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SOME NEW ESTIMATES OF PRECISION OF CUSA-HUYGENS AND HUYGENS APPROXIMATIONS

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In Memory of Professor D.S. Mitrinović

In this paper we present some new upper bounds of the Cusa-Huygens and the Huygens approximations. Bounds are obtained in the forms of some polynomial and some rational functions.

1. INTRODUCTION

In this paper it is considered the following Cusa-Huygens inequality

$$\frac{3\sin x}{2+\cos x} < x < \frac{2}{3}\sin x + \frac{1}{3}\tan x, \tag{1}$$

for $x \in \left(0, \frac{\pi}{2}\right)$, as shown in [**1**], [**2**] and [**3**]. Let us emphasize that the following approximation:

$$x \approx \frac{3\sin x}{2+\cos x}, \tag{2}$$

for $x \in (0, \pi]$, was first surmised in the De Cusa's *Opera book*, see [**4**] and [**6**]. Approximation stated above will be called the Cusa-Huygens approximation.

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Let us consider *the error of the* CUSA-HUYGENS *approximation* as the following function:

$$R(x) = x - \frac{3\sin x}{2+\cos x}, \tag{3}$$

for $x \in [0, \pi]$. One estimate of the precision of the CUSA-HUYGENS approximation is given by the following statement of LING ZHU:

Theorem 1. [**7**] *It is true that*:

$$\frac{1}{180}x^5 < x - \frac{3\sin x}{2+\cos x}, \tag{4}$$

and

$$\frac{1}{2100}x^7 < x - \frac{3\sin x}{2+\cos x}\left(1 + \frac{(1-\cos x)^2}{9(3+2\cos x)}\right), \tag{5}$$

for $x \in (0, \pi]$. *Moreover,* $1/180$ *and* $1/2100$ *are the best constants in the previous inequalities, respectively.*

The results of the previous theorem are corrections of the Theorem 3.4.20 from monograph [**1**]. This important discovery and the resulting corrections took place in 2018, almost half a century after the publication of classics [**7**].

In this paper we consider also the following HUYGENS*'s approximation*:

$$x \approx \frac{2}{3}\sin x + \frac{1}{3}\tan x, \tag{6}$$

for $x \in \left(0, \frac{\pi}{2}\right)$. Estimates of *the errors function of* HUYGENS *approximation* $Q(x) = \frac{2}{3}\sin x + \frac{1}{3}\tan x - x$, for $x \in \left(0, \frac{\pi}{2}\right)$, are achieved by use of some polynomial functions and some rational functions. Necessary theoretical basis for that research are stated in the following section.

2. PRELIMINARIES

Double sided Taylor approximations

Let us introduce some notation and the basic claims that shall be used according to the papers [**8**] and [**9**]. Let us begin from real function $f : (a, b) \longrightarrow \mathbb{R}$ for which there are the finite values $f^{(k)}(a+) = \lim\limits_{x \to a+} f^{(k)}(x)$, $k = 0, 1, \ldots, n$, for $n \in \mathbb{N}_0$. Here we use the notation $T_n^{f,\,a+}(x)$ for TAYLOR polynomial of order n, for $n \in \mathbb{N}_0$, for function $f(x)$ defined in right neighbourhood of a:

$$T_n^{f,\,a+}(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a+)}{k!}(x-a)^k. \tag{7}$$

We shall call $T_n^{f,\,a+}(x)$ *the first* TAYLOR *approximation in the right neighbourhood of* a [**8**]. For $n\in\mathbb{N}_0$, we define *the remainder of first* TAYLOR *approximation in the right neighbourhood of* a by $R_n^{f,\,a+}(x) = f(x) - T_n^{f,\,a+}(x)$. In the paper [**8**] are considered the polynomials:

$$\mathbb{T}_n^{f;\,a+,\,b-}(x) = \begin{cases} T_{n-1}^{f,\,a+}(x) + \dfrac{1}{(b-a)^n} R_{n-1}^{f,\,a+}(b-)(x-a)^n & : \quad n \geq 1 \\ f(b-) & : \quad n = 0, \end{cases} \tag{8}$$

and determined as*the second* TAYLOR *approximation in right neighbourhood of* a, for $n\in\mathbb{N}_0$, [**8**]. Then the following statement is true.

Theorem 2. *Let us assume that* $f:(a,b)\longrightarrow\mathbb{R}$, *and that* n *is natural number such that there exist* $f^{(k)}(a+)$, *for* $k\in\{0,1,2,\ldots,n\}$. *Let us assume that* $f^{(n)}(x)$ *is increasing on* (a,b). *Then for every* $x\in(a,b)$ *following inequality is true*:

$$T_n^{f,\,a+}(x) < f(x) < \mathbb{T}_n^{f;\,a+,\,b-}(x). \tag{9}$$

At that, if $f^{(n)}(x)$ *is decreasing over* (a,b), *then reversed inequality from* (9) *is true.*

The previous statement we call the *Theorem on double-sided* TAYLOR*'s approximations* in [**8**] and [**9**], i.e. *Theorem WD* in [**26**]-[**30**]. Let us emphasize that the proof of this Theorem (i.e. Theorem 2 in [**10**]) is based on L'HOSPITAL's rule for the monotonicity. A similar method is used in proving some related theorems in [**11**], [**12**] and [**13**], which were previously published. Further, the following claims are true.

Proposition 1. [**8**] *Let* $f:(a,b)\longrightarrow\mathbb{R}$ *be such real a function that there exist the first and the second* TAYLOR *approximation in the right neighbourhood of* a, *for some* $n\in N_0$. *Then,*

$$\operatorname{sgn}\Big(\mathbb{T}_n^{f,\,a+,\,b-}(x) - \mathbb{T}_{n+1}^{f,\,a+,\,b-}(x)\Big) = \operatorname{sgn}\Big(f(b-) - T_n^{f,\,a+}(b)\Big), \tag{10}$$

for every $x\in(a,b)$.

Theorem 3. [**8**] *Let* $f:(a,b)\longrightarrow\mathbb{R}$ *be a real analytic function with the power series*:

$$f(x) = \sum_{k=0}^{\infty} c_k (x-a)^k, \tag{11}$$

where $c_k\in\mathbb{R}$ *and* $c_k\geq 0$ *for every* $k\in\mathbb{N}_0$. *Then,*

$$\begin{gathered} T_0^{f,\,a+}(x) \leq \ldots \leq T_n^{f,\,a+}(x) \leq T_{n+1}^{f,\,a+}(x) \leq \ldots \\ \ldots \leq f(x) \leq \ldots \\ \ldots \leq \mathbb{T}_{n+1}^{f;\,a+,\,b-}(x) \leq \mathbb{T}_n^{f;\,a+,\,b-}(x) \leq \ldots \leq \mathbb{T}_0^{f;\,a+,\,b-}(x), \end{gathered} \tag{12}$$

for every $x\in(a,b)$. *If* $c_k\in\mathbb{R}$ *and* $c_k\leq 0$ *for every* $k\in\mathbb{N}_0$, *then the reversed inequality is true.*

Inequality for Bernoulli numbers

Let $(\mathbf{B}_k)$ be the sequence of Bernoulli numbers as it is usually considered, for example see [**14**]. In this paper we use the following well the known inequality for Bernoulli numbers as given by D. A'niello in [**15**]:

$$\frac{2(2n)!}{\pi^{2n}}\frac{1}{2^{2n}-1} < \mathbf{B}_{2n} < \frac{2(2n)!}{\pi^{2n}}\frac{1}{2^{2n}-2}. \tag{13}$$

The previous inequality can be rewritten in the equivalent form

$$2\frac{2^{2n}}{\pi^{2n}} < \frac{2^{2n}(2^{2n}-1)|\mathbf{B}_{2n}|}{(2n)!} < 2\frac{2^{2n}}{\pi^{2n}}\frac{2^{2n}-1}{2^{2n}-2}, \tag{14}$$

for $n \in \mathbb{N}$, and it shall be used in the next section.

3. THE MAIN RESULTS

3.1 The case of Cusa-Huygens approximation

In this section we determinate some upper bounds of one estimation of error of the Cusa-Huygens approximation.

In connection with inequality (4), we consider the following statements.

Lemma 1. *The function*

$$h(t) = \frac{30\sin t + 15\cos t \sin t}{4\cos^2 t + 22\cos t + 19} : [0,\pi] \longrightarrow \mathbb{R} \tag{15}$$

has:

1. *exactly one maximum on* $(0,\pi)$ *at the point*

$$t_1 = \pi - \arccos\left(1 - \frac{\sqrt[3]{98+42\sqrt{105}}}{14} + \frac{4}{\sqrt[3]{98+42\sqrt{105}}}\right) = 2.73210... \tag{16}$$

and the numerical value of the function $h(t)$ *in the point* t_1 *is*

$$h(t_1) = 2.95947...; \tag{17}$$

2. *exactly one inflection point on the interval* $(0,\pi)$

$$t_2 = \pi - \arccos\frac{35-3\sqrt{21}}{28} = 2.43258... \tag{18}$$

and the numerical value of the function $h(t)$ *at the point* t_2 *is*

$$h(t_2) = 2.63119.... \tag{19}$$

Proof. Based on the first and the second derivatives of the function $h(t)$:

$$h'(t) = \frac{210\cos^3 t + 630\cos^2 t + 810\cos t + 375}{(4\cos^2 t + 22\cos t + 19)^2} \tag{20}$$

and

$$h''(t) = \frac{30(28\cos^2 t + 70\cos t + 37)\,(\cos t - 1)^2 \sin t}{(4\cos^2 t + 22\cos t + 19)^3}, \tag{21}$$

the statements **1.** and **2.** are true. □

Lemma 2. *The equation*

$$h(t) = t, \tag{22}$$

has exactly one solution

$$t_0 = 2.83982... \tag{23}$$

in $(0, \pi)$.

Proof. We have $h(0) = 0$ and $h(\pi) = 0$. The function $h(t)$ is strictly increasing on $(0, t_1)$ and strictly decreasing on (t_1, π). The function $h(t)$ is convex on $(0, t_2)$ and concave on the interval (t_2, π). Let us note that

$$h(t_1) > t_1 \quad \text{and} \quad h(t_2) > t_2.$$

Therefore there exists exactly one solution of the equation $h(t) = t$ in (t_1, π) with the numerical value $t_0 = 2.83982\ldots$. □

Lemma 3. *The function*

$$f(t) = \frac{t - \dfrac{3\sin t}{2 + \cos t}}{t^5} : (0, \pi) \longrightarrow \mathbb{R} \tag{24}$$

has exactly one maximum at $t_0 = 2.83982...$ *and the numerical value of the function* $f(t)$ *in the point of the maximum is*

$$M_1 = f(t_0) = 0.010756\ldots. \tag{25}$$

Proof. The statement follows from the first derivative

$$f'(t) = \frac{30\sin t + 15\cos t \sin t - (4\cos^2 t + 22\cos t + 19)t}{(2 + \cos t)^2 t^6} \tag{26}$$

directly and using the previous two lemmas. □

Let us denote

$$m_1 = \frac{1}{M_1} = \frac{1}{0.010756\ldots} = 92.96406\ldots. \tag{27}$$

Then, based on the previous three lemmas and the result of the paper Ling Zhu [**7**] we have the following statement.

Theorem 4. *The following inequalities are true*

$$\frac{1}{180}x^5 < x - \frac{3\sin x}{2+\cos x} \leq \frac{1}{m_1}x^5 = \frac{1}{92.96406...}x^5, \tag{28}$$

for $x \in (0, \pi]$.

The above consideration on estimates of the precision of the Cusa-Huygens approximation may be further generalized by determining the Maclaurin series of the Cusa-Huygens function:

$$\theta(x) = \frac{3\sin x}{2+\cos x} : [0,\pi] \longrightarrow \mathbb{R}. \tag{29}$$

Next, in the connection with inequality (5) we consider the following statements.

Lemma 4. *The function*

$$\kappa(\tau) = \frac{294\sin\tau + 217\cos\tau\sin\tau + 14\cos^2\tau\sin\tau}{2\cos^3\tau + 78\cos^2\tau + 258\cos\tau + 187} : [0,\pi] \longrightarrow \mathbb{R} \tag{30}$$

has

1. *exactly one maximum on* $(0,\pi)$ *at the point*

$$\tau_1 = 2.79340... \tag{31}$$

and the numerical value of the function κ *at the point* τ_1 *is*

$$\kappa(\tau_1) = 2.97564...; \tag{32}$$

2. *exactly one inflection point on* $(0,\pi)$

$$\tau_2 = 2.55459... \tag{33}$$

and the numerical value of the function κ *at the point* τ_2 *is*

$$\kappa(\tau_2) = 2.71423.... \tag{34}$$

Proof. Based on the first and the second derivatives of the function $\kappa(\tau)$

$$\kappa'(\tau) = \frac{658\cos^5\tau + 6076\cos^4\tau + 41776\cos^3\tau + 96236\cos^2\tau + 95606\cos\tau + 35273}{(2\cos^3\tau + 78\cos^2\tau + 258\cos\tau + 187)^2} \tag{35}$$

and

$$\kappa''(\tau) = \frac{14(94\cos^4\tau - 1648\cos^3\tau - 23700\cos^2\tau - 46207\cos\tau - 23039)\,(\cos\tau - 1)^3\sin\tau}{(2\cos^3\tau + 78\cos^2\tau + 258\cos\tau + 187)^3} \tag{36}$$

the statements **1.** and **2.** are true. □

Lemma 5. *The equation*

$$\kappa(\tau)=\tau, \tag{37}$$

has exactly one solution

$$\tau_0=2.87934... \tag{38}$$

in $(0,\pi)$.

Proof. We have $\kappa(0)=0$ and $\kappa(\pi)=0$. The function $\kappa(\tau)$ is strictly increasing on $(0,\tau_1)$ and strictly decreasing on the interval (τ_1,π). The function $\kappa(\tau)$ is convex on $(0,\tau_2)$ and concave on (τ_2,π). Let us note that

$$\kappa(\tau_1)>\tau_1 \quad \text{and} \quad \kappa(\tau_2)>\tau_2.$$

Therefore then exists exactly one solution of the equation $\kappa(\tau)=\tau$ in (τ_1,π) with the numerical value $\tau_0=2.87934\ldots$. □

Lemma 6. *The function*

$$g(\tau)=\frac{\tau-\dfrac{3\sin\tau}{2+\cos\tau}\left(1+\dfrac{(1-\cos\tau)^2}{9(3+2\cos\tau)}\right)}{\tau^7}:(0,\pi)\longrightarrow\mathbb{R} \tag{39}$$

has exactly one maximum in at point $\tau_0=2.87934...$ *and the numerical value of the function* $g(\tau)$ *at the point of the maximum is*

$$M_2=g(\tau_0)=0.001112... \tag{40}$$

Proof. The statement follows from the first derivative

$$g'(\tau)=\frac{294\sin\tau+\cos\tau\sin\tau(217+14\cos\tau)-(2\cos^3\tau+78\cos^2\tau+258\cos\tau+187)\tau}{3(3+2\cos\tau)^2\tau^8} \tag{41}$$

directly and the previous two lemmas. □

Let us denote

$$m_2=\frac{1}{M_2}=899.04062\ldots. \tag{42}$$

Then, based on the previous three lemmas and the result of Ling Zhu [**7**] we have the following statement.

Theorem 5. *The following inequalities are true*

$$\frac{1}{2100}x^7<x-\frac{3\sin x}{2+\cos x}\left(1+\frac{(1-\cos x)^2}{9(3+2\cos x)}\right)\leq\frac{1}{m_2}x^7=\frac{1}{899.04062...}x^7, \tag{43}$$

for $x\in(0,\pi]$.

The above consideration may be further generalized using the MACLAURIN series of the function:

$$\Theta(x) = \frac{3\sin x}{2+\cos x}\left(1+\frac{(1-\cos x)^2}{9(3+2\cos x)}\right) : [0,\pi] \longrightarrow \mathbb{R}. \tag{44}$$

3.2 The Case of Huygens approximation

In this part we determine some upper bounds of one estimate of the error of the HUYGENS approximation. The results stated in preliminarily section are applied to the function:

$$\varphi(x) = \frac{2}{3}\sin x + \frac{1}{3}\tan x : \left(0, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}, \tag{45}$$

that we shall call *the* HUYGENS *function.*

Some polynomial bounds of the HUYGENS function. Let us start from the well-known power series, see [**14**]

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1}, \tag{46}$$

where $x \in \mathbb{R}$ and

$$\tan x = \sum_{k=0}^{\infty} \frac{2^{2k+2}(2^{2k+2}-1)|\mathbf{B}_{2k+2}|}{(2k+2)!} x^{2k+1}, \tag{47}$$

where $|x| < \frac{\pi}{2}$. Based on the previous two power series, it follows:

$$\varphi(x) = x + \frac{1}{20}x^5 + \frac{1}{56}x^7 + \frac{7}{960}x^9 + \frac{3931}{1330560}x^{11} + \ldots = \sum_{k=0}^{\infty} a_k x^{2k+1}, \tag{48}$$

with coefficients

$$a_k = \frac{2(-1)^k}{3(2k+1)!} + \frac{2^{2k+2}(2^{2k+2}-1)|\mathbf{B}_{2k+2}|}{3(2k+2)!}, \tag{49}$$

where $x \in \left(0, \frac{\pi}{2}\right)$ and $k \in \mathbb{N}_0$. Based on inequalities (14), it follows that

$$a_k > 0, \tag{50}$$

for $k \in \mathbb{N}_0$. From there comes that, based on Theorem 3, the following claim about some polynomial inequalities for the HUYGENS function is true.

Theorem 6. *Let there be given the function* $\varphi(x) = \sum_{k=0}^{\infty} a_k x^{2k+1} : \left(0, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}$*, with coefficients* a_k *determined with* (49)*, and let* $c \in \left(0, \frac{\pi}{2}\right)$ *be fixed. Then for* $x \in (0, c)$*, it holds*:

$$\begin{aligned} T_0^{\varphi,\,0+}(x) < \ldots < T_n^{\varphi,\,0+}(x) < T_{n+1}^{\varphi,\,0+}(x) < \ldots \\ \ldots < \frac{2}{3}\sin x + \frac{1}{3}\tan x < \ldots \\ \ldots < \mathbb{T}_{n+1}^{\varphi;\,0+,\,c-}(x) < \mathbb{T}_n^{\varphi;\,0+,\,c-}(x) < \ldots < \mathbb{T}_0^{\varphi;\,0+,\,c-}(x). \end{aligned} \tag{51}$$

Example 1. *Let us introduce some examples of the inequalities obtained for* $n = 0, 1, 2, 3, 4, 5$.

$\boldsymbol{n = 0}$**:** *Let* $c \in \left(0, \frac{\pi}{2}\right)$ *be fixed. Then for* $x \in (0, c)$*, it holds*:

$$0 = T_0^{\varphi,\,0+}(x) < \frac{2}{3}\sin x + \frac{1}{3}\tan x < \mathbb{T}_0^{\varphi;\,0+,\,c-}(x) = \frac{2}{3}\sin c + \frac{1}{3}\tan c. \tag{52}$$

$\boldsymbol{n = 1}$**:** *Let* $c \in \left(0, \frac{\pi}{2}\right)$ *be fixed. Then for* $x \in (0, c)$*, it holds*:

$$x = T_1^{\varphi,\,0+}(x) < \frac{2}{3}\sin x + \frac{1}{3}\tan x < \mathbb{T}_1^{\varphi;\,0+,\,c-}(x) = \frac{\frac{2}{3}\sin c + \frac{1}{3}\tan c}{c}\,x. \tag{53}$$

$\boldsymbol{n = 2, 3, 4}$**:** *Let* $c \in \left(0, \frac{\pi}{2}\right)$ *be fixed. Then for* $x \in (0, c)$ *holds*:

$$x = T_n^{\varphi,\,0+}(x) < \frac{2}{3}\sin x + \frac{1}{3}\tan x < \mathbb{T}_n^{\varphi;\,0+,\,c-}(x) = x + \frac{\frac{2}{3}\sin c + \frac{1}{3}\tan c}{c^n}\,x^n, \tag{54}$$

for $n = 2, 3, 4$.

$\boldsymbol{n=5}$**:** *Let* $c\in\left(0,\frac{\pi}{2}\right)$ *be fixed. Then for* $x\in(0,c)$*, it holds*:

(55)
$$x+\frac{1}{20}x^5=T_5^{\varphi,\,0+}(x)<\frac{2}{3}\sin x+\frac{1}{3}\tan x<\mathbb{T}_5^{\varphi;\,0+,\,c-}(x)=\frac{\frac{2}{3}\sin c+\frac{1}{3}\tan c}{c^5}x^5.$$

From the previous Theorem, directly follows the estimate of the function of error of HUYGENS approximation $Q(x)=\frac{2}{3}\sin x+\frac{1}{3}\tan x-x$ with previously considered polynomial functions.

Theorem 7. *Let there be given the function* $\varphi(x)=\sum_{k=0}^{\infty}a_kx^{2k+1}:\left(0,\frac{\pi}{2}\right)\longrightarrow\mathbb{R}$*, with coefficients* (a_k) *determined with* (49)*, and let* $c\in\left(0,\frac{\pi}{2}\right)$ *be fixed. Then for* $x\in(0,c)$ *holds*:

$$\begin{array}{c}T_0^{\varphi,\,0+}(x)-x<\ldots<T_n^{\varphi,\,0+}(x)-x<T_{n+1}^{\varphi,\,0+}(x)-x<\ldots\\ \ldots<\frac{2}{3}\sin x+\frac{1}{3}\tan x-x<\ldots\\ \ldots<\mathbb{T}_{n+1}^{\varphi;\,0+,\,c-}(x)-x<\mathbb{T}_n^{\varphi;\,0+,\,c-}(x)-x<\ldots<\mathbb{T}_0^{\varphi;\,0+,\,c-}(x)-x.\end{array}\tag{56}$$

Some rational bounds for the HUYGENS function. In this section are considered some series for tangent function obtained from well known series for the cotangent function [**14**]:

$$\begin{aligned}\cot x&=\frac{1}{x}-\sum_{k=0}^{\infty}\frac{2^{2k+2}|\mathbf{B}_{2k+2}|}{(2k+2)!}x^{2k+1}\\&=\frac{1}{x}-\frac{1}{3}x-\frac{1}{45}x^3-\frac{2}{945}x^5-\ldots\end{aligned}\tag{57}$$

which converges for $0<|x|<\pi$. From the previous series we conclude that

$$\begin{aligned}\tan x&=\frac{1}{\frac{\pi}{2}-x}-\sum_{k=0}^{\infty}\frac{2^{2k+2}|\mathbf{B}_{2k+2}|}{(2k+2)!}\left(\frac{\pi}{2}-x\right)^{2k+1}\\&=\frac{1}{\frac{\pi}{2}-x}-\frac{1}{3}\left(\frac{\pi}{2}-x\right)-\frac{1}{45}\left(\frac{\pi}{2}-x\right)^3-\frac{2}{945}\left(\frac{\pi}{2}-x\right)^5-\ldots\end{aligned}\tag{58}$$

for $0<\left|\frac{\pi}{2}-x\right|<\pi$, which holds for $x\in\left(0,\frac{\pi}{2}\right)$. From there $\phi(x)=\tan x-\frac{1}{\frac{\pi}{2}-x}$ determines real analytic function on $\left(0,\frac{\pi}{2}\right)$. Let us notice that M. NENEZIĆ and

L. Zhu obtained the following series in [**30**]:

$$\tan x = \frac{1}{\frac{\pi}{2}-x} + \sum_{k=0}^{\infty} b_k x^k, \tag{59}$$
$$= \frac{1}{\frac{\pi}{2}-x} - \frac{2}{\pi} + \left(1-\frac{4}{\pi^2}\right)x - \frac{8}{\pi^3}x^2 + \left(\frac{1}{3}-\frac{16}{\pi^4}\right)x^3 - \frac{32}{\pi^5}x^4 + \dots$$

for $x \in \left(0, \frac{\pi}{2}\right)$, with coefficients

$$b_k = \begin{cases} -\dfrac{2}{\pi} & : \ k = 0 \\ \dfrac{2^{k+1}(2^{k+1}-1)|\mathbf{B}_{k+1}|}{(k+1)!} - \dfrac{2^{k+1}}{\pi^{k+1}} & : \ k > 0, \end{cases} \tag{60}$$

for $k \in \mathbb{N}_0$. Let us introduce the sequence

$$\beta_k = (-1)^{k-1} b_k = \begin{cases} \dfrac{2^{k+1}}{\pi^{k+1}} & : \quad k = 2\ell \\ \dfrac{2^{k+1}(2^{k+1}-1)|\mathbf{B}_{k+1}|}{(k+1)!} - \dfrac{2^{k+1}}{\pi^{k+1}} & : \ k = 2\ell - 1, \end{cases} \tag{61}$$

for $\ell \in \mathbb{N}_0$. Based on the inequality (14) the following statement is easily checked.

Lemma 7. *For fixed* $x \in \left(0, \frac{\pi}{2}\right)$ *and sequence* (β_k)*, it holds*:

$$\beta_k x^k > 0, \quad \lim_{k\to\infty} \beta_k x^k = 0, \tag{62}$$

and $(\beta_k x^k)$ *is strictly monotonically decreasing.*

Based on the Leibniz alternating series test, the next statement about some rational inequalities for the tangent function follows .

Theorem 8. *Let there be given the function*:

$$\phi(x) = \tan x - \frac{1}{\frac{\pi}{2}-x} = \sum_{k=0}^{\infty} (-1)^{k-1}\beta_k x^k : \left(0, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}, \tag{63}$$

with coefficients (β_k) *determined by* (61)*. Then for* $x \in \left(0, \frac{\pi}{2}\right)$ *holds*:

$$\frac{1}{\frac{\pi}{2}-x} + T_0^{\phi,0+}(x) < \dots < \frac{1}{\frac{\pi}{2}-x} + T_{2n}^{\phi,0+}(x) < \frac{1}{\frac{\pi}{2}-x} + T_{2n+2}^{\phi,0+}(x) < \dots$$
$$< \tan x < \tag{64}$$
$$\dots < \frac{1}{\frac{\pi}{2}-x} + T_{2n+1}^{\phi,0+}(x) < \frac{1}{\frac{\pi}{2}-x} + T_{2n-1}^{\phi,0+}(x) < \dots < \frac{1}{\frac{\pi}{2}-x} + T_1^{\phi,0+}(x).$$

Furthermore, let us consider the function

$$(65)\quad \psi(x)=\varphi(x)-\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}=\frac{2}{3}\sin x+\frac{1}{3}\tan x-\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}=\sum_{k=0}^{\infty}c_k x^k:\Big(0,\frac{\pi}{2}\Big)\longrightarrow\mathbb{R},$$

with coefficients (c_k) given by

$$(66)\qquad c_k=\begin{cases}-\dfrac{2}{3\pi} & : \; k=0\\[2mm] \dfrac{2(-1)^{\frac{k-1}{2}}}{3k!}+\dfrac{2^{k+1}(2^{k+1}-1)|\mathbf{B}_{k+1}|}{3(k+1)!}-\dfrac{2^{k+1}}{3\pi^{k+1}} & : \; k=2j+1\\[2mm] -\dfrac{2^{k+1}}{3\pi^{k+1}} & : \; k=2j+2,\end{cases}$$

for $j\in\mathbb{N}_0$. Let us introduce the sequence

$$(67)\qquad \gamma_k=(-1)^{k-1}c_k=\begin{cases}\dfrac{2}{3\pi} & : \; k=0\\[2mm] \dfrac{2(-1)^{\frac{k-1}{2}}}{3k!}+\dfrac{2^{k+1}(2^{k+1}-1)|\mathbf{B}_{k+1}|}{3(k+1)!}-\dfrac{2^{k+1}}{3\pi^{k+1}} & : \; k=2j+1\\[2mm] \dfrac{2^{k+1}}{3\pi^{k+1}} & : \; k=2j+2,\end{cases}$$

for $j\in\mathbb{N}_0$. By applying symbolic, algebra system, we can determine the initial part of the power series of $\psi(x)$, for example up to the sixth degree:

$$(68)\quad \psi(x)=-\frac{2}{3\pi}+\Big(1-\frac{4}{3\pi^2}\Big)x-\frac{8}{3\pi^3}x^2-\frac{16}{3\pi^4}x^3-\frac{32}{3\pi^5}x^4+\Big(\frac{1}{20}-\frac{64}{3\pi^6}\Big)x^5-\frac{128}{3\pi^7}x^6+\ldots$$

Based on inequality (14) the following statement can be simply checked.

Lemma 8. *For fixed* $x\in\Big(0,\frac{\pi}{2}\Big)$ *and sequence* (γ_k)*, it holds*:

$$(69)\qquad \gamma_k x^k>0 \;(\textit{for } k>3),\qquad \lim_{k\to\infty}\gamma_k x^k=0,$$

and $(\gamma_k x^k)\downarrow$ *is strictly monotonically decreasing.*

Based on the Leibniz alternating series test, follows the statement about some rational inequalities for the Huygens function.

Theorem 9. *Let there be given the function*:

$$(70)\qquad \psi(x)=\varphi(x)-\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}=\sum_{k=0}^{\infty}(-1)^{k-1}\gamma_k x^k:\Big(0,\frac{\pi}{2}\Big)\longrightarrow\mathbb{R},$$

with coefficients (γ_k) *determined by* (66). *Then for* $x\in\left(0,\frac{\pi}{2}\right)$, *it holds*:

$$\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_4^{\psi,0+}(x)<\ldots<\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_{2n}^{\psi,0+}(x)<\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_{2n+2}^{\psi,0+}(x)<\ldots$$

$$<\frac{2}{3}\sin x+\frac{1}{3}\tan x< \tag{71}$$

$$\ldots<\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_{2n+1}^{\psi,0+}(x)<\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_{2n-1}^{\psi,0+}(x)<\ldots<\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_3^{\psi,0+}(x).$$

Example 2. *Let us introduce some examples of inequalities obtained for* $n=2,3$.

$\boldsymbol{n=2}$**:** *For* $x\in\left(0,\frac{\pi}{2}\right)$, *it holds*:

$$\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}-\frac{2}{3\pi}+\left(1-\frac{4}{3\pi^2}\right)x-\frac{8}{3\pi^3}x^2-\frac{16}{3\pi^4}x^3-\frac{32}{3\pi^5}x^4=$$

$$=\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_4^{\psi,0+}(x)<$$

$$<\frac{2}{3}\sin x+\frac{1}{3}\tan x< \tag{72}$$

$$<\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_3^{\psi,0+}(x)=$$

$$=\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}-\frac{2}{3\pi}+\left(1-\frac{4}{3\pi^2}\right)x-\frac{8}{3\pi^3}x^2-\frac{16}{3\pi^4}x^3.$$

$\boldsymbol{n=3}$**:** *For* $x\in\left(0,\frac{\pi}{2}\right)$, *it holds*:

$$\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}-\frac{2}{3\pi}+\left(1-\frac{4}{3\pi^2}\right)x-\frac{8}{3\pi^3}x^2-\frac{16}{3\pi^4}x^3-\frac{32}{3\pi^5}x^4+\left(\frac{1}{20}-\frac{64}{3\pi^6}\right)x^5-\frac{128}{3\pi^7}x^6$$

$$=\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_6^{\psi,0+}(x)<$$

$$<\frac{2}{3}\sin x+\frac{1}{3}\tan x< \tag{73}$$

$$<\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_5^{\psi,0+}(x)=$$

$$=\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}-\frac{2}{3\pi}+\left(1-\frac{4}{3\pi^2}\right)x-\frac{8}{3\pi^3}x^2-\frac{16}{3\pi^4}x^3-\frac{32}{3\pi^5}x^4+\left(\frac{1}{20}-\frac{64}{3\pi^6}\right)x^5.$$

Remark 10. *For* $x\in\left(0,\frac{\pi}{2}\right)$, *it holds*:

$$\frac{2}{3}\sin x+\frac{1}{3}\tan x<\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_2^{\psi,0+}(x)<\frac{1}{3}\frac{1}{\frac{\pi}{2}-x}+T_1^{\psi,0+}(x). \tag{74}$$

From the previous Theorem it simply follows the error function of Huygens approximation $Q(x) = \frac{2}{3}\sin x + \frac{1}{3}\tan x - x$ with previously considered rational functions.

Theorem 11. *Let there be given the function:*

$$\psi(x) = \phi(x) - \frac{1}{3}\frac{1}{\frac{\pi}{2}-x} = \sum_{k=0}^{\infty}(-1)^{k-1}\gamma_k x^k : \left(0, \frac{\pi}{2}\right) \longrightarrow \mathbb{R}, \tag{75}$$

with coefficients (γ_k) *given by* (66). *Then for* $x \in \left(0, \frac{\pi}{2}\right)$, *it holds*

$$\begin{array}{c} \frac{1}{3}\frac{1}{\frac{\pi}{2}-x} + T_4^{\psi,0+}(x) - x < \ldots < \frac{1}{3}\frac{1}{\frac{\pi}{2}-x} + T_{2n}^{\psi,0+}(x) - x < \frac{1}{3}\frac{1}{\frac{\pi}{2}-x} + T_{2n+2}^{\psi,0+}(x) - x < \ldots \\ < \frac{2}{3}\sin x + \frac{1}{3}\tan x - x < \\ \ldots < \frac{1}{3}\frac{1}{\frac{\pi}{2}-x} + T_{2n+1}^{\psi,0+}(x) - x < \frac{1}{3}\frac{1}{\frac{\pi}{2}-x} + T_{2n-1}^{\psi,0+}(x) - x < \ldots < \frac{1}{3}\frac{1}{\frac{\pi}{2}-x} + T_3^{\psi,0+}(x) - x. \end{array} \tag{76}$$

4. CONCLUSION

Based on inequalities (4) and (5), stated by Zhu [**7**], using elementary analysis we have obtained in Theorems 4 and 5 two new double inequalities which can be used to estimate some polynomial bounds of the error function of the Cusa-Huygens approximation. With Theorem 7 we determined some bounds of the error function of the Huygens approximation using polynomial functions and with Theorem 11 we determined some bounds of the error function of the Huygens approximation using rational functions. Let us emphasize that by Theorem 8 we gave some bounds of tangent function by use of rational functions which can be applied to other parts of Theory of analytical inequalities. Lastly, let us notice that the proofs of the considered inequalities can be also obtained by applying some methods and algorithms presented in papers [**16**], [**17**], [**18**], [**19**]-[**26**], [**31**]-[**34**] and in dissertation [**35**]. One automatic theorem prover related to some classes of inequalities, such as those presented in this paper, is currently being developed by our project team [**36**]. We expect that in near future some classes of problems related to analytic inequalities will be automatically proven by use of such software.

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