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LAPLACE TRANSFORM OF ANALYTIC COMPOSITE FUNCTIONS VIA BELL'S POLYNOMIALS

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Bell polynomials have already been used in many different fields, ranging from number theory to operator theory. Here another application in Laplace Transform (LT) theory is shown, describing a method for computing the LT of nested functions. A table containing the first few values of complete Bell polynomials is shown, and a program for deriving the next ones is given. A program for approximating the Laplace Transform of composed analytic functions is also presented.

1. INTRODUCTION

In this short note we want to disprove the common view that there is no formula for the definite integration of composite functions. In what follows, we show that, as in the case of the derivative of nested functions, it is possible to use the Bell's polynomials [**1, 5, 6, 8, 11**] in order to obtain the desired result.
The Bell's polynomials are also exploited in very different frameworks, which range from number theory [**9, 13, 14**] to operators theory [**12**], and from differential equations [**8**] to integral transforms [**3, 10**]. It is almost useless to recall how much is used the Laplace Transform (LT) in the theory of ordinary and partial differential equations and their applications to Analysis and Mathematical Physics problems, ranging form signals analysis to imagine processing. Applications of the LT, and of the even more useful Fourier Transform are recalled e.g. in [**2**].
We use the classical definition

$$\mathcal{L}(f) := \int_0^\infty \exp(-s\,t) f(t)\,dt = F(s)\,.$$

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The LT converts a function of a real variable t (representing the time) to a function of a complex variable s (representing the complex frequency).
The LT can be applied to local integrable functions on $[0, +\infty)$. It converges in each half plane $Re(s) > a$, where a is a constant (called the convergece abscissa), depending on the growth behavior of $f(t)$.
Owing the importance of this transformation in the solution of the most diverse differential problems, a large number of LT, together with the respective antitransforms, are reported in literature (see e.g. [**7**]).
However, excluding some particular cases, we have not found in literature a general method to compute the LT of composite analytic functions, moreover we have found on the Internet the common opinion that such a formula has not been found. This seems to be strange since, as in the case of the derivative of nested functions problem, the Bell's polynomials could be used.
To this aim, considering the Taylor's expansion of a fixed analytic function, as the coefficients of this expansion are expressed in terms of Bell's polynomials computed at the initial point, we approximate the integration problem by a series expansion. Obviously, if the considered integral is convergent, the obtained series is convergent too.
In the last sections the same methodology is applied to the case of the LT of nested functions, starting from the easier case of the LT of a nested exponential function. The result is a Laurent expansion approximating the considered LT. It is worth to note that only in very few cases our results can be compared with the LT of special nested functions present in literature, as it is shown in equations (7)-(8) and (9)-(10). The main problem is to construct a table of Bell's polynomials, which exhibit a highly increasing number of addends, but their evaluation at a fixed point does not present any difficult problem, using the modern computer programs.
To this aim, we show in the Appendix I the first few values of the complete Bell polynomials, obtained by the program reported in Appendix II. These polynomials are applied in the LT of nested exponential functions.
In Appendix III the computer program used for the LT of general nested functions is shown. All these results were obtained by the second and third authors by using Mathematica©.

2. RECALLING THE BELL POLYNOMIALS

Consider the composite function $\Phi(t) := f(g(t))$, where $x = g(t)$ and $y = f(x)$, are differentiable functions (up to the considerad order), defined in suitable intervals of the real axis, so that $\Phi(t)$ can be differentiated n times with respect to t, by using the chain rule.
Here, and in what follows, we use the notations

$$\Phi_m := D_t^m \Phi(t), \qquad f_h := D_x^h f(x)|_{x=g(t)}, \qquad g_k := D_t^k g(t).$$

Then the n-th derivative of $\Phi(t)$ is represented by

$$\Phi_n = Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{k=1}^{n} A_{n,k}(g_1, g_2, \ldots, g_n)\, f_k, \tag{1}$$

where the Y_n denote the Bell polynomials, and the coefficients $A_{n,k}$, for all $k = 1, \ldots, n$, are polynomials of the variables $g_1, g_2, \ldots, g_n$, are homogeneous polynomials of degree k and *isobaric* of weight n (i.e. they are a linear combination of monomials $g_1^{k_1} g_2^{k_2} \cdots g_n^{k_n}$ whose weight is constantly given by $k_1 + 2k_2 + \ldots + nk_n = n$).

The Bell polynomials (1) satisfy the recurrence relation

$$\begin{cases} Y_0 := f_1; \\ Y_{n+1}(f_1, g_1; \ldots; f_n, g_n; f_{n+1}, g_{n+1}) = \\ = \displaystyle\sum_{k=0}^{n} \binom{n}{k} Y_{n-k}(f_2, g_1; f_3, g_2; \ldots; f_{n-k+1}, g_{n-k}) g_{k+1}. \end{cases}$$

and they are given explicitly by Faà di Bruno's formula (although this formula has a very high computational complexity)

$$Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{\pi(n)} \frac{n!}{r_1! r_2! \ldots r_n!}\, f_r \left[\frac{g_1}{1!}\right]^{r_1} \left[\frac{g_2}{2!}\right]^{r_2} \cdots \left[\frac{g_n}{n!}\right]^{r_n},$$

where the sum runs over all partitions $\pi(n)$ of the integer n, r_i denotes the number of parts of size i, and $r = r_1 + r_2 + \cdots + r_n$ denotes the number of parts of the considered partition [**11**].

The $A_{n,k}$ coefficients satisfy the recursion $\forall\, n$

$$A_{n+1,1} = g_{n+1}\,, \qquad A_{n+1,n+1} = g_1^{n+1}\,,$$

and, $\forall\, k = 1, 2, \ldots, n-1$

$$A_{n+1,k+1}(g_1, g_2, \ldots, g_{n+1}) = \sum_{h=0}^{n-k} \binom{n}{h} A_{n-h,k}(g_1, g_2, \ldots, , g_{n-h})\, g_{h+1}.$$

Remark 1. Actually the coefficients $A_{n,k}$ depend only by the values $g_1, g_2, \ldots, g_{n-k+1}$, so that the above equation, shifting the indexes, writes

$$B_{n,k}(g_1, g_2, \ldots, g_{n-k+1}) = \sum_{h=0}^{n-k} \binom{n-1}{h} B_{n-h-1,k-1}(g_1, g_2, \ldots, g_{n-k-h+1})\, g_{h+1}\,,$$

where we have used the traditional notation for the partial Bell polynomials

$$A_{n,k}(g_1, g_2, \ldots, g_n) = B_{n,k}(g_1, g_2, \ldots, g_{n-k+1})\,,$$

so that equation (1) writes

$$Y_n(f_1, g_1; f_2, g_2; \ldots; f_n, g_n) = \sum_{k=1}^{n} B_{n,k}(g_1, g_2, \ldots, g_{n-k+1})\, f_k\,.$$

3. DEFINITE INTEGRALS OF COMPOSITE FUNCTIONS

Under the conditions of Sect. 2, let $f(g(t))$ be a composite function analytic in a neighborhood of the origin, so that it is expressed by the Taylor's expansion

$$f(g(t)) = \sum_{n=0}^{\infty} a_n \frac{t^n}{n!}\,, \quad a_n = D_t^n[f(g(t))]_{t=0}\,. \tag{2}$$

According to the preceding equations, it results

$$\begin{aligned} & a_0 = f(\mathring{g}_0)\,, \\ & a_n = D_t^n[f(g(t))]_{t=0} = \sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \ldots, \mathring{g}_{n-k+1})\; \mathring{f}_k\,, \quad (n \geq 1)\,, \end{aligned} \tag{3}$$

where

$$\mathring{f}_k := D_x^k f(x)|_{x=g(0)}, \qquad \mathring{g}_h := D_t^h g(t)|_{t=0}.$$

We can state the following

Theorem 2. *Consider now the definite integral*

$$\int_0^b f(g(t))\, dt\,,$$

with $0 < b \leq +\infty$ *, then, assuming the limit value, for a convergent integral, when* $b = +\infty$*, it results*

$$\int_0^b f(g(t))dt = b\, f(\mathring{g}_0) + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \ldots, \mathring{g}_{n-k+1})\; \mathring{f}_k \right) \frac{b^{n+1}}{(n+1)!}\,. \tag{4}$$

Proof. Using equation (3) for the coefficient of Taylor's expansion (2) we find

$$\begin{aligned} \int_0^b f(g(t))dt &= b\, f(\mathring{g}_0) + \sum_{n=1}^{\infty} \int_0^b \sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \ldots, \mathring{g}_{n-k+1})\; \mathring{f}_k\; \frac{t^n}{n!}\, dt = \\ &= b\, f(\mathring{g}_0) + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \ldots, \mathring{g}_{n-k+1})\; \mathring{f}_k \right) \int_0^b \frac{t^n}{n!}\, dt\,, \end{aligned}$$

where we have used the uniform convergence of Taylor's series. Furthermore, when $b = +\infty$, convergence of the series in equation (4) - depending on the numerical values of the coefficients appearing into brackets - is certainly ensured in case of a convergent integral since the second member in equation (4) is just a different expression of the first member. □

4. LAPLACE TRANSFORMS OF COMPOSITE FUNCTIONS

The above results, applied in the case of a LT, give the result

Theorem 3. *Considering a composite function $f(g(t))$ analytic in a neighborhood of the origin, for its LT the following equation holds*

$$\int_0^{+\infty} f(g(t)) e^{-ts} dt =$$

$$= \frac{f(\mathring{g}_0)}{s} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}) \, \mathring{f}_k \right) \frac{1}{s^{n+1}} \, . \tag{5}$$

Proof. Representing the coefficients of Taylor's expansion in (2) in terms of Bell polynomials, and using the uniform convergence of Taylor's series, we find

$$\int_0^{+\infty} f(g(t)) e^{-ts} dt =$$

$$= \frac{f(\mathring{g}_0)}{s} + \sum_{n=1}^{\infty} \int_0^{+\infty} \sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}) \, \mathring{f}_k \, \frac{t^n}{n!} \, e^{-ts} dt =$$

$$= \frac{f(\mathring{g}_0)}{s} + \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}) \, \mathring{f}_k \right) \int_0^{+\infty} \frac{t^n}{n!} \, e^{-ts} dt \, .$$

Then the result follows using the LT of powers. □

4.1. THE PARTICULAR CASE OF THE EXPONENTIAL FUNCTION

In the particular case when $f(x) = e^x$, that is considering the function $e^{g(t)}$, and assuming $g(0) = 0$, the we have the more simple form

$$\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}) \, \mathring{f}_k =$$

$$\sum_{k=1}^{n} B_{n,k}(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_{n-k+1}) = B_n(\mathring{g}_1, \mathring{g}_2, \dots, \mathring{g}_n) \, ,$$

where the B_n are the *complete Bell polynomials.* It results $B_0(g_0) := f(g_0)$, and the first few values of B_n, for $n = 1, 2, \ldots, 5$, are given by

$$\begin{aligned}
B_1(g_1) &= g_1, \\
B_2(g_1, g_2) &= g_1^2 + g_2, \\
B_3(g_1, g_2, g_3) &= g_1^3 + 3g_1g_2 + g_3, \\
B_4(g_1, g_2, g_3, g_4) &= g_1^4 + 6g_1^2g_2 + 4g_1g_3 + 3g_2^2 + g_4, \\
B_5(g_1, g_2, g_3, g_4, g_5) &= g_1^5 + 10g_1^3g_2 + 15g_1g_2^2 + 10g_1^2g_3 + 10g_2g_3 + 5g_1g_4 + g_5\,.
\end{aligned} \tag{6}$$

Further values are reported in the Appendix I, at the end of this article.

The values of the complete Bell polynomials for particular parameter choices can be found in [**9**].

The complete Bell polynomials satisfy the identity (see e.g. [**8**])

$$B_{n+1}(g_1, \ldots, g_{n+1}) = \sum_{k=0}^{n} \binom{n}{k} B_{n-k}(g_1, \ldots, g_{n-k})\, g_{k+1}.$$

In this case equation (5) reduces to

$$\int_0^{+\infty} \exp(g(t))\, e^{-ts} dt = \frac{\exp(\mathring{g}_0)}{s} + \sum_{n=1}^{\infty} B_n(\mathring{g}_1, \mathring{g}_2, \ldots, \mathring{g}_n)\, \frac{1}{s^{n+1}}\ .$$

In what follows we show the approximation of the LT of nested functions. It were obtained by the second author by using the computer algebra program Mathematica©.

4.2. EXAMPLES

We start considering the case of the LT of nested exponential functions

• Let $f(x) = e^x$ and $g(t) = \sin t$. Then $g_1 = 1, g_2 = 0, g_3 = -1, g_4 = 0$, and in general $g_{2h} = 0, g_{2h+1} = (-1)^h$, $h = 1, 2, 3, \ldots$.
According to equation (6) it results

$B_1(1) = 1$, $B_2(1, 0) = 1$, $B_3(1, 0, -1) = 0$, $B_4(1, 0, -1, 0) = -3$, $B_5(1, 0, -1, 0, 1) = -8$.

Then

$$\int_0^{+\infty} \exp(\sin t)\, e^{-ts} dt = \frac{1}{s} + \frac{1}{s^2} + \frac{1}{s^3} - \frac{3}{s^5} - \frac{8}{s^6} + O\left(\frac{1}{s^7}\right)\ .$$

• Consider the Bessel function $g(t) := J_1(t)$ and the LT of the corresponding exponential function. We find

$$\int_0^{+\infty} \exp(J_1(t))\, e^{-ts} dt = \frac{1}{s} + \frac{1}{2s^2} + \frac{1}{4s^3} - \frac{3}{4s^4} - \frac{11}{16s^5} - \frac{19}{32s^6} +$$

$$\frac{91}{64s^7} + \frac{701}{128s^8} + \frac{953}{256s^9} - \frac{15245}{512s^{10}} + O\left(\frac{1}{s^{11}}\right)\ .$$

• Let $g(t) = -\arctan(t)$, we find

$$\int_0^{+\infty} \exp(-\arctan(t))\, e^{-ts} dt = \frac{1}{s} - \frac{1}{s^2} + \frac{1}{s^3} + \frac{1}{s^4} - \frac{7}{s^5} - \frac{5}{s^6} + \frac{145}{s^7} + \frac{5}{s^8} - \frac{6095}{s^9} - \frac{5815}{s^{10}} + O\left(\frac{1}{s^{11}}\right) .$$

• Consider the complete elliptic integral of the second kind $g(t) := E(t)$ and the LT of the corresponding exponential function. We find

$$\int_0^{+\infty} \exp(E(t))\, e^{-ts} dt = \frac{e^{\pi/2}}{s} - \frac{\pi}{8s^2} + \frac{\pi^2 - 3\pi}{64s^3} - \frac{\pi^3 - 9\pi^2 + 30\pi}{512s^4} + \frac{\pi^4 - 18\pi^3 + 147\pi^2 - 525\pi}{4096s^5} + O\left(\frac{1}{s^6}\right) .$$

4.3. THE GENERAL CASE

Examples of the above method for approximating the LT of general nested functions are reported in what follows.

• Assuming $f(x) = \log(x)$, $g(t) = \cosh(t)$, it results

$$\int_0^{+\infty} \log[\cosh(t)]\, e^{-ts} dt = \frac{1}{2s}\left[\psi\left(\frac{1}{2} + \frac{s}{4}\right) - \psi\left(\frac{s}{4}\right)\right] - \frac{1}{s^2} \ , \tag{7}$$

and using our approximation, we find

$$\int_0^{+\infty} \log[\cosh(t)]\, e^{-ts} dt = = \frac{1}{s^3} - \frac{2}{s^5} + \frac{16}{s^7} - \frac{272}{s^9} + \frac{7936}{s^{11}} + O\left(\frac{1}{s^{12}}\right) . \tag{8}$$

• Considering the Bessel function $f(x) = J_0(x)$, $a > 0$ and $g(t) = at^2$, it results

$$\int_0^{+\infty} J_0(at^2)\, e^{-ts} dt = \frac{\pi s}{16a}\left\{\left[J_{1/4}(s^2/8a) + Y_{1/4}(s^2/8a)\right]^2\right\}, \ \ \Re s > 0 \, , \tag{9}$$

and using our approximation, we find

$$\int_0^{+\infty} J_0(at^2) e^{-ts} dt = = \frac{1}{s} - \frac{6a^2}{s^5} + \frac{630a^4}{s^9} - \frac{207900a^6}{s^{13}} + \frac{141891750a^8}{s^{17}} + O\left(\frac{1}{s^{21}}\right) . \tag{10}$$

- Assuming $f(x) = \cos(x)$, $g(t) = ae^{-t}$, it results

$$\int_0^{+\infty} \cos[ae^{-t}]\, e^{-ts} dt = \frac{\cos(a)}{s} + \frac{a\sin(a)}{s^2} - \frac{a[a\cos(a) + \sin(a)]}{s^3} +$$

$$\frac{a[3a\cos(a) + (1-a^2)\sin(a)]}{s^4} + \frac{a[(a^3 - 7a)\cos(a) + (6a^2 - 1)\sin(a)]}{s^5} +$$

$$\frac{a[(15a - 10a^3)\cos(a) + (1 - 25a^2 + a^4)\sin(a)]}{s^6} + O\left(\frac{1}{s^7}\right) .$$

- Assuming $f(x) = \operatorname{arcsinh}(x)$, $g(t) = at^3$, it results

$$\int_0^{+\infty} \operatorname{arcsinh}(at^3)\, e^{-ts} dt =$$

$$= \frac{6a}{s^4} - \frac{60480a^3}{s^{10}} + \frac{98075577600a^5}{s^{16}} + O\left(\frac{1}{s^{17}}\right) .$$

- Assuming $f(x) = \arctan(x)$, $g(t) = \log(1+t)$, it results

$$\int_0^{+\infty} \arctan[\log(1+t)]\, e^{-ts} dt = \frac{1}{s^2} - \frac{1}{s^3} + \frac{6}{s^5} - \frac{22}{s^6} - \frac{30}{s^7} +$$

$$+ \frac{952}{s^8} - \frac{5656}{s^9} + \frac{9952}{s^{10}} - \frac{508320}{s^{11}} + O\left(\frac{1}{s^{12}}\right) .$$

- Assuming $f(x) = \operatorname{arcsinh}(x)$, $g(t) = t+1$, it results

$$\int_0^{+\infty} \operatorname{arcsinh}(t+1)\, e^{-ts} dt = \frac{\operatorname{arcsinh}(1)}{s} + \frac{1}{2\sqrt{2}s^2} - \frac{3}{8\sqrt{2}s^3} +$$

$$+ \frac{19}{32\sqrt{2}s^4} - \frac{189}{128\sqrt{2}s^5} + \frac{2601}{512\sqrt{2}s^6} + O\left(\frac{1}{s^7}\right) .$$

A lot of further examples can be constructed using the above method and the most of them have not a close expression in terms of special functions.

5. GRAPHICAL DISPLAY IN TWO KNOWN CASES

5.1. TEST CASE #1

Let us consider the function $l(t) = \log[\cosh(t)]$. As per the expression (7), the Laplace transform of $l(t)$ is given by:

$$L(s) = \frac{1}{2s}\left[\psi\left(\frac{1}{2}+\frac{s}{4}\right) - \psi\left(\frac{s}{4}\right)\right] - \frac{1}{s^2}\ , \tag{11}$$

for $\Re s > 0$.

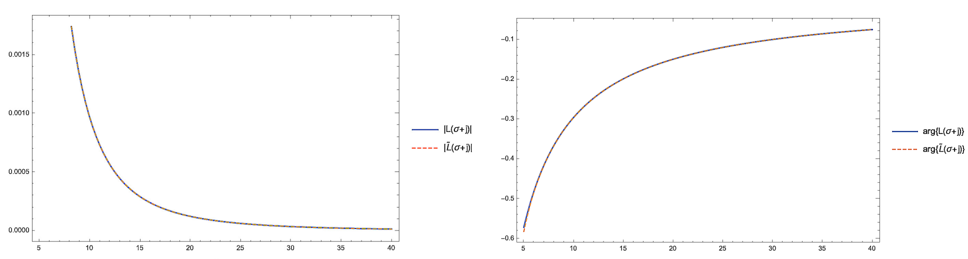
By applying the approximation methodology presented in this study, we have demonstrated in (8) that:

$$L(s) \simeq \tilde{L}(s) = \frac{1}{s^3} - \frac{2}{s^5} + \frac{16}{s^7} - \frac{272}{s^9} + \frac{7936}{s^{11}}\ , \tag{12}$$

so that, by inverse Laplace transformation, one can readily conclude that:

$$\tilde{l}(t) \simeq \left(\frac{t^2}{2} - \frac{t^4}{12} + \frac{t^6}{45} - \frac{17t^8}{2520} + \frac{31t^{10}}{14175}\right) H(t)\ ,$$

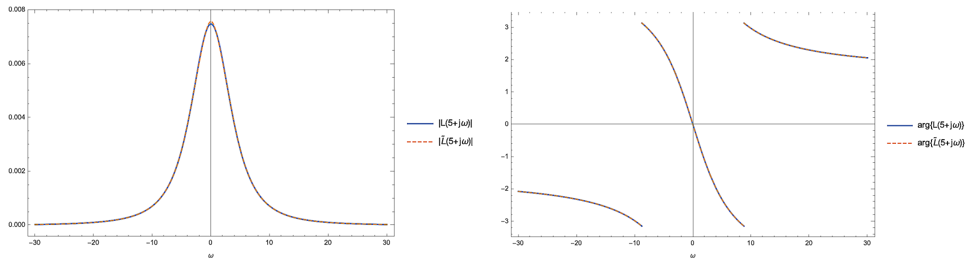
with $H(\cdot)$ denoting the classical Heaviside distribution.



(a) (b)

Figure 1: Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \log[\cosh(t)]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\,\omega$ with $\omega = 1$.

The distributions of $L(s)$ and $\tilde{L}(s)$ along the cut sections $\omega = \Im s = 1$ and $\sigma = \Re s = 5$ are reported in Figures 1 and 2, respectively. As it can be noticed, the agreement between the exact transform (11) and the relevant approximation (12) is very good especially as $s \to +\infty$. Conversely, the functions $l(t)$ and $\tilde{l}(t)$ tend to match for $t \to 0^+$ as one would expect from theory (see Figure 3).



(a) (b)

Figure 2: Magnitude (a) and argument (b) of the Laplace transform of $l(t) = \log[\cosh(t)]$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\,\omega$ with $\sigma = 5$.

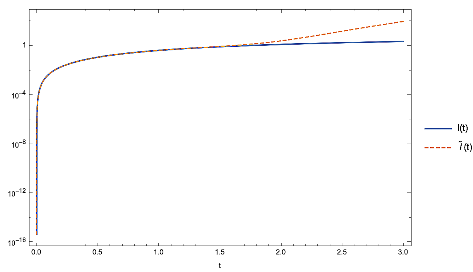


Figure 3: Distribution of $l(t) = \log[\cosh(t)]$ and the relevant approximant $\tilde{l}(t)$.

5.2. TEST CASE #2

Let us consider the function $l(t) = J_0(t^2)$. As per the expression (9), the Laplace transform of $l(t)$ is given by:

$$L(s) = \frac{\pi s}{16}\left\{\left[J_{1/4}(s^2/8)\right]^2 + \left[Y_{1/4}(s^2/8)\right]^2\right\}\,, \tag{13}$$

for $\Re s > 0$.

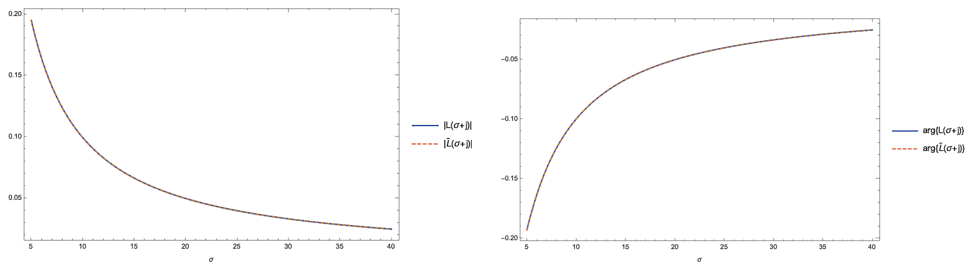
By applying the approximation methodology presented in this study, we can demonstrate, on the basis of (10), that:

$$L(s) \simeq \tilde{L}(s) = \frac{1}{s} - \frac{6}{s^5} + \frac{630}{s^9} - \frac{207900}{s^{13}} + \frac{141891750}{s^{17}} - \frac{164991726900}{s^{21}}\,, \tag{14}$$

so that, by inverse Laplace transformation, one can readily conclude that:

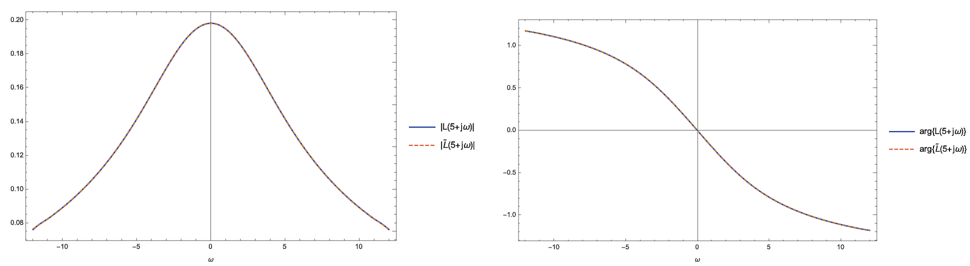
$$\tilde{l}(t) \simeq \left(1 - \frac{t^4}{4} + \frac{t^8}{64} - \frac{t^{12}}{2304} + \frac{t^{16}}{147456} - \frac{t^{20}}{14745600}\right) H(t)\,,$$

with $H(\cdot)$ denoting the classical Heaviside distribution.



(a) (b)

Figure 4: Magnitude (a) and argument (b) of the Laplace transform of $l(t) = J_0(t^2)$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\,\omega$ with $\omega = 1$.



(a) (b)

Figure 5: Magnitude (a) and argument (b) of the Laplace transform of $l(t) = J_0(t^2)$ as evaluated through the approximant $\tilde{L}(s)$ and the rigorous analytical expression $L(s)$ for $s = \sigma + \mathrm{i}\,\omega$ with $\sigma = 5$.

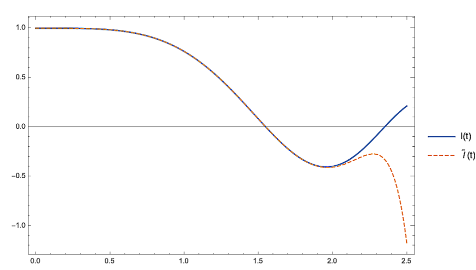


Figure 6: Distribution of $l(t) = J_0(t^2)$ and the relevant approximant $\tilde{l}(t)$.

The distributions of $L(s)$ and $\tilde{L}(s)$ along the cut sections $\omega = \Im s = 1$ and $\sigma = \Re s = 5$ are reported in Figures 4 and 5, respectively. As it can be noticed, the agreement between the exact transform (13) and the relevant approximation (14) is very good especially as $s \to +\infty$. Conversely, the functions $l(t)$ and $\tilde{l}(t)$ tend to match for $t \to 0^+$ as one would expect from theory (see Figure 6).

Appendix I

$$B_1 = g_1,$$

$$B_2 = g_1^2 + g_2,$$

$$B_3 = g_1^3 + 3g_1g_2 + g_3,$$

$$B_4 = g_1^4 + 6g_1^2g_2 + 4g_1g_3 + 3g_2^2 + g_4,$$

$$B_5 = g_1^5 + 10g_1^3g_2 + 15g_1g_2^2 + 10g_1^2g_3 + 10g_2g_3 + 5g_1g_4 + g_5,$$

$$\begin{aligned} B_6 = {} & g_1^6 + 15g_1^4g_2 + 45g_1^2g_2^2 + 15g_2^3 + 20g_1^3g_3 + 60g_1g_2g_3 + 10g_3^2 + 15g_1^2g_4 + \\ & 15g_2g_4 + 6g_1g_5 + g_6, \end{aligned}$$

$$\begin{aligned} B_7 = {} & g_1^7 + 21g_1^5g_2 + 105g_1^3g_2^2 + 105g_1g_2^3 + 35g_1^4g_3 + 210g_1^2g_2g_3 + 105g_2^2g_3 + \\ & 70g_1g_3^2 + 35g_1^3g_4 + 105g_1g_2g_4 + 35g_3g_4 + 21g_1^2g_5 + 21g_2g_5 + 7g_1g_6 + g_7, \end{aligned}$$

$$\begin{aligned} B_8 = {} & g_1^8 + 28g_1^6g_2 + 210g_1^4g_2^2 + 420g_1^2g_2^3 + 105g_2^4 + 56g_1^5g_3 + 560g_1^3g_2g_3 + \\ & 840g_1g_2^2g_3 + 280g_1^2g_3^2 + 280g_2g_3^2 + 70g_1^4g_4 + 420g_1^2g_2g_4 + 210g_2^2g_4 + \\ & 280g_1g_3g_4 + 35g_4^2 + 56g_1^3g_5 + 168g_1g_2g_5 + 56g_3g_5 + 28g_1^2g_6 + 28g_2g_6 + \\ & 8g_1g_7 + g_8, \end{aligned}$$

$$\begin{aligned} B_9 = {} & g_1^9 + 36g_1^7g_2 + 378g_1^5g_2^2 + 1260g_1^3g_2^3 + 945g_1g_2^4 + 84g_1^6g_3 + 1260g_1^4g_2g_3 + \\ & 3780g_1^2g_2^2g_3 + 1260g_2^3g_3 + 840g_1^3g_3^2 + 2520g_1g_2g_3^2 + 280g_3^3 + 126g_1^5g_4 + \\ & 1260g_1^3g_2g_4 + 1890g_1g_2^2g_4 + 1260g_1^2g_3g_4 + 1260g_2g_3g_4 + 315g_1g_4^2 + \\ & 126g_1^4g_5 + 756g_1^2g_2g_5 + 378g_2^2g_5 + 504g_1g_3g_5 + 126g_4g_5 + 84g_1^3g_6 + \\ & 252g_1g_2g_6 + 84g_3g_6 + 36g_1^2g_7 + 36g_2g_7 + 9g_1g_8 + g_9, \end{aligned}$$

$$\begin{aligned} B_{10} = {} & g_1^{10} + 45g_1^8g_2 + 630g_1^6g_2^2 + 3150g_1^4g_2^3 + 4725g_1^2g_2^4 + 945g_2^5 + 120g_1^7g_3 + \\ & 2520g_1^5g_2g_3 + 12600g_1^3g_2^2g_3 + 12600g_1g_2^3g_3 + 2100g_1^4g_3^2 + 12600g_1^2g_2g_3^2 + \\ & 6300g_2^2g_3^2 + 2800g_1g_3^3 + 210g_1^6g_4 + 3150g_1^4g_2g_4 + 9450g_1^2g_2^2g_4 + 3150g_2^3g_4 + \\ & 4200g_1^3g_3g_4 + 12600g_1g_2g_3g_4 + 2100g_3^2g_4 + 1575g_1^2g_4^2 + 1575g_2g_4^2 + \\ & 252g_1^5g_5 + 2520g_1^3g_2g_5 + 3780g_1g_2^2g_5 + 2520g_1^2g_3g_5 + 2520g_2g_3g_5 + \\ & 1260g_1g_4g_5 + 210g_1^4g_6 + 1260g_1^2g_2g_6 + 630g_2^2g_6 + 840g_1g_3g_6 + 210g_4g_6 + \\ & 120g_1^3g_7 + 126g_5^2 + 360g_1g_2g_7 + 120g_3g_7 + 45g_1^2g_8 + 45g_2g_8 + 10g_1g_9 + g_{10}\,. \end{aligned}$$

Appendix II

ntot = 10;
$B_0 = 1$;
Do [$B_{n+1} = \sum_{k=0}^{n} Binomial\,[n,k] * B_{n-k} * g_{k+1}, \{n, 0, ntot, 1\}$];
Do [$B_n = \mathrm{Expand}[B_n, \{n, 0, ntot, 1\}]$;
Do [Print [“B”$_n$, “ = ”, B_n], $\{n, 0, ntot, 1\}$];

Appendix III

In this appendix we report the Mathematica© code used to evaluate the approximation of the LT of general nested functions through Bell's polynomials.

```
In[]:= g[t_] = Cosh[t];

In[]:= f[x_] = Log[x];

In[]:= g0[n_] = SeriesCoefficient[n! g[t], {t, 0, n}]

Out[]= { 1  Mod[n, 2] == 0 && n ≥ 0
       { 0  True

In[]:= f0[n_] = SeriesCoefficient[n! f[x], {x, g[0], n}]

Out[]= { (-1)^(1+n) Gamma[n]  n ≥ 1
       { 0                    True
```

In[]:= $\mathcal{A}[\mathrm{n\_}] := \sum_{k=1}^{n} \mathrm{BellY}[n, k, \mathrm{Table}[g0[m], \{m, 1, n-k+1\}]] \times f0[k]$

```
In[]:= N = 10;

In[]:= Table[g0[n], {n, 0, N}]

Out[]= {1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1}

In[]:= Table[f0[n], {n, 0, N}]

Out[]= {0, 1, -1, 2, -6, 24, -120, 720, -5040, 40320, -362880}

In[]:= Table[A[n], {n, 0, N}]

Out[]= {0, 0, 1, 0, -2, 0, 16, 0, -272, 0, 7936}
```

In[]:= $\mathcal{L}[\mathrm{s\_}] = \frac{f[g[0]]}{s} + \sum_{n=1}^{N} \frac{\mathcal{A}[n]}{s^{n+1}}$

Out[]= $\frac{7936}{s^{11}} - \frac{272}{s^9} + \frac{16}{s^7} - \frac{2}{s^5} + \frac{1}{s^3}$

6. CONCLUSION

We have shown a method for approximating the integral of analytic composite functions. We started from the Taylor expansion of the considered function in a neighborhood of the origin. Since the coefficients con be expressed in terms of Bell's

polynomials, the integral is reduced to the computation of an approximanting series, which obviously converges if the integral is convergent.
Then this methodology has been applied to the case of the LT of an analytic composite function, starting from the case of analytic nested exponential functions. To this aim, a table of the used complete Bell polynomials is reported in Appendix I, and the relevant computer program is shown in Appendix II. In the last section the LT of a general analytic nested funtion is considered, and the used program is described in Appendix III. We want to stess that even we dealt with a basic subject, we have not found in literature any general method for approximating this type of LT's, a gap which, in our opinion, deserves to be overcome.

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