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FOUR TYPES OF VARIANT EULER HARMONIC SUMS

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We aim to investigate the four types of variant Euler harmonic sums. Also, as corollaries, we provide particular examples of our core findings, some of whose further instances are evaluated in terms of basic and well-known functions as well as certain mathematical constants. We explore relevant linkages between our results and those of other previously established studies. An examination of a specific case of one result shows a relationship to series involving zeta functions, which is also a popular area of research.

1. INTRODUCTION

Certain required functions, numbers and notations are recalled and given. The gamma function Γ is given by

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt \quad (\Re(z) > 0). \tag{1}$$

The Beta function $B(\mu,\, \nu)$ is given as follows (see, e.g., [29, p. 8]):

$$B(\mu,\, \nu) = \begin{cases} \displaystyle\int_0^1 t^{\mu-1}(1-t)^{\nu-1}\, dt & (\Re(\mu) > 0,\ \Re(\nu) > 0) \\ \dfrac{\Gamma(\mu)\,\Gamma(\nu)}{\Gamma(\mu+\nu)} & (\mu,\, \nu \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0})\,. \end{cases} \tag{2}$$

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The psi (or digamma) function ψ is defined by

$$\psi(z) := \frac{d}{dz}\log\Gamma(z) = \frac{\Gamma'(z)}{\Gamma(z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}).$$

The polygamma function $\psi^{(k)}(z)$ is defined by

$$\begin{aligned}\psi^{(k)}(z) &:= \frac{d^k}{dz^k}\{\psi(z)\} = (-1)^{k+1}\, k! \sum_{r=0}^{\infty} \frac{1}{(r+z)^{k+1}} \\ &= (-1)^{k+1}\, k!\, \zeta(k+1, z) \quad (k \in \mathbb{N},\ z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}),\end{aligned} \tag{3}$$

where $\psi^{(0)}(z) = \psi(z)$, and $\zeta(s, z)$ is the generalized (or Hurwitz) zeta function defined by

$$\zeta(s, z) := \sum_{j=0}^{\infty} \frac{1}{(j+z)^s} \quad (\Re(s) > 1,\ z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}),$$

and $\zeta(s, 1) =: \zeta(s)$ is the Riemann zeta function. It has the recurrence

$$\psi^{(k)}(z+1) = \psi^{(k)}(z) + \frac{(-1)^k\, k!}{z^{k+1}} \quad (k \in \mathbb{Z}_{\geqslant 0}).$$

Here and in the following, let $\mathbb{C}$, $\mathbb{R}$, $\mathbb{R}_{>0}$, $\mathbb{Z}$, and $\mathbb{N}$ denote the sets of complex numbers, real numbers, positive real numbers, integers, and positive integers, respectively. Also let $\mathbb{Z}_{\geqslant \ell}$ and $\mathbb{Z}_{\leqslant \ell}$ denote the sets of integers whose elements are greater than or equal to ℓ and less than or equal to some $\ell \in \mathbb{Z}$. For more properties and formulas of the above functions, one may refer to [29, Sections 1.1, 1.3 and 2.2, 2.3].

The generalized harmonic numbers $H_n^{(s)}$ of order s are defined by

$$H_n^{(s)} := \sum_{k=1}^{n} \frac{1}{k^s} \quad (n \in \mathbb{N},\ s \in \mathbb{C}),$$

where $H_n^{(1)} =: H_n$ $(n \in \mathbb{N})$ are harmonic numbers, and $H_0^{(s)} = 0 = H_0$. Here and elsewhere, an empty sum is assumed to be nil. The following relations are recalled:

$$H_n = \gamma + \psi(n+1) \quad (n \in \mathbb{Z}_{\geqslant 0}), \tag{4}$$

where γ is the Euler-Mascheroni constant (see, e.g., [29, Section 1.2]);

$$H_n^{(m+1)} = \zeta(m+1) + \frac{(-1)^m}{m!}\psi^{(m)}(n+1) \quad (m \in \mathbb{N},\ n \in \mathbb{Z}_{\geqslant 0}) \tag{5}$$

(see, e.g., [2, Eq. (1.25)] and (3)). Equations (4) and (5) are utilized to define extended harmonic numbers $H_\eta^{(m)}$ of order $m \in \mathbb{N}$ with index $\eta \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ by (see [28])

$$H_\eta^{(m)} := \begin{cases} \gamma + \psi(\eta+1) & (m = 1), \\ \zeta(m) + \frac{(-1)^{m-1}}{(m-1)!}\psi^{(m-1)}(\eta+1) & (m \in \mathbb{Z}_{\geqslant 2}). \end{cases}$$

A generalized binomial coefficient $\binom{s}{t}$ $(s,\, t \in \mathbb{C})$ is defined, in terms of the gamma functions, by

$$\binom{s}{t} := \frac{\Gamma(s+1)}{\Gamma(t+1)\Gamma(s-t+1)} \quad (s,\, t \in \mathbb{C}). \tag{6}$$

Euler found the following identity in 1742, and it has a lengthy history (see, e.g., [6, p. 252 et seq.]):

$$\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^2} = \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k}{k^2} = \zeta(3). \tag{7}$$

The identity (7) is a particular instance of the following more generalized Euler sum (see, e.g., [7], [10], [12], [27]):

$$2 \sum_{k=1}^{\infty} \frac{H_k}{(k+1)^m} = m\,\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(k+1)\zeta(m-k) \quad (m \in \mathbb{Z}_{\geqslant 2})\,, \tag{8}$$

or, equivalently,

$$2 \sum_{k=1}^{\infty} \frac{H_k}{k^m} = (m+2)\,\zeta(m+1) - \sum_{k=1}^{m-2} \zeta(k+1)\zeta(m-k) \quad (m \in \mathbb{Z}_{\geqslant 2})\,. \tag{9}$$

During his interaction with Goldbach starting in 1742, Euler initiated a series of investigations for the *linear* harmonic sums (see, e.g., [10, 12]):

$$\mathtt{S}(p,q) := \sum_{n=1}^{\infty} \frac{H_n^{(p)}}{n^q} \quad (p \in \mathbb{N},\, q \in \mathbb{Z}_{\geqslant 2})\,. \tag{10}$$

Euler's study, which Nielsen completed in 1906 (see [20]), revealed that the linear harmonic sums in (10) may be determined in the following situations: $p = 1$; $p = q$; $p + q$ odd; $p + q$ even, but with only the pair (p, q) being the set $\{(2, 4), (4.2)\}$. Of these particular instances, in the ones with $p \neq q$, if $\mathtt{S}(p, q)$ is determined, then $\mathtt{S}(q, p)$ may be evaluated by means of the symmetry relation

$$\mathtt{S}(p,q) + \mathtt{S}(q,p) = \zeta(p)\,\zeta(q) + \zeta(p+q) \tag{11}$$

and vice versa.

The *nonlinear* harmonic sums include at least two (generalized) harmonic number products. The works of Flajolet and Salvy (see [12]) marked the beginning of the systematic study and definition of nonlinear harmonic sums (also known as nonlinear Euler sums). Put $R = (r_1, \ldots, r_\ell)$ be a partition of an integer r into ℓ summands, so that $r = r_1 + \cdots + r_\ell$ and $r_1 \leq r_2 \leq \cdots \leq r_\ell$. The Euler sum of index R, q is defined by

$$\mathtt{S}(R;q) := \sum_{n=1}^{\infty} \frac{H_n^{(r_1)}\, H_n^{(r_2)} \cdots H_n^{(r_\ell)}}{n^q},$$

where the quantity $q + r_1 + \cdots + r_\ell$ is called the weight, the quantity ℓ is the degree. In partitions, repeating summands are represented by powers for brevity, for instance,

$$\mathtt{S}(1^2, 2^3, 7; q) = \mathtt{S}(1, 1, 2, 2, 2, 7; q) = \sum_{n=1}^{\infty} \frac{H_n^2 \left\{H_n^{(2)}\right\}^3 H_n^{(7)}}{n^q}.$$

In the vast mathematical literature, many researchers have undertaken research on Euler, Euler-type sums, and other versions of these sums using a variety of techniques (see, e.g., [2], [4], [5], [7], [8], [10], [12], [13], [21], [25], [26], [27], [28], [30] and the references therein).

The main purpose of this paper is to explore the following variants of the Euler harmonic sums: For $n \in \mathbb{Z}_{\geqslant 0}$, $m \in \mathbb{N}$, and $p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$,

$$\sum_{k=1}^{\infty} \frac{H_k}{(n+k+1)^m \binom{n+k}{k}}, \quad \sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(n+k+1)^m \binom{n+k}{k}},$$

$$\sum_{k=1}^{\infty} \frac{H_k}{k(p+n+k)^m \binom{n+k}{k}} \quad \text{and} \quad \sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{k(p+n+k)^m \binom{n+k}{k}}.$$

In addition, as corollaries, we present specific cases of our primary discoveries, some of whose further particular instances are assessed in terms of elementary and well-known functions as well as certain mathematical constants. We discuss pertinent connections between our findings and those of other known ones. Investigation of a particular instance of one result reveals a connection to series involving zeta functions, which is also an interesting and useful research topic.

2. PRELIMINARY RESULTS

The following lemma recalls some required properties for the gamma, psi and polygamma functions for easy reference.

Lemma 1. *The followings hold true:*

(i)

$$\Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin(\pi z)} \quad (z \in \mathbb{C} \setminus \mathbb{Z}). \tag{12}$$

(ii) $\Gamma(z)$ *and* $\psi(z)$ *are meromorphic functions on the whole complex* z*-plane with simple poles at* $z = -k$ $(k \in \mathbb{Z}_{\geqslant 0})$ *with their respective residues given by*

$$\operatorname*{Res}_{z=-k} \Gamma(z) = \lim_{z \to -k} (z+k)\Gamma(z) = \frac{(-1)^k}{k!} \quad (k \in \mathbb{Z}_{\geqslant 0}) \tag{13}$$

and

$$\operatorname*{Res}_{z=-k} \psi(z) = \lim_{z \to -k} (z+k)\psi(z) = -1 \quad (k \in \mathbb{Z}_{\geqslant 0}). \tag{14}$$

(iii) *The Laurent expansion for* $\psi(z)$ *at* $z=-k$ $(k\in\mathbb{Z}_{\geqslant 0})$ *is given by*

$$\psi(z)=-\frac{1}{z+k}+\psi(k+1)+\sum_{n=2}^{\infty}\alpha_n\,(z+k)^{n-1}, \tag{15}$$

where

$$\alpha_n=(-1)^n\,\zeta(n)+H_k^{(n)}. \tag{16}$$

(iv) *The Laurent expansion for the polygamma function* $\psi^{(\ell)}(z)$ *at* $z=-k$ $(k\in\mathbb{Z}_{\geqslant 0})$ *is given by*

$$\psi^{(\ell)}(z)=\frac{(-1)^{\ell+1}\,\ell!}{(z+k)^{\ell+1}}+\sum_{n=\ell}^{\infty}\{n\}_\ell\,\alpha_{n+1}\,(z+k)^{n-\ell}\quad(\ell\in\mathbb{N}), \tag{17}$$

where $\{\lambda\}_\ell$ $(\lambda\in\mathbb{C})$ *is the falling factorial defined by*

$$\{\lambda\}_\ell:=\begin{cases}1 & (\ell=0)\\ \lambda(\lambda-1)\cdots(\lambda-\ell+1) & (\ell\in\mathbb{N}),\end{cases}$$

and α_n *are given as in* (16).

(v)

$$\psi(z+m)=\psi(z)+\sum_{j=0}^{m-1}\frac{1}{z+j}\quad(m\in\mathbb{Z}_{\geqslant 0})$$

and

$$\psi^{(n)}(z+m)=\psi^{(n)}(z)+(-1)^n\,n!\sum_{j=0}^{m-1}\frac{1}{(z+j)^{n+1}}\quad(m,\,n\in\mathbb{Z}_{\geqslant 0})\,.$$

Proof. One may refer to [29, pp. 4 and 24] and [18, Section 1.2]. Equation (17) can be derived by differentiating both sides of (15) ℓ-times. Reference [12] contains formulas (15) and (17), which can also be located in a box at the top of page 20. □

Lemma 2. *Let* $k\in\mathbb{Z}_{\geqslant 0}$. *Then*

$$\lim_{z\to -k}\frac{\psi(z)}{\Gamma(z)}=(-1)^{k-1}\,k!; \tag{18}$$

$$\begin{aligned}\lim_{z\to -k}\frac{\psi^2(z)-\psi'(z)}{\Gamma(z)}&=-\lim_{z\to -k}\frac{d}{dz}\left\{\frac{\psi(z)}{\Gamma(z)}\right\}\\&=2\,(-1)^{k-1}\,k!\,\psi(k+1);\end{aligned} \tag{19}$$

$$\begin{aligned}\lim_{z\to -k}\frac{\psi^3(z)-3\,\psi(z)\,\psi'(z)+\psi^{(2)}(z)}{\Gamma(z)}&=\lim_{z\to -k}\frac{d^2}{dz^2}\left\{\frac{\psi(z)}{\Gamma(z)}\right\}\\&=3\,(-1)^k\,k!\left\{\zeta(2)+H_k^{(2)}-\psi^2(k+1)\right\}.\end{aligned} \tag{20}$$

Proof. One finds from (13) and (14) that

$$\lim_{z\to -k}\frac{\psi(z)}{\Gamma(z)}=\lim_{z\to -k}\frac{(z+k)\,\psi(z)}{(z+k)\,\Gamma(z)}=(-1)^{k-1}\,k!,$$

which proves (18).

One may use (15) and (17) to obtain

$$\begin{aligned}\psi^2(z)-\psi'(z)&=-\frac{2\,\psi(k+1)}{z+k}+O(1)+O(z+k)\quad(z\to -k)\\&=-\frac{2\,\psi(k+1)}{z+k}+O(1)+o(1)\quad(z\to -k)\\&=-\frac{2\,\psi(k+1)}{z+k}+O(1)\quad(z\to -k).\end{aligned}\tag{21}$$

Employing (12) and (21), one can get

$$\begin{aligned}&\lim_{z\to -k}\frac{\psi^2(z)-\psi'(z)}{\Gamma(z)}=\lim_{z\to -k}\frac{\sin(\pi z)\,\Gamma(1-z)}{\pi}\,\left\{\psi^2(z)-\psi'(z)\right\}\\&=\frac{\Gamma(1+k)}{\pi}\lim_{z\to -k}\frac{\sin(\pi z)}{z+k}\,(z+k)\left\{\psi^2(z)-\psi'(z)\right\}\\&=\frac{k!}{\pi}\,\pi\,\cos(\pi k)\,\left\{-2\,\psi(k+1)\right\},\end{aligned}$$

which justifies (19).

One may use (15) and (17) to derive

$$\psi^3(z)-3\,\psi(z)\,\psi'(z)+\psi^{(2)}(z)=\frac{3\left\{\alpha_2-\psi^2(k+1)\right\}}{z+k}+O(1)\quad(z\to -k).$$

Now, a similar technique as in getting (19) may verify (20). □

The next two theorems are recalled (see [5]).

Theorem 3. *Let $x\in\mathbb{R}\setminus\mathbb{Z}_{\leqslant -1}$ and $m\in\mathbb{N}$. Then*

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{x}{k}=\frac{(-1)^m}{m!}\frac{\partial^m}{\partial z^m}\frac{\Gamma(x+1)\Gamma(z)}{\Gamma(z+x)}\bigg|_{z=1}.\tag{22}$$

Theorem 4. *Let $m,\,n\in\mathbb{N}$. Then*

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^{n-1}}{(n+k+1)^{m+1}\binom{n+k}{k}}&=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}(H_n-H_{n-k})\\&\quad-\frac{(-1)^m}{m!}\frac{\partial}{\partial x}\left\{\frac{\partial^m}{\partial z^m}\frac{\Gamma(x+1)\Gamma(z)}{\Gamma(z+x)}\right\}\bigg|_{\substack{x=n\\z=1}}.\end{aligned}\tag{23}$$

3. MAIN RESULTS

This section establishes our main findings.

Theorem 5. *Let $x \in \mathbb{R} \setminus \mathbb{Z}_{\leqslant -1}$, $z \in \mathbb{R} \setminus \mathbb{Z}_{\leqslant 0}$, $m \in \mathbb{N}$, and $n \in \mathbb{Z}_{\geqslant 0}$. Then*

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{H_k}{(n+k+1)^{m+1}\binom{n+k}{k}} \\
&\quad = \frac{(-1)^n}{2} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^m} \binom{n}{k} \left(H_{n-k}^2 + H_{n-k}^{(2)} - H_n^{(2)} - H_n^2 \right) \\
&\qquad - \frac{(-1)^{m+n}}{2\,m!} \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial^m}{\partial z^m} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(z+x)} \right\} \Bigg|_{\substack{x=n \\ z=1}} \\
&\qquad + \frac{(-1)^{m+n} H_n}{m!} \frac{\partial}{\partial x} \left\{ \frac{\partial^m}{\partial z^m} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(z+x)} \right\} \Bigg|_{\substack{x=n \\ z=1}}.
\end{aligned} \tag{24}$$

Proof. Let $P(x)$ be the left and right members of (22). Then the use of (6) can write $P(x)$ as follows:

$$P(x) := \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m\, k!} P_k(x) \quad \text{and} \quad P_k(x) := \frac{\Gamma(x+1)}{\Gamma(x-k+1)}. \tag{25}$$

Also

$$P(x) = \frac{(-1)^m}{m!} \frac{\partial^m}{\partial z^m} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(z+x)} \Bigg|_{z=1}. \tag{26}$$

We first show that the series $P(x)$ in (25) can be differentiated term-by-term for any point $x \in \mathbb{R} \setminus \mathbb{Z}_{\leqslant -1}$. Note that

$$P_k(x) = \frac{1}{\Gamma(x-k+1)} \cdot \Gamma(x+1),$$

whose first factor is an entire function and the second factor is analytic on $\mathbb{R} \setminus \mathbb{Z}_{\leqslant -1}$. Also, for each fixed $x \in \mathbb{R} \setminus \mathbb{Z}_{\leqslant -1}$,

$$\frac{d}{dx} P_k(x) = P_k'(x) = -\frac{\Gamma(x+1)}{\Gamma(x-k+1)} \left\{ \psi(x-k+1) - \psi(x+1) \right\}.$$

By using asymptotic expansions for the ratio of gamma functions (see, e.g., [29, p. 7]) and the psi function (see, e.g., [29, p. 36]), one may obtain

$$\begin{aligned}
P_k'(x) &= O\left(x^k\right) \left\{ \log(x-k+1) - \log(x+1) + O\left(\frac{1}{x}\right) \right\} \\
&= O\left(x^k\right) \left\{ O\left(\frac{k}{x}\right) + O\left(\frac{1}{x}\right) \right\} \\
&= O\left(x^{k-1}\right) \quad (|x| \to \infty).
\end{aligned}$$

Thus, there exists $M > 1$ so large that

$$P_k'(x) = O\left(x^{k-1}\right) = O\left(x^k\right) \quad (|x| > M). \tag{27}$$

Let x_0 be any point in $\mathbb{R} \setminus \mathbb{Z}_{\leqslant -1}$. One can choose $\delta > 0$ so small that

$$(x_0 - \delta, x_0 + \delta) \subseteq [x_0 - \delta, x_0 + \delta] \subseteq \mathbb{R} \setminus \mathbb{Z}_{\leqslant -1}.$$

Now consider three cases: (i) $(x_0 - \delta, x_0 + \delta) \subseteq [x_0 - \delta, x_0 + \delta] \subseteq [-M, M]$. Since $P_k'(x)$ is continuous on the bounded closed interval $[x_0 - \delta, x_0 + \delta]$, $|P_k'(x)| \leqslant L_1$ for some $L_1 > 0$ and for all $x \in [x_0 - \delta, x_0 + \delta]$. Thus one may find

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k-1}}{k^m\, k!} P_k'(x) \right| \leqslant L_1 \sum_{k=1}^{\infty} \frac{1}{k!} = L_1(e-1)$$

for all $x \in [x_0 - \delta, x_0 + \delta]$. In view of Weierstrass M-test, the series $\sum\limits_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m\, k!} P_k'(x)$ converges uniformly on $(x_0 - \delta, x_0 + \delta)$. Therefore the series $P(x)$ is termwise differentiable at any point $x_0 \in (-M, M)$.

(ii) $(x_0 - \delta, x_0 + \delta) \subseteq [x_0 - \delta, x_0 + \delta] \subseteq (-\infty, -M) \cup (M, \infty)$. Let $\eta := \max\{|x_0 - \delta|, |x_0 + \delta|\}$. One may use (27) to see that

$$\sum_{k=1}^{\infty} \left| \frac{(-1)^{k-1}}{k^m\, k!} P_k'(x) \right| \leqslant L_2 \sum_{k=1}^{\infty} \frac{|x|^k}{k!} \leqslant L_2 \sum_{k=1}^{\infty} \frac{\eta^k}{k!} = L_2\, e^{\eta}$$

for some $L_2 > 0$ and for all $x \in [x_0 - \delta, x_0 + \delta]$. As in (ii), the series $P(x)$ can be term-by-term differentiation at any point $x_0 \in (-\infty, -M) \cup (M, \infty)$.

(iii) $x_0 = M$. Then either $(x_0 - \delta, x_0 + \delta) \subset [M - \delta, M + \delta] \subseteq (0, M) \cup (M, \infty)$ or $(x_0 - \delta, x_0 + \delta) \subset [-M - \delta, -M + \delta] \subseteq (-\infty, -M] \cup (-M, 0)$. Proof of this case leaves to the interested reader.

One therefore has

$$P'(x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m\, k!} \frac{\Gamma(x+1)}{\Gamma(x-k+1)} \left\{\psi(x+1) - \psi(x-k+1)\right\}. \tag{28}$$

Putting $x = n$ $(n \in \mathbb{N})$ in (28) and using (4) affords

$$\begin{aligned} P'(n) &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m\, k!} \frac{\Gamma(n+1)}{\Gamma(n-k+1)} \left\{\psi(n+1) - \psi(n-k+1)\right\} \\ &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^m} \binom{n}{k} (H_n - \gamma) - \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^m} \binom{n}{k} \psi(n-k+1). \end{aligned} \tag{29}$$

The following identity is known (see [5, Example 3.7]):

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n-k+1)=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}(H_{n-k}-\gamma)+(-1)^{n+1}\sum_{k=0}^{\infty}\frac{1}{(k+n+1)^{m+1}\binom{n+k}{k}}. \tag{30}$$

Employing (30) in (29) yields

$$P'(n)=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}(H_n-H_{n-k})+(-1)^{n}\sum_{k=0}^{\infty}\frac{1}{(k+n+1)^{m+1}\binom{n+k}{k}}, \tag{31}$$

which is found to be equivalent to (23).

Term-by-term differentiation of $P'(x)$ in (28), which can be confirmed using the preceding procedure, produces

$$\begin{aligned}P''(x)&=\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{x}{k}\big[\{\psi(x+1)-\psi(x-k+1)\}^2\\&\qquad+\{\psi'(x+1)-\psi'(x-k+1)\}\big]\\&=\big\{\psi^2(x+1)+\psi'(x+1)\big\}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{x}{k}\\&\quad-2\psi(x+1)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{x}{k}\psi(x-k+1)\\&\quad+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{x}{k}\big\{\psi^2(x-k+1)-\psi'(x-k+1)\big\}.\end{aligned}$$

Setting $x=n$ ($n\in\mathbb{N}$) and using (4) and (5) gives

$$\begin{aligned}P''(n)&=\big\{(H_n-\gamma)^2+\zeta(2)-H_n^{(2)}\big\}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\\&\quad-2\,(H_n-\gamma)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n-k+1)\\&\quad+\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^2(n-k+1)-\psi'(n-k+1)\big\}.\end{aligned} \tag{32}$$

Employing (4) and (5), we obtain

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^2(n-k+1)-\psi'(n-k+1)\big\}\\
&\quad=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{(H_{n-k}-\gamma)^2+H_{n-k}^{(2)}-\zeta(2)\big\}\\
&\qquad+\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^2(n-k+1)-\psi'(n-k+1)\big\}.
\end{aligned}\tag{33}$$

We get

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^2(n-k+1)-\psi'(n-k+1)\big\}\\
&\quad=\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}\,n!}{k^m\,k!}\,\frac{\psi^2(n-k+1)-\psi'(n-k+1)}{\Gamma(n-k+1)},
\end{aligned}$$

which, upon setting $k-n-1=k'$ and dropping the prime on k, with the aid of (19), offers

$$\begin{aligned}
&\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^2(n-k+1)-\psi'(n-k+1)\big\}\\
&\quad=\sum_{k=0}^{\infty}\frac{(-1)^{n+k}\,n!}{(k+n+1)^m\,(k+n+1)!}\lim_{z\to-k}\frac{\psi^2(z)-\psi'(z)}{\Gamma(z)}\\
&\quad=2\,(-1)^{n+1}\sum_{k=0}^{\infty}\frac{n!}{(k+n+1)^m\,(k+n+1)!}\,k!\,\psi(k+1)\\
&\quad=2\,(-1)^{n+1}\sum_{k=0}^{\infty}\frac{\psi(k+1)}{(k+n+1)^{m+1}\,\binom{n+k}{k}}.
\end{aligned}\tag{34}$$

Putting (34) in (33) provides

$$\begin{aligned}
&\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^2(n-k+1)-\psi'(n-k+1)\big\}\\
&\quad=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{(H_{n-k}-\gamma)^2+H_{n-k}^{(2)}-\zeta(2)\big\}\\
&\qquad+2\,(-1)^{n+1}\sum_{k=0}^{\infty}\frac{H_k-\gamma}{(k+n+1)^{m+1}\,\binom{n+k}{k}}.
\end{aligned}\tag{35}$$

Setting (30) and (35) in (32) gives

$$
\begin{aligned}
P''(n) =& \left\{(H_n-\gamma)^2+\zeta(2)-H_n^{(2)}\right\}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k} \\
&-2\,(H_n-\gamma)\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}(H_{n-k}-\gamma) \\
&+2\,(-1)^n\,H_n\sum_{k=0}^{\infty}\frac{1}{(k+n+1)^{m+1}\binom{n+k}{k}} \\
&+\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\left\{(H_{n-k}-\gamma)^2+H_{n-k}^{(2)}-\zeta(2)\right\} \\
&+2\,(-1)^{n+1}\sum_{k=0}^{\infty}\frac{H_k}{(k+n+1)^{m+1}\binom{n+k}{k}}.
\end{aligned}
\tag{36}
$$

From (31) and (36), we derive

$$
\begin{aligned}
P''(n)-2\,H_n\,P'(n) =& \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\left(H_{n-k}^2+H_{n-k}^{(2)}-H_n^2-H_n^{(2)}\right) \\
&+2\,(-1)^{n+1}\sum_{k=0}^{\infty}\frac{H_k}{(k+n+1)^{m+1}\binom{n+k}{k}}.
\end{aligned}
\tag{37}
$$

Finally, (26) is used in the left member of (37) to yield the desired result (24). □

The next corollary provides a proof of the Euler's classical formula (8) as the particular case of (24) when $n=0$.

Corollary 6. *Let $m\in\mathbb{Z}_{\geqslant 2}$. Then*

$$
2\sum_{k=1}^{\infty}\frac{H_k}{(k+1)^m}=m\,\zeta(m+1)-\sum_{k=1}^{m-2}\zeta(k+1)\,\zeta(m-k).
$$

Proof. Setting $n=0$ in (24) may yield

$$
\sum_{k=0}^{\infty}\frac{H_k}{(k+1)^{m+1}}=\frac{(-1)^{m+1}}{2m!}\frac{\partial^2}{\partial x^2}\left\{\frac{\partial^m}{\partial z^m}\frac{\Gamma(x+1)\Gamma(z)}{\Gamma(z+x)}\right\}\Bigg|_{\substack{x=0\\ z=1}}.
\tag{38}
$$

Interchanging the order of differentiations with respect to x and z in (38), which can be guaranteed since

$$
\frac{\partial^2}{\partial x^2}\left\{\frac{\partial^m}{\partial z^m}\frac{\Gamma(x+1)\Gamma(z)}{\Gamma(z+x)}\right\}\quad\text{and}\quad\frac{\partial^m}{\partial z^m}\left\{\frac{\partial^2}{\partial x^2}\frac{\Gamma(x+1)\Gamma(z)}{\Gamma(z+x)}\right\}
$$

are analytic and so continuous on $x \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$ and $z \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, replacing m by $m-1$, and noticing that

$$\frac{d^2}{dx^2} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(z+x)}\bigg|_{x=0} = \psi^2(z) + 2\gamma\psi(z) - \psi'(z) + \gamma^2 + \frac{\pi^2}{6},$$

we obtain

$$\begin{aligned}
2\sum_{k=1}^{\infty} \frac{H_k}{(k+1)^m} &= \frac{(-1)^m}{(m-1)!} \frac{\partial^{m-1}}{\partial z^{m-1}} \left[\psi^2(z) + 2\gamma\psi(z) - \psi'(z)\right]_{z=1} \\
&= \frac{(-1)^m}{(m-1)!} \left\{ \sum_{k=0}^{m-1} \binom{m-1}{k} \psi^{(k)}(1)\psi^{(m-k-1)}(1) + 2\gamma\psi^{(m-1)}(1) - \psi^{(m)}(1) \right\} \\
&= \frac{(-1)^m}{(m-1)!} \left\{ \sum_{k=1}^{m-2} \binom{m-1}{k} \psi^{(k)}(1)\psi^{(m-k-1)}(1) - \psi^{(m)}(1) \right. \\
&\quad \left. + \underbrace{2\psi(1)\psi^{((m-1))}(1) + 2\gamma\psi^{(m-1)}(1)}_{0} \right\},
\end{aligned}$$

which, upon using (5), yields the desired result. □

Theorem 7. *Let $n \in \mathbb{Z}_{\geqslant 0}$ and $m \in \mathbb{N}$. Then*

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(n+k+1)^{m+1}\binom{n+k}{k}} = \frac{(-1)^{n-1}}{3} \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k^m} \binom{n}{k} \\
&\times \left\{ H_n^3 + 2H_n^{(3)} + 3H_n H_n^{(2)} - H_{n-k}^3 - 2H_{n-k}^{(3)} - 3H_{n-k}H_{n-k}^{(2)} \right\} \\
&+ \frac{(-1)^{m+n}}{m!} \left\{ \left(H_n^2 + H_n^{(2)}\right) F_1(n,m) - H_n\, F_2(n,m) + \frac{1}{3}\, F_3(n,m) \right\},
\end{aligned} \tag{39}$$

where

$$F_1(n,m) := \frac{\partial}{\partial x} \left\{ \frac{\partial^m}{\partial z^m} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z)} \right\}\bigg|_{\substack{x=n \\ z=1}},$$

$$F_2(n,m) := \frac{\partial^2}{\partial x^2} \left\{ \frac{\partial^m}{\partial z^m} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z)} \right\}\bigg|_{\substack{x=n \\ z=1}},$$

and

$$F_3(n,m) := \frac{\partial^3}{\partial x^3} \left\{ \frac{\partial^m}{\partial z^m} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z)} \right\}\bigg|_{\substack{x=n \\ z=1}};$$

Proof. As in the proof of Theorem 5, let $P(x)$ be the same as in (25). Then we differentiate $P(x)$ with respect to x three times term-by-term. Then putting $x = n$

($n \in \mathbb{N}$) in the $P^{(3)}(x)$, with the aid of (4) and (5), we find

$$
\begin{aligned}
P^{(3)}(n) = \big\{&(H_n-\gamma)^3 + 3(H_n-\gamma)(\zeta(2)-H_n^{(2)}) \\
&+ 2(H_n^{(3)}-\zeta(3))\big\}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k} \\
&- 3\Big\{(H_n-\gamma)^2 + (\zeta(2)-H_n^{(2)})\Big\}\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\psi(n-k+1) \\
&+ 3(H_n-\gamma)\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^2(n-k+1)-\psi'(n-k+1)\big\} \\
&- \sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^3(n-k+1)-3\psi(n-k+1)\psi'(n-k+1) \\
&\qquad + \psi''(n-k+1)\big\}.
\end{aligned}
\tag{40}
$$

Here, we consider

$$
\begin{aligned}
&\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^3(n-k+1)-3\psi(n-k+1)\psi'(n-k+1) \\
&\qquad + \psi''(n-k+1)\big\} \\
&= \sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{(H_{n-k}-\gamma)^3 - 3(H_{n-k}-\gamma)(\zeta(2)-H_{n-k}^{(2)}) \\
&\qquad + 2\,H_{n-k}^{(3)} - 2\,\zeta(3)\big\} \\
&+ \sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^3(n-k+1)-3\psi(n-k+1)\psi'(n-k+1) \\
&\qquad + \psi''(n-k+1)\big\}.
\end{aligned}
$$

As in getting (35), we use (20) to obtain

$$
\begin{aligned}
&\sum_{k=n+1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^3(n-k+1)-3\psi(n-k+1)\psi'(n-k+1) \\
&\qquad + \psi''(n-k+1)\big\} \\
&= 3\,(-1)^n\sum_{k=0}^{\infty}\frac{H_k^{(2)} - H_k^2 + \zeta(2) + 2\gamma\,H_k - \gamma^2}{(k+n+1)^{m+1}\binom{k+n}{k}}
\end{aligned}
$$

We therefore have

$$\sum_{k=1}^{\infty}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{\psi^3(n-k+1)-3\psi(n-k+1)\psi'(n-k+1)$$
$$+\psi''(n-k+1)\big\}$$
$$=\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{(H_{n-k}-\gamma)^3-3(H_{n-k}-\gamma)(\zeta(2)-H_{n-k}^{(2)})$$
$$+2\,H_{n-k}^{(3)}-2\,\zeta(3)\big\}$$
$$+3\,(-1)^n\sum_{k=0}^{\infty}\frac{H_k^{(2)}+\zeta(2)-(H_k-\gamma)^2}{(k+n+1)^{m+1}\binom{k+n}{k}}. \tag{41}$$

Employing (30), (35) and (41) in (40), we find

$$P^{(3)}(n)=\big\{(H_n-\gamma)^3+3(H_n-\gamma)(\zeta(2)-H_n^{(2)})$$
$$+2(H_n^{(3)}-\zeta(3))\big\}\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}$$
$$-3\Big\{(H_n-\gamma)^2+(\zeta(2)-H_n^{(2)})\Big\}\Bigg[\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}(H_{n-k}-\gamma)$$
$$+(-1)^{n+1}\sum_{k=0}^{\infty}\frac{1}{(k+n+1)^{m+1}\binom{n+k}{k}}\Bigg]$$
$$+3(H_n-\gamma)\Bigg[\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{(H_{n-k}-\gamma)^2+H_{n-k}^{(2)}-\zeta(2)\big\}$$
$$+2\,(-1)^{n+1}\sum_{k=0}^{\infty}\frac{H_k-\gamma}{(k+n+1)^{m+1}\binom{n+k}{k}}\Bigg]$$
$$-\sum_{k=1}^{n}\frac{(-1)^{k-1}}{k^m}\binom{n}{k}\big\{(H_{n-k}-\gamma)^3-3(H_{n-k}-\gamma)(\zeta(2)-H_{n-k}^{(2)})$$
$$+2\,H_{n-k}^{(3)}-2\,\zeta(3)\big\}$$
$$-3\,(-1)^n\sum_{k=0}^{\infty}\frac{H_k^{(2)}-H_k^2+\zeta(2)+2\gamma\,H_k-\gamma^2}{(k+n+1)^{m+1}\binom{k+n}{k}}.$$

Finally, using (31), (32), and the expression $P^{(3)}(n)$ just obtained, as in getting the result in Theorem 5, we can readily establish (39). □

A particular case of (39) when $n=0$ produces the identity in Corollary 8.

Corollary 8. *Let $m \in \mathbb{N}$. Then*

$$\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(k+1)^{m+1}} = \frac{(-1)^m}{3\,m!}\, F_3(0,m)$$

$$= \frac{(m+1)(m+2)}{3}\,\zeta(m+3) - \sum_{j=1}^{m-1} (j+1)\,\zeta(j+2)\,\zeta(m+1-j) + \frac{1}{m}\sum_{\ell=1}^{m-1} (m-\ell)\,\zeta(m-\ell+1) \sum_{j=1}^{\ell-1} \zeta(j+1)\,\zeta(\ell-j+1). \tag{42}$$

Also

$$\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{k^{m+1}} = \frac{(-1)^m}{3\,m!}\, F_3(0,m) + (m+2)\,\zeta(m+3) - \sum_{j=1}^{m} \zeta(j+1)\,\zeta(m+2-j)$$

$$= \frac{(m+2)(m+4)}{3}\,\zeta(m+3) - \zeta(2)\,\zeta(m+1) - 2\sum_{j=2}^{m} \zeta(j+1)\,\zeta(m+2-j) - \sum_{j=1}^{m-1} j\,\zeta(j+2)\,\zeta(m+1-j) + \frac{1}{m}\sum_{\ell=1}^{m-1} (m-\ell)\,\zeta(m-\ell+1) \sum_{j=1}^{\ell-1} \zeta(j+1)\,\zeta(\ell-j+1), \tag{43}$$

where

$$F_3(0,m) = \frac{\partial^3}{\partial x^3}\left\{ \frac{\partial^m}{\partial z^m} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z)} \right\}\Bigg|_{\substack{x=0\\ z=1}}.$$

Proof. Setting $n = 0$ (39) gives

$$\sum_{k=0}^{\infty} \frac{H_k^2 - H_k^{(2)}}{(k+1)^{m+1}} = \frac{(-1)^m}{3\,m!}\, F_3(0,m),$$

where

$$F_3(0,m) = \frac{d^{m-1}}{dz^{m-1}}\left\{ \frac{d}{dz}\frac{\partial^3}{\partial x^3} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z)}\Bigg|_{x=0} \right\}\Bigg|_{z=1}.$$

We obtain

$$\frac{d}{dz}\left\{ \frac{\partial^3}{\partial x^3} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z)}\Bigg|_{x=0} \right\} = -3\,\left(\gamma^2 + \zeta(2)\right)\,\psi'(z) + 3\gamma\,\psi^{(2)}(z) - \psi^{(3)}(z) + 3\,\psi(z)\,\psi^{(2)}(z) + 3\,\left(\psi'(z)\right)^2 - 6\gamma\,\psi(z)\,\psi'(z) - 3(\psi(z))^2\,\psi'(z).$$

Note that

$$\frac{d^\ell}{dz^\ell} f(z) := \frac{d^\ell}{dz^\ell}(\psi(z))^2 = \sum_{j=0}^{\ell} \binom{\ell}{j} \psi^{(j)}(z)\, \psi^{(\ell-j)}(z) \quad (\ell \in \mathbb{Z}_{\geqslant 0}), \tag{44}$$

which, upon putting $z = 1$ and using (4) and (5), yields

$$\begin{aligned} \frac{d^\ell}{dz^\ell}(\psi(z))^2\Big|_{z=1} &= \sum_{j=0}^{\ell} \binom{\ell}{j} \psi^{(j)}(1)\, \psi^{(\ell-j)}(1) \\ &= 2\gamma\,(-1)^\ell\, \ell!\, \zeta(\ell+1) + (-1)^\ell\, \ell! \sum_{j=1}^{\ell-1} \zeta(j+1)\, \zeta(\ell-j+1). \end{aligned}$$

Employing , with the aid of (4), (5) and (44), we derive

$$\begin{aligned} \frac{\partial^m}{\partial z^m}\left\{ \frac{\partial^3}{\partial x^3} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z)}\Big|_{x=0} \right\} &= -3\,\left(\gamma^2 + \zeta(2)\right)\, \psi^{(m)}(z) + 3\gamma\, \psi^{(m+1)}(z) \\ &- \psi^{(m+2)}(z) + 3 \sum_{j=0}^{m-1} \binom{m-1}{j} \psi^{(2+j)}(z)\, \psi^{(m-1-j)}(z) \\ &+ 3 \sum_{j=0}^{m-1} \binom{m-1}{j} \psi^{(j+1)}(z)\, \psi^{(m-j)}(z) \\ &- 6\gamma \sum_{j=0}^{m-1} \binom{m-1}{j} \psi^{(j)}(z)\, \psi^{(m-j)}(z) \\ &- 3 \sum_{\ell=0}^{m-1} \binom{m-1}{\ell} f^{(\ell)}(z)\, \psi^{(m-\ell)}(z), \end{aligned}$$

which, upon setting $z = 1$, yields

$$\begin{aligned} \frac{(-1)^m}{3\,m!} F_3(0,m) &= \zeta(2)\, \zeta(m+1) + \frac{(m+1)(m+2)}{3}\, \zeta(m+3) \\ &- \frac{1}{m} \sum_{j=0}^{m-2} (j+1)(j+2)\, \zeta(3+j)\, \zeta(m-j) \\ &- \frac{1}{m} \sum_{j=0}^{m-1} (j+1)\,(m-j)\, \zeta(j+2)\, \zeta(m+1-j) \\ &+ \frac{1}{m} \sum_{\ell=1}^{m-1} (m-\ell)\, \zeta(m-\ell+1) \sum_{j=1}^{\ell-1} \zeta(j+1)\, \zeta(\ell-j+1). \end{aligned}$$

Finally, the last expression may be simplified to yield the desired result (42).

Using (9), we may obtain

$$\sum_{k=0}^{\infty}\frac{H_k^2-H_k^{(2)}}{(k+1)^{m+1}}=\sum_{k=1}^{\infty}\frac{H_k^2-H_k^{(2)}}{k^{m+1}}-(m+2)\,\zeta(m+3)+\sum_{k=1}^{m}\zeta(k+1)\,\zeta(m+2-k)\quad(m\in\mathbb{N}). \tag{45}$$

Employing (45) in (42) produces (43). □

Theorem 9. *Let $p\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$, $x\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant -1}$ and $m\in\mathbb{N}$. Then*

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(p+k)^{m+1}}\binom{x}{k}=\frac{(-1)^m}{m!}\frac{\partial^m}{\partial s^m}\frac{\Gamma(x+1)\Gamma(s)}{\Gamma(x+s+1)}\bigg|_{s=p}. \tag{46}$$

Proof. Using (1), one may find

$$\frac{1}{t^{m+1}}=\frac{1}{m!}\int_0^{\infty}u^m e^{-t\,u}\,du\quad(m\in\mathbb{Z}_{\geqslant 0},\ \Re(t)>0)\,. \tag{47}$$

Employing (47), one can obtain

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(p+k)^{m+1}}\binom{x}{k}=\frac{1}{m!}\sum_{k=0}^{\infty}(-1)^k\binom{x}{k}\int_0^{\infty}u^m e^{-(p+k)u}\,du.$$

Here and in the following, $\Re(p)>0$ is assumed. Interchanging the order of integration and summation, one may get

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k}{(p+k)^{m+1}}\binom{x}{k}&=\frac{1}{m!}\int_0^{\infty}u^m e^{-pu}\sum_{k=0}^{\infty}(-1)^k\binom{x}{k}e^{-ku}du\\&=\frac{1}{m!}\int_0^{\infty}u^m\,e^{-pu}(1-e^{-u})^x\,du.\end{aligned} \tag{48}$$

In order to verify the above term-by-term integration, let

$$g_k(u):=u^m\,e^{-pu}\,(-1)^k\binom{x}{k}e^{-ku}\quad(u\in\mathbb{R}_{>0},\,k\in\mathbb{Z}_{\geqslant 0})\,.$$

Note that

$$\left|\binom{x}{k}\right|\leqslant\frac{|x|(|x|+1)\cdots(|x|+k-1)}{k!}=\frac{1}{\Gamma(|x|)}\frac{\Gamma(k+|x|)}{\Gamma(k+1)}.$$

Holding x fixed and using the asymptotic expansion of ratio of gamma functions (see, e.g., [29, p. 7]), we find

$$\left|\binom{x}{k}\right|=O\left(k^{|x|-1}\right)\quad(k\to\infty).$$

That is, there exist $M \in \mathbb{R}_{>0}$ and $N \in \mathbb{Z}_{\geqslant 2}$ such that

$$\left|\binom{x}{k}\right| \leqslant M\, k^{|x|-1} \quad \text{for all } \ k \in \mathbb{Z}_{\geqslant N},$$

in particular,

$$\left|\binom{x}{k}\right| \leqslant M \quad \text{for all } \ k \in \mathbb{Z}_{\geqslant N} \ \text{ and } \ |x| < 1.$$

Thus we find that, for $|x| < 1$ and $\Re(p) > 0$,

$$\begin{aligned}\sum_{k=N}^{\infty} |g_k(u)| &\leqslant M\, u^m\, e^{-\Re(p)\, u} \sum_{k=N}^{\infty} e^{-ku} \\ &= M\, u^m\, e^{-\Re(p)\, u}\, \frac{e^{-Nu}}{1-e^{-u}} \\ &= M\, u^m\, e^{-\Re(p)\, u}\, \frac{e^{-(N-1)u}}{e^u - 1}.\end{aligned}$$

Since $e^u - 1 \geqslant u$ for all $u \in \mathbb{R}_{>0}$, we obtain

$$\sum_{k=N}^{\infty} |g_k(u)| \leqslant M\, u^{m-1}\, e^{-(\Re(p)+N-1)u}$$

and, with the aid of (47),

$$\begin{aligned}\int_0^{\infty} \sum_{k=N}^{\infty} |g_k(u)|\, du &\leqslant M \int_0^{\infty} u^{m-1}\, e^{-(\Re(p)+N-1)u}\, du \\ &= \frac{M\,(m-1)!}{(\Re(p)+N-1)^m} < \infty.\end{aligned}$$

Now, by employing the Lebesgue dominated convergence theorem (see, e.g., [14, p. 53]), (48) may be justified.

Making the change of variable $1 - e^{-u} = y$, we derive that

$$\begin{aligned}\sum_{k=0}^{\infty} \frac{(-1)^k}{(p+k)^{m+1}} \binom{x}{k} &= \frac{(-1)^m}{m!} \int_0^1 y^x (1-y)^{p-1} \log^m(1-y)\, dy \\ &= \frac{(-1)^m}{m!} \int_0^1 y^x \frac{\partial^m}{\partial s^m} (1-y)^{s-1}\big|_{s=p} dy \\ &= \frac{(-1)^m}{m!} \frac{\partial^m}{\partial s^m} B(x+1, s)\big|_{s=p},\end{aligned}$$

where $B(x+1, s)$ is the Beta function in (2).

We thus proved the following identity:

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(p+k)^{m+1}} \binom{x}{k} = \frac{(-1)^m}{m!} \frac{\partial^m}{\partial s^m} \frac{\Gamma(x+1)\Gamma(s)}{\Gamma(x+s+1)}\bigg|_{s=p} \tag{49}$$

$$(\Re(p) > 0,\ |x| < 1,\ m \in \mathbb{N})\,.$$

One can observe that both sides of (49) are analytic functions of both variables p and x in the wider domains $p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ and $x \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}$. Finally, by the principle of analytic continuation, the desired identity (46) can hold true for the given domains. □

Corollary 10. *Let $p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ and $m \in \mathbb{Z}_{\geqslant 0}$. Then*

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(H_k - 2H_{2k})\binom{2k}{k}}{4^k (p+k)^{m+1}} \\ = \frac{\sqrt{\pi}\,(-1)^m}{m!} \frac{d^m}{ds^m} \frac{\Gamma(s)}{\Gamma(s+\frac{1}{2})} \left\{\psi\left(\tfrac{1}{2}\right) - \psi\left(s+\tfrac{1}{2}\right)\right\}\Big|_{s=p}.\end{aligned} \tag{50}$$

Proof. We can get (50) by differentiating both sides of (46), with respect to x, and then setting $x = -\frac{1}{2}$ in the resultant identity, and using several formulas for gamma and ψ functions such as

$$\Gamma\left(\tfrac{1}{2} - k\right) = \sqrt{\pi}\,(-1)^k \frac{2^{2k}\,k!}{(2k)!} \quad (k \in \mathbb{Z}_{\geqslant 0}),$$

$$\psi\left(\tfrac{1}{2}\right) - \psi\left(\tfrac{1}{2} - k\right) = H_k - 2\,H_{2k} \quad (k \in \mathbb{Z}_{\geqslant 0})$$

and

$$\psi\left(\tfrac{1}{2} - k\right) = \psi\left(\tfrac{1}{2} + k\right) \quad (k \in \mathbb{Z}_{\geqslant 0}).$$

□

Theorem 11. *Let $p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, $m \in \mathbb{Z}_{\geqslant 0}$, and $n \in \mathbb{Z}_{\geqslant 0}$. Then*

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(-1)^n}{k(p+n+k)^{m+1}\binom{n+k}{k}} &= \sum_{k=0}^{n} \frac{(-1)^k}{(p+k)^{m+1}} \binom{n}{k} \left(H_n - H_{n-k}\right) \\ &\quad - \frac{(-1)^m}{m!} \frac{\partial}{\partial x} \frac{\partial^m}{\partial s^m} \frac{\Gamma(x+1)\Gamma(s)}{\Gamma(x+s+1)}\Bigg|_{\substack{x=n\\ s=p}}.\end{aligned} \tag{51}$$

Proof. Denote both sides of (46) by $Q(x)$. Then, differentiating the left member of (46), with respect to x, gives

$$Q'(x) = \sum_{k=0}^{\infty} \frac{(-1)^k}{(p+k)^{m+1}} \binom{x}{k} (\psi(x+1) - \psi(x-k+1)),$$

which, upon setting $x = n$, yields

$$Q'(n) = \psi(n+1) \sum_{k=0}^{n} \frac{(-1)^k}{(p+k)^{m+1}} \binom{n}{k} - \sum_{k=0}^{\infty} \frac{(-1)^k}{(p+k)^{m+1}} \binom{n}{k} \psi(n-k+1). \tag{52}$$

Split the second sum in (52) into two parts as follows:

$$\sum_{k=0}^{\infty}\frac{(-1)^k}{(p+k)^{m+1}}\binom{n}{k}\psi(n-k+1)=\sum_{k=0}^{n}\frac{(-1)^k}{(p+k)^{m+1}}\binom{n}{k}(H_{n-k}-\gamma)$$
$$+\sum_{k=n+1}^{\infty}\frac{(-1)^k}{(p+k)^{m+1}}\binom{n}{k}\psi(n-k+1).$$

Letting $k-n-1=k'$ and then dropping the prime on k, and using (18), we get

$$\begin{aligned}&\sum_{k=n+1}^{\infty}\frac{(-1)^k}{(p+k)^{m+1}}\binom{n}{k}\psi(n-k+1)\\&=(-1)^{n+1}\sum_{k=0}^{\infty}\frac{(-1)^k}{(p+n+k+1)^{m+1}}\frac{n!}{(n+k+1)!}\frac{\psi(-k)}{\Gamma(-k)}\\&=\sum_{k=0}^{\infty}\frac{(-1)^n}{(p+n+k+1)^{m+1}(n+k+1)\binom{n+k}{k}}.\end{aligned}$$

Thus,

$$\begin{aligned}\sum_{k=0}^{\infty}\frac{(-1)^k}{(p+k)^{m+1}}\binom{n}{k}\psi(n-k+1)&=\sum_{k=0}^{n}\frac{(-1)^k}{(p+k)^{m+1}}\binom{n}{k}(H_{n-k}-\gamma)\\&\quad+\sum_{k=0}^{\infty}\frac{(-1)^n}{(p+n+k+1)^{m+1}(n+k+1)\binom{n+k}{k}}.\end{aligned}\tag{53}$$

Finally, substituting (53) for the second sum (52), and differentiating the right member of (46), with respect to x, setting $x=n$, and matching the two expressions, we may obtain the desired result (51). □

Corollary 12. *Let $p\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$ and $m\in\mathbb{Z}_{\geqslant 0}$. Then*

$$\sum_{k=1}^{\infty}\frac{1}{k(p+k)^{m+1}}=\frac{\gamma}{p^{m+1}}+\frac{1}{p^{m+1}}\sum_{j=0}^{m}\frac{(-1)^j p^j}{j!}\psi^{(j)}(p+1).\tag{54}$$

Proof. Setting $n=0$ in (51) can yield (54). □

Theorem 13. *Let $p\in\mathbb{C}\setminus\mathbb{Z}_{\leqslant 0}$ and $m,\,n\in\mathbb{Z}_{\geqslant 0}$. Then*

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{(-1)^n H_{k-1}}{k(p+n+k)^{m+1}\binom{n+k}{k}}\\&=\frac{1}{2}\sum_{k=0}^{n}\frac{(-1)^k}{(p+k)^{m+1}}\binom{n}{k}\{H_n^2+H_n^{(2)}-H_{n-k}^2-H_{n-k}^{(2)}\}\\&\quad+\frac{(-1)^m}{2\,m!}\frac{\partial^m}{\partial s^m}\left[\left(\frac{\partial^2}{\partial x^2}-2H_n\frac{\partial}{\partial x}\right)\frac{\Gamma(x+1)\Gamma(s)}{\Gamma(x+s+1)}\right]_{\substack{x=n\\s=p}}.\end{aligned}\tag{55}$$

Proof. A similar technique of the proof of Theorem 5 may be used. The details are omitted. □

Corollary 14. *Let $p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ and $m \in \mathbb{Z}_{\geqslant 0}$. Then*

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{H_{k-1}}{k(p+k)^{m+1}} &= \frac{1}{2}\Big\{(\gamma+\psi(p+1))^2+\zeta(2)-\psi'(p+1)\Big\} \\ &+ \frac{1}{2}\sum_{\ell=1}^{m} \frac{(-1)^{\ell}}{\ell!\,p^{m-\ell+1}}\Big\{2\gamma\,\psi^{(\ell)}(p+1)-\psi^{(\ell+1)}(p+1) \\ &\qquad + \sum_{j=0}^{\ell}\binom{\ell}{j}\psi^{(j)}(p+1)\psi^{(\ell-j)}(p+1)\Big\}. \end{aligned} \tag{56}$$

Proof. Setting $n=0$ in (55) offers

$$\sum_{k=1}^{\infty} \frac{H_{k-1}}{k(p+k)^{m+1}} = \frac{(-1)^m}{2\,m!}\left[\frac{\partial^m}{\partial s^m}\left\{\frac{\partial^2}{\partial x^2}\frac{\Gamma(x+1)\Gamma(s)}{\Gamma(x+s+1)}\right\}\bigg|_{x=0}\right]\bigg|_{s=p}.$$

Also we may have

$$\Gamma(s)\,\frac{\partial^2}{\partial x^2}\frac{\Gamma(x+1)}{\Gamma(x+s+1)}\bigg|_{x=0} = \frac{h(s)}{s},$$

where

$$h(s) := \gamma^2+\zeta(2)+2\gamma\,\psi(s+1)+(\psi(s+1))^2-\psi'(s+1).$$

Then one may get the desired identity (56). □

Theorem 15. *Let $p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$, $m \in \mathbb{N}$ and $n \in \mathbb{Z}_{\geqslant 0}$. Then*

$$\begin{aligned}\sum_{k=2}^{\infty} \frac{H_{k-1}^2-H_{k-1}^{(2)}}{k(p+n+k)^m\binom{n+k}{k}} &= \frac{(-1)^n}{3}\sum_{k=0}^{n}\frac{(-1)^k}{(p+k)^m}\binom{n}{k} \\ &\times\left\{H_n^3+2H_n^{(3)}+3H_nH_n^{(2)}-H_{n-k}^3-2H_{n-k}^{(3)}-3H_{n-k}H_{n-k}^{(2)}\right\} \\ &+ \frac{(-1)^{m+n}\big(H_n^2+H_n^{(2)}\big)}{(m-1)!}\frac{\partial}{\partial x}\left[\frac{\partial^m}{\partial z^m}\frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z+1)}\right]_{\substack{x=n\\z=p}} \\ &- \frac{(-1)^{m+n}H_n}{(m-1)!}\frac{\partial^2}{\partial x^2}\left[\frac{\partial^m}{\partial z^m}\frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z+1)}\right]_{\substack{x=n\\z=p}} \\ &+ \frac{(-1)^{m+n}}{3(m-1)!}\frac{\partial^3}{\partial x^3}\left[\frac{\partial^m}{\partial z^m}\frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z+1)}\right]_{\substack{x=n\\z=p}}. \end{aligned} \tag{57}$$

Proof. The proof would parallel that of Theorem 7. The details are omitted. □

Corollary 16. *Let $p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}$ and $m \in \mathbb{N}$. Then*

$$\sum_{k=2}^{\infty} \frac{H_{k-1}^2 - H_{k-1}^{(2)}}{k(p+k)^m} = \sum_{\ell=0}^{m} \binom{m}{\ell} (-1)^{m-\ell} \frac{(m-\ell)!}{z^{m-\ell+1}} g^{(\ell)}(p),$$

where

$$\begin{aligned} g(z) = &- \gamma^3 - 3\gamma\,\zeta(2) - 2\zeta(3) - 3\{\gamma^2 + \zeta(2)\}\,\psi(z+1) \\ &+ 3\gamma\,\psi'(z+1) - \psi^{(2)}(z+1) \\ &+ 3\psi(z+1)\,\psi'(z+1) - 3\gamma\,\psi^2(z+1) - \psi^3(z+1) \end{aligned}$$

and, for $\ell \in \mathbb{N}$,

$$\begin{aligned} g^{(\ell)}(z) = &- 3\{\gamma^2 + \zeta(2)\}\,\psi^{(\ell)}(z+1) + 3\gamma\,\psi^{(\ell+1)}(z+1) - \psi^{(\ell+2)}(z+1) \\ &+ 3\sum_{j=0}^{\ell} \binom{\ell}{j} \psi^{(j+1)}(z+1)\,\psi^{(\ell-j)}(z+1) \\ &- 3\gamma \sum_{j=0}^{\ell} \binom{\ell}{j} \psi^{(j)}(z+1)\,\psi^{(\ell-j)}(z+1) \\ &- \sum_{k=0}^{\ell} \binom{\ell}{k} \left\{ \sum_{j=0}^{k} \binom{k}{j} \psi^{(j)}(z+1)\,\psi^{(k-j)}(z+1) \right\} \psi^{(\ell-k)}(z+1). \end{aligned}$$

Proof. Setting $n = 0$ (57) gives

$$\sum_{k=1}^{\infty} \frac{H_{k-1}^2 - H_{k-1}^{(2)}}{k(p+k)^m} = \frac{(-1)^m}{3(m-1)!} \left[\frac{d^m}{dz^m} \left\{ \frac{\partial^3}{\partial x^3} \frac{\Gamma(x+1)\Gamma(z)}{\Gamma(x+z+1)} \bigg|_{x=0} \right\} \right]_{z=p}.$$

□

4. PARTICULAR CASES AND REMARKS

This section demonstrates certain particular instances of our main findings along with pertinent comments (if any).

Example 1. From (43), one may find that

$$\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{k^{m+1}} = \mathtt{S}(1^2; m+1) - \mathtt{S}(2, m+1) \quad (m \in \mathbb{N}),$$

which is a combination of a nonlinear harmonic sum and a linear harmonic sum and can be evaluated in terms of Riemann zeta functions for any $m \in \mathbb{N}$. The simple one is

$$\sum_{k=1}^{\infty} \frac{H_k^2 - H_k^{(2)}}{k^2} = \mathtt{S}(1^2; 2) - \mathtt{S}(2, 2) = \frac{5}{2}\,\zeta(4).$$

As noted in the paragraph between (10) and (11), the linear Euler sum $\mathtt{S}(2, m+1)$ is determined in terms of Riemann zeta functions only when $m = 1$, $m = 5$, and m is even. So is $\mathtt{S}(1^2; m+1)$ for $m = 1$, $m = 5$, and m is even. For example,

$$\mathtt{S}(2,2) = \frac{7}{4}\,\zeta(4)$$

and

$$\mathtt{S}(1^2;2) = \frac{17}{4}\,\zeta(4). \tag{58}$$

The identity in (58) was observed by E. Au-Yeung. The $\mathtt{S}(1^2; m+1)$ for $m = 1$, $m = 5$, and m is even was evaluated in terms of Riemann zeta functions by Borwein et al. [7] who used the Eulerian beta integral in (2) and, also, by Flajolet and Salvy [12] who applied residue calculus to ψ expansions such as (15) and (17).

Example 2. Setting $p = 1$ and $m = 0$ in (50) yields an interesting sum which involves harmonic numbers and binomial coefficients:

$$\sum_{k=1}^{\infty} \frac{(2H_{2k} - H_k)\binom{2k}{k}}{(k+1)\,4^{k+1}} = 1.$$

Putting $m = 1$ in (50) offers

$$\begin{aligned}\sum_{k=1}^{\infty} \frac{(2H_{2k} - H_k)\binom{2k}{k}}{(p+k)^2\,4^k} &= \frac{\Gamma(\frac{1}{2})\Gamma(p)}{\Gamma(p+\frac{1}{2})} \\ &\quad\times \left[\{\psi(p) - \psi(p+\tfrac{1}{2})\}\{\psi(\tfrac{1}{2}) - \psi(p+\tfrac{1}{2})\} - \psi'(p+\tfrac{1}{2})\right] \\ &\qquad \left(p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0},\ p \neq \tfrac{1-2k}{2}\ (k \in \mathbb{N})\right).\end{aligned} \tag{59}$$

The particular case of (59) when $p = \frac{1}{2}$ gives

$$\sum_{k=1}^{\infty} \frac{(2H_{2k} - H_k)\binom{2k}{k}}{(k+\frac{1}{2})^2\,4^k} = \pi\left(4\log^2 2 - \tfrac{\pi^2}{6}\right).$$

Extended parametric harmonic sums involving H_{qk} $(q \in \mathbb{N})$ were investigated in [25].

Example 3. Setting $m = 0$ in (54) produces a known identity for the psi function (see, e.g., [29, p. 24]):

$$\sum_{k=1}^{\infty} \frac{p}{k(p+k)} = \gamma + \psi(p+1) \quad \left(p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant -1}\right).$$

By applying

$$\frac{1}{k} = \frac{1}{k+p}\,\frac{1}{1-\frac{p}{k+p}} = \sum_{j=0}^{\infty} \frac{p^j}{(k+p)^{j+1}} \quad (|p| < |k+p|)$$

to the left member of (54), we may obtain

$$\sum_{k=1}^{\infty} \frac{1}{k(k+p)^{m+1}} = \sum_{j=0}^{\infty} p^j \, \zeta(m+j+2, p+1) \quad (|p| < |1+p|)$$

and

$$\sum_{j=0}^{\infty} p^j \, \zeta(m+j+2, p+1) = \frac{\gamma}{p^{m+1}} + \frac{1}{p^{m+1}} \sum_{j=0}^{m} \frac{(-1)^j p^j}{j!} \psi^{(j)}(p+1) \tag{60}$$

$$(p \in \mathbb{C} \setminus \mathbb{Z}_{\leqslant 0}, \ m \in \mathbb{Z}_{\geqslant 0}, \ |p| < |1+p|) \, .$$

Setting $p = 1$ in (60) provides

$$\sum_{j=2}^{\infty} \{\zeta(m+j) - 1\} = m + 1 - \sum_{k=1}^{m} \zeta(k+1) \quad (m \in \mathbb{Z}_{\geqslant 0}) \, . \tag{61}$$

Putting $m = 0$ in (61) offers

$$\sum_{j=2}^{\infty} \{\zeta(j) - 1\} = 1. \tag{62}$$

In fact, Shallit and Zikan [22] revealed that a relatively traditional (more than two centuries old) theorem of Christian Goldbach (1690–1764), which was given in a letter dated 1729 from Goldbach to Daniel Bernoulli (1700–1782):

$$\sum_{\eta \in E} (\eta - 1)^{-1} = 1 \quad \left(E := \left\{ n^k \,\middle|\, n, \, k \in \mathbb{Z}_{\geqslant 2} \right\}\right)$$

is turned out to be the elegant form (62).

The research topic of series involving zeta functions such as (60), (61), and (62) has been popularly investigated by many researchers who have presented closed form expressions of a variety of series involving zeta functions and given applications (see, e.g., [29, Chapter 3] and the references cited therein; for recent ones, see also [1], [2], [9], [11], [25]).

Example 4. Setting $p = \frac{1}{2}$ in (56) produces

$$\begin{aligned}
\sum_{k=2}^{\infty} \frac{H_{k-1}}{k\left(k - \frac{1}{2}\right)^{m+1}} &= 2 \, \log^2 2 - \zeta(2) \\
&+ (-1)^{m+1} \, 2^m \sum_{\ell=1}^{m} (-1)^{\ell} \, 2^{-\ell} \Big\{ (\ell+1) \, (1 - 2^{\ell+2}) \, \zeta(\ell+2) \\
&+ 4 \, \log 2 \, (2^{\ell+1} - 1) \, \zeta(\ell+1) \\
&+ \sum_{j=1}^{\ell-1} (2^{j+1} - 1) \, (2^{\ell-j+1} - 1) \, \zeta(j+1) \, \zeta(\ell-j+1) \Big\}.
\end{aligned}$$

5. FURTHER REMARKS

The Stirling numbers $s(n,k)$ of the first kind are defined by the generating functions (see, e.g., [29, Section 1.6]):

$$z(z-1)\cdots(z-n+1) = (-1)^n\,(-z)_n = \sum_{k=0}^{n} s(n,k)\,z^k$$

and

$$\{\log(1+z)\}^k = k!\sum_{n=k}^{\infty} s(n,k)\,\frac{z^n}{n!} \quad (|z|<1),$$

where $(-z)_n$ denotes the familiar Pochhammer symbol. Shen [24] recognized certain useful relationships between the Stirling numbers $s(n,k)$ and the harmonic numbers such as

$$\begin{aligned}
s(n,2) =&(-1)^n\,(n-1)!\,H_{n-1};\\
s(n,3) =&(-1)^{n+1}\,\frac{(n-1)!}{2}\,\left[(H_{n-1})^2 - H_{n-1}^{(2)}\right];\\
s(n,4) =&(-1)^{n}\,\frac{(n-1)!}{6}\,\left[(H_{n-1})^3 - 3H_{n-1}\,H_{n-1}^{(2)} + 2H_{n-1}^{(3)}\right];\\
s(n,5) =&(-1)^{n+1}\,\frac{(n-1)!}{24}\\
&\cdot\left[(H_{n-1})^4 + 8\,H_{n-1}\,H_{n-1}^{(3)} - 6\,(H_{n-1})^2\,H_{n-1}^{(2)} + 3\,\left(H_{n-1}^{(2)}\right)^2 - 6\,H_{n-1}^{(4)}\right].
\end{aligned}$$

Shen [24] evaluated several linear and nonlinear harmonic sums. By utilizing these relationships, it is possible to express the left-hand sides of the identities found in Theorems 5, 7, 13, and 15 in terms of the Stirling numbers of the first kind $s(n,k)$. For example, we have

$$\sum_{k=1}^{\infty}\frac{H_k}{k(n+k+1)^{m+1}\binom{n+k}{k}} = (-1)^{n+1}n!\sum_{k=n+1}^{\infty}\frac{(-1)^k s(k-n+1,2)}{k!(k+1)^{m+1}},$$

and

$$\sum_{k=2}^{\infty}\frac{H_{k-1}^2 - H_{k-1}^{(2)}}{k(p+n+k)^{m}\binom{n+k}{k}} = 2(-1)^{n+1}n!\sum_{k=n+3}^{\infty}\frac{(-1)^k s(k-n,3)}{k!(p+k)^{m}}.$$

Our further research can be motivated by the formula

$$\frac{d^k}{d\alpha^k}\binom{\alpha}{n} = k!\binom{\alpha}{n}\zeta_n(\{1\}_k;\alpha-n+1),$$

which was proven by Kaneko et al. [15] (see (3.24)).

Closed forms of certain results, such as (24), can be obtained by utilizing established identities (refer to [16, 17, 23]).

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