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ON COMPLETE MONOTONICITY OF THREE PARAMETER MITTAG-LEFFLER FUNCTION

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Dedicated to Memory of Professor Bogoljub Stanković
on the Occasion of his 95th Birth Anniversary.

Using the Bernstein theorem we prove the complete monotonicity of the three parameter Mittag–Leffler function $E^{\gamma}_{\alpha,\beta}(-w)$ for $w \geq 0$ and suitably constrained parameters α, β and γ.

1. INTRODUCTION

The three parameter generalization of the Mittag-Leffer function [**6**], in the literature devoted to the fractional calculus and its applications disseminated also as the Prabhakar function, was introduced by T. R. Prabhakar [**26**] as

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{r\geq 0} \frac{(\gamma)_r}{r!\,\Gamma(\alpha r+\beta)} z^r, \qquad z \in \mathbb{C}, \, \Re(\alpha) > 0,$$

where the Pochhammer symbol $(\gamma)_r$ denotes $\Gamma(\gamma+r)/\Gamma(\gamma)$. For $\beta = \gamma = 1$ it reduces to the standard Mittag-Leffler function $E_\alpha(z)$ considered firstly in [**19, 20, 21**], while for $\gamma = 1$ becomes the two parameter Mittag–Leffler function $E_{\alpha,\beta}(z)$ studied by Wiman in 1905 [**40**]. From the other point of view the three parameter

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Mittag-Leffler function is considered a representative of the plethora of generalized hypergeometric functions - namely a special type of the Wright Ψ-function [**2, 41, 42**] or the multivariate Mittag-Leffler functions [**23**].

Properties of functions belonging to the Mittag-Leffler family, among them those concerning complete monotonicity (CM) for the first time studied in classical papers by H. Pollard [**25**] and W. R. Schneider [**34**], are interesting both for mathematicians and physicists. For the first of these communities various Mittag-Leffler functions are important objects in the theory and applications of special functions [**22, 35, 36**], fractional calculus and fractional differential equations [**6, 16**], while the representatives of the second community successfully use Mittag-Leffler functions to describe phenomena exhibiting the memory-dependent time evolution, like dielectric relaxation [**3, 11**] and/or viscoelasticity [**10, 16, 29, 30, 31, 32**]. The CM enables using the Bernstein theorem (see for instance [**39**, Theorem 12a], [**33**, Theorem 1.4]) which concerns correspondence between CM functions and Laplace transforms of probability measures on $\mathbb{R}_+$. A physical interpretation comes from the time decay patterns which, if given by CM functions of time, may be represented as effects of summing up elementary Debye (exponential) decays. The examples are provided by phenomenological relaxation patterns belonging to the class of Havriliak–Negami-like models used to describe experimental data [**3, 7, 12**].

Properties of the function $t \mapsto \mathscr{E}^{\gamma}_{\alpha,\beta}(t;x) := t^{\beta-1}E^{\gamma}_{\alpha,\beta}(-xt^{\alpha})$, bearing the name Prabhakar kernel [**4**], are recently the subject of extensive investigations [**1, 5, 17, 38**]. These efforts result, among others, in showing that this function is CM for $x, t \geq 0$ and parameters range of $0 < \alpha\gamma \leq \beta \leq 1$ with $0 < \alpha, \beta, \gamma \leq 1$. Proving the CM property of $\mathscr{E}^{\gamma}_{\alpha,\beta}(t;x)$ the authors of [**1, 17, 38**] have treated this function as a whole and have not investigated the CM of the function $E^{\gamma}_{\alpha,\beta}(-w)$ itself, leaving a gap between their results and the classical ones collected in the papers of Pollard [**25**] and Schneider [**34**] who demonstrated that the Mittag-Leffler $E_{\alpha}(-w)$ function for $w \geq 0$, $0 < \alpha \leq 1$ and Wiman's function $E_{\alpha,\beta}(-w)$ for $w \geq 0$, $0 < \alpha \leq 1$ and $\beta \geq \alpha$ are CM, respectively. These observations have motivated us to investigate the CM behavior of $E^{\gamma}_{\alpha,\beta}(-w)$ clarifying the meaning of both conditions, $0 < \alpha\gamma \leq \beta \leq 1$ and $0 < \beta \leq 1$, obtained in [**1, 17, 38**] as ensuring CM of $\mathscr{E}^{\gamma}_{\alpha,\beta}(t;x)$. *Ergo*, our task is to prove the CM property of $E^{\gamma}_{\alpha,\beta}(-w)$ for $w \geq 0$ and to precise the most general range of constrains which are to be put on parameters α, β and γ.

The paper is organized as follows. In section 2 we present our main result, Theorem 2.1, which comprises, using the Bernstein theorem, the proof of the CM character of the three parameter Mittag–Leffler function $E^{\gamma}_{\alpha,\beta}(-w)$. Giving, in section 3, explicit representation of $E^{\gamma}_{\alpha,\beta}(-w)$ for rational $\alpha \in (0, 1]$ in terms of the Laplace transform of generalized hypergeometric function enables us to show the same result in an alternative way. The section 4 closes the paper with concluding comments.

2. THE MAIN RESULT

This section addresses the problem of the conditions to be imposed in order to establish the CM of the three parameter Mittag-Leffler function. Our proof that the three parameter Mittag-Leffler function is CM for defined range of the parameters α, β, γ is inspired by and generalizes H. Pollard's paper [**25**]. To achieve the result we use the Bernstein theorem and besides of proving the CM derive the constraints which the parameters have to obey.

Theorem 2.1. *The three parameter generalized Mittag-Leffler function* $E^{\gamma}_{\alpha,\beta}(-w)$ *is completely monotone* (CM) *for* $0 < \alpha \leq 1$, $\gamma > 0$, *and* $\beta \geq \alpha\gamma$.

Proof. First we will show that $E^{\gamma}_{\alpha,\beta}(-w)$, for $w > 0$ and $\alpha, \beta, \gamma > 0$, can be expressed as the Laplace transform of $\sigma : u \mapsto u^{-1-\frac{1}{\alpha}} g^{\gamma}_{\alpha,\beta}(u^{-1/\alpha})$; $u > 0$, where

$$g^{\gamma}_{\alpha,\beta}(y) = \frac{\alpha}{\Gamma(\gamma)} \frac{y^{-\beta}}{2\pi\,\mathrm{i}} \int_L \mathrm{e}^{yz}\, z^{\alpha\gamma-\beta}\, \mathrm{e}^{-z^{\alpha}}\, \mathrm{d}z \tag{2.1}$$

with L being the Bromwich contour with $\Re(z) > 0$. To begin with we consider Prabhakar's result [**26**, Eq. (2.5)]

$$\mathscr{L}[\mathscr{E}^{\gamma}_{\alpha,\beta}(t;x)](s) = \frac{s^{\alpha\gamma-\beta}}{(x+s^{\alpha})^{\gamma}}, \qquad \Re(s), \Re(\beta) > 0;\ |s| > |x|^{1/\Re(\alpha)}.$$

Its inverse gives the Mittag-Leffler function

$$E^{\gamma}_{\alpha,\beta}(-xt^{\alpha}) = \frac{t^{1-\beta}}{2\pi\mathrm{i}} \int_{L_s} \mathrm{e}^{st}\, \frac{s^{\alpha\gamma-\beta}}{(s^{\alpha}+x)^{\gamma}} \mathrm{d}s\,,$$

where L_s is the Bromwich contour with $\Re(s) > 0$. Changing *mutatis mutandis* the variable $st = \xi^{1/\alpha}$ and making use of the familiar Gamma-function formula

$$(\xi + xt^{\alpha})^{-\gamma} = \frac{1}{\Gamma(\gamma)} \int_0^{\infty} \mathrm{e}^{-(\xi+xt^{\alpha})u}\, u^{\gamma-1}\, \mathrm{d}u\,, \tag{2.2}$$

we arrive at

$$E^{\gamma}_{\alpha,\beta}(-xt^{\alpha}) = \frac{1}{2\pi\mathrm{i}\,\alpha} \int_{L_{\xi}} \mathrm{e}^{\xi^{1/\alpha}}\, \xi^{\frac{\alpha\gamma-\beta+1}{\alpha}-1} \left[\frac{1}{\Gamma(\gamma)} \int_0^{\infty} \mathrm{e}^{-(\xi+xt^{\alpha})u}\, u^{\gamma-1}\, \mathrm{d}u\right] \mathrm{d}\xi. \tag{2.3}$$

These steps are legitimate as by the substitution $\xi = (st)^{\alpha}$ the integration contour L_s transforms into L_{ξ} and obviously $\Re(\xi) = |s|^{\alpha} t^{\alpha} \cos\big(\alpha \arg(s)\big) > 0$. In this way the convergence of integral (2.2) is controlled. Both integrals in (2.3) converge absolutely so we are allowed to change the order of integration. This gives

$$E^{\gamma}_{\alpha,\beta}(-xt^{\alpha}) = \frac{1}{2\pi\mathrm{i}\,\alpha\Gamma(\gamma)} \int_0^{\infty} \mathrm{e}^{-xt^{\alpha}u}\, u^{\gamma-1} \left[\int_{L_{\xi}} \mathrm{e}^{\xi^{1/\alpha}-u\xi}\, \xi^{\frac{\alpha\gamma-\beta+1}{\alpha}-1}\, \mathrm{d}\xi\right] \mathrm{d}u.$$

In the second integral we change the integration variable $\xi \to u^{-1}z^{\alpha}$. Next introduce the function $g^{\gamma}_{\alpha,\beta}(y)$ defined in (2.1) and rewrite the formula above as

$$E^{\gamma}_{\alpha,\beta}(-xt^{\alpha}) = \frac{1}{\alpha}\int_0^{\infty} \mathrm{e}^{-xt^{\alpha}u}\, u^{-1-1/\alpha}\, g^{\gamma}_{\alpha,\beta}(u^{-1/\alpha})\,\mathrm{d}u. \tag{2.4}$$

Setting $xt^{\alpha} = w$ completes the claim we have begun with.

According to (2.4) $E^{\gamma}_{\alpha,\beta}(-w)$ is given as the Laplace transform and, due to the Bernstein theorem, represents a CM function **iff** the integrand in RHS of (2.4) is non-negative on $\mathbb{R}_+$. Thus, if we write down (2.1) as

$$g^{\gamma}_{\alpha,\beta}(y) = \frac{\alpha}{\Gamma(\gamma)\, y^{\beta}}\, f^{\gamma}_{\alpha,\beta}(y), \tag{2.5}$$

it is enough to show the non-negativity of the function $f^{\gamma}_{\alpha,\beta}(y)$ for $y > 0$. The latter, as seen from (2.1), is defined as the inverse Laplace transform so

$$\int_0^{\infty} \mathrm{e}^{-yz}\, f^{\gamma}_{\alpha,\beta}(y)\,\mathrm{d}y = z^{\alpha\gamma-\beta}\,\mathrm{e}^{-z^{\alpha}}. \tag{2.6}$$

Reminding again the Bernstein theorem we see that if the RHS of (2.6) is CM then $f^{\gamma}_{\alpha,\beta}(y)$ must be non-negative on $\mathbb{R}_+$. Recall that the product of CM functions also obeys this property. Due to Pollard's result [**24**] the stretched exponential $\mathrm{e}^{-z^{\alpha}}$ is CM for $0 < \alpha \le 1$, whereas $z^{\alpha\gamma-\beta}$ is obviously CM if $\beta \ge \alpha\gamma$, where is $\gamma > 0$ which we assume throughout. Hence, for this parameter range $f^{\gamma}_{\alpha,\beta}(y) \ge 0$ on $\mathbb{R}_+$, so does *a fortiori* $\sigma : u \mapsto u^{-1-1/\alpha} g^{\gamma}_{\alpha,\beta}(u^{-\frac{1}{\alpha}})$; $u > 0$ for $g^{\gamma}_{\alpha,\beta}(y)$ introduced in (2.5).

Resumming: by (2.4) with $xt^{\alpha} = w > 0$, (2.1) and $f^{\gamma}_{\alpha,\beta}(y) \ge 0$ for $y > 0$ we deduce that $E^{\gamma}_{\alpha,\beta}(-w)$ is the Laplace transform of a non-negative function and thus is CM which statement completes the proof. □

Remark 2.2. *$E^{\gamma}_{\alpha,\beta}(-w)$ is* CM *for the parameter range $0 < \alpha \le 1$, $\gamma > 0$, and $\beta \ge \alpha\gamma$. We point out that the restriction $\beta \le 1$ appearing in* [**1, 17, 38**] *is needed if one wants to explain the* CM *of the Prabhakar kernel $\mathscr{E}^{\gamma}_{\alpha,\beta}(t;1)$ as a result of* CM *substitution $w \to t^{\alpha}$ in $E^{\gamma}_{\alpha,\beta}(-w)$ and next multiplying it by $t^{\beta-1}$ being* CM *for $0 < \beta < 1$. Thus the* CM *property of $\mathscr{E}^{\gamma}_{\alpha,\beta}(t;1)$ arises from two facts:* **(i)** *composition $f \circ g(x)$ of* CM *functions f, g leads to the* CM *of the result;* **(ii)** *the product of* CM *functions gives* CM *result as well. We emphasize that for just mentioned construction the* CM *of $E^{\gamma}_{\alpha,\beta}(-w)$ is necessary as it enables us to realize the step* **(i)**. *In our approach the condition $\beta \le 1$ is* not needed *because it plays no role for* CM *of $E^{\gamma}_{\alpha,\beta}(-w)$ itself. The condition $\beta \ge \alpha\gamma$ which appears in our proof fully coincides with those known as leading to the* CM *of the standard Mittag-Leffler function $E_{\alpha}(-w) = E^{1}_{\alpha,1}(-w)$, $0 < \alpha \le 1$* [**25**], *and of the Wiman generalization of the Mittag-Leffler function $E_{\alpha,\beta}(-w) = E^{1}_{\alpha,\beta}(-w)$ proved to be* CM *for $0 < \alpha \le 1$, $\beta \ge \alpha$,* [**34**]. *This shows that* Theorem 2.1 *provides us with the result being more general than consequences emerging from the* CM *property of the Prabhakar kernel $\mathscr{E}^{\gamma}_{\alpha,\beta}(t;1)$.*

3. THE FUNCTION $F^{\gamma}_{\alpha,\beta}$ - WHAT WE DO KNOW ABOUT IT?

Involving apparatus of the generalized hypergeometric functions we shall establish explicit formula for the function $f^{\gamma}_{\alpha,\beta}$. Assuming that the parameter α is a rational taken from the interval $(0, 1]$ we will show that $f^{\gamma}_{\alpha,\beta}$ may be expressed in terms of the Meijer G function which in the case under consideration boils down to the Wright Ψ function. To infer this result we employ the Laplace and the Mellin transform techniques in conjunction with appropriate transformation properties of the Meijer G function. We note that the restriction for α to be rational is justified two-fold: **(iii)** giving it up we are forced to apply much more complicated formalism of the Fox H–function (consult [**13**] for further details) which is redundant for physical applications as the latter always use rational values of parameters and **(iv)** the Meijer G and the generalized hypergeometric ${}_pF_q$ functions are built into computer algebra systems which makes to work with them much easier and effective, both in symbolic and numerical calculations.

To proceed further recall that:

1. The Meijer G function is defined through the Mellin-Barnes integral [**2, 18, 27**]

$$G^{m,n}_{p,q}\left(z \,\middle|\, \begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right) = \frac{1}{2\pi\mathrm{i}}\int_G \frac{\prod_{j=1}^{m}\Gamma(\beta_j+s)\ \prod_{j=1}^{n}\Gamma(1-\alpha_j-s)}{\prod_{j=m+1}^{q}\Gamma(1-\beta_j-s)\ \prod_{j=n+1}^{p}\Gamma(\alpha_j+s)}\, z^{-s}\,\mathrm{d}s,$$

   valid for any $z \neq 0$ and where empty products are taken to be equal 1. Also, $0 \leq m \leq q$; $0 \leq n \leq p$ while for a list of complex parameters the shorthand $(a_t) = a_1, a_2, \ldots, a_t$ is used. The integration contour G separates the poles of $\Gamma(\beta_j + s)$ from those of $\Gamma(1 - \alpha_j - s)$, for details consult [**27**].

2. The series representation of the generalized hypergeometric function reads as

$${}_pF_q\left(\begin{matrix}a_1,\ldots,a_p\\b_1,\ldots,b_q\end{matrix}; x\right) = \sum_{r\geq 0}\frac{(a_1)_r\ldots(a_p)_r}{(b_1)_r\ldots(b_q)_r}\frac{x^r}{r!}.$$

   For the convergence conditions see [**27**].

3. The Mellin transform of $f(y)$ denoted as $\hat{f}(s), s \in \mathbb{C}$ (or symbolically by $\mathscr{M}[\cdot](s)$) is, together with its inverse, defined as

$$\hat{f}(s) = \mathscr{M}[f(y)](s) = \int_0^{\infty} y^{s-1} f(y)\,\mathrm{d}y, \qquad f(y) = \frac{1}{2\pi\mathrm{i}}\int_{M_s} y^{-s}\hat{f}(s)\,\mathrm{d}s,$$

   where M_s denotes the Bromwich contour with $\Re(s) > 0$.

In what follows we will use tools of the Mellin transform technique. By [**15**, Theorem 1] (see also [**39**, Theorem 9a] regarding that issue) $\mathscr{M}[f(y)](s)$ absolutely converges on the vertical strip $a < \Re(s) < b$ with $a < 1$ **iff**

$$\mathscr{M}[f(y)](s) = \frac{1}{\Gamma(1-s)} \mathscr{M}\big[\mathscr{L}[f(y)](z)\big](1-s), \qquad \Re(z) > 0. \tag{3.7}$$

Due to (2.6), the Laplace transform $\mathscr{L}[f^{\gamma}_{\alpha,\beta}(y)](z)$ equals $z^{\alpha\gamma-\beta}\,\mathrm{e}^{-z^{\alpha}}$ which in conjunction with (3.7) leads to

$$\hat{f}^{\,\gamma}_{\alpha,\beta}(s) = \frac{1}{\alpha\,\Gamma(1-s)}\,\Gamma\left(\gamma + \frac{1-\beta-s}{\alpha}\right).$$

For rational $\alpha = l/k \in (0,1]$ we invert this Mellin transform. Introducing the variable $\sigma = \gamma/k + (1-\beta-s)/l$ we get

$$\begin{aligned} f^{\gamma}_{l/k,\beta}(y) &= \frac{1}{2\pi \mathrm{i}} \frac{k}{y^{1+\frac{l}{k}\gamma-\beta}} \int_{M_\sigma} \frac{\Gamma(k\sigma)\, y^{l\sigma}}{\Gamma(l\sigma + \beta - \frac{l}{k}\gamma)}\,\mathrm{d}\sigma \\ &= \frac{\sqrt{lk}}{(2\pi)^{\frac{k-l}{2}}} \frac{l^{\frac{l}{k}\gamma-\beta}}{y^{1+\frac{l}{k}\gamma-\beta}} \frac{1}{2\pi\mathrm{i}} \int_{M_\sigma} \frac{\prod_{j=0}^{k-1}\Gamma(\frac{j}{k}+\sigma)}{\prod_{j=0}^{l-1}\Gamma(\frac{j+\beta-l\gamma/k}{l}+\sigma)} \left(\frac{l^l}{k^k y^l}\right)^{-\sigma} \mathrm{d}\sigma \qquad (3.8)\\ &= \frac{\sqrt{lk}}{(2\pi)^{\frac{k-l}{2}}} \frac{l^{\frac{l}{k}\gamma-\beta}}{y^{1+\frac{l}{k}\gamma-\beta}}\, G^{k,0}_{l,k}\left(\frac{l^l}{k^k y^l}\,\middle|\, \begin{matrix}\Delta(l,\beta-\frac{l}{k}\gamma)\\ \Delta(k,0)\end{matrix}\right) \end{aligned}$$

where $\Delta(n,a)$ stands for the list $a/n, (a+1)/n, \ldots, (a+n-1)/n$. To get the second line the Gauss multiplication formula for the Gamma function, *viz.* $\Gamma(nz) = (2\pi)^{(1-n)/2}\, n^{nz-1/2} \prod_{j=0}^{n-1}\Gamma(z+j/n)$, ($n$ positive integer), was used, while the third line came from the definition in the item 1 above as its special case $G^{k,0}_{l,k}\big(y\big|\,{}^{(a_p)}_{(b_q)}\big)$.

Making use of [**28**, Eq. (2.2.1.19)] for $\nu = \beta - l\gamma/k$ and $a = 1$ we calculate the Laplace transform of the RHS of (3.8)

$$\frac{\sqrt{lk}(l^{\frac{l}{k}\gamma-\beta})}{(2\pi)^{\frac{k-l}{2}}} \int_0^\infty e^{-zy} y^{-(1+\frac{l}{k}\gamma-\beta)}\, G^{k,0}_{l,k}\left(\frac{l^l}{k^k y^l}\,\middle|\, \begin{matrix}\Delta(l,\beta-\frac{l}{k}\gamma)\\ \Delta(k,0)\end{matrix}\right) \mathrm{d}y = z^{\frac{l}{k}\gamma-\beta} e^{-z^{l/k}}. \tag{3.9}$$

As shown in section 2 the RHS above is CM for $\beta \ge l\gamma/k$ which implies that $G^{k,0}_{l,k}\left(\frac{l^l}{k^k y^l}\,\middle|\, \begin{matrix}\Delta(l,\beta-\frac{l}{k}\gamma)\\ \Delta(k,0)\end{matrix}\right)$ is non-negative for this range of parameters. Thus we are ready to formulate the corollary

Corollary 3.3. *Let $\alpha = l/k$, $l \le k$ positive integers, $\beta, \gamma > 0$, $g^{\gamma}_{l/k,\beta}(y)$ is defined by (2.5) and $w > 0$. Then the three parameter Mittag-Leffler function $E^{\gamma}_{l/k,\beta}(-w)$*

a) is given by the Laplace transform of a function non-negative on $\mathbb{R}_+$

$$\frac{l}{k}\, E^{\gamma}_{l/k,\beta}(-w) = \mathscr{L}\left[u^{-1-k/l} g^{\gamma}_{l/k,\beta}\left(u^{-k/l}\right)\right](w),$$

b) is CM for $\beta \geq l\gamma/k$.

Proof. Specifying $\alpha = l/k$ in Eqs. (2.5) and (3.9) we have

$$h_{l,k}(u) := \frac{k}{l} u^{-1-\frac{k}{l}} g^{\gamma}_{\frac{l}{k},\beta}(u^{-\frac{k}{l}}) = \frac{l^{\frac{l}{k}\gamma-\beta}}{(2\pi)^{\frac{k-l}{2}}} \frac{\sqrt{lk}}{\Gamma(\gamma)} u^{\gamma-1} G^{k,0}_{l,k}\left(\frac{l^l u^k}{k^k}\middle| \begin{matrix} \Delta(l,\beta-\frac{l}{k}\gamma) \\ \Delta(k,0)\end{matrix}\right).$$

According to [**27**, Eq. (2.24.3.1)] and by the argument inversion property of the Meijer G function [**27**, Eq. (8.2.2.14)], its Laplace transform reduces to

$$\mathscr{L}[h_{l,k}(u)](w) = \frac{k^{\gamma} l^{\frac{l}{k}\gamma+\frac{1}{2}-\beta} w^{-\gamma}}{\Gamma(\gamma)(2\pi)^{k-\frac{1+l}{2}}}\, G^{k,k}_{k,l+k}\left(\frac{w^k}{l^l}\middle| \begin{matrix} \Delta(k,1) \\ \Delta(k,\gamma), \Delta(l, 1+\frac{l}{k}\gamma-\beta)\end{matrix}\right). \tag{3.10}$$

The Meijer G function may be expressed in terms of the finite sum of generalized hypergeometric functions [**27**, Eq.(8.2.2.3)]

$$G^{m,n}_{p,q}\left(z\middle|\begin{matrix}(\alpha_p)\\(\beta_q)\end{matrix}\right) = \sum_{k=1}^{m} \frac{\left[\prod\limits_{j=1}^{m}\Gamma(\beta_j-\beta_k)\right]^{\star} \prod\limits_{j=1}^{n}\Gamma(1+\beta_k-\alpha_j)}{\prod\limits_{j=n+1}^{p}\Gamma(\alpha_j-\beta_k)\prod\limits_{j=m+1}^{q}\Gamma(1+\beta_k-\beta_j)} z^{\beta_k}$$
$$\times\, {}_pF_{q-1}\left(\begin{matrix}1+\beta_k-(\alpha_p)\\ 1+\beta_k-(\beta_q)'\end{matrix}; (-1)^{p-m-n} z\right),$$

where the shorthands $\left[\prod_{j=1}^{m}\Gamma(\beta_j-\beta_k)\right]^{\star} := \prod_{j=1}^{k-1}\Gamma(\beta_j-\beta_k)\prod_{j=k+1}^{m}\Gamma(\beta_j-\beta_k)$ and $c_k-(c_q)' := c_k-c_1,\ldots,c_k-c_{k-1},c_k-c_{k+1},\ldots,c_k-c_q$ are used. This relation and the Gauss multiplication formula, if adopted to RHS of (3.10), lead to

$$\mathscr{L}[h_{l,k}(u)](w) = \sum_{j=0}^{k-1}\frac{(\gamma)_j(-w)^j}{j!\Gamma(\beta+\frac{l}{k}j)}\, {}_{1+k}F_{l+k}\left(\begin{matrix}1, \Delta(k,\gamma+j)\\ \Delta(k,1+j), \Delta(l,\beta+\frac{l}{k}j)\end{matrix}; \frac{(-w)^k}{l^l}\right).$$

Using the series expansion of the generalized hypergeometric functions, (3.7), and the splitted summation formula for double sums $\sum_{j=0}^{m-1}\sum_{r\geq 0} a_{mr+j} = \sum_{r\geq 0} a_r$, we arrive at $\mathscr{L}[h_{l,k}(u)](w) = E^{\gamma}_{l/k,\beta}(-w)$. Applying previously shown non-negativity of $h_{l,k}(u)$ and the Bernstein theorem ends the proof. □

Remark 3.4. *The function* $f^{\gamma}_{\alpha,\beta}$ *is closely related to the* confluent Wright function [**2, 41, 42**]

$$ {}_0\Psi_1\big(-;(\beta,\alpha);z\big) = \sum_{r\geq 0}\frac{z^r}{r!\,\Gamma(\alpha r+\beta)}, \qquad z\in\mathbb{C}.$$

Using the lines of proof following the procedure exposed in the proof of Corollary 3.3. *it may be shown (we leave it to the interested reader) that if $l \leq k$ are positive integers then for all $y > 0$ we get*

$$f^{\gamma}_{l/k,\beta}(y) = y^{\beta - \frac{l}{k}\gamma - 1}\, {}_0\Psi_1\Big(-; \big(\beta - \tfrac{l}{k}\gamma, -\tfrac{l}{k}\big); -y^{-\frac{l}{k}}\Big).$$

Here we would like to recall that the confluent Wright function *was in detail discussed by Bogoljub Stanković* [**37**] *who, in particular, showed the non-negativity of* ${}_0\Psi_1\Big(-; \big(\beta - \tfrac{l}{k}\gamma, -\tfrac{l}{k}\big); -y^{-\frac{l}{k}}\Big)$ *on* $\mathbb{R}_+$ *and* $\beta - \frac{l}{k}\gamma \geq 0$. *The authors are thankful to his mastery in exposing this topic by a complex analytical approach in a highly elegant way.*

4. CONCLUDING REMARKS

Our proof of the CM property of $E^{\gamma}_{\alpha,\beta}(-w)$ is rooted in H. Pollard's elegant and hitting home considerations. The latter, although devoted solely to the one parameter Mittag-Leffler function $E_\alpha(-w)$, admit natural generalization to the other functions belonging to the Mittag-Leffler family. Here we would like to emphasize the paramount importance of the $E^{\gamma}_{\alpha,\beta}(-w)$ and its properties in ongoing studies of affined functions which frequently reduce to the $E^{\gamma}_{\alpha,\beta}(-w)$ modified by some prefactors and/or parameter-dependent substitutions. Complete monotonicity of the three parameter Mittag-Leffler function and its relatives plays a pivoting role in understanding the time behavior of physical processes whose evolution is governed by equations involving fractional derivatives [**14**]. As shown in studies of equations proposed to describe either non-Debye relaxations or anomalous diffusion physically acceptable solutions to these equations have to satisfy requirements closely related, or even encoded in complete monotonicity [**4, 8, 9**] which makes methods developed in the theory of CM functions more and more important tool of mathematical physics.

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