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SUMMATION FORMULAE INVOLVING MULTIPLE HARMONIC NUMBERS

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By means of the generating function approach, we derive several summation formulae involving multiple harmonic numbers $H_{n,\ell}(\sigma)$, as well as other combinatorial numbers named after Bernoulli, Euler, Bell, Genocchi and Stirling.

1. INTRODUCTION AND MOTIVATION

The harmonic numbers and the alternating ones are well–known that are defined by

$$H_0 = 0 \quad \text{and} \quad H_n = \sum_{k=1}^{n} \frac{1}{k} \quad \text{for} \quad n \geq 1;$$
$$\mathcal{H}_0 = 0 \quad \text{and} \quad \mathcal{H}_n = \sum_{k=1}^{n} \frac{(-1)^k}{k} \quad \text{for} \quad n \geq 1;$$

as well as their generating functions

$$\sum_{n=0}^{\infty} H_n x^n = \frac{-\ln(1-x)}{1-x} \quad \text{and} \quad \sum_{n=0}^{\infty} \mathcal{H}_n x^n = \frac{-\ln(1+x)}{1-x}.$$

For their wide applications in combinatorics, number theory and computer science (for example, the analysis of algorithms), properties and identities that

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involve them have been explored by various methods (cf. [**1, 3, 7**]). Further generalizations can be found in the papers [**4, 5, 8, 9, 10**].

By introducing a variable σ, we can unify the both numbers by

$$H_0(\sigma) = 0 \quad \text{and} \quad H_n(\sigma) = \sum_{k=1}^{n} \frac{\sigma^k}{k} \quad \text{for} \quad n \geq 1$$

with the generating function

$$\sum_{n=0}^{\infty} H_n(\sigma) x^n = \frac{-\ln(1-x\sigma)}{1-x}.$$

In this paper, we shall examine the following harmonic–like numbers

$$H_{n,0}(\sigma) \equiv 1 \quad \text{and} \quad H_{n,\ell}(\sigma) = \sum_{\substack{k_1,k_2,\cdots,k_\ell \geq 1 \\ k_1+k_2+\cdots+k_\ell \leq n}} \frac{\sigma^{k_1+k_2+\cdots+k_\ell}}{k_1 k_2 \cdots k_\ell} \quad \text{for} \quad n \geq 1.$$

When $\ell = 1$ and $\sigma = \pm 1$, they will reduce to H_n and $\mathcal{H}_n$, respectively.

Classifying according to the sum $m = k_1 + k_2 + \cdots + k_\ell$, we have

$$H_{n,\ell}(\sigma) = \sum_{m=1}^{n} G_{m,\ell}(\sigma) \quad \text{where} \quad G_{m,\ell}(\sigma) := \sum_{\substack{k_1,k_2,\cdots,k_\ell \geq 1 \\ k_1+k_2+\cdots+k_\ell = m}} \frac{\sigma^m}{k_1 k_2 \cdots k_\ell}.$$

Since the generating function of $G_{m,\ell}(\sigma)$ is equal to

$$\sum_{m=0}^{\infty} G_{m,\ell}(\sigma) x^m = \ln^\ell \frac{1}{1-x\sigma},$$

the generating function for the sequence $H_{n,\ell}(\sigma)$ of their partial sums results in

$$\sum_{n=0}^{\infty} H_{n,\ell}(\sigma) x^n = \frac{\{-\ln(1-x\sigma)\}^\ell}{1-x}.$$

Let $[x^n]f(x)$ stand for the coefficient of x^n in the formal power series $f(x)$. Then we have the following expression

$$H_{n,\ell}(\sigma) = [x^n]\frac{\{-\ln(1-x\sigma)\}^\ell}{1-x} = \sum_{m=1}^{n} \frac{\ell!}{m!} \begin{bmatrix} m \\ \ell \end{bmatrix} \sigma^m,$$

where the signless Stirling numbers of the first kind is given by the generating function

$$\sum_{m=0}^{\infty} \frac{\ell!}{m!} \begin{bmatrix} m \\ \ell \end{bmatrix} x^m = \ln^\ell \frac{1}{1-x}.$$

When $\sigma = 1$, Cheon and El-Mikkawy [**2**] found not only the generating function

$$\sum_{n=0}^{\infty} H_{n,\ell}(1)x^n = \frac{\big\{-\ln(1-x)\big\}^{\ell}}{1-x},$$

but also the following explicit expression

$$H_{n,\ell}(1) = \begin{bmatrix} n+1 \\ \ell+1 \end{bmatrix} \frac{\ell!}{n!}.$$

Unfortunately, for $H_{n,\ell}(-1)$, there does not exist such an elegant expression.

By making use of Riordan arrays, Cheon and El-Mikkawy [**2**] proved four summation formulae about the multiple harmonic numbers $H_{n,\ell}(1)$. Examining carefully the structure of their sums, we find that they fit into the following general scheme about formal power series.

Observe that the numbers $H_{n,\ell}(\sigma)$ form a Riordan array generated by the formal power series pair

$$\Big(\frac{1}{1-x}, \ln\frac{1}{1-x\sigma}\Big).$$

If $\Lambda(x)$ is another formal power series

$$\Lambda(x) := \sum_{\ell=0}^{\infty} \lambda_\ell x^\ell$$

then we can evaluate the sum

$$\text{(1)} \qquad \sum_{\ell=0}^{\infty} \lambda_\ell H_{n,\ell}(\sigma) = \sum_{\ell=0}^{\infty} \lambda_\ell [x^n] \frac{\{-\ln(1-x\sigma)\}^{\ell}}{1-x} = [x^n]\frac{\Lambda(-\ln(1-x\sigma))}{1-x}.$$

According to this scheme, we shall establish seven classes of summation formulae about $H_{n,\ell}(\sigma)$, including the aforementioned four identities of Cheon and El-Mikkawy [**2**, Theorem 3.2]. Then the paper will end in Section 3 with a comment about a composite sum involving the derangement numbers.

Throughout the paper, the following well–known classical numbers (cf. Comtet [**6**, §1.14]) will be used without being recalled:

- Bernoulli numbers B_n with the generating function

$$\sum_{k=0}^{\infty} \frac{B_k}{k!} x^k = \frac{x}{e^x - 1}.$$

- Euler numbers E_n with the generating function

$$\sum_{k=0}^{\infty} \frac{E_k}{k!} x^k = \frac{2e^x}{e^{2x} + 1}.$$

- Genocchi numbers G_n with the generating function

$$\sum_{k=1}^{\infty} \frac{G_k}{k!} x^k = \frac{2x}{e^x + 1}.$$

- Bell numbers $\mathcal{B}_n$ with the generating function

$$\sum_{k=0}^{\infty} \frac{\mathcal{B}_k}{k!} x^k = e^{e^x - 1}.$$

- The derangement numbers D_n with the generating function

$$\sum_{k=0}^{\infty} \frac{D_k}{k!} x^k = \frac{e^{-x}}{1 - x}.$$

- The Stirling number of the second kind with the generating function

$$\sum_{n \geq k} \frac{x^n}{n!} S(n,k) = \frac{(e^x - 1)^k}{k!}.$$

In order to reduce lengthy expressions, we shall make use of the following notations. As usual, the logical function is defined by $\chi(\text{true}) = 1$ and $\chi(\text{false}) = 0$. For a real number x, the smallest integer $\geq x$ and the greatest integer $\leq x$ will be denoted $\lceil x \rceil$ and $\lfloor x \rfloor$, respectively. When m is a natural number, $i \equiv_m j$ stands for that "i is congruent to j modulo m".

2. SEVEN CLASSES OF SUMMATION FORMULAE

In this section, we prove seven summation theorems, where all the formulae are valid for $n \geq 1$ because there are always the same initial value "1" for the results corresponding to the trivial case $n = 0$. To our knowledge, all the formulae displayed in this section are new except for those being explicitly given by references.

We start with the following summation theorem.

Theorem 1. *For the sum defined by*

$$\mathbb{A}_n(\tau, \sigma) := \sum_{\ell=0}^{n} \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma)$$

the following identity holds

$$\mathbb{A}_n(\tau, \sigma) = \sum_{k=0}^{n} \binom{\tau + k - 1}{k} \sigma^k.$$

Two particular cases are highlighted below, where the former integrates the first two formulae given by Cheon and El-Mikkawy [**2**, Theorem 3.2: i & ii]:

$$\mathbb{A}_n(\pm 1,\ 1) = \begin{cases} n+1, & \text{“}+\text{”}; \\ 0, & \text{“}-\text{”}; \end{cases}$$

$$\mathbb{A}_n(\pm 1, -1) = \begin{cases} 1-\chi(n \equiv_2 1), & \text{“}+\text{”}; \\ 2, & \text{“}-\text{”}. \end{cases}$$

Proof. Consider the exponential function $\Lambda(x) = e^{x\tau}$ in (1). Then we have immediately

$$\sum_{\ell=0}^{n} \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma) = [x^n] \frac{(1-x\sigma)^{-\tau}}{1-x} = \sum_{k=0}^{n} \binom{\tau+k-1}{k} \sigma^k. \qquad \square$$

The next theorem is a variant of Theorem 1.

Theorem 2. *For the sum defined by*

$$\mathbb{B}_n(\tau, \sigma) := \sum_{\ell=1}^{n} \frac{\tau^\ell}{(\ell-1)!} H_{n,\ell}(\sigma)$$

the following identity holds

$$\mathbb{B}_n(\tau, \sigma) = \tau \sum_{k=0}^{n} \binom{\tau+k-1}{k} \sigma^k H_{n-k}(\sigma).$$

We record two special cases, where the first one corresponding to “$-$” is due to Cheon and El-Mikkawy [**2**, Theorem 3.2: iii]:

$$\mathbb{B}_n(\pm 1,\ 1) = \begin{cases} (n+1)H_n - n, & \text{“}+\text{”}; \\ \dfrac{-1}{n}, & \text{“}-\text{”}; \end{cases}$$

$$\mathbb{B}_n(\pm 1, -1) = \begin{cases} \dfrac{1}{2}\big\{\mathcal{H}_n + (-1)^n H_n\big\}, & \text{“}+\text{”}; \\ -\mathcal{H}_n - \mathcal{H}_{n-1}, & \text{“}-\text{”}. \end{cases}$$

Proof. Analogously let $\Lambda(x) = \tau x e^{x\tau}$ in (1). Then we can evaluate

$$\begin{aligned} \sum_{\ell=1}^{n} \frac{\tau^\ell}{(\ell-1)!} H_{n,\ell}(\sigma) &= [x^n] \frac{\tau}{(1-x\sigma)^\tau} \times \frac{\ln(1-x\sigma)}{x-1} \\ &= \tau \sum_{k=0}^{n} \binom{\tau+k-1}{k} \sigma^k H_{n-k}(\sigma). \qquad \square \end{aligned}$$

Theorem 3. *For the sum defined by*

$$\mathbb{C}_n(\tau,\sigma) := \sum_{\ell=0}^{n} B_\ell \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma)$$

the following identities hold

$$\mathbb{C}_n(1,\sigma) = \frac{H_{n+1}(\sigma)}{\sigma} - H_n(\sigma),$$
$$\mathbb{C}_n(-1,\sigma) = \frac{H_{n+1}(\sigma)}{\sigma}.$$

In particular for $\sigma = \pm 1$, we deduce further, where the former with the minus sign "$-$" is equivalent to Cheon and El-Mikkawy [**2**, Theorem 3.2: iv]:

$$\mathbb{C}_n(\pm 1, 1) = \begin{cases} \dfrac{1}{n+1}, & \text{“}+\text{”}; \\ H_{n+1}, & \text{“}-\text{”}; \end{cases}$$
$$\mathbb{C}_n(\pm 1, -1) = \begin{cases} -\mathcal{H}_n - \mathcal{H}_{n+1}, & \text{“}+\text{”}; \\ -\mathcal{H}_{n+1}, & \text{“}-\text{”}. \end{cases}$$

Proof. Choose $\Lambda(x) = \frac{\tau x}{e^{x\tau}-1}$, the generating function of Bernoulli numbers. We get from (1)

$$\sum_{\ell=0}^{n} B_\ell \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma) = [x^n] \frac{\tau}{(1-x\sigma)^{-\tau} - 1} \times \frac{\ln(1-x\sigma)}{x-1}.$$

For $\tau = 1$, we can determine the coefficient

$$[x^n] \frac{1-x\sigma}{x\sigma} \times \frac{\ln(1-x\sigma)}{x-1} = \frac{H_{n+1}(\sigma)}{\sigma} - H_n(\sigma).$$

Instead, when $\tau = -1$, the coefficient becomes

$$[x^n] \frac{1}{x\sigma} \times \frac{\ln(1-x\sigma)}{x-1} = \frac{H_{n+1}(\sigma)}{\sigma}. \qquad \square$$

Theorem 4. *For the sum defined by*

$$\mathbb{D}_n(\tau,\sigma) := \sum_{\ell=0}^{n} S(\ell,m) \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma)$$

the following identities hold

$$\mathbb{D}_n(1,\sigma) = \frac{\sigma^m}{m!} \sum_{k=m}^{n} \binom{k-1}{m-1} \sigma^k,$$
$$\mathbb{D}_n(-1,\sigma) = \frac{(-\sigma)^m}{m!} \chi(m \le n).$$

When $\sigma = \pm 1$, they can be restated as follows:

$$\mathbb{D}_n(\pm 1, 1) = \begin{cases} \dfrac{1}{m!}\dbinom{n}{m}, & \text{“}+\text{”}; \\ \dfrac{(-1)^m}{m!}\chi(m \le n), & \text{“}-\text{”}; \end{cases}$$

$$\mathbb{D}_n(\pm 1, -1) = \begin{cases} \dfrac{(-1)^m}{m!}\displaystyle\sum_{k=m}^{n}\binom{k-1}{m-1}(-1)^k, & \text{“}+\text{”}; \\ \dfrac{\chi(m \le n)}{m!}, & \text{“}-\text{”}. \end{cases}$$

Proof. Let $\Lambda(x) = \frac{(e^{x\tau}-1)^m}{m!}$ be the exponential generating function of Stirling numbers of the second kind. We have from (1)

$$\sum_{\ell=0}^{n} S(\ell, m)\frac{\tau^\ell}{\ell!}H_{n,\ell}(\sigma) = [x^n]\frac{\{(1-x\sigma)^{-\tau}-1\}^m}{m!(1-x)}.$$

For $\tau = 1$, we can determine the coefficient

$$[x^n]\frac{(x\sigma)^m(1-x\sigma)^{-m}}{m!(1-x)} = \frac{\sigma^m}{m!}\sum_{k=m}^{n}\binom{k-1}{m-1}\sigma^k.$$

Instead, when $\tau = -1$, the coefficient becomes

$$[x^n]\frac{(-x\sigma)^m}{m!(1-x)} = \frac{(-\sigma)^m}{m!}\chi(m \le n). \qquad \square$$

Theorem 5. *For the sum defined by*

$$\mathbb{E}_n(\tau, \sigma) := \sum_{\ell=0}^{n} \mathcal{B}_\ell \frac{\tau^\ell}{\ell!}H_{n,\ell}(\sigma)$$

the following identities hold

$$\mathbb{E}_n(1, \sigma) = 1 + \sum_{k=1}^{n}\sum_{i=1}^{k}\binom{k-1}{i-1}\frac{\sigma^k}{i!},$$

$$\mathbb{E}_n(-1, \sigma) = \sum_{k=0}^{n}\frac{(-\sigma)^k}{k!}.$$

For $\sigma = \pm 1$, these formulae can be further simplified into

$$\mathbb{E}_n(\pm 1, 1) = \begin{cases} 1 + \sum_{k=1}^{n} \frac{1}{k!}\binom{n}{k}, & \text{“}+\text{”}; \\ \frac{D_n}{n!}, & \text{“}-\text{”}; \end{cases}$$

$$\mathbb{E}_n(\pm 1, -1) = \begin{cases} 1 + \sum_{k=1}^{n} \sum_{i=1}^{k} \frac{(-1)^k}{i!}\binom{k-1}{i-1}, & \text{“}+\text{”}; \\ \sum_{k=0}^{n} \frac{1}{k!}, & \text{“}-\text{”}. \end{cases}$$

Proof. Consider $\Lambda(x) = \exp\big(e^{x\tau} - 1\big)$, the generating function of Bell numbers. We can express the sum in (1) as

$$\sum_{\ell=0}^{n} \mathcal{B}_\ell \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma) = [x^n]\frac{\exp\{(1-x\sigma)^{-\tau} - 1\}}{1-x}.$$

For $\tau = 1$, we can determine the coefficient

$$[x^n]\frac{\exp(\frac{x\sigma}{1-x\sigma})}{1-x} = \sum_{k=0}^{n} [x^k]\exp(\frac{x\sigma}{1-x\sigma}).$$

Instead, when $\tau = -1$, the coefficient becomes

$$[x^n]\frac{\exp(-x\sigma)}{1-x} = \sum_{k=0}^{n} \frac{(-\sigma)^k}{k!}. \qquad \square$$

Theorem 6. *For the sum defined by*

$$\mathbb{F}_n(\tau, \sigma) := \sum_{\ell=0}^{n} G_\ell \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma)$$

the following identities hold

$$\mathbb{F}_n(1, \sigma) = H_n(\sigma) - \sum_{k=1}^{n} \frac{\sigma^k}{2^k} H_{n-k}(\sigma),$$

$$\mathbb{F}_n(-1, \sigma) = -\sum_{k=0}^{n} \frac{\sigma^k}{2^k} H_{n-k}(\sigma).$$

Observe that

$$\sum_{k=1}^{n} \frac{\sigma^k}{2^k} H_{n-k}(\sigma) = \sum_{k=1}^{n-1} \frac{\sigma^k}{2^k} \sum_{i=1}^{n-k} \frac{\sigma^i}{i} = \sum_{i=1}^{n-1} \frac{\sigma^i}{i} \sum_{k=1}^{n-i} \frac{\sigma^k}{2^k}$$

$$= \sum_{i=1}^{n-1} \frac{\sigma^i}{i}\left(\frac{\frac{\sigma}{2} - (\frac{\sigma}{2})^{n-i+1}}{1-\frac{\sigma}{2}}\right).$$

When $\sigma = \pm 1$, the formulae in Theorem 6 can stated explicitly as follows:

$$\mathbb{F}_n(\pm 1, 1) = \begin{cases} \displaystyle\sum_{k=1}^{n} \frac{2^k}{2^n k}, & \text{“}+\text{”}; \\ \displaystyle -2H_n + \sum_{k=1}^{n} \frac{2^k}{2^n k}, & \text{“}-\text{”}; \end{cases}$$

$$\mathbb{F}_n(\pm 1, -1) = \begin{cases} \displaystyle\frac{4}{3}\mathcal{H}_n - \frac{(-1)^n}{3} \sum_{k=1}^{n} \frac{2^k}{2^n k}, & \text{“}+\text{”}; \\ \displaystyle -\frac{2}{3}\mathcal{H}_n - \frac{(-1)^n}{3} \sum_{k=1}^{n} \frac{2^k}{2^n k}, & \text{“}-\text{”}. \end{cases}$$

Proof. Let $\Lambda(x) = \frac{2x\tau}{1+e^{x\tau}}$ be the generating function of Genocchi numbers. We have from (1)

$$\sum_{\ell=0}^{n} G_\ell \frac{\tau^\ell}{\ell!} H_{n,\ell}(\sigma) = [x^n] \frac{2\tau}{1 + (1 - x\sigma)^{-\tau}} \times \frac{\ln(1 - x\sigma)}{x - 1}.$$

For $\tau = 1$, we can determine the coefficient

$$[x^n] \frac{2(1 - x\sigma)}{2 - x\sigma} \times \frac{\ln(1 - x\sigma)}{x - 1} = H_n(\sigma) - \sum_{k=1}^{n} \frac{\sigma^k}{2^k} H_{n-k}(\sigma).$$

Instead, when $\tau = -1$, the coefficient becomes

$$[x^n] \frac{-2}{2 - x\sigma} \times \frac{\ln(1 - x\sigma)}{x - 1} = -\sum_{k=0}^{n} \frac{\sigma^k}{2^k} H_{n-k}(\sigma). \qquad \square$$

Theorem 7. *For the sum defined by*

$$\mathbb{G}_n(\sigma) := \sum_{\ell=0}^{n} \frac{E_\ell}{\ell!} H_{n,\ell}(\sigma)$$

the following identities hold

$$G_n(1) = \chi(n \not\equiv_4 3) \frac{(-1)^{\lfloor \frac{n}{4} \rfloor}}{2^{\lfloor \frac{n}{2} \rfloor}},$$

$$G_n(-1) = \frac{4}{5} + \frac{(-1)^{\lfloor \frac{n}{4} \rfloor}}{5 \cdot 2^{\lfloor \frac{n}{2} \rfloor}} \times \begin{cases} 1, & n \equiv_4 0; \\ 1, & n \equiv_4 1; \\ -3, & n \equiv_4 2; \\ 2, & n \equiv_4 3. \end{cases}$$

Proof. Finally specify $\Lambda(x)=\frac{2e^x}{1+e^{2x}}$ by the generating function of Euler numbers in (1), where another variable τ is suppressed because this function is even. Then the corresponding sum can be reformulated as

$$\sum_{\ell=0}^{n}\frac{E_\ell}{\ell!}H_{n,\ell}(\sigma)=[x^n]\frac{2(1-x\sigma)}{(1-x)\big\{1+(1-x\sigma)^2\big\}}.$$

When $\sigma=1$, by making use of partial fractions

$$\frac{2}{1+(1-x)^2}-\frac{1}{1-x}=\frac{\mathbf{i}}{1+\mathbf{i}-x}-\frac{\mathbf{i}}{1-\mathbf{i}-x}-\frac{1}{1-x},$$

we can extract 'the coefficient of x^n'

$$\begin{aligned}[x^n]\frac{2}{1+(1-x)^2}&=\frac{\mathbf{i}}{(1+\mathbf{i})^{n+1}}-\frac{\mathbf{i}}{(1-\mathbf{i})^{n+1}}\\&=2^{-\frac{n+1}{2}}\Big\{\mathbf{i}e^{-\frac{n+1}{4}\pi\mathbf{i}}-\mathbf{i}e^{\frac{n+1}{4}\pi\mathbf{i}}\Big\}.\end{aligned}$$

Then the first formula for $G_n(1)$ follows from the simplification

$$\Big\{\mathbf{i}e^{-\frac{n+1}{4}\pi\mathbf{i}}-\mathbf{i}e^{\frac{n+1}{4}\pi\mathbf{i}}\Big\}=(-1)^{\lfloor\frac{n}{4}\rfloor}\times\begin{cases}\sqrt{2}, & n\equiv_4 0;\\ 2, & n\equiv_4 1;\\ \sqrt{2}, & n\equiv_4 2;\\ 0, & n\equiv_4 3.\end{cases}$$

Alternatively for $\sigma=-1$, we can decompose the rational fraction

$$\frac{2(1+x)}{(1-x)\big\{1+(1+x)^2\big\}}=\frac{4}{5(1-x)}+\frac{2-\mathbf{i}}{5(1+x+\mathbf{i})}+\frac{2+\mathbf{i}}{5(1+x-\mathbf{i})}$$

and determine 'the coefficient of x^n'

$$\begin{aligned}[x^n]\frac{2(1+x)}{(1-x)\big\{1+(1+x)^2\big\}}&=\frac{4}{5}+\frac{(-1)^n(2-\mathbf{i})}{5(1+\mathbf{i})^{n+1}}+\frac{(-1)^n(2+\mathbf{i})}{5(1-\mathbf{i})^{n+1}}\\&=\frac{4}{5}+\frac{(-1)^n}{5\cdot 2^{\frac{n+1}{2}}}+\Big\{(2+\mathbf{i})e^{\frac{n+1}{4}\pi\mathbf{i}}+(2-\mathbf{i})e^{-\frac{n+1}{4}\pi\mathbf{i}}\Big\}.\end{aligned}$$

Then the second formula for $G_n(-1)$ follows from the simplification

$$\Big\{(2+\mathbf{i})e^{\frac{n+1}{4}\pi\mathbf{i}}+(2-\mathbf{i})e^{-\frac{n+1}{4}\pi\mathbf{i}}\Big\}=(-1)^{\lceil\frac{n}{4}\rceil}\times\begin{cases}\sqrt{2}, & n\equiv_4 0;\\ 2, & n\equiv_4 1;\\ 3\sqrt{2}, & n\equiv_4 2;\\ 4, & n\equiv_4 3.\end{cases}$$

□

3. CONCLUDING COMMENTS

There should be a number of further choices for the formal power series $\Lambda(x)$ in (1), that may lead us to more summation formulae. For instance, Specifying in (1) by the generating function $\Lambda(x) = \frac{e^{-x\tau}}{1-x\tau}$ of the derangement numbers, we have the sum

$$\sum_{\ell=0}^{n} D_\ell \frac{\tau^\ell}{\ell!} H_n(\ell) = [x^n] \frac{(1-x\sigma)^\tau}{(1-x)\{1+\tau\ln(1-x\sigma)\}}.$$

When $\tau = \sigma = 1$, we can evaluate the sum

$$\sum_{\ell=0}^{n} \frac{D_\ell}{\ell!} H_{n,\ell}(1) = [x^n] \frac{1}{1+\ln(1-x)} = \sum_{k=0}^{n} \frac{k!}{n!} \begin{bmatrix} n \\ k \end{bmatrix}.$$

However, for other values of τ and σ, the corresponding sums don't admit such nice expressions.

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