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A CHARACTERISATION OF COMPLETENESS OF b-FUZZY METRIC SPACES AND NONLINEAR CONTRACTIONS

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The purpose of this paper is to present a common fixed point theorem for a pair of R-weakly commuting mappings defined on b-fuzzy metric spaces satisfying nonlinear contractive conditions of Boyd-Wong type, obtained in D. W. Boyd, J. S. W. Wong: *On nonlinear contractions,* Proc. Amer. Math. Soc. **20** (1969), 458–464.

1. INTRODUCTION AND PRELIMINARIES

Schweizer and Sklar have defined statistical metric spaces (see [**9**]). Following this definition Kramosil and Michalek have defined fuzzy metric spaces (see [**7**]).

Definition 2.1. [**9**] *A binary operation* $T : [0,1] \times [0,1] \to [0,1]$ *is continuous* t*-norm if* T *satisfies the following conditions:*

(a) T *is commutative and associative;*

(b) T *is continuous;*

(c) $T(a,1) = a$ *for all* $a \in [0,1]$*;*

(d) $T(a,b) \le T(c,d)$ *whenever* $a \le c$ *and* $b \le d$*, and* $a,b,c,d \in [0,1]$.

Examples of t-norm are $T(a,b) = \min\{a,b\}$ and $T(a,b) = ab$.

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Definition 2.2. [**13**] *A fuzzy set A in X is a function with domain X and values in* $[0,1]$.

As a natural extension of fuzzy metric spaces (see [**7**]) and b-metric spaces (see [**1**], [**12**]) S. Sedghi and N. Shobe defined b-fuzzy metric spaces.

Definition 2.3. [**10**] *A 3-tuple $(X, M, *)$ is called a b-fuzzy metric space if X is an arbitrary set, T is a continuous t-norm and M is a fuzzy set on $X^2 \times (0,\infty)$ satisfying the following conditions, for all $x, y, z \in X$, $s, t > 0$ and $b \geq 1$ be a given real number,*

(Fb-1) $M(x,y,t) > 0$;
(Fb-2) $M(x,y,t) = 1$ *if and only if* $x = y$;
(Fb-3) $M(x,y,t) = M(y,x,t)$;
(Fb-4) $T(M(x,y,\frac{t}{b}), M(y,z,\frac{s}{b})) \leq M(x,y,t+s)$;
(Fb-5) $M(x,y,\cdot) : (0,\infty) \to [0,1]$ *is continuous;*

Function M is called a b-fuzzy metric on X.

It is easy to show that every fuzzy metric space is a b-fuzzy metric space for $b = 1$. Converse is not true. For examples of b-fuzzy metric spaces and b-fuzzy metric spaces that are not fuzzy metric spaces see [**10**] and [**3**].

Definition 2.4. [**10**] *A function $f : \mathbb{R} \to \mathbb{R}$ is called b-nondecreasing if $x > by$ implies $f(x) \geq f(y)$, for each $x, y \in \mathbb{R}$.*

Lemma 2.1. [**10**] *Let (X, M, T) be a b-fuzzy metric space. Then $M(x,y,\cdot)$ is b-nondecreasing function for all $x, y \in X$.*

Definition 2.5. [**10**] *Let (X, M, T) be a b-fuzzy metric space and $r \in (0,1)$, $t > 0$ and $x \in X$. The set $B(x,r,t) = \{y \in X : M(x,y,t) > 1 - r\}$ is called an open ball with centre x and radius r with respect to t.*

Remark 2.1. [**10**] *Every open ball $B(x,r,t)$ is an open set.*

Remark 2.2. [**10**] *Let (X, M, T) be a b-fuzzy metric space. Define $\tau = \{A \subseteq X :$ for every $x \in A$ there exist $t > 0$ and $r \in (0,1)$ such that $B(x,r,t) \subset A\}$. Then τ is a topology on X.*

Definition 2.6. [**10**] *Let (X, M, T) be a b-fuzzy metric space.*

(i) *A sequence $\{x_n\}_n$ in X is said to be convergent to $x \in X$ if for every $t > 0$ and $\varepsilon > 0$ there exists positive integer N such that $M(x_n, x, t) > 1 - \varepsilon$ whenever $n \geq N$.*

(ii) *A sequence $\{x_n\}_n$ in X is called Cauchy sequence if, for every $t > 0$ and $\varepsilon > 0$ there exists positive integer N such that $M(x_n, x_m, t) > 1 - \varepsilon$ whenever $n, m \geq N$.*

(iii) *A b-fuzzy metric space is said to be complete if every Cauchy sequence in X is convergent to a point in X.*

Lemma 2.2. [10] *If (X, M, T) is a b-fuzzy metric space and sequence $\{x_n\}$ converges to x in X, then:*

(i) x is unique;

(ii) $\{x_n\}$ is a Cauchy sequence in X.

Remark 2.3. *Let (X, M, T) be a b-fuzzy metric space. Notice that a sequence $\{x_n\}$ from X converges to a point $x \in X$ if and only if $\lim_{n\to\infty} M(x_n, x, t) = 1$.*

Lemma 2.3. [11] *If (X, M, T) is a b-fuzzy metric space and sequence $\{x_n\}$ converges to x in X, then*

$$M\left(x, y, \frac{t}{b}\right) \le \limsup_{n\to+\infty} M(x_n, y, t) \le M(x, y, bt),$$
$$M\left(x, y, \frac{t}{b}\right) \le \liminf_{n\to+\infty} M(x_n, y, t) \le M(x, y, bt).$$

For more results see [**4**] [**5**], [**6**] and [**8**].

2. MAIN RESULTS

Definition 3.7. *Let (X, M, T) be a b-fuzzy metric space and $A \subseteq X$. Closure of the set A is the smallest closed set containing A, denoted by $\bar{A}$.*

Definition 3.8. *Let (X, M, T) be a b-fuzzy metric space and $r \in (0, 1), t > 0$ and $x \in X$. The set $B[x, r, t] = \{y \in X : M(x, y, t) \ge 1 - r\}$ is called a closed ball with centre x and radius r with respect to t.*

Definition 3.9. *Let (X, M, T) be a b-fuzzy metric space. A collection $\{F_n\}_{n\in\mathbb{N}}$ is said to have b-fuzzy diameter zero if for each $r \in (0, 1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that $M(x, y, t) > 1 - r$ for all $x, y \in F_{n_0}$.*

Theorem 3.1. *A b-fuzzy metric space (X, M, T) is complete if and only if every nested sequence $\{F_n\}_{n\in\mathbb{N}}$ of nonempty closed sets with b-fuzzy diameter zero have nonempty intersection.*

Proof. Suppose that the given condition is satisfied. Let us prove that (X, M, T) is complete. Let $\{x_n\}$ be a Cauchy sequence in X. Set $B_n = \{x_k : k \ge n\}$ and $F_n = \overline{B_n}$, then $\{F_n\}$ has b-fuzzy diameter zero. Indeed, for given $s \in (0, 1)$ we can choose $r \in (0, 1)$ such that $T(1 - r, T(1 - r, 1 - r)) > 1 - s$. Since $\{x_n\}$ is Cauchy sequence, there exists $n_0 \in \mathbb{N}$ such that $M\left(x_n, x_m, \frac{t}{4b^2}\right) > 1 - r$ for all $m, n \ge n_0$. Therefore, $M\left(x, y, \frac{t}{4b^2}\right) > 1 - r$ for all $x, y \in B_{n_0}$.

Let $x, y \in F_{n_0}$. Then there exist sequences $\{x_n^1\}$ and $\{y_n^1\}$ in B_{n_0} such that $x_n^1 \to x$ and $y_n^1 \to y$. Thus, $x_n^1 \in B\left(x, r, \frac{t}{4b^2}\right)$ and $y_n^1 \in B\left(y, r, \frac{t}{4b^2}\right)$ for n sufficiently large. We have that

$$
\begin{aligned}
M(x,y,t) &\geq T\left(M\left(x,x_n^1,\frac{t}{2b}\right),M\left(x_n^1,y,\frac{t}{2b}\right)\right)\\
&\geq T\left(M\left(x,x_n^1,\frac{t}{2b}\right),T\left(M\left(x_n^1,y_n^1,\frac{t}{4b^2}\right),M\left(y_n^1,y,\frac{t}{4b^2}\right)\right)\right)\\
&\geq T\left(M\left(x,x_n^1,\frac{t}{2b}\right),T(1-r,1-r)\right)
\end{aligned}
$$

Since $M(x,y,\cdot)$ is b-nondecreasing and $\frac{t}{2b} > b\cdot\frac{t}{4b^2}$ it follows that $M\left(x,x_n^1,\frac{t}{2b}\right) \geq M\left(x,x_n^1,\frac{t}{4b^2}\right) > 1-r$. From previous we get

$$M(x,y,t) > T\left(1-r,T(1-r,1-r)\right) > 1-s$$

Thus, $M(x,y,t) > 1-s$ for all $x,y \in F_{n_0}$ i.e. $\{F_n\}$ has b-fuzzy diameter zero and by hypothesis $\bigcap_{n\in\mathbb{N}} F_n$.

Take $x \in \bigcap_{n\in\mathbb{N}} F_n$. We show that $x_n \to x$. Then, for $r \in (0,1)$ and $t>0$ there exists $n_1 \in \mathbb{N}$ such that $M(x_n,x,t) > 1-r$ for all $n \geq n_1$. Thus, $M(x_n,x,t) \to 1$ as $n \to \infty$ for each $t>0$, i.e. $x_n \to x$. Therefore, (X,M,T) is complete.

Conversely, suppose that (X,M,T) is complete and $\{F_n\}_{n\in\mathbb{N}}$ is a nested sequence of nonempty closed sets with b-fuzzy diameter zero. For each $n \in \mathbb{N}$ choose a point $x_n \in F_n$. We show that $\{x_n\}$ is a Cauchy sequence. Indeed, since $\{F_n\}_{n\in\mathbb{N}}$ has b-fuzzy diameter zero, for $t>0$ and $r \in (0,1)$ there exists $n_0 \in \mathbb{N}$ such that $M(x,y,t) > 1-r$ for all $x,y \in F_{n_0}$. Since $\{F_n\}$ is nested sequence, it follows that $M(x_n,x_m,t) > 1-r$ for all $n,m \geq n_0$. Thus, $\{x_n\}$ is a Cauchy sequence. Since (X,M,T) is complete, $x_n \to x$ for some $x \in X$. It follow that $x \in \overline{F_n} = F_n$ for every n, i.e. $x \in \bigcap_{n\in\mathbb{N}} F_n$. □

Remark 3.4. *The element $x \in \bigcap_{n\in\mathbb{N}} F_n$ is unique. Indeed, if we suppose that there are two elements $x,y \in \bigcap_{n\in\mathbb{N}} F_n$, since $\{F_n\}$ has b-fuzzy diameter zero, for arbitrary fixed $t>0$ it follows that $M(x,y,t) > 1-\frac{1}{n}$ for each $n \in \mathbb{N}$. This implies $M(x,y,t)=1$, i.e. $x=y$.*

Definition 3.10. *Let (X,M,T) be a b-fuzzy metric space. Let the mapping $\delta_A(t) : (0,\infty) \to [0,1]$ be defined as*

$$\delta_A(t) = \inf_{x,y\in A} \sup_{\varepsilon< t} M(x,y,\varepsilon).$$

The constant $\delta_A = \sup_{t>0} \delta_A(t)$ is called b-fuzzy diameter of set A.

Definition 3.11. *If $\delta_A = 1$ the set A is called bF-strongly bounded.*

Lemma 3.4. *Let (X,M,T) be a b-fuzzy metric space. A set $A \subseteq X$ is bF-strongly bounded if and only if for each $r \in (0,1)$ there exists $t>0$ such that $M(x,y,t) > 1-r$ for all $x,y \in A$.*

Proof. The proof follows from the definitions of sup A and inf A of non-empty sets. □

Definition 3.12. [11] *Let (X, M, T) be a b-fuzzy metric space and let f and g be self-mappings of X. The mappings f and g will be said to be R-weakly commuting if there exists some positive real number R such that*

$$M(f(g(x)), g(f(x)), Rt) \geq M(f(x), g(x), t) \tag{1}$$

for all $t > 0$ and each $x \in X$.

Throughout this paper we will consider b-fuzzy metric spaces that are not fuzzy metric spaces i.e. $b > 1$, satisfying the next condition.

$$M(x, y, 0) = \lim_{t \to 0+} M(x, y, t) = 0 \quad \text{for } x \neq y \tag{2}$$

Lemma 3.5. *Let (X, M, T) be a b-fuzzy metric space, $b > 1$, which satisfies (2). Let $\varphi : (0, \infty) \to (0, \infty)$ be a continuous function which satisfies $\varphi(t) < \frac{t}{b}$ for all $t > 0$. If for $x, y \in X$ it holds that $M(x, y, \varphi(t)) \geq M(x, y, t)$ for all $t > 0$ then $x = y$.*

Proof. First note that from $\varphi(t) < \frac{t}{b}$, by induction we get that $\varphi^n(t) < \frac{t}{b^n}$. From previous it follows that $\lim_{n \to \infty} \varphi^n(t) = 0$ for all $t \geq 0$ and $b > 1$.

Let us suppose that $M(x, y, \varphi(t)) \geq M(x, y, t)$ and $x \neq y$. From this condition, by induction, we have that $M(x, y, \varphi^n(t)) \geq M(x, y, t)$. Taking limit as $n \to \infty$, we get that $M(x, y, t) = 0$ for all $t > 0$, which is a contradiction with (Fb-1) i.e. $x = y$. □

Theorem 3.2. *Let (X, M, T) be a complete b-fuzzy metric space with $b > 1$, which satisfies (2) and let f and g be R-weakly commuting self-mappings on X, g is a continuous function, $g(X)$ is fF-strongly bounded set and $g(X) \subseteq f(X)$, satisfying the condition*

$$M(g(x), g(y), \varphi(t)) \geq M(f(x), f(y), t) \tag{3}$$

for some continuous function $\varphi : (0, \infty) \to (0, \infty)$, which satisfies $\varphi(t) < t$ for all $t > 0$. Then f and g have a unique common fixed point.

Proof. Let $x_0 \in X$ be an arbitrary point. Since $g(X) \subseteq f(X)$, there exists a $x_1 \in X$ such that $g(x_0) = f(x_1)$. By induction, a sequence $\{x_n\}$ can be chosen such that $g(x_n) = f(x_{n+1})$.

Let us consider nested sequence of nonempty closed sets defined by

$$F_n = \overline{\{gx_n, gx_{n+1}, \ldots\}}, \ n \in \mathbb{N}.$$

We shall prove that the family $\{F_n\}_{n \in \mathbb{N}}$ has b-fuzzy diameter zero.

In this sense, let $r \in (0,1)$ and $t > 0$ be arbitrary. From $F_k \subseteq \overline{g(X)}$ it follows that F_k is a bF-strongly bounded set for arbitrary $k \in \mathbb{N}$. It means that there exists $t_0 > 0$ such that

$$(4) \qquad M(x, y, t_0) > 1 - r \quad \text{for all} \quad x, y \in F_k.$$

From $\lim_{n\to\infty} \varphi^n(t_0) = 0$ we conclude that there exists $m \in \mathbb{N}$ such that $\varphi^m(t_0) < t$. Let $n = m + k$ and $x, y \in F_n$ be arbitrary. There exist sequences $\{gx_{n(i)}\}$, $\{gx_{n(j)}\}$ in F_n $\quad (n(i), n(j) \geq n \quad i, j \in \mathbb{N})$ such that $\lim_{i\to\infty} gx_{n(i)} = x$ and $\lim_{j\to\infty} gx_{n(j)} = y$.

From (3) we have

$$M(gx_{n(i)}, gx_{n(j)}, \varphi(t)) \geq M(fx_{n(i)}, fx_{n(j)}, t) = M(gx_{n(i)-1}, gx_{n(j)-1}, t).$$

Thus, by induction we get

$$M(gx_{n(i)}, gx_{n(j)}, \varphi^m(t)) \geq M(gx_{n(i)-m}, gx_{n(j)-m}, t)$$

Since $\varphi^m(t_0) < t < bt$ and because $M(x, y, \cdot)$ is a b-non-decreasing function, from the last inequalities it follows that

$$(5) \quad M(gx_{n(i)}, gx_{n(j)}, t) \geq M(gx_{n(i)}, gx_{n(j)}, \varphi^m(t_0)) \geq M(gx_{n(i)-m}, gx_{n(j)-m}, t_0)$$

As $\{gx_{n(i)-m}\}$, $\{gx_{n(j)-m}\}$ are sequences in F_k from (4) it follows that

$$M(gx_{n(i)-m}, gx_{n(j)-m}, t_0) > 1 - r$$

for all $i, j \in \mathbb{N}$.

Finally, from previous and (6) we conclude that $M(gx_{n(i)}, gx_{n(j)}, t) > 1 - r$ for all $i, j \in \mathbb{N}$. Taking liminf as $j \to \infty$ we get that

$$M(gx_{n(i)}, y, bt) > 1 - r$$

for all $t > 0$ and $x, y \in F_n$.

Taking liminf as $i \to \infty$ it follows that $M(x, y, b^2 t) > 1 - r$, for all $t > 0$ for all $x, y \in F_n$. From previous it follows that $M(x, y, t) > 1 - r$, for all $t > 0$ for all $x, y \in F_n$ i.e. family $\{F_n\}_{n\in\mathbb{N}}$ has b-fuzzy diameter zero.

Applying Theorem 3.1 we conclude that this family has nonempty intersection, which consists of exactly one point z. Since the family $\{F_n\}_{n\in\mathbb{N}}$ has b-fuzzy diameter zero and $z \in F_n$ for all $n \in \mathbb{N}$ then for each $r \in (0,1)$ and each $t > 0$ there exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ hold

$$M(gx_n, z, t) > 1 - r.$$

From the last it follows that for each $r \in (0,1)$ hold

$$\lim_{n\to\infty} M(gx_n, z, t) > 1 - r.$$

Taking that $r \to 0$ we get

$$\lim_{n\to\infty} M(gx_n, z, t) = 1$$

i.e. $\lim_{n\to\infty} gx_n = z$. From the definition of sequence $\{fx_n\}$ it follows that $\lim_{n\to\infty} fx_n = z$.

Let us prove that z is a common fixed point of mappings f and g. From condition (1) we have that for all $t > 0$ holds

$$M(f(g(x_n)), g(f(x_n)), Rt) \geq M(f(x_n), g(x_n), t).$$

For previous we get that for all $t > 0$ holds

$$M(f(g(x_n)), g(f(x_n)), Rt) \geq T\left(M\left(f(x_n), z, \frac{t}{b}\right), M\left(z, g(x_n), \frac{t}{b}\right)\right).$$

Since $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = z$, taking $\liminf$ when $n \to \infty$ and using Lemma 2.3 we get that for all $t > 0$ it holds that

$$\liminf_{n\to\infty} M(f(g(x_n)), g(z), bRt) \geq 1$$

i. e.

$$\liminf_{n\to\infty} M(f(g(x_n)), g(f(x_n)), t) = 1.$$

Similarly, using Lemma 2.3 we can prove that for all $t > 0$ it holds that

$$\limsup_{n\to\infty} M(f(g(x_n)), g(f(x_n)), t) = 1.$$

From previous we get that for all $t > 0$ it holds that

$$\lim_{n\to\infty} M(f(g(x_n)), g(f(x_n)), t) = 1.$$

Since g is continuous, we get that

$$\lim_{n\to\infty} f(g(x_n)) = \lim_{n\to\infty} g(f(x_n)) = g(\lim_{n\to\infty} f(x_n)) = g(z).$$

From the inequalities (3) follows that

$$M(g(x_n), g(g(x_n)), \varphi(t)) \geq M(f(x_n), f(g(x_n)), t)$$

for all $t > 0$. Similarly as in the previous part, using Lemma 2.3 and taking $\liminf$ ($\limsup$) as $n \to \infty$, we get

$$M(z, g(z), \varphi(t)) \geq M(z, g(z), t)$$

for all $t > 0$. Applying Lemma 3.5 we conclude that $g(z) = z$.

Since $g(X) \subseteq f(X)$, there exists $z_1 \in X$ such that $f(z_1) = g(z) = z$. From starting condition we have that

$$M(g(g(x_n)), g(z_1), \varphi(t)) \geq M(f(g(x_n)), f(z_1), t)$$

holds for all $t > 0$. Using Lemma 2.3 and taking $\liminf$ ($\limsup$) as $n \to \infty$, we get

$$M(z, g(z_1), \varphi(t)) \geq M(z, z, t) = 1$$

for all $t > 0$. From $\varphi(t) < \frac{t}{b}$, i.e. $t > b\varphi(t)$, since $M(x, y, \cdot)$ is b-nondecreasing it follows that $M(z, g(z_1), t) \geq M(z, g(z_1), \varphi(t)) = 1$ for all $t > 0$. From previous it follows that $M(z, g(z_1), t) = 1$ for all $t > 0$. i.e. $g(z_1) = z$.

For arbitrary $t > 0$ there exists $t_1 > 0$ such that $t = Rt_1$. From $f(z_1) = z$, $g(z_1) = z$ we get

$$\begin{aligned} M(g(z), f(z), t) &= M(g(z), f(z), Rt_1) = M(g(f(z_1)), f(g(z_1)), Rt_1) \\ &\geq M(f(z_1), g(z_1), t_1) = M(z, z, t_1) = 1 \end{aligned}$$

from where it follows that $f(z) = g(z) = z$.

Let us prove that z is a unique common fixed point. For this purpose let us suppose that there exists another common fixed point, denoted by u. From the starting condition, for all $t > 0$ it follows that

$$M(g(z), g(u), \varphi(t)) \geq M(f(z), f(u), t)$$

i.e.

$$M(z, u, \varphi(t)) \geq M(z, u, t).$$

Finally, applying Lemma 3.5 it follows that $z = u$. This completes the proof.

□

Example 3.1. *Let (X, M, T) be a complete b-fuzzy metric space $d(x, y) = |x - y|$ with $M(x, y, t) = e^{-\frac{|x-y|^2}{t}}$ and $X = [0, +\infty) \subset \mathbb{R}$. Let*

$$f(x) = 2x, \quad g(x) = \frac{x}{1+x}, \quad g(X) = [0, 1) \subset X = f(X)$$

and

$$\varphi(t) = \begin{cases} \frac{t}{3+t}, & 0 < t \leq 1 \\ \frac{t}{4}, & t \geq 1 \end{cases}$$

We shall prove that all the conditions of Theorem 3.2 are satisfied, too. Because $g(f(x)) = \frac{2x}{1+2x}$ and $f(g(x)) = \frac{2x}{1+x}$ we conclude that $f(x)$ and $g(x)$ are not commuting mappings, but they are R-weakly commuting for $R = 1$. We have that for all $x \geq 0$ follow

$$|f(g(x)) - g(f(x))| = \frac{2x^2}{(1+x)(1+2x)}$$

and

$$|f(x) - g(x)| = \frac{x + 2x^2}{1+x}.$$

Since $\frac{2x^2}{(1+x)(1+2x)} \leq \frac{x+2x^2}{1+x}$ and e^{-s} is decreasing function, we have that

$$M(f(g(x)), g(f(x)), t) \geq M(f(x), g(x), t)$$

for all $x, t \geq 0$ *i.e.* $f(x)$ *and* $g(x)$ *are R-weakly commuting for* $R = 1$.

We shall prove that the condition (3) *is satisfied, too. Note that for all* $x, y \in X$ *we have that* $\frac{1}{(1+x)^2(1+y)^2} \leq 1$. *We will consider two possibilities.*

If $0 < t \leq 1$, *since* $3 + t \leq 4$, *we have*

$$\frac{\left|\frac{x-y}{(1+x)(1+y)}\right|^2}{\frac{t}{3+t}} = \frac{(3+t)|x-y|^2}{t(1+x)^2(1+y)^2} \leq \frac{4|x-y|^2}{t}.$$

Since e^{-s} *is decreasing function, it follow that, for* $0 < t \leq 1$

$$M(g(x), g(y), \varphi(t)) \geq M(f(x), f(y), t).$$

If $t \geq 1$, *we have*

$$\frac{\left|\frac{x-y}{(1+x)(1+y)}\right|^2}{\frac{t}{4}} = \frac{4|x-y|^2}{t(1+x)^2(1+y)^2} \leq \frac{4|x-y|^2}{t}.$$

Since e^{-s} *is decreasing function, it follow that, for* $t \geq 1$

$$M(g(x), g(y), \varphi(t)) \geq M(f(x), f(y), t).$$

From the last inequalities we conclude that the condition (3) *is satisfied. Since* φ *satisfies all the conditions of Theorem 3.2, we get that* $f(x)$ *and* $g(x)$ *have a unique common fixed point. It is easy to see that this point is* $x = 0$.

One consequence of previous theorem is the following corollary.

Corollary 3.1. *Let* (X, M, T) *be a complete b-fuzzy metric space with* $b > 1$, *which satisfies* (2). *Let* g *be a continuous function on* X, *such that* $g(X)$ *is bF-strongly bounded set and* $g(X) \subseteq X$, *satisfying the condition*

$$M(g(x), g(y), \varphi(t)) \geq M(x, y, t) \tag{6}$$

for some continuous function $\varphi : (0, \infty) \to (0, \infty)$, *which satisfies* $\varphi(t) < \frac{t}{b}$ *for all* $t > 0$. *Then* g *has a unique fixed point.*

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