Applicable Analysis and Discrete Mathematics
available online at http://pefmath.etf.rs

Appl. Anal. Discrete Math. **x** (xxxx),xxx–xxx.
https://doi.org/10.2298/AADM220603003M

SMOOTH SQUARED, TRIANGULAR, AND HEXAGONAL BARGRAPHS

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In this paper, we find an explicit formula for the generating function for the number of smooth squared (triangular, hexagonal) bargraphs according to the perimeter and number of columns. In particular, we show that the number of smooth squared, triangular, and hexagonal bargraphs with perimeter $2n$ (resp. n, $2n$) is asymptotic to $\frac{c_s r_s^{1-n}}{\sqrt{\pi n^3}}$ (resp. $\frac{c_t r_t^{1-n}}{\sqrt{\pi n^3}}$, $\frac{c_h}{\sqrt{\pi n^3}}\sqrt{2}^{n+2}$), where $r_s=\frac{1+\sqrt[3]{181+24\sqrt{78}}}{12}-\frac{23}{12\sqrt[3]{181+24\sqrt{78}}}$, r_t is the smallest positive root of the polynomial $p^{16}-2p^{14}+p^{12}-2p^{11}-2p^{10}+2p^9+4p^8-5p^6-2p^5+p^4-2p^3-2p^2+1$ and c_s,c_t,c_h are three constants, as $n\mapsto\infty$.

1. INTRODUCTION

An *animal* on a two-dimensional lattice is an edge-connected set of basic two-dimensional polygon-like cells on a two-dimensional lattice, where the connectivity of two cells is defined by having a common edge. An example of animals are squared polyominoes or just *polyominoes*. These are animals that are made by edge-connected squares on the squared lattice and are well-studied in discrete geometry, statistical physics, and combinatorics. For instance, they are used in the enumeration of graphs [22], modeling the mechanics of macro-molecules [28, 31], percolation processes [10], and cell growth processes [15, 24, 29]. In combinatorics, the main problem is to enumerate lattice animals of a specific size n; see [2, 4, 5, 12, 21, 30, 32] and references therein. Since the enumeration of general animals is well-known as

2020 Mathematics Subject Classification. 05A05, 05A15.
Keywords and Phrases. Bargraphs, Hexagonal bargraphs, Smooth bargraphs, Squared bargraphs, Triangular bargraphs

a hard problem, researchers restricted the study of animals with various conditions such as directional convexity and/or directional growth in squared and hexagonal lattices. See [2, 3, 6, 8, 11, 13, 14, 23] for a few examples of the squared lattice and [2,16,17,20,25,33] for the hexagonal lattice, where several important subsets of animals including *bargraphs*, *column-convex animals,* and *convex animals* are studied. Although a natural extension to these works is to explore the animals residing on the triangular lattice [2, 26], we were not able to find much work in this direction. Therefore, in this paper, we study *smooth bargraphs* in squared, triangular, and hexagonal lattices.

1.1 Squared bargraphs

Let $\mathcal{S} = \mathbb{Z}^2$ be the two-dimensional plane, where the angle between x-axis and y-axis is $\frac{\pi}{2}$ counter clock-wise and we mark these axes with integer points. Then, we partition $\mathcal{S}$ with squares, each having two edges parallel to the x-axis, two edges parallel to the y-axis, and the edges of length one. We refer to this dimensional plane as a *squared lattice* and to these squared building blocks as *squared cells* or simply *cells*. Let the line $x = j$ (resp. $y = j$) denote the line parallel with y-axis (resp. x-axis), intersecting x-axis (resp. y-axis) at j.

A *squared bargraph* B or simple *bargraph* is an animal in $\mathcal{S}$ with m columns, where the j-th column of B has a_j cells between the lines $x = j - 1$ and $x = j$ and between the lines $y = 0$ and $y = a_j$. A cell in B with at least one edge in common with a cell in $B^c := \mathcal{S} \setminus B$ is referred to as a *boundary cell* of B and the common edge is referred to as a *boundary edge*. The *perimeter* of B is the total number of its boundary edges. For example, Figure 1 presents a bargraph with 16 cells and a perimeter 28.

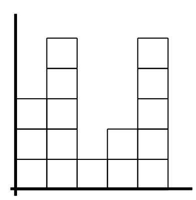

Figure 1: A squared bargraph

Bargraphs referred to as wall polyominoes or skylines [19], have recently been studied from several directions and have led to various refined enumerations; see [27] and references therein. The enumeration of bargraphs by their perimeter is studied in [9]. In [7] the inner site-perimeter (the number of all cells in the bargraph that have at least one boundary edge) of bargraphs is considered.

1.2 Triangular bargraphs

Let $\mathcal{T}$ be the two-dimensional plane, where the angle between x-axis and y-axis is $\frac{\pi}{3}$ counter clock-wise and we mark these axes with integer points. Then, we partition $\mathcal{T}$ with equilateral triangles; each with a horizontal edge parallel to the x-axis and the sides of length one. We refer to this dimensional plane as a *triangular lattice* and to these triangular building blocks as *triangular cells* or simply *cells*. Let the line $x = j$ (resp. $y = j$) denote the line parallel with y-axis (resp. x-axis), intersecting x-axis (resp. y-axis) at j.

A *triangular bargraph* B or simple *bargraph* is an animal in $\mathcal{T}$ with m columns, where the j-th column of B has a_j cells between the lines $x = j - 1$ and $x = j$ and between the lines $y = 0$ and $y = \lceil a_j/2 \rceil$. Depending on the orientation of the top cell of B, we identify two types of columns as seen in Figure 2. A cell in B with at

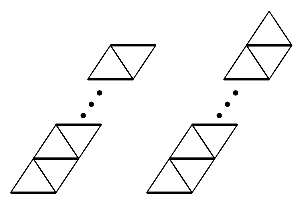

Figure 2: From left to right, columns of type one and two

least one edge in common with a cell in $B^c := \mathcal{T} \setminus B$ is referred to as a *boundary cell* of B and the common edge is referred to as a *boundary edge*. The *perimeter* of B is the total number of its boundary edges. For example, Figure 3 presents a triangular bargraph with 23 cells and perimeter 15.

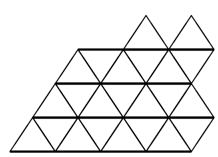

Figure 3: A triangular bargraph

There is a limited research on triangular bargraphs. However, in [26], the examination of the number of triangular bargraphs, as well as other various families of triangular animals, according to the perimeter is considered.

1.3 Hexagonal bargraphs

A *hexagonal lattice* $\mathcal{H}$ is a regular tiling of $\mathbb{R}^2$ such that exactly three hexagons meet at each vertex and the vertices of each hexagon are (a, b), $(a + 1, b)$, $(a + 2, b + 1)$,

$(a+1,b+2)$, $(a,b+2)$, and $(a-1,b+1)$, for some $a,b\in\mathbb{Z}$ such that the vertices of the leftmost lower hexagon in the first quarter of the plane $\mathbb{R}^2$ are $(1,0)$, $(2,0)$, $(3,1)$, $(2,2)$, $(1,2)$, and $(0,1)$. We refer to these hexagons as *hexagonal cells* or simply *cells*. Let the line $x=j$ (resp. $y=j$) denote the line parallel with y-axis (resp. x-axis), intersecting x-axis (resp. y-axis) at j. A *hexagonal bargraph* B or simple *bargraph* is an animal in $\mathcal{H}$ with m columns, where the j-th column of B has a_j cells between the lines $x=2j-2$ and $x=2j+1$, and between the lines $y=0$ and $y=2a_j$ when j is odd, $y=1$ and $y=2a_j+1$ when j is even.

A cell in B with at least one edge in common with a cell in $B^c:=\mathcal{H}\setminus B$ is referred to as a *boundary cell* of B and the common edge is referred to as a *boundary edge*. The *perimeter* of B is the total number of its boundary edges. For example, Figure 4 presents a hexagonal bargraph with 9 cells and perimeter 32.

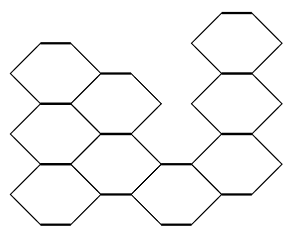

Figure 4: A hexagonal bargraph

As we discussed in the introduction, on hexagonal animals several important subsets of animals including *bargraphs*, *column-convex animals*, and *convex animals* are considered (see [2, 16, 17, 20, 25, 33]).

1.4 Smooth bargraphs

A squared (triangular, hexagonal) bargraph B is said to be *smooth* if the difference between the number of cells in two adjacent columns is either 0 or 1. In other words, if the highest point in j-th column of B lies at line $y=a_j$ then $|a_j-a_{j-1}|=0,1$. The paper aims to find an explicit formula for the generating function for the number of smooth squared (triangular, hexagonal) bargraphs with respect to their perimeter. More precisely, we show the following result.

Theorem 1. *We have (1) the generating function for the number of smooth squared bargraphs with at least one column according to the perimeter and the number of columns is given by*

$$\begin{aligned}S(p,q)&=\frac{p^2q(1+p^2-p^2q-5p^4q)}{2(1-p^2)(1-2p^2q-p^4q)^2}-\frac{p^2q\sqrt{(1-p^2q)^2-4p^6q^2}}{2(1-2p^2q-p^4q)^2}\\&=qp^4+(q^2+q)p^6+(q^3+3q^2+q)p^8+(q^4+6q^3+3q^2+q)p^{10}+\cdots.\end{aligned}$$

Moreover, the number of smooth squared bargraphs with perimeter $2n$ in the squared lattice is asymptotic to

$$\frac{\sqrt{r^2-4r+3}(1840r^2+336r+1065)r^{1-n}}{16\sqrt{\pi n^3}}$$

with $r=\frac{1+\sqrt[3]{181+24\sqrt{78}}}{12}-\frac{23}{12\sqrt[3]{181+24\sqrt{78}}}$ *as* $n\mapsto\infty$.

(2) the generating function for the number of smooth triangular bargraphs with at least one column according to the perimeter and the number of columns is given by

$$T(p,q)=\frac{qp^3(1+p^5q-p^6q)}{f(p,q)}\left((p+1)^2qpu_0-(p+1)p^6q^2+\frac{1-pq-3p^2q-p^3q}{1-p}\right),$$

where $f(p,q)=((p^2-1)p^6q^2-(p^3+p^2+2p+1)pq+1)(p^8q^2-p^6q^2-p^4q-2p^3q-2p^2q+1)$ *and*

$$u_0=\frac{(p^2-1)p^6q^2-(p+1)p^2q+1-\sqrt{((p^2-1)p^6q^2-(p+1)p^2q+1)^2-4q^2p^5(p+1)}}{2p^3q(p+1)}.$$

Moreover, the number of smooth triangular bargraphs with perimeter n is asymptotic to

$$\frac{(r^5(r-1)^2-r+1)\sqrt{A_r}r^{1-n}}{2B_r\sqrt{2\pi n^3}}$$

with

$$\begin{aligned}A_r=&2r^{12}-2r^{14}-5r^{11}-6r^{10}+7r^9+16r^8-25r^6-11r^5+6r^4-13r^3-14r^2+8,\\B_r=&2r^{15}-2r^{14}-2r^{13}+4r^{12}-3r^{11}-2r^{10}-2r^9+3r^8+4r^7-9r^6-r^5\\&+3r^4+r^3-2r^2-3r+2,\end{aligned}$$

as $n\mapsto\infty$, *where* $r\approx 0.540323274$ *is the smallest positive root of the polynomial* $p^{16}-2p^{14}+p^{12}-2p^{11}-2p^{10}+2p^9+4p^8-5p^6-2p^5+p^4-2p^3-2p^2+1$.

(3) The generating function for the number of smooth hexagonal bargraphs with at least one column according to the perimeter and number of columns is given by

$$H(p,q)=\frac{p^6q(1+p^2)}{1-p^6q^2(1+p^2)^2}\left(\frac{1+p^4q(1+p^2)}{1-p^4}-\frac{p^6q^2C(p^8q^2)}{1-p^4qC(p^8q^2)}\right),$$

where $C(x)=\frac{1-\sqrt{1-4x}}{2x}$ *is the generating function for the Catalan numbers* $\frac{1}{n+1}\binom{2n}{n}$. *Moreover, the number of smooth triangular bargraphs with perimeter $2n$, is asymptotic to*

$$\frac{(304+215\sqrt{2})\sqrt{2}^{n+2}}{\sqrt{\pi n^3}}.$$

The proof of this theorem is given in the next three sections.

2. SMOOTH SQUARED BARGRAPHS

Let $S(p,q)$ be the generating function for the number of nonempty smooth squared bargraphs according to the perimeter and number of columns. In order to obtain an explicit formula for $S(p,q)$, we define $S_a(p,q)$ to be the generating function for the number of smooth squared bargraphs according to the perimeter and number of columns such that the first column has exactly a square cells. Note that each smooth squared bargraph such that the first column has exactly a cells can be decomposed as either (1) has exactly one column, or (2) has at least two columns where the second column has exactly $j \in \{a-1, a, a+1\}$ cells, see Figure 5. Thus,

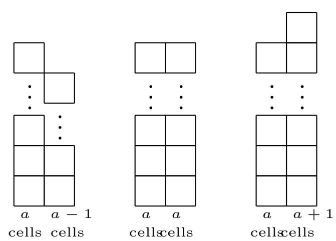

Figure 5: Decomposition of a smooth squared bargraph such that the first column has exactly a cells

$$S_a(p,q) = p^{2a+2}q + p^4qS_{a-1}(p,q) + p^2qS_a(p,q) + p^2qS_{a+1}(p,q), \tag{1}$$

for all $a \geq 1$, where $S_0(p,q)$ is defined to be 0. Define

$$S(p,q;u) = \sum_{a\geq 1} S_a(p,q)u^{a-1}.$$

Then, by multiplying (1) by u^{a-1} and summing over $a \geq 1$, we obtain

$$\begin{aligned} S(p,q;u) = {} & \frac{p^4q}{1-p^2u} + p^4quS(p,q;u) + p^2qS(p,q;u) \\ & + \frac{p^2q}{u}(S(p,q;u) - S_1(p,q)), \end{aligned} \tag{2}$$

This functional equation that can be solved by the kernel method [1]. To do so, we re-arrange (2) as

$$\left(1 - p^4qu - p^2q - \frac{p^2q}{u}\right) S(p,q;u) = \frac{p^4q}{1-p^2u} - \frac{p^2q}{u}S_1(p,q). \tag{3}$$

By setting

$$u = u_0(p,q) = \frac{1 - p^2q - \sqrt{(1-p^2q)^2 - 4p^6q^2}}{2p^4q}$$

into (3), we have

$$S_1(p,q) = \frac{p^2u_0(p,q)}{1 - p^2u_0(p,q)} = \frac{2p^4q + p^2q - 1 + \sqrt{(1-p^2q)^2 - 4p^6q^2}}{2(1 - 2p^2q - p^4q)}.$$

Hence, by (3) with $u = 1$, we have the following result.

Theorem 2. *The generating function for the number of smooth squared bargraphs with at least one column according to the perimeter and number of columns is given by*

$$\begin{aligned} S(p,q) &= \frac{p^2q(1+p^2-p^2q-5p^4q)}{2(1-p^2)(1-2p^2q-p^4q)^2} - \frac{p^2q}{2(1-2p^2q-p^4q)^2}\sqrt{(1-p^2q)^2-4p^6q^2} \\ &= qp^4 + (q^2+q)p^6 + (q^3+3q^2+q)p^8 + (q^4+6q^3+3q^2+q)p^{10} + \cdots . \end{aligned}$$

By singularity analysis (see for instance [18]), we see that the coefficient of p^n in

$$S(\sqrt{p},1) = \frac{p(1-5p^2)}{2(1-p)(1-2p-p^2)^2} - \frac{p}{2(1-2p-p^2)^2}\sqrt{(1-p)^2-4p^3},$$

namely, the number of smooth squared bargraphs with perimeter $2n$, is asymptotic to

$$\frac{\sqrt{r^2-4r+3}(1840r^2+336r+1065)r^{1-n}}{16\sqrt{\pi n^3}}$$

with $r = \frac{1}{12}\sqrt[3]{181+24\sqrt{78}} - \frac{23}{12\sqrt[3]{181+24\sqrt{78}}} + \frac{1}{12}$, as $n \mapsto \infty$.

3. SMOOTH TRIANGULAR BARGRAPHS

Let $T(p,q)$ be the generating function for the number of nonempty smooth triangular bargraphs according to the perimeter and number of columns. In order to obtain an explicit formula for $T(p,q)$, we define $T_a^{(i)}(p,q)$, $i = 1,2$, to be the generating function for the number of smooth triangular bargraphs according to the perimeter and number of columns such that the first column has exactly a cells and its type is i (see Figure 2). Define $T^{(i)}(p,q;u) = \sum_{a\geq 1} T_a^{(i)}(p,q)u^{a-1}$ for $i = 1,2$. Clearly,

$$T(p,q) = T^{(1)}(p,q;1) + T^{(2)}(p,q;1). \tag{4}$$

By considering the number of cells in the second column (if it exists) of a smooth triangular bargraph such that the column has a type 1 and exactly a cells (see Figures 6), we obtain

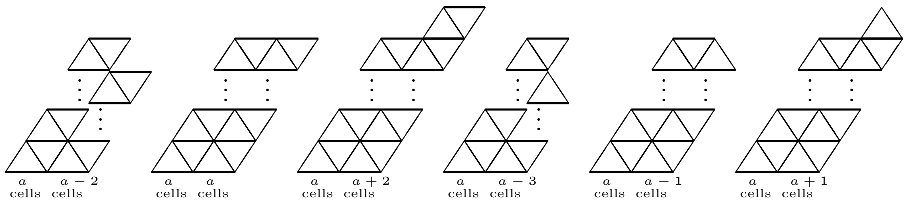

Figure 6: Decomposition of a smooth triangular bargraph such that the first column has type 1 and exactly a cells

$$\begin{aligned}T_a^{(1)}(p,q) &= p^{a+2}q + p^4qT_{a-2}^{(1)}(p,q) + p^2qT_a^{(1)}(p,q) + p^2qT_{a+2}^{(1)}(p,q)\\ &\quad + p^4qT_{a-3}^{(2)}(p,q) + p^2qT_{a-1}^{(2)}(p,q) + p^2qT_{a+1}^{(2)}(p,q).\end{aligned}$$

By considering the number of cells in the second column (if it exists) in a smooth triangular bargraph such that the column has a type 2 and exactly a cells (see Figures 6), we obtain

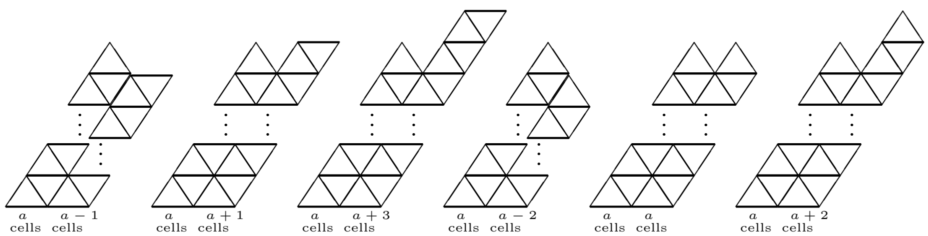

Figure 7: Decomposition of a smooth triangular bargraph such that the first column has type 2 and exactly a cells

$$\begin{aligned}T_a^{(2)}(p,q) &= p^{a+2}q + p^3qT_{a-1}^{(1)}(p,q) + p^3qT_{a+1}^{(1)}(p,q) + p^3qT_{a+3}^{(1)}(p,q)\\ &\quad + p^3qT_{a-2}^{(2)}(p,q) + p^3qT_{a}^{(2)}(p,q) + p^3qT_{a+2}^{(2)}(p,q).\end{aligned}$$

By multiplying $T_a^{(1)}(p,q)$ (respectively, $T_a^{(2)}(p,q)$) by $u^{(a-2)/2}$ (respectively, $u^{(a-1)/2}$) and summing over all $a = 2b \geq 2$ (respectively, $a = 2b+1 \geq 1$) (here, we define $T_0^{(1)}(p,q) = T_0^{(2)}(p,q) = 0$), we obtain

$$\begin{aligned}T^{(1)}(p,q;u) &= \frac{p^4q}{1-p^2u} + p^4quT^{(1)}(p,q;u) + p^2qT^{(1)}(p,q;u)\\ &\quad + \frac{p^2q}{u}(T^{(1)}(p,q;u) - T^{(1)}(p,q;0)) + p^4quT^{(2)}(p,q;u)\\ &\quad + p^2qT^{(2)}(p,q;u) + \frac{p^2q}{u}(T^{(2)}(p,q;u) - T^{(2)}(p,q;0)) \qquad (5)\end{aligned}$$

and

$$\begin{aligned}T^{(2)}(p,q;u) &= \frac{p^3q}{1-p^2u} + p^3quT^{(1)}(p,q;u) + p^3qT^{(1)}(p,q;u)\\ &\quad + \frac{p^3q}{u}(T^{(1)}(p,q;u) - T^{(1)}(p,q;0)) + p^3quT^{(2)}(p,q;u)\\ &\quad + p^3qT^{(2)}(p,q;u) + \frac{p^3q}{u}(T^{(2)}(p,q;u) - T^{(2)}(p,q;0)).\end{aligned}\tag{6}$$

By considering $p\cdot(5)$-(6), we obtain

$$T^{(2)}(p,q;u) = \frac{p^3q(1-p^2)}{(1-p^2u)(1-p^3qu+p^5qu)} + \frac{p(1+p^2qu-p^4qu)}{2-p^3qu+p^5qu}T^{(1)}(p,q;u).\tag{7}$$

Hence, (5) can be written as

$$\begin{aligned}&\frac{-p^2q + (p^8q^2 - p^6q^2 - p^3q - p^2q + 1)u - p^3q(p+1)u^2}{u(1-p^3qu+p^5qu)}T^{(1)}(p,q;u)\\ &= -\frac{p^2q}{u}T^{(1)}(p,q;0) + \frac{p^4q((p^5q-p^3q-1)(p^3q-pq-1) - q^2p^6(p^2-1)^2u)}{(1-p^2u)(1-p^3qu+p^5qu)}.\end{aligned}\tag{8}$$

By setting

$$\begin{aligned}u_0 = u(p,q) &= \frac{p^8q^2 - p^6q^2 - p^3q - p^2q + 1}{2p^3q(p+1)}\\ &\quad - \frac{\sqrt{(p^8q^2 - p^6q^2 - p^3q - p^2q + 1)^2 - 4q^2p^5(p+1)}}{2p^3q(p+1)}\end{aligned}$$

into (8), we have

$$\begin{aligned}T^{(1)}(p,q;0) &= \frac{p^2q(p+1)(p^6q - p^5q - 1)}{p^8q^2 - p^6q^2 - p^4q - p^3q - 2p^2q - pq + 1}u_0\\ &\quad + \frac{p^4q(p^8q^2 - p^7q^2 - p^6q^2 + p^5q^2 - p^5q - p^2q + pq + q + 1)}{p^8q^2 - p^6q^2 - p^4q - p^3q - 2p^2q - pq + 1}.\end{aligned}$$

Hence, by (8) and then by (7), we obtain explicit formulas for the generating function $T^{(1)}(p,q;u)$ and $T^{(2)}(p,q;u)$, which, by (4), leads to the following result.

Theorem 3. *The generating function for the number of smooth triangular bargraphs with at least one column according to the perimeter and number of columns is given by*

$$\begin{aligned}T(p,q) &= \frac{qp^3(1+p^5q-p^6q)}{f(p,q)}\left((p+1)^2qpu_0 - (p+1)p^6q^2 + \frac{1-pq-3p^2q-p^3q}{1-p}\right),\\ &= qp^3 + qp^4 + q(q+1)p^5 + q(3q+1)p^6 + q(q^2+5q+1)p^7 + \cdots,\end{aligned}$$

where $f(p,q) = (p^8q^2 - p^6q^2 - p^4q - p^3q - 2p^2q - pq + 1)(p^8q^2 - p^6q^2 - p^4q - 2p^3q - 2p^2q + 1)$.

By singularity analysis (see for instance [18]), we see that the coefficient of p^n in

$$T(p,1) = \frac{p(p^6-p^5-1)\sqrt{(p^8-p^6-p^3-p^2+1)^2-4p^5(p+1)}}{2(p+1)^2(p^6-2p^5+2p^4-2p^3+p^2-2p+1)(p^7-p^6-p^3-2p+1)}$$
$$+\frac{p(p^6-p^5-1)(p^9-p^8+p^7-p^6-p^4-4p^3+p^2-p+1)}{2(p+1)^2(p^7-p^6-p^3-2p+1)(p-1)(p^6-2p^5+2p^4-2p^3+p^2-2p+1)},$$

namely the number of smooth triangular bargraphs with perimeter n, is asymptotic to

$$\frac{(r^5(r-1)^2-r+1)\sqrt{A_r}r^{1-n}}{2B_r\sqrt{2\pi n^3}} \approx \frac{3013.3562752348\cdot r^{1-n}}{4\sqrt{\pi n^3}}$$

with

$$\begin{aligned}
A_r =& 2r^{12}-2r^{14}-5r^{11}-6r^{10}+7r^9+16r^8-25r^6-11r^5+6r^4-13r^3-14r^2+8,\\
B_r =& 2r^{15}-2r^{14}-2r^{13}+4r^{12}-3r^{11}-2r^{10}-2r^9+3r^8+4r^7-9r^6-r^5\\
&+3r^4+r^3-2r^2-3r+2,
\end{aligned}$$

as $n\mapsto\infty$, where $r\approx 0.540323274$ is the smallest positive root of the polynomial $p^{16}-2p^{14}+p^{12}-2p^{11}-2p^{10}+2p^9+4p^8-5p^6-2p^5+p^4-2p^3-2p^2+1$.

4. SMOOTH HEXAGONAL BARGRAPHS

Let $H(p,q)$ be the generating function for the number of nonempty smooth hexagonal bargraphs according to the perimeter and number of columns. In order to obtain an explicit formula for $H(p,q)$, we define $H_a(p,q)$ to be the generating function for the number of smooth hexagonal bargraphs according to the perimeter and number of columns such that the first column has exactly a cells. By considering the number of cells in the second column of a hexagonal bargraph such that the number of cells in the first column is exactly a (see Figure 8), we obtain

$$H_a(p,q) = qp^{4a+2} + qp^6H'_{a-1}(p,q) + qp^4H'_a(p,q)$$

and

$$H'_a(p,q) = qp^{4a+2} + qp^4H_a(p,q) + qp^2H_{a+1}(p,q),$$

where $H'_a(p,q)$ is the generating function for the number of nonempty shifted smooth hexagonal bargraphs according to the perimeter and number of columns. A *shifted smooth hexagonal bargraph* is a nonempty smooth hexagonal bargraph after removing its first column.

Define $H(p,q;u)=\sum_{a\geq1}H_a(p,q)u^{a-1}$ and $H'(p,q;u)=\sum_{a\geq1}H'_a(p,q)u^{a-1}$. Then, by multiplying the above recurrences by u^{a-1} and summing over $a\geq1$, we

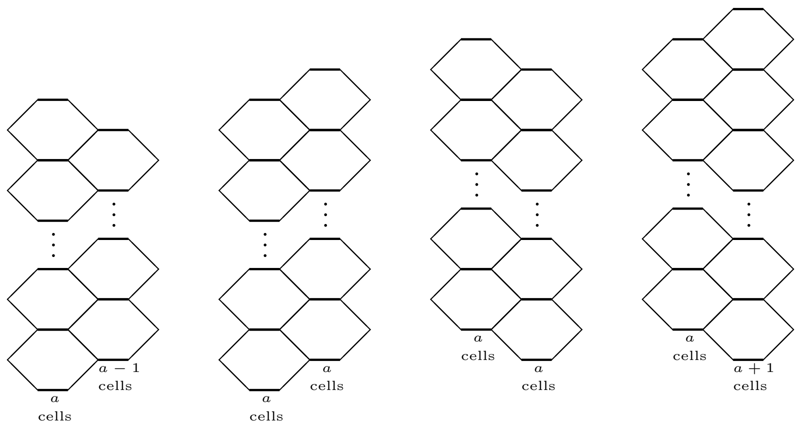

Figure 8: Decomposition of a smooth hexagonal bargraph such that the first column has exactly a cells

obtain

$$\begin{aligned} H(p,q;u) &= \frac{qp^6}{1-p^4u} + qp^6uH'(p,q;u) + qp^4H'(p,q;u), \\ H'(p,q;u) &= \frac{qp^6}{1-p^4u} + qp^4H(p,q;u) + \frac{qp^2}{u}(H(p,q;u) - H(p,q;0)). \end{aligned}$$

By expressing $H'(p,q;u)$ in terms of $H(p,q;u)$, and then substituting it into the second equation, we have

$$\frac{-p^6q^2 + (1-2p^8q^2)u - p^{10}q^2u^2}{p^4qu(1+p^2u)} H(p,q;u) = -\frac{p^2q}{u}H(p,q;0) + \frac{p^2(1+p^4q+p^6qu)}{1-p^4u^2}. \tag{9}$$

Let $C(x) = \frac{1-\sqrt{1-4x}}{2x}$ be the generating function for the Catalan numbers $\frac{1}{n+1}\binom{2n}{n}$. By setting $u_0 = u(p,q) = p^6q^2C^2(p^8q^2)$ into (9), we have

$$H(p,q;0) = \frac{p^6qC(p^8q^2)}{1-p^4qC(p^8q^2)}.$$

Hence, by (9), we have the following result.

Theorem 4. *The generating function for the number of smooth hexagonal bargraphs with at least one column according to the perimeter and number of columns is given by*

$$H(p,q) = \frac{p^6q(1+p^2)}{1-p^6q^2(1+p^2)^2}\left(\frac{1+p^4q(1+p^2)}{1-p^4} - \frac{p^6q^2C(p^8q^2)}{1-p^4qC(p^8q^2)}\right).$$

By singularity analysis (see for instance [18]), we see that the coefficient of p^n in

$$H(\sqrt{p},1)=\frac{p^3(3+p+2p^2-p^3-8p^4-8p^5-5p^6-2p^7)}{2(1-p^2)(1-p^3(1+p)^2)^2}-\frac{p^3(1+p)(1+p^2(1+p))\sqrt{1-4p^4}}{2(1-p^3(1+p)^2)^2},$$

namely the number of smooth triangular bargraphs with perimeter n, is asymptotic to $\frac{304+215\sqrt{2}}{\sqrt{\pi n^3}}\sqrt{2}^{n+2}$.

Acknowledgements. The author would like to thank the anonymous referee for carefully reading the paper and giving helpful comments and suggestions.

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(Received 03. 06. 2022.)
(Revised 18. 01. 2024.)