{d} \binom{k}{j} j!S(d,j)\lambda^j(\lambda+1)^{k-j}. \tag{18}$$

Now, let us consider the following general problem concerning $Y(d,k;\lambda)$.

Theorem 1. *Let* d *be a nonnegative integer. Then*

(19)
$$Y(d,k;\lambda) = \{x_0(\lambda)k^d + x_1(\lambda)k^{d-1} + x_2(\lambda)k^{d-2} + \cdots + x_{d-1}(\lambda)k\}(\lambda+1)^{k-d},$$

where the coefficients $x_{d-i}(\lambda)$ $(i = 1, 2, \ldots d)$ *can be explicitly given by*

$$x_{d-i}(\lambda) = \sum_{j=i}^{d} s(j,i)S(d,j)\lambda^j(\lambda+1)^{d-j}. \tag{20}$$

Proof. The case for $d = 0$ is trivial. So let us consider the case for $d \geq 1$. By (18) we have

$$\begin{aligned} Y(d,k;\lambda) &= \sum_{j=1}^{d} \binom{k}{j} S(d,j)j!\lambda^j(\lambda+1)^{k-j} \\ &= \sum_{j=1}^{d} (k)_j S(d,j)\lambda^j(\lambda+1)^{k-j}. \end{aligned}$$

Using (12) we have

$$Y(d,k;\lambda) = \sum_{j=1}^{d} S(d,j)\lambda^j(\lambda+1)^{k-j} \sum_{i=1}^{j} k^i s(j,i).$$

Interchanging the order of summations yields

$$\begin{aligned} Y(d,k;\lambda) &= \sum_{i=1}^{d} k^i \sum_{j=i}^{d} s(j,i)S(d,j)\lambda^j(\lambda+1)^{k-j} \\ &= (\lambda+1)^{k-d} \sum_{i=1}^{d} k^i \sum_{j=i}^{d} s(j,i)S(d,j)\lambda^j(\lambda+1)^{d-j}, \end{aligned}$$

which leads to (19). □

Taking $\lambda = 1$ in Theorem 1 we immediately obtain the following corollary.

Corollary 1. *Let d be a nonnegative integer. Then*

$$B(d,k) = (k^d + x_1 k^{d-1} + x_2 k^{d-2} + \cdots + x_{d-1}k)2^{k-d}, \tag{21}$$

where the coefficients $x_{d-i}(i = 1, 2, \ldots, d-1)$ are given by

$$x_{d-i} = \sum_{j=i}^{d} s(j,i)S(d,j)2^{d-j}. \tag{22}$$

Remark 3. *Eq.*(22) *shows that the coefficients $x_{d-i}(i = 1, 2, \ldots, d-1)$ can be explicitly expressed in terms of two kinds of Stirling numbers, which gives an answer to the open problem* (I) *in Section* 1.

Remark 4. *By* (22), *we can easily calculate that*

$$\begin{aligned} B(0,k) &= 2^k, \\ B(1,k) &= k \cdot 2^{k-1}, \\ B(2,k) &= (k^2 + k)2^{k-2}, \\ B(3,k) &= (k^3 + 3k^2)2^{k-3}, \\ B(4,k) &= (k^4 + 6k^3 + 3k^2 - 2k)2^{k-4}. \end{aligned}$$

*In fact, both of the formulas for $B(3,k)$ and $B(4,k)$ were given in [**12**]. In [**12**], Simsek was aware of the fact that the numbers x_j, $j = 1, 2, \ldots, d-1$, are integers and d is a positive integer since $x_3 = -2$. Therefore there is a typing error in the expression of open problems, so it would be more appropriate to specify in Remark 4 that x_j, $j = 1, 2, \ldots, d-1$, are integers and d is a positive integer.*

Theorem 1 gives an explicit expression for $Y(d,k;\lambda)$. We further obtain a recursive formula for $Y(d,k;\lambda)$ below. First we need the following identity.

Lemma 1. *If d is a positive integer, then*

$$\sum_{\nu=0}^{d-1} s(d,d-\nu)Y(d-\nu,k;\lambda) = \binom{k}{d} d!\lambda^d(\lambda+1)^{k-d}. \tag{23}$$

Proof. We have

$$\begin{aligned}\frac{1}{d!}\sum_{\nu=0}^{d-1} s(d,d-\nu)Y(d-\nu,k;\lambda) &= \frac{1}{d!}\sum_{\nu=0}^{d-1} s(d,d-\nu)\sum_{j=0}^{k}\binom{k}{j} j^{d-\nu}\lambda^j \\ &= \frac{1}{d!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j\sum_{\nu=1}^{d} s(d,\nu)j^{\nu} \\ &= \frac{1}{d!}\sum_{j=0}^{k}\binom{k}{j}\lambda^j (j)_d \\ &= \sum_{j=0}^{k}\binom{k}{j}\binom{j}{d}\lambda^j = \sum_{j=0}^{k}\binom{k}{d}\binom{k-d}{k-j}\lambda^j \\ &= \binom{k}{d}\lambda^d(\lambda+1)^{k-d},\end{aligned}$$

which implies that (23) is true. □

Since $s(d,0)=0$ for $d \geq 1$, (23) can be rewritten as

$$\sum_{\nu=0}^{d} s(d,d-\nu)Y(d-\nu,k;\lambda) = \binom{k}{d} d!\lambda^d(\lambda+1)^{k-d}. \tag{24}$$

By using Identity (24), we immediately have the following theorem.

Theorem 2. *We have the recurrence relation:*

$$Y(0,k;\lambda) = (\lambda+1)^k,$$

$$Y(d,k;\lambda) = \binom{k}{d} d!\lambda^d(\lambda+1)^{k-d} - \sum_{\nu=1}^{d} s(d,d-\nu)Y(d-\nu,k;\lambda), \quad d \geq 1. \tag{25}$$

Taking $\lambda = 1$ we obtain a recursive formula for $B(d,k)$.

Corollary 2. *If d is a nonnegative integer, then the numbers $B(d,k)$ can be computed recursively by*

$$B(0,k) = 2^k,$$

$$B(d,k) = \frac{2^{k-d}}{m_0}\binom{k}{d} - \sum_{\nu=1}^{d}\frac{m_\nu}{m_0}B(d-\nu,k), \quad d \geq 1, \tag{26}$$

where the coefficients

$$m_\nu = \frac{1}{d!} s(d, d-\nu), \quad \nu = 0, 1, \ldots, d. \tag{27}$$

Remark 5. *In fact, Simsek [**12**] obtained the recursive formula* (26)*. He pointed out that all* $m_\nu \in \mathbb{Q}$*,* $\nu = 0, 1, \ldots, d$*. However, he did not give the explicit formula for the coefficients* m_ν $(\nu = 0, 1, \ldots, d)$*.*

Let $f_d(x;\lambda) = \sum_{k=1}^\infty Y(d,k;\lambda)x^k$ be a generating function of $Y(d,k;\lambda)$. Then we have the following theorem.

Theorem 3. *If* d *is a nonnegative integer, then*

$$f_d(x;\lambda) = \begin{cases} \frac{(\lambda+1)x}{1-(\lambda+1)x}, & d = 0, \\ \sum_{j=1}^d j! S(d,j) \frac{(\lambda x)^j}{(1-(\lambda+1)x)^{j+1}}, & d \geq 1. \end{cases} \tag{28}$$

Proof. If $d = 0$, we have

$$f_0(x;\lambda) = \sum_{k=1}^\infty Y(0,k;\lambda)x^k = \sum_{k=1}^\infty (\lambda+1)^k x^k = \frac{(\lambda+1)x}{1-(\lambda+1)x}. \tag{29}$$

If $d > 0$, then by (19) we have

$$\begin{aligned} f_d(x;\lambda) &= \sum_{k=1}^\infty Y(d,k;\lambda)x^k \\ &= \sum_{k=1}^\infty ((\lambda+1)x)^k \sum_{i=1}^d k^i \sum_{j=i}^d s(j,i) S(d,j) \left(\frac{\lambda}{\lambda+1}\right)^j \\ &= \sum_{k=1}^\infty ((\lambda+1)x)^k \sum_{j=1}^d S(d,j) \left(\frac{\lambda}{\lambda+1}\right)^j \sum_{i=1}^j k^i s(j,i) \\ &= \sum_{k=1}^\infty ((\lambda+1)x)^k \sum_{j=1}^d S(d,j) \left(\frac{\lambda}{\lambda+1}\right)^j (k)_j \end{aligned}$$

Interchanging the order of summations we obtain

$$\begin{aligned} f_d(x;\lambda) &= \sum_{j=1}^d S(d,j) \left(\frac{\lambda}{\lambda+1}\right)^j \sum_{k=1}^\infty ((\lambda+1)x)^k (k)_j \\ &= \sum_{j=1}^d S(d,j) \left(\frac{\lambda}{\lambda+1}\right)^j j! \frac{((\lambda+1)x)^j}{(1-(\lambda+1)x)^{j+1}} \\ &= \sum_{j=1}^d j! S(d,j) \frac{(\lambda x)^j}{(1-(\lambda+1)x)^{j+1}}. \end{aligned}$$

The desired result is derived. □

Let us consider the generating function of $f_d(x;\lambda)$.

Theorem 4. *We have*

$$\sum_{k=0}^{\infty} f_k(x;\lambda)\frac{t^k}{k!} = \frac{x(1+\lambda e^t)}{1-x-\lambda x e^t}. \tag{30}$$

Proof. By (28) we have

$$\begin{aligned}\sum_{k=0}^{\infty} f_k(x;\lambda)\frac{t^k}{k!} &= \frac{(\lambda+1)x}{1-(\lambda+1)x} + \sum_{k=1}^{\infty}\frac{t^k}{k!}\sum_{j=1}^{k} j!S(k,j)\frac{(\lambda x)^j}{(1-(\lambda+1)x)^{j+1}} \\ &= \frac{(\lambda+1)x}{1-(\lambda+1)x} + \sum_{j=1}^{\infty} j!\frac{(\lambda x)^j}{(1-(\lambda+1)x)^{j+1}}\sum_{k=j}^{\infty} S(k,j)\frac{t^k}{k!}.\end{aligned}$$

Using (14) we have

$$\begin{aligned}\sum_{k=0}^{\infty} f_k(x;\lambda)\frac{t^k}{k!} &= \frac{(\lambda+1)x}{1-(\lambda+1)x} + \sum_{j=1}^{\infty}\frac{(\lambda x)^j}{(1-(\lambda+1)x)^{j+1}}(e^t-1)^j \\ &= \frac{(\lambda+1)x}{1-(\lambda+1)x} + \frac{1}{1-(\lambda+1)x}\left\{\frac{1}{1-\frac{\lambda x}{1-(\lambda+1)x}(e^t-1)} - 1\right\}.\end{aligned}$$

After simplifying the right-hand side of the last identity, we arrive at (30). □

By the generating function (30) we find the recurrence relation of $f_d(x;\lambda)$.

Theorem 5. *The sequence of the functions $f_d(x;\lambda)$ $(d = 0,1,\ldots)$ satisfies the following recurrence relation:*

$$f_0(x;\lambda) = \frac{(\lambda+1)x}{1-(\lambda+1)x},$$

$$f_d(x;\lambda) = \frac{\lambda x}{1-(\lambda+1)x}\left\{1+\sum_{j=0}^{d-1}\binom{d}{j} f_j(x;\lambda)\right\}, \quad d \geq 1. \tag{31}$$

Proof. Multiplying both sides of (30) by $1-x-\lambda x e^t$ and equating the coefficient of t^d, we obtain (31). □

Taking $\lambda = 1$ we naturally find the function $f_d(x) = f_d(x;1)$ which was proposed in the open problem (II).

Corollary 3. *The $f_d(x)$ can be explicitly computed by*

$$f_d(x) = \sum_{k=1}^{\infty} B(d,k)x^k = \begin{cases} \frac{2x}{1-2x}, & d=0, \\ \sum_{j=1}^{d} j!S(d,j)\frac{x^j}{(1-2x)^{j+1}}, & d \geq 1. \end{cases} \tag{32}$$

Corollary 4. *The sequence of the functions $f_d(x)$ $(d = 0, 1, \ldots)$ can be computed by the following recurrence relation:*

$$f_0(x) = \frac{2x}{1-2x},$$

(33)
$$f_d(x) = \frac{x}{1-2x}\left\{1 + \sum_{j=0}^{d-1}\binom{d}{j} f_j(x)\right\}, \quad d \geq 1.$$

Remark 6. *Corollaries* 3 *and* 4 *give the explicit expression and the recursive formula of $f_d(x)$, respectively, which answer the open problem (II) in Section* 1.

3.2. The Relation between $f_d(x;\lambda)$ and the Eulerian polynomials

Recall the following classical result [**4, 5, 7**]

$$\sum_{k=0}^{\infty} k^n x^k = \frac{A_n(x)}{(1-x)^{n+1}}, \quad |x| < 1,$$

where $A_n(x)$ is the Eulerian polynomial of degree n, and it may be explicitly written by

$$A_n(x) = \sum_{k=1}^{n} A(n,k)x^k, \quad A_0(x) = 1.$$

In the above, $A(n,k)$ are known as Eulerian numbers and they can be expressed by

$$A(n,k) = \sum_{i=1}^{k}(-1)^i\binom{n+1}{i}(k-i)^n.$$

The Eulerian polynomials $A_n(x)$ can be defined by the exponential generating function

(34)
$$\sum_{n=0}^{\infty} A_n(x)\frac{t^n}{n!} = \frac{1-x}{1-xe^{t(1-x)}}.$$

From (34) one may easily obtain

$$A_0(x) = 1,\ A_1(x) = x,\ A_2(x) = x^2 + x,\ A_3(x) = x^3 + 4x^2 + x,\ \ldots$$

The Eulerian polynomials are closely related to the Apostol-Bernoulli numbers [**2**]:

(35)
$$A_n(x) = -\frac{(1-x)^{n+1}}{n+1}\mathcal{B}_{n+1}(x),$$

where $\mathcal{B}_n(x)$ are the Apostol-Bernoulli numbers [**1, 9, 10, 11, 16**] defined by

(36)
$$\frac{t}{xe^t - 1} = \sum_{n=0}^{\infty}\mathcal{B}_n(x)\frac{t^n}{n!}.$$

Theorem 6. *If d is a nonnegative integer, then*

$$f_d(x;\lambda) = \begin{cases} \frac{(\lambda+1)x}{1-(\lambda+1)x}, & d = 0, \\ \frac{(1-x)^d}{(1-(\lambda+1)x)^{d+1}} A_d\left(\frac{\lambda x}{1-x}\right), & d \geq 1, \end{cases} \tag{37}$$

where $A_d(x)$ is the Eulerian polynomial of degree d.

Proof. By (30) we have

$$\sum_{k=0}^{\infty} f_k(x;\lambda)\frac{t^k}{k!} = \frac{1}{x-1}\frac{1}{\frac{\lambda x}{1-x}e^t - 1} - 1.$$

According to (36), if $d = 0$, then

$$f_0(x;\lambda) = \frac{1}{x-1}\mathcal{B}_1\left(\frac{\lambda x}{1-x}\right) - 1 = \frac{(\lambda+1)x}{1-(\lambda+1)x}, \tag{38}$$

and if $d \geq 1$, then

$$f_d(x;\lambda) = \frac{1}{(d+1)(x-1)}\mathcal{B}_{d+1}\left(\frac{\lambda x}{1-x}\right). \tag{39}$$

Replacing x by $\frac{\lambda x}{1-x}$ in (35) yields

$$A_d\left(\frac{\lambda x}{1-x}\right) = -\frac{(1-(\lambda+1)x)^{d+1}}{(d+1)(1-x)^{d+1}}\mathcal{B}_{d+1}\left(\frac{\lambda x}{1-x}\right). \tag{40}$$

Combining (38), (39) with (40) we immediately arrive at (37). □

Taking $\lambda = 1$ we establish the relation between $f_d(x)$ and the Eulerian polynomials.

Corollary 5. *If d is a nonnegative integer, then*

$$f_d(x) = \begin{cases} \frac{2x}{1-2x}, & d = 0, \\ \frac{(1-x)^d}{(1-2x)^{d+1}} A_d\left(\frac{x}{1-x}\right), & d \geq 1. \end{cases} \tag{41}$$

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REFERENCES

1. T.M. Apostol: *On the Lerch zeta function.* Pacific J. Math., **1** (1951), 161–167.
2. K.N. Boyadzhiev: *Apostol-Bernoulli functions, derivative polynomials and Eulerian polynomials.* Adv. Appl. Discrete Math., **1** (2008), 109–122.
3. K.N. Boyadzhiev: *Binomial transform and the backward difference.* http://arxiv.org/abs/1410.3014v2.
4. L. Carlitz: *Eulerian numbers and polynomials.* Math. Magazine, **32** (1959), 247–260.
5. L. Comtet: *Advanced Combinatorics, the Art of Finite and Infinite Expansions.* D. Reidel Publishing Co., Dordrecht, 1974.
6. R. Golombek: *Aufgabe 1088.* El. Math., **49** (1994), 126–127.
7. L.C. Hsu, J.S. Shiue: *On certain summation problems and generalizations of Eulerian polynomials and numbers.* Discrete Math., **204** (1999), 237–247.
8. W.P. Johnson: *The curious history of Faà di Bruno's formula.* Amer. Math. Monthly, **109** (2002), 217–234.
9. M.S. Kim, S. Hu: *Sums of products of Apostol-Bernoulli numbers.* Ramanujan J., **28** (2012), 113–123.
10. Q.-M. Luo, H.M. Srivastava: *Some generalizations of the Apostol-Bernoulli and Apostol-Euler polynomials.* J. Math. Anal. Appl., **308** (2005), 290–302.
11. F. Qi, B.-N. Guo: *Two nice determinantal expressions and a recurrence relation for the Apostol-Bernoulli polynomials.* J. Indones. Math. Soc., **23** (2017), 81–87.
12. Y. Simsek: *New families of special numbers for computing negative order Euler numbers and related numbers and polynomials.* Appl. Anal. Discrete Math., **12** (2018), 1–35.
13. Y. Simsek: *Computation methods for combinatorial sums and Euler type numbers related to new families of numbers.* to appear Math. Meth. Appl. Sci., (2017), DOI: 10.1002/mma.4143.
14. M.Z. Spivey: *Combinatorial sums and finite differences.* Discrete Math., **307** (2007), 3130–3146.
15. A.M. Xu, Z.D. Cen: *A unified approach to some recurrence sequences via Faà di Bruno's formula.* Comput. Math. Appl., **62** (2011), 253–260.
16. A.M. Xu, Z.D. Cen: *Some identities involving exponential functions and Stirling numbers and applications.* J. Comput. Appl. Math., **260** (2014), 201–207.

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