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DETERMINANT EVALUATION OF BANDED TOEPLITZ MATRICES VIA BIVARIATE POLYNOMIAL FAMILIES

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We define three kinds banded Toeplitz matrices via with the upper and lower bandwidths "$\mp x$" and "$\mp y$". The determinant evaluation is explicitly given for three kinds banded Toeplitz matrices via bivariate Tribonacci and Delannoy polynomials by using generating function approach and recurrence relations. Moreover perturbed versions of each kinds of the banded Toeplitz matrices by a 2×2 general square matrix at the upper right corner will be explicitly computed.

1. INTRODUCTION AND PRELIMINARY RESULTS

In general, a banded Toeplitz matrix T_n of order n has the form for $k, r < n$

$$T_n = \begin{bmatrix} t_0 & \cdots & t_{-r} & & \\ \vdots & \ddots & \cdots & \ddots & \\ t_k & & \ddots & & t_{-r} \\ & \ddots & & \ddots & \vdots \\ & & t_k & \cdots & t_0 \end{bmatrix},$$

where the coefficients t_i, $i = -r, \ldots, k$, being complex numbers.

Many special cases of the banded matrices such as Toeplitz matrices, symmetric Toeplitz matrices, especially tri-diagonal matrices, etc., have been studied

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by many authors. Since their inverses are frequently used, explicitly and effectively finding inverses of them are important. Tridiagonal matrices and Toeplitz matrices are of great importance in both mathematics (cf. [**5, 6, 13, 24**]) and physics (cf. [**1, 17**]). Especially these kind of matrices have been frequently used in various application areas ranging from engineering to economics (see [**3, 4, 7, 9, 20**]) as well as in the computation of special functions, PDEs and number theory (see [**2, 8, 11, 12, 19, 23**]). Various features of tridiagonal matrices are used to solve the systems of linear equations that arise from these applications and many authors (for example, [**9, 10, 15, 24**]) have studied various tridiagonal matrices and their properties such as LU decompositions, determinants and inverses.

There are well-known examples of banded Toeplitz matrices whose entries consist of indetermines. Determinants of such Toeplitz matrices generate well known polynomial families. By these kinds of relations, one can derive interesting properties of polynomial families or well known number sequences via linear algebra and vice versa. We may give an example for such a relation between tridiagonal Toeplitz matrices and Chebyshev polynomials as shown

$$\det \begin{bmatrix} 2x & -1 & & \\ 1 & 2x & \ddots & \\ & \ddots & \ddots & -1 \\ & & 1 & 2x \end{bmatrix} = U_n\left(x\right),$$

where $U_n\left(x\right)$'s are the Chebyshev polynomials of the second kind. For more relations between well-known second order polynomial sequences and certain determinants of Toeplitz or *perturbed Toeplitz matrices* and useful applications of such relations, we refer to [**14, 16, 21, 22**] and the references therein.

Recently Kurmanbek, Amanbek and Erlangga (for more details we refer to [**18**] and the references therein) considered a recent open problem and evaluated the determinant of the two pentadiagonal matrices

$$A_n := \left[a_{i-j}\right]_{1\le i,j\le n} : a_k = \begin{cases} 1, & k = 0, \pm 1, 2, 1-n; \\ 0, & \text{otherwise}; \end{cases}$$

$$B_n := \left[b_{i-j}\right]_{1\le i,j\le n} : b_k = \begin{cases} 1, & k = 0, \pm 1, 2, 2-n; \\ 0, & \text{otherwise}; \end{cases}$$

and then they proved that

$$\det A_n = \begin{cases} 1, & n \equiv_4 0; \\ 2, & n \equiv_4 1; \\ -1, & n \equiv_4 2; \\ 0, & n \equiv_4 3; \end{cases} \quad \text{and} \quad \det B_n = \begin{cases} 0, & n \equiv_4 0; \\ 2, & n \equiv_4 1; \\ 3, & n \equiv_4 2; \\ 1, & n \equiv_4 3. \end{cases}$$

It would be valuable to note that both A_n and B_n are banded Toeplitz matrices.

Here and for later use, $i \equiv_m j$ stands for "i is congruent to j modulo m". For a real number α, we shall also make use of $\lfloor \alpha \rfloor$ for the maximum integer $\leq \alpha$.

Motivated by the determinant evaluations of certain Toeplitz and perturbed Toeplitz matrices, we define new three Toeplitz matrices dependent on indetermines. In the next section, we will present two of them and the explicit formulae for their determinants *combinatorially* as sums of the product of two binomial coefficients weighted with the indetermines which we call *bivariate Tribonacci polynomials*. Moreover we shall investigate generalizations of these two Toeplitz matrices which are perturbed by the 2×2 square matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to each matrix at the upper right corner. By obtaining their LU-decompositions explicitly, we shall formulate their determinants.

In Section 3, we will present a third kind of Toeplitz matrix and the explicit formulae for its determinant *combinatorially* via bivariate *Delannoy polynomials* whose coefficients are the Delannoy numbers. Also we shall investigate generalization of the third kind of Toeplitz matrices which are perturbed by the 2×2 square matrix $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ to each matrix at the upper right corner. By obtaining their LU-decompositions explicitly, we shall formulate their determinants. In the last section, we present an another generalization of the third Toeplitz matrix with an additional parameter. Comparing results derived from the last two sections, we will derive new combinatorial formulas as double binomial sums for general second order linear recurrences. As applications, we derive new combinatorial representations for Tribonacci numbers and a general second order recurrence relation, especially for Fibonacci, Pell numbers and Chebyshev polynomials.

2. TWO TOEPLITZ MATRICES AND THEIR DETERMINANTS

As mentioned in the previous section, we present two Toeplitz matrices and then evaluate their determinants combinatorially via sums of products of binomial coefficients. We will use the generating function method and the Laplace expansion of determinant to prove our claims in this section.

Two Toeplitz matrices are defined as

$$\begin{aligned} \mathrm{C}_n(x,y) &= \left[c_{i-j}\right]_{1\leq i,j\leq n}: \ c_0 = x,\ c_{-1} = y,\ c_1 = -y,\ c_2 = x, \\ \mathrm{F}_n(x,y) &= \left[f_{i-j}\right]_{1\leq i,j\leq n}: \ f_0 = x,\ f_{-1} = y,\ f_1 = y,\ f_2 = -x. \end{aligned}$$

Clearly the matrix $\mathrm{C}_n(x,y)$ has the form

$$\mathrm{C}_n(x,y) = \begin{bmatrix} x & y & & & & \\ -y & x & y & & & \\ x & -y & x & \ddots & & \\ & x & \ddots & \ddots & y & \\ & & \ddots & -y & x & y \\ & & & x & -y & x \end{bmatrix}.$$

Then we shall state our first main claims that the determinants of the matrices $\mathrm{C}_n(x,y)$ and $F_n(x,y)$ are evaluated by

$$\begin{aligned}\mathbf{C}_n(x,y) &= \det \mathrm{C}_n(x,y) = Q_n(x,y),\\ \mathbf{F}_n(x,y) &= \det \mathrm{F}_n(x,y) = Q_n(x,y\sqrt{-1}),\end{aligned}$$

where the bivariate polynomial $Q_n(x,y)$ is defined as

$$Q_n(x,y) = \sum_{i+2j+3k=n} \binom{i+j+k}{i,j,k} x^{i+k} y^{2j+2k},$$

which could be also rewritten as

$$Q_n(x,y) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2k}{j} \binom{k}{j} 2^j x^{n-2k} y^{2k}.$$

Throughout the paper, for the sake of brevity, we will frequently use the shortened notations Q_n, H_n, T_n and V_n instead of $Q_n(x,y)$, $Q_n(y,x)$, $Q_n\left(y, x\sqrt{-1}\right)$ and $Q_n\left(x, y\sqrt{-1}\right)$, respectively.

Define the generating function of the polynomials Q_n as

$$\mathcal{Q}(z) = \sum_{n\geq 0} Q_n z^n.$$

Then we have the following result.

Lemma 1.

$$\mathcal{Q}(z) = \frac{1}{1 - xz - y^2z^2 - xy^2z^3}.$$

Proof. Consider

$$
\begin{aligned}
\mathcal{Q}(z) &= \sum_{n\geq 0}\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}\sum_{j=0}^{\lfloor \frac{n}{2}\rfloor}\binom{n-2k}{j}\binom{k}{j}2^j x^{n-2k}y^{2k}z^n \\
&= \sum_{0\leq j\leq k}\binom{k}{j}2^j (yt)^{2k}\sum_{n\geq 0}\binom{n-2k}{j}(xz)^{n-2k} \\
&= \sum_{0\leq j\leq k}\binom{k}{j}\frac{2^j (yt)^{2k}(xz)^j}{(1-xz)^{j+1}} \\
&= \sum_{0\leq j}\frac{\left(2xy^2z^3\right)^j}{(1-xz)^{j+1}(1-y^2z^2)^{j+1}} \\
&= \frac{1}{(1-xz)(1-y^2z^2)}\sum_{0\leq j}\frac{\left(2xy^2z^3\right)^j}{(1-xz)^j(1-y^2z^2)^j} \\
&= \frac{1}{(1-xz)(1-y^2z^2)-2xy^2z^3} \\
&= \frac{1}{1-xz-y^2z^2-xy^2z^3},
\end{aligned}
$$

which completes the proof. □

Therefore by Lemma 1 and the definitions of $H_n,\ T_n$ and V_n, we present the collected list of generating functions as shown

$$
\begin{aligned}
\mathcal{Q}(z) &= \sum_{n\geq 0}Q_n z^n = \frac{1}{1-xz-y^2z^2-xy^2z^3}, \\
\mathcal{H}(z) &= \sum_{n\geq 0}H_n z^n = \frac{1}{1-yz-x^2z^2-yx^2z^3}, \\
\mathcal{T}(z) &= \sum_{n\geq 0}T_n z^n = \frac{1}{1-yz+x^2z^2+yx^2z^3}, \\
\mathcal{V}(z) &= \sum_{n\geq 0}V_n z^n = \frac{1}{1-xz+y^2z^2+xy^2z^3}.
\end{aligned}
$$

By the generating functions of the polynomials $Q_n, H_n,\ T_n$ and V_n, we derive their recurrence relation for further use.

Lemma 2. *For $n>2$,*

(i)

$$Q_n = xQ_{n-1}+y^2Q_{n-2}+xy^2Q_{n-3},$$

with the initials $Q_0=1,\ Q_1=x,\ Q_2=x^2+y^2$ and $Q_3=x^3+3xy^2$.

(ii)

$$H_n = yH_{n-1} + x^2H_{n-2} + yx^2H_{n-3},$$

with the initials $H_0 = 1,\ H_1 = y,\ H_2 = x^2 + y^2$ *and* $H_3 = y^3 + 3yx^2$.

(iii)

$$T_n = yT_{n-1} - x^2T_{n-2} - yx^2T_{n-3},$$

with the initials $T_0 = 1, T_1 = y, T_2 = y^2 - x^2$ *and* $T_3 = y^3 - 3x^2y$.

(iv)

$$V_n = xV_{n-1} - y^2V_{n-2} - xy^2V_{n-3},$$

with the initials $V_0 = 1,\ V_1 = x,\ V_2 = x^2 - y^2$ *and* $V_3 = x^3 - 3xy^2$.

Lemma 3. $(n \in \mathbb{N})$

$$\mathbf{C}_n(x, y) = Q_n(x, y).$$

Proof. To prove the claim, we show that $\mathbf{C}_n(x, y)$ and $Q_n(x, y)$ satisfy the same recurrence relation with the same initials. For computing $\mathbf{C}_n(x, y)$, we will use the Laplace expansion of the determinant. First expanding $\mathbf{C}_n(x, y)$ according to the first row entries x and y, respectively, and then expanding the second consequent determinant corresponding to the second entry y according to the first column gives us

$$\mathbf{C}_n(x, y) = x\mathbf{C}_{n-1}(x, y) + y^2\mathbf{C}_{n-2}(x, y) + xy^2\mathbf{C}_{n-3}(x, y).$$

On the other hand, by Lemma 2, the recurrence relation of $Q_n(x, y)$ is

$$Q_n(x, y) = xQ_{n-1}(x, y) + y^2Q_{n-2}(x, y) + xy^2Q_{n-3}(x, y).$$

Since both the recurrece relations and the initials terms of $\mathbf{C}_n(x, y)$ and $Q_n(x, y)$ are equal, we derive that

$$\mathbf{C}_n(x, y) = Q_n(x, y),$$

as claimed. □

Now we give two auxiliary results for later use. As a consequence of Lemma 3 and by the definitions of $\{Q_n, T_n\}$, we have the following result without proof.

Corollary 4. *The polynomials* Q_n *and* T_n *satisfy the recurrences*

$$Q_n = xQ_{n-1} + y^2Q_{n-2} + xy^2Q_{n-3},$$

and

$$T_n = yT_{n-1} - x^2T_{n-2} - yx^2T_{n-3}.$$

By the definitions of $\{Q_n, T_n\}$ and properties of the binomial coefficients, we have the following result without proof.

Lemma 5. *For $n > 2$*

$$Q_n T_{n+1} + x Q_{n+1} T_{n-1} = y \left[x Q_{n-1} T_n + Q_n T_n \right]$$

and for $n > 1$

$$Q_n T_{n+2} + x Q_{n+1} T_n = y \left[x Q_{n-1} T_{n+1} + Q_n T_{n+1} \right].$$

As consequences of this section, we will derive interesting results. The well known Tribonacci numbers t_n are defined as

$$t_n = t_{n-1} + t_{n-2} + t_{n-2},$$

where $t_0 = t_1 = 1$ and $t_2 = 2$.

The generating function of Tribonacci numbers is given as

$$\sum_{n \geq 0} t_n z^n = \frac{1}{1 - z - z^2 - z^3}.$$

Therefore, we can derive the following interesting combinatorial representation for the Tribonacci numbers.

Corollary 6. *For $n > 0$,*

$$t_n = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2k}{j} \binom{k}{j} 2^j.$$

From Lemma 1, we have that the generating function of the polynomials $Q_n(x, y)$ is

$$\mathcal{Q}(z) = \frac{1}{1 - xz - y^2 z^2 - x y^2 z^3}.$$

If we take $x = y = 1$ in the generating function of $Q_n(x, y)$, it is reduced to the generating function of Tribonacci numbers. Thus we write

$$Q_n(1,1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n-2k}{j} \binom{k}{j} 2^j = t_n,$$

as claimed.

One can derive similar combinatorial representations for Tribonacci-like sequences for the other polynomials H_n, T_n and V_n.

2.1 The matrix $\mathrm{C}_n(x, y; a, b, c, d)$

In this section we shall investigate the perturbed generalizations of the Toeplitz matrix $\mathrm{C}_n(x, y)$ which are defined by adding the 2×2 square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to each matrix at the upper right corner.

We define the matrix $\mathrm{R}_n(x,y)$ as shown

$$\mathrm{R}_n(x,y) := \begin{bmatrix} x & y & 0 & \cdots & & \cdots & 0 \\ x & -y & x & y & & & \vdots \\ 0 & x & -y & \ddots & \ddots & & \\ \vdots & & x & \ddots & x & y & \vdots \\ & & & \ddots & -y & x & 0 \\ \vdots & & & & x & -y & y \\ 0 & \cdots & & \cdots & 0 & x & x \end{bmatrix}$$

or, via the matrix $\mathrm{C}_n(x,y)$,

$$\mathrm{R}_n(x,y) = \begin{bmatrix} x & y & & 0 & \cdots & 0 \\ x & & & & & \vdots \\ 0 & & & \mathrm{C}_{n-1}(x,y) & & 0 \\ \vdots & & & & & y \\ 0 & \cdots & & 0 & x & x \end{bmatrix}.$$

Denote $\det \mathrm{R}_n(x,y)$ by $\mathbf{R}_n(x,y)$. Then we have the following result to give the relationship between $\mathbf{R}_n(x,y)$ and the polynomial T_n without proof. Its proof follows by expanding $\mathbf{R}_n(x,y)$ with respect to the first row and then expanding the consequent determinant with respect to its last row.

Lemma 7. *For $n>0$*

$$\mathbf{R}_{n+3}(x,y) = (-1)^{n+1}\left[T_{n+1} + 2yT_n + y^2 T_{n-1}\right].$$

We are going to define a unit lower triangular matrix $L = [L_{i,j}]_{1\le i,j\le n}$, that has the only nonzero elements on the main diagonal, the first and the second subdiagonals. Its entries are explicitly given by

$$\begin{aligned} L_{i,i} &= 1, & 1 \le i \le n; \\ L_{i,i-1} &= -\frac{y\left(xQ_{i-3}+Q_{i-2}\right)}{Q_{i-1}}, & 1 \le i \le n-1; \\ L_{i,i-2} &= \frac{xQ_{i-3}}{Q_{i-2}}, & 1 \le i \le n; \end{aligned}$$

together with the following exceptional element

$$L_{n,n-1} = -\frac{y\left(xQ_{n-3}+Q_{n-2}\right) + cx^2\left(yT_{n-5}+T_{n-4}\right) + axT_{n-3}}{Q_{n-1} + cx\left(yT_{n-4}+T_{n-3}\right) + aT_{n-2}}.$$

For $n=6$, we have

$$L=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ -\frac{yQ_0}{Q_1} & 1 & 0 & 0 & 0 & 0\\ \frac{xQ_0}{Q_1} & -\frac{y(xQ_0+Q_1)}{Q_2} & 1 & 0 & 0 & 0\\ 0 & \frac{xQ_1}{Q_2} & -\frac{y(xQ_1+Q_2)}{Q_3} & 1 & 0 & 0\\ 0 & 0 & \frac{xQ_2}{Q_3} & -\frac{y(xQ_2+Q_3)}{Q_4} & 1 & 0\\ 0 & 0 & 0 & \frac{xQ_3}{Q_4} & L_{6,5} & 1 \end{bmatrix},$$

where $L_{6,5}=-\frac{y(xQ_3+Q_4)+cx^2(yT_1+T_2)+axT_3}{Q_5+cx(yT_2+T_3)+aT_4}$.

We define also an upper triangular matrix $U_n=[U_{i,j}]_{1\le i,j\le n}$, that has the only nonzero elements on the diagonal, the first superdiagonal, and the last two columns, which are given by

$$\begin{aligned} U_{i,i} &= \tfrac{Q_i}{Q_{i-1}}, && 1\le i\le n-2;\\ U_{i,i+1} &= y, && 1\le i\le n-3;\\ U_{i,n-1} &= \tfrac{1}{Q_{i-1}}\left[cx\left(yT_{i-3}+T_{i-2}\right)+aT_{i-1}\right], && 1\le i\le n-3;\\ U_{i,n} &= \tfrac{1}{Q_{i-1}}\left[dx\left(yT_{i-3}+T_{i-2}\right)+bT_{i-1}\right], && 1\le i\le n-2; \end{aligned}$$

and the following amended exceptional entries

$$\begin{aligned} U_{n-2,n-1} &= y+\tfrac{1}{Q_{n-3}}\left[cx\left(yT_{n-5}+T_{n-4}\right)+aT_{n-3}\right];\\ U_{n-1,n-1} &= \tfrac{1}{Q_{n-2}}\left[Q_{n-1}+cx\left(yT_{n-4}+T_{n-3}\right)+aT_{n-2}\right];\\ U_{n-1,n} &= y+\tfrac{1}{Q_{n-2}}\left[dx\left(yT_{n-4}+T_{n-3}\right)+bT_{n-2}\right]; \end{aligned}$$

and

$$\begin{aligned} U_{n,n} = {} & \frac{1}{Q_{n-1}+cx\left(yT_{n-4}+T_{n-3}\right)+aT_{n-2}}\\ & \times\left[Q_n+bT_{n-1}+x^{n-2}\left(ad-bc\right)\right.\\ & \left.+\,x\left(a+d\right)\left(yT_{n-3}+T_{n-2}\right)-\left(-1\right)^n cx^2\mathbf{R}_{n-1}\right], \end{aligned}$$

where $\mathbf{R}_n$ is defined as before. Recall that as usual we assume that the empty sum is 0.

For instance, if $n=6$, we have

$$U=\begin{bmatrix} \frac{Q_1}{Q_0} & y & 0 & 0 & \frac{a}{Q_0} & \frac{b}{Q_0}\\ 0 & \frac{Q_2}{Q_1} & y & 0 & \frac{cx+aT_1}{Q_1} & \frac{dx+bT_1}{Q_1}\\ 0 & 0 & \frac{Q_3}{Q_2} & y & \frac{cx(y+T_1)+aT_2}{Q_2} & \frac{dx(y+T_1)+bT_2}{Q_2}\\ 0 & 0 & 0 & \frac{Q_4}{Q_3} & y+\frac{cx(yT_1+T_2)+aT_3}{Q_3} & \frac{dx(yT_1+T_2)+bT_3}{Q_3}\\ 0 & 0 & 0 & 0 & \frac{Q_5}{Q_4}+\frac{cx(yT_2+T_3)+aT_4}{Q_4} & y+\frac{dx(yT_2+T_3)+bT_4}{Q_4}\\ 0 & 0 & 0 & 0 & 0 & U_{6,6} \end{bmatrix},$$

where

$$U_{6,6} = \frac{Q_6 + bT_5 + x^4 (ad - bc) + x (a + d) (yT_3 + T_4) - cx^2 \mathbf{R}_5}{Q_5 + cx (yT_2 + T_3) + aT_4}$$

and $\mathbf{R}_5 = y \left(5x^2 - 4y^2\right).$

Then we have the following result.

Theorem 8. *With the above defined triangular matrices L and U, the matrix $\mathrm{C}_n(x, y; a, b, c, d)$ admits the following LU-decomposition*

$$\mathrm{C}_n(x, y; a, b, c, d) = LU.$$

Proof. Let $W := LU$. We show in details that $W = \mathrm{C}_n(x, y; a; b; c; d)$.

First, we start to verify the entries along the four diagonals.

i) Case $i = j$ for $1 \le i \le n$

- $i = j < n - 1$:

$$\begin{aligned} W_{i,i} &= \sum_{k=1}^{n} L_{ik} U_{ki} = L_{i,i} U_{i,i} + L_{i,i-1} U_{i-1,i} + L_{i,i-2} U_{i-2,i} \\ &= \frac{Q_i}{Q_{i-1}} - \frac{y (xQ_{i-3} + Q_{i-2})}{Q_{i-1}} y \\ &= \frac{1}{Q_{i-1}} \left[Q_i - y^2 (xQ_{i-3} + Q_{i-2})\right], \end{aligned}$$

which, by using the recurrence relation of $\{Q_n\}$, that is,

$$Q_n = xQ_{n-1} + y^2 Q_{n-2} + xy^2 Q_{n-3},$$

gives us

$$W_{i,i} = \frac{xQ_{i-1}}{Q_{i-1}} = x.$$

- $i = j = n-1$: We shall prove that $W_{n-1,n-1} = x$. Consider

$$\begin{aligned}
W_{n-1,n-1} &= L_{n-1,n-3}U_{n-3,n-1} + L_{n-1,n-2}U_{n-2,n-1} + L_{n-1,n-1}U_{n-1,n-1} \\
&= \frac{xQ_{n-4}}{Q_{n-3}}\left[\frac{1}{Q_{n-4}}\left(cx\left(yT_{n-6}+T_{n-5}\right)+aT_{n-4}\right)\right] \\
&- \frac{y\left(xQ_{n-4}+Q_{n-3}\right)}{Q_{n-2}}\left[y+\frac{1}{Q_{n-3}}\left(cx\left(yT_{n-5}+T_{n-4}\right)+aT_{n-3}\right)\right] \\
&+ \frac{1}{Q_{n-2}}\left[Q_{n-1}+cx\left(yT_{n-4}+T_{n-3}\right)+aT_{n-2}\right] \\
&= \frac{x}{Q_{n-3}}\left[cx\left(yT_{n-6}+T_{n-5}\right)+aT_{n-4}\right] \\
&- \frac{y\left(xQ_{n-4}+Q_{n-3}\right)}{Q_{n-2}Q_{n-3}}\left[cx\left(yT_{n-5}+T_{n-4}\right)+aT_{n-3}\right] \\
&+ \frac{1}{Q_{n-2}}\left[cx\left(yT_{n-4}+T_{n-3}\right)+aT_{n-2}\right] \\
&- \frac{y^2\left(xQ_{n-4}+Q_{n-3}\right)}{Q_{n-2}} + \frac{Q_{n-1}}{Q_{n-2}}.
\end{aligned}$$

By the recursion of Q_n, since

$$\frac{Q_{n-1}}{Q_{n-2}} - \frac{y^2\left(xQ_{n-4}+Q_{n-3}\right)}{Q_{n-2}} = x,$$

we write

$$\begin{aligned}
W_{n-1,n-1} = x &+ \frac{x}{Q_{n-3}}\left[cx\left(yT_{n-6}+T_{n-5}\right)+aT_{n-4}\right] \\
&- \frac{y\left(xQ_{n-4}+Q_{n-3}\right)}{Q_{n-2}Q_{n-3}}\left[cx\left(yT_{n-5}+T_{n-4}\right)+aT_{n-3}\right] \\
&+ \frac{1}{Q_{n-2}}\left[cx\left(yT_{n-4}+T_{n-3}\right)+aT_{n-2}\right].
\end{aligned}$$

In that case, to prove our main claim $W_{n-1,n-1} = x$, we have to prove the equality

$$\begin{aligned}
&\frac{x}{Q_{n-3}}\left[cx\left(yT_{n-6}+T_{n-5}\right)+aT_{n-4}\right] \\
&- \frac{y\left(xQ_{n-4}+Q_{n-3}\right)}{Q_{n-2}Q_{n-3}}\left[cx\left(yT_{n-5}+T_{n-4}\right)+aT_{n-3}\right] \\
&+ \frac{1}{Q_{n-2}}\left[cx\left(yT_{n-4}+T_{n-3}\right)+aT_{n-2}\right] \\
&= 0 \qquad (1)
\end{aligned}$$

After clearing denominators and some rearrangements and by the results of Lemma 5, the claimed sum above is computed as 0. Thus we have the claim $W_{n-1,n-1} = x$.

- $i = j = n$:

$$W_{n,n} = L_{n,n-2}U_{n-2,n} + L_{n,n-1}U_{n-1,n} + L_{n,n}U_{n,n},$$

which could be similarly proven by using Lemmas 3 and 5.

ii) Case $j = i + 1$ for $1 \le i < n$

- $1 \le i \le n - 3$:

$$W_{i,i+1} = \sum_{k=1}^{n} L_{i,k}U_{k,i+1} = L_{i,i}U_{i,i+1} = y.$$

- $i = n - 2$:

$$\begin{aligned} W_{n-2,n-1} &= \sum_{k=1}^{n} L_{n-2,k}U_{k,n-1} \\ &= L_{n-2,n-4}U_{n-4,n-1} + L_{n-2,n-3}U_{n-3,n-1} + L_{n-2,n-2}U_{n-2,n-1} \\ &= \frac{x}{Q_{n-4}} \left[cx \left(yT_{n-7} + T_{n-6}\right) + aT_{n-5}\right] \\ &- \frac{y\left(xQ_{n-5} + Q_{n-4}\right)}{Q_{n-3}Q_{n-4}} \left[cx \left(yT_{n-6} + T_{n-5}\right) + aT_{n-4}\right] \\ &+ y + \frac{1}{Q_{n-3}} \left[cx \left(yT_{n-5} + T_{n-4}\right) + aT_{n-3}\right], \end{aligned}$$

which, by (1), gives us the claimed result.

$i = n - 1$:

$$\begin{aligned} W_{n-1,n} &= L_{n-1,n-1}U_{n-1,n} + L_{n-1,n-2}U_{n-2,n} + L_{n-1,n-3}U_{n-3,n} \\ &= y + \frac{1}{Q_{n-2}} \left[dx \left(yT_{n-4} + T_{n-3}\right) + bT_{n-2}\right] \\ &- \frac{y\left(xQ_{n-4} + Q_{n-3}\right)}{Q_{n-2}Q_{n-3}} \left[dx \left(yT_{n-5} + T_{n-4}\right) + bT_{n-3}\right] \\ &+ \frac{x}{Q_{n-3}} \left[dx \left(yT_{n-6} + T_{n-5}\right) + bT_{n-4}\right], \end{aligned}$$

which, by (1), gives us

$$W_{n-1,n} = y.$$

iii) Case $i = j + 1$ for $1 < i \le n$

- $1 < i \le n - 1$:

$$\begin{aligned} W_{i,i-1} &= \sum_{k=1}^{n} L_{i,k}U_{k,i-1} = L_{i,i-1}U_{i-1,i-1} + L_{i,i}U_{i,i-1} \\ &= \frac{xQ_{i-3}}{Q_{i-2}} \times y - \frac{y\left(xQ_{i-3} + Q_{i-2}\right)}{Q_{i-1}} \times \frac{Q_{i-1}}{Q_{i-2}} \\ &= -y. \end{aligned}$$

$i = n$:

$$\begin{aligned}
W_{n,n-1} &= L_{n,n-2}U_{n-2,n-1} + L_{n,n-1}U_{n-1,n-1} \\
&= \frac{xQ_{n-3}}{Q_{n-2}}\left[y + \frac{1}{Q_{n-3}}\left(cx\left(yT_{n-5}+T_{n-4}\right)+aT_{n-3}\right)\right] \\
&- \frac{y\left(xQ_{n-3}+Q_{n-2}\right)+cx^2\left(yT_{n-5}+T_{n-4}\right)+axT_{n-3}}{Q_{n-1}+cx\left(yT_{n-4}+T_{n-3}\right)+aT_{n-2}} \\
&\times \left[\frac{Q_{n-1}+cx\left(yT_{n-4}+T_{n-3}\right)+aT_{n-2}}{Q_{n-2}}\right] \\
&= \frac{xQ_{n-3}}{Q_{n-2}}y - \frac{y\left(xQ_{n-3}+Q_{n-2}\right)}{Q_{n-2}} \\
&= -y.
\end{aligned}$$

iv) Case $i = j+2$ for $2 < i \le n$

- $2 < i < n$:

$$W_{i,i-2} = \sum_{k=1}^{n} L_{i,k}U_{k,i-2} = L_{i,i-2}U_{i-2,i-2} = y.$$

- $i = n$:

$$W_{n,n-2} = \sum_{k=1}^{n} L_{n,k}U_{k,n-2} = L_{n,n-2}U_{n-2,n-2} = y.$$

Then it is much easier to check the four perturbing entries at the upper right corner:

$$\begin{aligned}
W_{1,n-1} &= \sum_{k=1}^{n} L_{1,k}U_{k,n-1} = L_{1,1}U_{1,n-1} = a, \\
W_{1,n} &= \sum_{k=1}^{n} L_{1,k}U_{k,n} = L_{1,1}U_{1,n} = b, \\
W_{2,n-1} &= \sum_{k=1}^{n} L_{2,k}U_{k,n-1} = L_{2,1}U_{1,n-1} + L_{2,2}U_{2,n-1} = -\frac{yz_0}{z_1}a + \frac{1}{z_1}\left(ay+cx\right) = c, \\
W_{2,n} &= \sum_{k=1}^{n} L_{2,k}U_{k,n} = L_{2,1}U_{1,n} + L_{2,2}U_{2,n} = -\frac{yz_0}{z_1}b + \frac{1}{x}\left(by+dx\right) = d.
\end{aligned}$$

Finally, the proof of Theorem 8 will be completed by observing that

$$W_{i,j} = \sum_{k=1}^{n} L_{i,k}U_{k,j} = L_{i,i}U_{i,j} + L_{i,i-1}U_{i-1,j} + L_{i,i-2}U_{i-2,j} = 0$$

when $i > j+2$ and $j > i+1$, except for the four entries at the upper right corner.

□

Consequently, we can evaluate the determinant $\det \mathrm{C}_n(x,y;a,b,c,d)$ in the following corollary.

Corollary 9.

$$\begin{aligned}\det \mathrm{C}_n(x,y;a,b,c,d) &= \det U \\ &= \prod_{k=1}^{n} U_{k,k} = U_{n,n}U_{n-1,n-1}\prod_{k=1}^{n-2}\frac{Q_k}{Q_{k-1}} \\ &= Q_n + bT_{n-1} + x^{n-2}(ad-bc) \\ &+ x(a+d)(yT_{n-3}+T_{n-2}) - (-1)^n cx^2\mathbf{R}_{n-1}.\end{aligned}$$

2.2 The matrix $\mathrm{F}_n(x,y;a,b,c,d)$

In this section we shall investigate the perturbed generalizations of the Toeplitz matrix $\mathrm{F}_n(x,y)$ which are defined by adding the 2×2 square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to each matrix at the upper right corner.

Define the matrix $G_n(x,y)$ of order n as shown

$$G_n(x,y) := \begin{bmatrix} x & y & 0 & \cdots & & \cdots & 0 \\ x & y & x & y & & & \vdots \\ 0 & -x & y & \ddots & \ddots & & \\ \vdots & & -x & \ddots & x & y & \vdots \\ & & & \ddots & y & x & 0 \\ \vdots & & & & -x & y & y \\ 0 & \cdots & & \cdots & 0 & -x & x \end{bmatrix}.$$

For the sake of brevity, we shall denote $\det G_n(x,y)$ by $\mathbf{G}_n(x,y)$. Then we have the following result to give the relationship between $\mathbf{G}_n(x,y)$ and the polynomials H_n. We will omit proof of the claims of this section that could be followed similar to Section 2.1.

Lemma 10. *For $n>1$,*

$$\mathbf{G}_{n+3}(x,y) = x^2\left[H_{n+1} + 2yH_n + y^2H_{n-1}\right].$$

We are going to define a unit lower triangular matrix $L_n = [L_{i,j}]_{1\le i,j\le n}$, that has the only nonzero elements on the main diagonal, the first and the second subdiagonals. Its entries are explicitly given by

$$\begin{aligned} L_{i,i} &= 1; \\ L_{i,i-1} &= \frac{y\left(xV_{i-3}+V_{i-2}\right)}{V_{i-1}}; \\ L_{i,i-2} &= -\frac{xV_{i-3}}{V_{i-2}} \end{aligned}$$

together with the following exceptional element

$$L_{n,n-1} = \frac{y\left(xV_{n-3}+V_{n-2}\right)+(-1)^n cx^2\left(yH_{n-5}+H_{n-4}\right)-(-1)^n axH_{n-3}}{V_{n-1}-(-1)^n cx\left(yH_{n-4}+H_{n-3}\right)+(-1)^n aH_{n-2}}.$$

We define also an upper triangular matrix $U_n = [U_{i,j}]_{1\le i,j\le n}$, that has the only nonzero elements on the diagonal, the first superdiagonal, and the last two columns, which are given by

$$\begin{aligned} U_{i,i} &= \frac{V_i}{V_{i-1}}, && 1\le i\le n-2; \\ U_{i,i+1} &= y, && 1\le i\le n-3; \\ U_{i,n-1} &= \frac{(-1)^i}{V_{i-1}}\left[cx\left(yH_{i-3}+H_{i-2}\right)-aH_{i-1}\right], && 1\le i\le n-3; \\ U_{i,n} &= \frac{(-1)^i}{V_{i-1}}\left[dx\left(yH_{i-3}+H_{i-2}\right)-bH_{i-1}\right], && 1\le i\le n-2; \end{aligned}$$

and

$$\begin{aligned} U_{n,n} = &\frac{1}{V_{n-1}+(-1)^n\left[aH_{n-2}-cx\left(yH_{n-4}+H_{n-3}\right)\right]} \\ &\times\left\{V_n-(-1)^n\left[bH_{n-1}-x^{n-2}\left(ad-bc\right)\right.\right. \\ &\left.\left.-x\left(a+d\right)\left(yH_{n-3}+H_{n-2}\right)+c\mathbf{G}_{n-1}\left(x,y\right)\right]\right\}, \end{aligned}$$

where as usual the empty sum is 0 and $\mathbf{G}_n\left(x,y\right)$ is defined as before.

For example, if $n=6$, then we have

$$U = \begin{bmatrix} \frac{V_1}{V_0} & y & 0 & 0 & -\frac{cx-a}{V_0} & -\frac{dx-b}{V_0} \\ 0 & \frac{V_2}{V_1} & y & 0 & \frac{cx-aH_1}{V_1} & \frac{dx-bH_1}{V_1} \\ 0 & 0 & \frac{V_3}{V_2} & y & -\frac{cx(y+H_1)-aH_2}{V_2} & -\frac{dx(y+H_1)-bH_2}{V_2} \\ 0 & 0 & 0 & \frac{V_4}{V_3} & y+\frac{cx(yH_1+H_2)-aH_3}{V_3} & \frac{dx(yH_1+H_2)-bH_3}{V_3} \\ 0 & 0 & 0 & 0 & \frac{V_5-(cx(yH_2+H_3)-aH_4)}{V_4} & y-\frac{dx(yH_2+H_3)-bH_4}{V_4} \\ 0 & 0 & 0 & 0 & 0 & U_{6,6} \end{bmatrix},$$

where the last diagonal element $U_{6,6}$ is

$$U_{6,6} = \frac{V_6-bH_5+x^4\left(ad-bc\right)+x\left(a+d\right)\left(yH_3+H_4\right)-c\mathbf{G}_5\left(x,y\right)}{V_5+aH_4-cx\left(yH_2+H_3\right)}.$$

Consequently we have the following result.

Theorem 11. *With the above defined triangular matrices L and U, the matrix* $\mathrm{F}_n(x,y;a,b,c,d)$ *admits the following LU-decomposition*

$$\mathrm{F}_n(x,y;a,b,c,d) = LU.$$

Consequently, we can evaluate the determinant $\det \mathrm{F}_n(x,y;a,b,c,d)$ in the following corollary.

Corollary 12.

$$\begin{aligned}\det \mathrm{F}_n(x,y;a,b,c,d) &= \det U \\ &= \prod_{k=1}^{n} U_{k,k} = U_{n,n}U_{n-1,n-1}\prod_{k=1}^{n-2}\frac{V_k}{V_{k-1}} \\ &= V_n - (-1)^n \left[bH_{n-1} - x^{n-2}(ad-bc)\right. \\ &\quad \left. - x(a+d)(yH_{n-3}+H_{n-2}) + c\mathbf{G}_{n-1}(x,y)\right].\end{aligned}$$

3. THE THIRD TOEPLITZ MATRIX

In this section we shall investigate a new kind of Toeplitz matrix and evaluate its determinant. For this purpose, it is interesting that we will need to define a kind of polynomials, namely bivariate Delannoy polynomials, whose coefficients will consist of Delannoy numbers. The results will be given could be proven similar to the previous results but we will omit them here. However we shall give generating functions of the bivariate Delannoy polynomials as auxiliary results for the reader's convenience. We also present interesting applications of our results. We will give new combinatorial representations for general second order linear recurrences. Especially we give combinatorial formulas for the Fibonacci and Pell numbers as well as the Chebyshev polynomials.

A Toeplitz matrix is defined as

$$\mathrm{E}_n(x,y) = \left[e_{i-j}\right]_{1\le i,j\le n}: \ e_0 = x,\ e_{-1} = y,\ e_1 = y,\ e_2 = (-1)^j x.$$

Clearly the matrix $\mathrm{E}_n(x,y)$ has the form

$$\mathrm{E}_n(x,y) = \begin{bmatrix} x & y & & & & & \\ y & x & y & & & & \\ x & y & x & \ddots & & & \\ & -x & y & \ddots & y & & \\ & & x & \ddots & x & y & \\ & & & \ddots & y & x & y \\ & & & & (-1)^n x & y & x \end{bmatrix}.$$

The Delannoy numbers $D(n,k)$ are defined as shown

$$D(n,k) = \sum_{j=0}^{n} \binom{n+k-j}{k} \binom{k}{j}.$$

The Delannoy numbers satisfy the recurrence relation

$$D(n,k) = D(n-1,k) + D(n,k-1) + D(n-1,k-1)$$

with $D(0,0) = 1$.

The generating function of the Delannoy numbers is

$$\sum_{n,k\geq 0} D(n,k)\, x^n y^k = \frac{1}{1-x-y-xy}.$$

The square array $D(n,k)$ for $n,k \geq 0$ reads as in the following Table.

1	1	1	1	1	**1**	$\cdots$
1	3	5	7	**9**	11	$\cdots$
1	5	13	**25**	41	61	$\cdots$
1	7	**25**	63	129	231	$\cdots$
1	**9**	41	129	321	681	$\cdots$
1	11	61	231	681	1683	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Table 1

Througout this section, we will focus on the antidiagonal entries, $D(n-k,k)$, of Table 1 given by. We define a kind of bivariate Delannoy polynomials whose coefficients are the Delannoy numbers $D(n-k,k)$ as shown

$$\Gamma_n(x,y) = \sum_{k=0}^{n} D(n-k,k)\, x^{2k} y^{2n-2k} (-1)^{n+k}$$

or clearly

$$\Gamma_n(x,y) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n-j}{k} \binom{k}{j} x^{2k} y^{2n-2k} (-1)^{n+k}.$$

We will frequently use the shortened notation Γ_n instead of $\Gamma_n(x,y)$. Also we denote the polynomial $\Gamma_n(y,x)$ by Φ_n for the sake of brevity.

For further use, define the generating functions of the Delannoy polynomials Γ_n and Φ_n as

$$\mathcal{D}(z) = \sum_{n\geq 0} \Gamma_n z^n \quad \text{and} \quad \mathcal{F}(z) = \sum_{n\geq 0} \Phi_n z^n.$$

Then we have the following result.

Lemma 13.
$$\mathcal{D}\left(z\right)=\frac{1}{1-\left(x^{2}-y^{2}\right)z+x^{2}y^{2}z^{2}}$$
and
$$\mathcal{F}\left(z\right)=\mathcal{D}\left(-z\right).$$

Proof. Consider

$$\begin{aligned}
&\sum_{n\geq0}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n-j}{k}\binom{k}{j}x^{2k}y^{2n-2k}\left(-1\right)^{n+k}z^{n}\\
&=\sum_{k,j\geq0}\binom{k}{j}x^{2k}y^{-2k}\left(-1\right)^{k}\left(-zy^{2}\right)^{j}\sum_{n\geq0}\binom{n-j}{k}\left(-zy^{2}\right)^{n-j}\\
&=\sum_{k,j\geq0}\binom{k}{j}\frac{x^{2k}y^{-2k}\left(-1\right)^{k}\left(-zy^{2}\right)^{k}}{\left(1+zy^{2}\right)^{k+1}}\left(-zy^{2}\right)^{j}\\
&=\frac{1}{\left(1+zy^{2}\right)}\sum_{j\geq0}\left(-zy^{2}\right)^{j}\sum_{k\geq0}\binom{k}{j}\frac{z^{k}x^{2k}}{\left(1+zy^{2}\right)^{k}}\\
&=\frac{1}{1-\left(x^{2}-y^{2}\right)z}\sum_{j\geq0}\frac{\left(-1\right)^{k}\left(xyz\right)^{2k}}{\left(1-\left(x^{2}-y^{2}\right)z\right)^{k}}\\
&=\frac{1}{1-\left(x^{2}-y^{2}\right)z+x^{2}y^{2}z^{2}},
\end{aligned}$$

as claimed.

By the definition of the polynomial Φ_n, we have that $\Phi_n=\Gamma_n\left(y,x\right)$ and by the generating function of $\Gamma_n\left(x,y\right)$, we write

$$\begin{aligned}
\mathcal{F}\left(z\right) &= \frac{1}{1-\left(y^{2}-x^{2}\right)z+x^{2}y^{2}z^{2}}=\frac{1}{1+\left(x^{2}-y^{2}\right)z+x^{2}y^{2}z^{2}}\\
&= \mathcal{D}\left(-z\right),
\end{aligned}$$

as claimed. □

Throughout this section, when we write $k\equiv_2 0,1$ for $k\geq0$, we will frequently assume that $k=2m$ and $k=2m+1$ for $m\geq0$, respectively.

Then we shall state our one of the main claims on the determinant of the matrix $E_n(x,y)$ as follows.

Theorem 14. *For $n>0$,*

$$\mathbf{E}_n(x,y)=\det\mathrm{E}_n(x,y)=\begin{cases}\Gamma_m\left(x,y\right), & n\equiv_2 0;\\ x\Gamma_m\left(x,y\right) & n\equiv_2 1.\end{cases}$$

Its proof could be derived by using the Laplace expansion similar to the earlier results given in the previous section.

Now we will present interesting applications of our results. The well known Pell numbers P_n are defined as

$$P_n = 2P_{n-1} + P_{n-2},$$

where $P_1 = 1$ and $P_2 = 2$.

The generating function of the Pell numbers is given as

$$\mathcal{P}(z) = \sum_{n \geq 0} P_n z^n = \frac{1}{1 - 2z - z^2}.$$

Thus we derive the following relations between the Pell numbers and the sums of Delannoy numbers.

Theorem 15. *For $n > 0$,*

$$P_{n+1} = \sum_{k=0}^{n} D(n-k, k) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n-j}{k} \binom{k}{j}$$

and

$$P_{2n+1} = \sum_{k=0}^{n} D(n-k, k) \, 4^{n-k} = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n-j}{k} \binom{k}{j} 4^{n-k}.$$

Proof. The generating function of Pell numbers is obtained from the generating functions of the Delannoy polynomials Γ_n by taking $x = 1$ and $y = \mathbf{i}$ (imaginary unit, $\mathbf{i} = \sqrt{-1}$), then by combining the above results we see that

$$\Gamma_n(1, \mathbf{i}) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n-j}{k} \binom{k}{j} = P_{n+1}.$$

$\square$

Similarly the generating function of odd indexed Pell numbers is obtained from the generating functions of the Delannoy polynomials Γ_n by taking $x = 1,\ y = 2\mathbf{i}$ (imaginary unit, $\mathbf{i} = \sqrt{-1}$) or $x = 2,\ y = \mathbf{i}$, then by combining the above results we see that

$$\Gamma_n(1, 2\mathbf{i}) = \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n-j}{k} \binom{k}{j} 4^k = P_{2n+1},$$

which is an interesting relation and a new representation for the Pell numbers according to our best literature acknowledgement.

We shall show the relation between the Chebyshev polynomials of the second kind $U_n(x)$ and the Delannoy polynomial $\Gamma_n(x, y)$ in the following result.

Theorem 16. *For $n>0$, the Chebysev polynomials of the second kind $U_n(x)$ has the representation*

$$\begin{aligned} U_n(x) &= \Gamma_n\left(\left(x+\sqrt{x^2+1}\right)^{\frac{1}{2}}, \left(x+\sqrt{x^2+1}\right)^{-\frac{1}{2}}\right) \\ &= \sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n-j}{k}\binom{k}{j}\left(x+\sqrt{x^2+1}\right)^{2k-n}(-1)^{n+k} \\ &= \sum_{k=0}^{n} D(n-k,k)\left(x+\sqrt{x^2+1}\right)^{2k-n}(-1)^{n+k}, \end{aligned}$$

where $\Gamma_n(x,y)$ is the Delannoy polynomial and $D(n-k,k)$ is the Delannoy number that are defined as before.

These relations can also be carried in determinantal representations conversely by the above results.

In that case, for example, we can derive for

$$\mathbf{E}_{2n}(w,w) = \det \mathrm{E}_{2n}(w,w) = \Gamma_n(w,w) = U_n(x),$$

where $w=\sqrt{x+\sqrt{x^2+1}}$.

Specifically,

$$\mathbf{E}_{2n}(z,z) = \det \mathrm{E}_{2n}(z,z) = \Gamma_n(z,z) = F_n,$$

where F_n is nth Fibonacci number and $z=\sqrt{\left(\mathbf{i}+\sqrt{3}\right)/2}$. Equivalently the relation just above could be given in terms of the generalized Delannoy polynomials as

$$F_n = \Gamma_n\left(\sqrt{\left(\sqrt{3}+\mathbf{i}\right)/2}, \sqrt{\left(-\mathbf{i}+\sqrt{3}\right)/2}\right)$$

or clearly

$$F_n = \mathbf{i}^{-n}\sum_{k=0}^{n}\sum_{j=0}^{n-k}\binom{n-j}{k}\binom{k}{j}\left(\frac{\mathbf{i}+\sqrt{3}}{2}\right)^{2k-2}(-1)^{n+k}.$$

In general, we define the general second order recursion $\{H_n\}$ as follows

$$H_{n+1} = aH_n + bH_{n-1}$$

with $H_0=0$ and $H_1=1$.

By considering above relations including the general Delannoy polynomials, we give a similar direction for the general sequence $\{H_n\}$ by the following result which generalizes the above results.

Theorem 17. *For $n > 0$, the general sequence $\{H_n\}$ has the representation*

$$H_{n+1} = \frac{1}{2^n} \sum_{k=0}^{n} \sum_{j=0}^{n-k} \binom{n-j}{k} \binom{k}{j} (a-\Delta)^k (a+\Delta)^{n-k},$$

where $\Delta = \sqrt{a^2 - 4b}$.

The above representations for the general second order recurrence $\{H_n\}$, especially Fibonacci, Pell numbers and Chebyshev polynomials are new according to our best literature acknowledgement.

3.3 The matrix $\mathrm{E}_n(x, y; a, b, c, d)$

In this section, we shall investigate the perturbed generalizations of the Toeplitz matrix $\mathrm{E}_n(x, y)$ which are defined by adding the 2×2 square matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to each matrix at the upper right corner.

As we mentioned before, we now evaluate LU-decomposition and the determinant of $\mathrm{E}_n(x, y; a, b, c, d)$. We will use the Delannoy polynomials while formulating entries of the matrices come from LU-decomposition and determinant of the matrix $E_n(x, y; a, b, c, d)$.

We are going to define a unit lower triangular matrix $L = [L_{i,j}]_{1 \le i,j \le n}$, that has the only nonzero elements on the main diagonal, the first and the second subdiagonals. Its entries are explicitly given by

$$\begin{aligned} L_{i,i} &= 1; \\ L_{i,i-1} &= \begin{cases} 0, & i \equiv_2 1 \quad \& \quad i \ne n; \\ \dfrac{1}{x\Gamma_{m-1}} \left[x^2 y \Gamma_{m-2} + \Gamma_{m-1}\right], & i \equiv_2 0 \quad \& \quad i \ne n; \end{cases} \\ L_{i,i-2} &= \begin{cases} 1, & i \equiv_2 1; \\ -x^2 \Gamma_{m-2}/\Gamma_{m-1}, & i \equiv_2 0; \end{cases} \end{aligned}$$

together with the following exceptional element

$$L_{n,n-1} = \begin{cases} \dfrac{(-1)^m a x \Gamma_{m-1}}{c x^{n-2} + \Gamma_m + (-1)^m a y \Gamma_{m-1}}, & n \equiv_2 1; \\ \\ \dfrac{c x^{n-2} + y\left[(x - (-1)^m a) x \Gamma_{m-2} + \Gamma_{m-1}\right]}{(x - (-1)^m a) \Gamma_{m-1}}, & n \equiv_2 0. \end{cases}$$

For example if $n = 7$, then we have

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{x} & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{x^2\Gamma_0}{\Gamma_1} & \frac{x^2y\Gamma_0+\Gamma_1}{x\Gamma_1} & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-x^2\Gamma_1}{\Gamma_2} & \frac{x^2y\Gamma_1+\Gamma_2}{x\Gamma_2} & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & -\frac{ax\Gamma_2}{cx^5+\Gamma_3-ay\Gamma_2} & 1 \end{bmatrix}.$$

We define also an upper triangular matrix $U_n = [U_{i,j}]_{1\le i,j\le n}$, that has the only nonzero elements on the diagonal, the first superdiagonal, and the last two columns, which are given by

$$\begin{aligned} U_{i,i} &= \begin{cases} x, & i \equiv_2 1 \quad \& \quad i< n-1; \\ \Gamma_m/x\Gamma_{m-1}, & i \equiv_2 0 \quad \& \quad i< n-1; \end{cases} \\ U_{i,i+1} &= y, \qquad i< n-2 \\ U_{i,n-1} &= \begin{cases} (-1)^m a, & i \equiv_2 1 \quad \& \quad i < n-2; \\ \left[cx^{i-1} + (-1)^m ay\Gamma_{m-1}\right]/x\Gamma_{m-1}, & i \equiv_2 0 \quad \& \quad i < n-1; \end{cases} \\ U_{i,n} &= \begin{cases} (-1)^m b, & i \equiv_2 1 \quad \& \quad i < n-1; \\ \left[dx^{i-1} + (-1)^m by\Gamma_{m-1}\right]/x\Gamma_{m-1}, & i \equiv_2 0 \quad \& \quad i < n-1; \end{cases} \end{aligned}$$

and the following amended exceptional entries

$$\begin{aligned} U_{n-2,n-1} &= \begin{cases} \dfrac{cx^{n-3} + (x-(-1)^m a)\, y\Gamma_{m-2}}{x\Gamma_{m-2}}, & n \equiv_2 0; \\ y - (-1)^m a, & n \equiv_2 1; \end{cases} \\ U_{n-1,n-1} &= \begin{cases} x - (-1)^m a, & n \equiv_2 0; \\ \dfrac{\Gamma_m + (-1)^m ay\Gamma_{m-1} + cx^{n-2}}{x\Gamma_{m-1}}, & n \equiv_2 1; \end{cases} \\ U_{n-1,n} &= \begin{cases} y - (-1)^m b, & n \equiv_2 0; \\ \dfrac{dx^{n-2} + (x+(-1)^m b)\, y\Gamma_{m-1}}{x\Gamma_{m-1}}, & n \equiv_2 1; \end{cases} \end{aligned}$$

and for $n \equiv_2 0$,

$$\begin{aligned} U_{n,n} &= \frac{1}{(x-(-1)^m a)\,\Gamma_{m-1}} \\ &\quad \times \big(\Gamma_m + ax\left[\Phi_{m-1} + y^2\Phi_{m-2}\right] \\ &\quad + (-1)^m \left\{by\Gamma_{m-1} - x^{n-2}\left[ad - bc + (-1)^m (cy - dx)\right]\right\}\big) \end{aligned}$$

and for $n \equiv_2 1$,

$$U_{n,n} = \frac{(x + (-1)^m b)\,\Gamma_m - (-1)^{\,m} x^{n-2}\,(ad - bc - (-1)^m\,cx)}{\Gamma_m + (-1)^m\,ay\Gamma_{m-1} + cx^{n-2}}.$$

For instance, we have

$$U_7 = \begin{bmatrix} x & y & 0 & 0 & 0 & a & b \\ 0 & \frac{\Gamma_1}{x\Gamma_0} & y & 0 & 0 & \frac{cx-ay\Gamma_0}{x\Gamma_0} & \frac{dx-by\Gamma_0}{x\Gamma_0} \\ 0 & 0 & x & y & 0 & -a & -b \\ 0 & 0 & 0 & \frac{\Gamma_2}{x\Gamma_2} & y & \frac{cx^3+ay\Gamma_1}{x\Gamma_1} & \frac{dx^3+by\Gamma_1}{x\Gamma_1} \\ 0 & 0 & 0 & 0 & x & a+y & b \\ 0 & 0 & 0 & 0 & 0 & \frac{\Gamma_3-ay\Gamma_2+cx^5}{x\Gamma_2} & \frac{dx^5+(x-b)y\Gamma_2}{x\Gamma_2} \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{(x-b)\Gamma_3+x^5(ad-bc+cx)}{\Gamma_3-ay\Gamma_2+cx^5} \end{bmatrix}.$$

Similar to the previous section, by the help of Lemma 13, we shall give the following result without proof.

Theorem 18. *With the above defined triangular matrices L and U, the matrix $\mathrm{E}_n(x, y; a, b, c, d)$ admits the following LU-decomposition*

$$\mathrm{E}_n(x, y; a, b, c, d) = LU.$$

Consequently, we can evaluate the determinant $\det \mathrm{E}_n(x, y; a, b, c, d)$ in the following corollary.

Corollary 19.

$$\begin{aligned} \det \mathrm{E}_n(x, y; a, b, c, d) &= \det U \\ &= \prod_{k=1}^{n} U_{k,k} = U_{n,n} U_{n-1,n-1} \prod_{k=1}^{n-2} U_{k,k} \\ &= U_{n,n} U_{n-1,n-1} \begin{cases} \Gamma_{m-1}, & n \equiv_2 0; \\ x\Gamma_{m-1}, & n \equiv_2 1. \end{cases} \end{aligned}$$

Especially,

$$\begin{aligned} \det \mathrm{E}_{2m}(x, y; a, b, c, d) = {}& \Gamma_m + ax\left(\Phi_{m-1} + y^2 \Phi_{m-2}\right) + (-1)^{\,m} \\ & \times \left\{by\Gamma_{m-1} - x^{n-2}\left[ad - bc + (-1)^{\,m}(cy - dx)\right]\right\} \end{aligned}$$

and

$$\begin{aligned} \det \mathrm{E}_{2m+1}(x, y; a, b, c, d) = {}& (x + (-1)^m b)\,\Gamma_m \\ & - (-1)^{\,m} x^{n-2}\,(ad - bc - (-1)^m\,cx). \end{aligned}$$

4. A GENERALIZATION OF THE THIRD TOEPLITZ MATRIX

We present a generalization of the Toeplitz matrix $\mathrm{E}_n(x,y)$ defined in the previous section and evaluate its determinant.

We define the Toeplitz matrix $\Psi_n(x,y,\tau)$ as follows

$$\Psi_n(x,y,\tau) = \left[\lambda_{i-j}\right]_{1\leq i,j\leq n} : \ \lambda_0 = x, \ \lambda_{-1} = y, \ \lambda_1 = y$$

and

$$\lambda_2 = \begin{cases} \tau & \text{if } i \text{ is even,} \\ x & \text{if } i \text{ is odd.} \end{cases}$$

When $\tau = (-1)^j x$, the matrix $\Psi_n(x,y,\tau)$ is reduced to the matrix $\mathrm{E}_n(x,y)$. As a special case of the general matrix, we will return to the matrix $\mathrm{E}_n(x,y)$. We evaluate the determinant of the matrix $\Psi_n(x,y,\tau)$ and we don't give its perturbed version. We clarify why we study the matrices $\Psi_n(x,y,\tau)$ and $\mathrm{E}_n(x,y)$ separately? Because the result related with $\Psi_n(x,y,\tau)$ would be different from the result related with $\mathrm{E}_n(x,y)$ since their forms although the general result includes the special result. The reason for this is that we formulate the determinants of these matrices by using different grouping their values.

For example, the matrix $\Psi_6(x,y,\tau)$ has the form

$$\Psi_6(x,y,\tau) = \begin{bmatrix} x & y & 0 & 0 & 0 & 0 \\ y & x & y & 0 & 0 & 0 \\ x & y & x & y & 0 & 0 \\ 0 & \tau & y & x & y & 0 \\ 0 & 0 & x & y & x & y \\ 0 & 0 & 0 & \tau & y & x \end{bmatrix}.$$

We only present our results related with the matrix $\Psi_n(x,y,\tau)$ without proof.

Theorem 20. *For $n \geq 0$,*

$$\begin{aligned} \det \Psi_{4n+1}(x,y,\tau) &= x \det \Psi_{4n}(x,y,\tau) \\ &= \sum_{k=0}^{n} \binom{n+k}{2k} \tau^{n-k} x^{n+1-k} y^{2(n-k)} \left(x^2-y^2\right)^{2k} \end{aligned}$$

and

$$\begin{aligned} \det \Psi_{4n-1}(x,y,\tau) &= x \det \Psi_{4n-2}(x,y,\tau) \\ &= \sum_{k=1}^{n} \binom{n+k-1}{2k-1} \tau^{n-k} x^{n+1-k} y^{2(n-k)} \left(x^2-y^2\right)^{2k-1}. \end{aligned}$$

If we take $\tau = -x$, the matrix $\Psi_n(x,y,\tau)$ equals the matrix $\mathrm{E}_n(x,y)$. In the previous section we already derive a formula for $\det \mathrm{E}_n(x,y)$. And we derive a formula for $\det \mathrm{E}_n(x,y)$ from Theorem 20 by taking $\tau=-x$. Then we equalize these formulas and get a combinatorial identity for later use. Since special case of $\det \mathrm{E}_n(x,y)$ equals to the Delannoy polynomials, we also obtain a new combinatorial representation for them by using the combinatorial identity we derived.

Therefore by taking $\tau=-x$ and expanding the powers of $\left(x^2-y^2\right)$ in the formulas of $\det \Psi_n$ and rearranging them, we have that for $n \geq 0$,

$$\begin{aligned}
&\det \Psi_{4n+1}(x,y,-x) = x \det \Psi_{4n}(x,y,-x)\\
&= x^{2n+1} y^{2n} (-1)^n \sum_{k=0}^{n} \sum_{j=0}^{2k} \binom{2k}{j} \binom{n+k}{2k} (-1)^{k+j} \left(xy^{-1}\right)^{2k-2j}
\end{aligned}$$

and

$$\begin{aligned}
&\det \Psi_{4n-1}(x,y,-x) = x \det \Psi_{4n-2}(x,y,-x)\\
&= \left(x^2-y^2\right) x^{2n-1} y^{2n-2} (-1)^{n-1} \sum_{k=0}^{n-1} \sum_{j=0}^{2k} \binom{n+k}{2k+1} \binom{2k}{j} (-1)^{k+j} \left(xy^{-1}\right)^{2k-2j}.
\end{aligned}$$

By combining these results and the results of Theorem 14, we reach at the following result.

Theorem 21. *For $n \geq 0$,*

$$\begin{aligned}
&\sum_{k=0}^{n} \sum_{j=0}^{2k} \binom{n+k}{2k} \binom{2k}{j} (-1)^{k+j} \left(xy^{-1}\right)^{2k-2j}\\
&= \sum_{k=0}^{2n} \sum_{j=0}^{2n-k} \binom{2n-j}{k} \binom{k}{j} \left(x^{-1}y\right)^{2k-2n} (-1)^{n+k}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{k=0}^{2n-1} \sum_{j=0}^{2n-1-k} \binom{2n-1-j}{k} \binom{k}{j} \left(xy^{-1}\right)^{2k} (-1)^{n+k}\\
&= \left(x^2y^{-2}-1\right) \sum_{k=0}^{n} \sum_{j=0}^{2k} \binom{2k}{j} \binom{n+k}{2k+1} (-1)^{k+j} \left(xy^{-1}\right)^{2k-2j+2n-2}.
\end{aligned}$$

Thus by using the equalities above, we have new combinatorial representations for the generalized Delannoy polynomials by the following result.

Corollary 22. *For $n > 0$,*

$$\Gamma_{2n}(x,y) = (-1)^n (xy)^{2n} \sum_{k=0}^{n} \sum_{j=0}^{2k} \binom{2k}{j} \binom{n+k}{2k} (-1)^{k+j} \left(xy^{-1}\right)^{2k-2j}$$

and

$$\Gamma_{2n}(x,y) = (-1)^{n+1} y^{4n-2} \left(x^2y^{-2}-1\right) \times \sum_{k=0}^{n}\sum_{j=0}^{2k}\binom{2k}{j}\binom{n+k}{2k+1}(-1)^{k+j}\left(xy^{-1}\right)^{2k-2j+2n-2}.$$

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REFERENCES

1. J. Borowska, L. Lacinska: *Eigenvalues of* 2*-tridiagonal Toeplitz matrix.*, J. Appl. Math. Comput. Mechanics **14** (2015), 11–17.

2. A. Bunse-Gerstner, R. Byers, V. Mehrmann: *A chart of numerical methods for structured eigenvalue problems.* SIAM J. Matrix Anal. Appl. **13** (1992), 419–453.

3. R. A. Bustos-Marún, E. A. Coronado, H. M. Pastawski: *Buffering plasmons in nanoparticle wave guides at the virtual-localized transition.* Phys. Rev. B **82** (2010), 035434.

4. B. K. Choudhury: *Diffusion of heat in multidimensional composite spherical body.* IMA J. Appl. Math. **78** (2013), 474–493.

5. W. Chu: *Spectrum and eigenvectors for a class of tridiagonal matrices.* Linear Algebra Appl. **582** (2019), 499–516.

6. W. Chu, X. Wang: *Eigenvectors of tridiagonal matrices of Sylvester type.* Calcolo **45** (2008), 217–233.

7. S. Dursun, A. M. Grigoryan: *Nonlinear l_2-by-3 transform for PAPR reduction in OFDM systems.* Computers & Electrical Engin. **36** (2010), 1055–1065.

8. C. F. Fischer, R. A. Usmani: *Properties of some tridiagonal matrices and their application to boundary value problems.* SIAM J. Numer. Anal. **6** (1969), 127–142.

9. P.-L. Giscard, S. J. Thwaite, D. Jaksch: *Evaluating matrix functions by resummations on graphs: The method of path-sums.* SIAM. J. Matrix Anal. & Appl. **34** (2013), 445–469.

10. W. W. Hager: *Applied numerical linear algebra, Prentice-Hall International Editions* Englewood Cliffs, NJ, 1988.

11. T. Hopkins, E. Kiliç: *An analytical approach: Explicit inverses of periodic tridiagonal matrices.* J. Comput. Appl. Math. **335** (2018), 207–226.

12. Z. Jiang, T-Y. Tam, Y. Wang: *Inversion of conjugate Toeplitz matrices and conjugate-Hankel matrices.* Linear and Multilinear Algebra, **65** (2017), 256–268.

13. E. Kiliç, P. Stanica: *The inverse of banded matrices.* J. Comput. Appl. Math. **237** (2013), 126–135.

14. E. Kiliç, P. Stanica: *Factorizations and representations of binary polynomial recurrences by matrix methods.* Rocky Mount. J. Math. **41** (2011), 1247–1264.

15. E. Kılıç: *Explicit formula for the inverse of a tridiagonal matrix by backward continued fractions.* Appl. Mat. Comput. **197** (2008), 345–357.

16. M. Merca: *A note on the determinant of a Toeplitz-Hessenberg matrix.* Special Matrices, **1** (2013), 10–16.

17. J. W. Kim: *Quasi-disjoint pentadiagonal matrix systems for the parallelization of compact finite-difference schemes and filters.* J. Comput. Physics **241** (2013), 168–194.

18. B. Kurmanbek, Y. Amanbek, Y. Erlangga: *A proof of Anđelić–Fonseca conjectures on the determinant of some Toeplitz matrices and their generalization.* Linear and Multilinear Algebra **70** (2022), 1563–1570

19. G. Meurant: *A review on the inverse of symmetric tridiagonal and block tridiagonal matrices.* SIAM J. Matrix Anal. Appl. **13** (1992), 707–728.

20. N. Perel, U. Yechiali: *The Israeli queue with a general group-joining policy.* Ann. Oper. Res. **317** (2022), 1–34

21. M. Püschel, J. M.F. Moura: *The algebraic approach to the discrete cosine and sine transforms and their fast algorithms.* SIAM J. Comput. **32** (2003), 1280–1316.

22. G. Strang: *The discrete cosine transform.* SIAM Rev. **41** (1999), 135–147.

23. S.-F. Xu: *On the Jacobi matrix inverse eigenvalue problem with mixed given data.* SIAM J. Matrix Anal. Appl. **17** (1996), 632–639.

24. W. C. Yueh, S. S. Cheng: *Explicit eigenvalues and inverses of tridiagonal Toeplitz matrices with four perturbed corners.* ANZIAM J. **49** (2008), 361–387.

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