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APPLICATION OF THE FINK IDENTITY TO JENSEN-TYPE INEQUALITIES FOR HIGHER ORDER CONVEX FUNCTIONS

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The focus of this paper is the application of the Fink identity in obtaining Jensen-type inequalities for higher order convex functions. In addition to the basic form, we establish superadditivity and monotonicity relations that correspond to the Jensen inequality in this setting. We also obtain the corresponding Lah-Ribarič inequality. The obtained results are valid for functions of even degree of convexity. With this method, we derive some new bounds for the differences of power means, as well as some new Hölder-type inequalities.

1. INTRODUCTION

At the very end of the twentieth century, Dragomir et al. [**4**], introduced the Jensen functional as the difference between the right-hand side and the left-hand side of the Jensen inequality, i.e.

$$\mathcal{J}_m(f,\mathbf{x},\mathbf{p}) = \sum_{i=1}^{m} p_i f(x_i) - P_m f\left(\frac{1}{P_m}\sum_{i=1}^{m} p_i x_i\right), \tag{1}$$

where $f : I \subset \mathbb{R} \to \mathbb{R}$ is a convex function, $\mathbf{x} = (x_1, x_2, \ldots, x_m) \subset I^m$, $\mathbf{p} = (p_1, p_2, \ldots, p_m) \subset \mathbb{R}_+^m$ and $P_m = \sum_{i=1}^{m} p_i > 0$. Clearly, with the above assumptions, $\mathcal{J}_m(f,\mathbf{x},\mathbf{p})$ is non-negative. However, Dragomir et al. [**4**], noticed another

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important property of the Jensen functional, that is, the superadditivity. More precisely, they established the relation

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}+\mathbf{q}) \geq \mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) + \mathcal{J}_m(f, \mathbf{x}, \mathbf{q}), \tag{2}$$

where $\mathbf{p}, \mathbf{q} \subset \mathbb{R}_+^m$. At that time, it wasn't yet quite obvious that this relation would become the starting point in establishing refinements of numerous inequalities closely related to the Jensen inequality. Namely, one of the trivial consequences of superadditivity is the so called monotonicity of the Jensen functional, i.e.

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) \geq \mathcal{J}_m(f, \mathbf{x}, \mathbf{q}) \geq 0, \tag{3}$$

whenever $\mathbf{p} \geq \mathbf{q}$, i.e. $p_i \geq q_i$, $i = 1, 2, \ldots, m$ (see also [**11**], p.717).

Some ten years ago, Krnić et al. [**9**], noticed that relation (3) provides mutual bounds for functional (1) expressed in terms of the adjoint non-weighted functional. Namely, with the above order between n-tuples, every n-tuple can be mutually bounded by constant n-tuples, whereby we take the minimum coordinate for the lower bound and the maximum coordinate for the upper bound. Consequently, Krnić et al. [**9**], established the following bounds for the Jensen functional

$$mp_{\max}\mathcal{I}_m(f, \mathbf{x}) \geq \mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) \geq mp_{\min}\mathcal{I}_m(f, \mathbf{x}), \tag{4}$$

where $p_{\min} = \min_{1\leq i \leq m} p_i$, $p_{\max} = \max_{1\leq i\leq m} p_i$ and

$$\mathcal{I}_m(f, \mathbf{x}) = \frac{\sum_{i=1}^m f(x_i)}{m} - f\left(\frac{\sum_{i=1}^m x_i}{m}\right).$$

Obviously, the lower bound in (4) provides the refinement, while the upper one yields the reverse of the Jensen inequality. Based on this property of $\mathcal{J}_m(f, \mathbf{x}, \mathbf{p})$, numerous improvements of some important inequalities (such as inequalities of Young and Hölder, means inequalities etc.) have been established (for more details, see [**8, 9**] and the references cited therein). Moreover, for a comprehensive inspection of the classical and new results in connection to the Jensen inequality, the interested reader is referred to monographs [**7, 11, 12**] and the references cited therein.

Nowadays, a lot of attention is paid to Jensen-type inequalities related to some other variants of convexity such as the higher-order convexity, operator convexity etc. In this paper, we deal with Jensen-type inequalities in companion with the higher order convexity. Recall that an n-convex function is defined via the n-th order divided difference. The n-th order divided difference of a function $f\colon [a, b] \to \mathbb{R}$ at mutually distinct points $t_0, t_1, ..., t_n \in [a, b]$ is defined recursively by

$$f[t_i] = f(t_i), \quad i = 0, ..., n,$$
$$f[t_0, ..., t_n] = \frac{f[t_1, ..., t_n] - f[t_0, ..., t_{n-1}]}{t_n - t_0}.$$

The above definition can be extended to cover the cases in which some or all points coincide (see, e.g. [**1, 12**]). Therefore, a function $f\colon [a, b] \to \mathbb{R}$ is said to be n-convex, $n \geq 0$, if $f[t_0, ..., t_n] \geq 0$ for all choices of $n+1$ distinct points $t_0, t_1, ..., t_n \in [a, b]$. On the other hand, if $f[t_0, ..., t_n] \leq 0$, f is said to be n-concave. The higher order convexity is indeed an extension of the mere convexity, 2-convex functions are just convex functions, 1-convex functions are increasing functions, while 0-convex functions are non-negative functions. The most important and, at the same time, the simplest characterization of the n-convexity asserts that if the n-th order derivative $f^{(n)}$ exists, then the function f is n-convex if and only if $f^{(n)} \geq 0$ (for more details, see [**1, 12**]).

The crucial role in establishing Jensen-type inequalities with respect to higher order convexity is played by certain integral representations of n-times differentiable functions. By using such transformations, the Jensen functional (1) takes a form which is suitable to study in higher order convexity setting. Initial steps in this direction have been made in [**13**], where the authors established several Jensen-type inequalities by virtue of the Montgomery identity (see [**11**]). Moreover, in our recent paper [**2**], we have extended results from [**13**] in view of superadditivity, monotonicity and mutual boundedness of the Jensen functional.

The starting point in this paper is another integral representation of an n-times differentiable function, the so called Fink identity. Let $n \in \mathbb{N}$ and let $a, b \in \mathbb{R}$. The Fink identity asserts that if the function $f : [a, b] \to \mathbb{R}$ is such that $f^{(n-1)}$ is absolutely continuous, then holds the identity

$$
\begin{aligned}
f(x) = & \frac{n}{b-a} \int_a^b f(t)dt + \frac{1}{b-a} \sum_{k=1}^{n-1} \Phi_{a,b}^k(x) \\
& + \frac{1}{(n-1)!(b-a)} \int_a^b R_n(x,s) f^{(n)}(s) ds,
\end{aligned}
\tag{5}
$$

where

$$
\Phi_{a,b}^k(x) = \frac{n-k}{k!} \left[f^{(k-1)}(b)(x-b)^k - f^{(k-1)}(a)(x-a)^k \right]
\tag{6}
$$

and

$$
R_n(x,s) = \begin{cases} (x-s)^{n-1}(s-a), & a \leq s \leq x \leq b \\ (x-s)^{n-1}(s-b), & a \leq x < s \leq b. \end{cases}
$$

For more details about the Fink identity, the reader is referred to [**5**].

Some basic Jensen-type inequalities for higher convex functions via the Fink identity have been recently established in [**3**], which will be discussed in the next section. Similarly to our recent paper [**2**], the main goal of the present article is to extend the results from [**3**], taking into account the discussed properties of the Jensen functional.

This paper consists of five sections. After the Introduction, in Section 2 we discuss and also give slight extensions of discrete Jensen-type inequalities derived in [**3**]. A special attention is paid to Jensen-type inequalities involving functions with even degree of convexity. In particular, we derive the corresponding version of the Lah-Ribarič inequality. In addition, we also show that in the case of the mere convexity, the obtained results coincide with already known Jensen-type inequalities given in this Introduction. Further, in Section 3 we derive superadditivity and monotonicity relations for the Jensen inequality with respect to the higher-order convexity. Namely, we establish Jensen-type inequalities for n-convex functions (n even) that correspond to relations (2), (3) and (4). Similarly to [**3**], the key idea of proving our results is a transformation of the Jensen functional via the Fink identity. In this way, we get a form of the functional that is suitable to study in described setting. The established results hold for functions of even degree of convexity since only in this case we are able to determine the sign of the integral appearing in transformed relations. We have already mentioned that in the case of $n = 2$ the obtained results coincide with the classical Jensen-type inequalities. Generally speaking, we are not able to determine the level of precision of the obtained inequalities, in some cases they are more accurate than the classical Jensen inequality, but they can also be weaker. As an application, in Section 4 we obtain mutual bounds for the differences of power means which can represent improvement of the monotonicity property of means in some cases. Finally, in Section 5 we derive several new Hölder-type inequalities based on the method developed in this paper.

2. ANOTHER INSIGHT INTO THE BASIC JENSEN-TYPE INEQUALITIES OBTAINED VIA THE FINK IDENTITY

The main objective of this section is a more detailed discussion and analysis of some results established by Butt et al. in their recent paper [**3**]. A key step in this approach is a transformation of the Jensen functional using the Fink identity (5). The following transformation of the Jensen functional has been derived in [**3**], however, we give it here in a form that will be more convenient in our future study. Moreover, we also give the corresponding proof since it has been omitted in [**3**] and since it will be important in our further analysis.

Theorem 1 (see also [**3**]). *Let $f : [a, b] \to \mathbb{R}$ be such that $f^{(n-1)}$ is absolutely continuous, let $\mathbf{x} = (x_1, x_2, \dots, x_m) \in [a, b]^m$ and let $\mathbf{p} = (p_1, p_2, \dots, p_m) \in \mathbb{R}_+^m$. Then holds the identity*

(7)
$$\begin{aligned}\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) = \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{m} p_i \Phi_{a,b}^k(x_i) - P_m \Phi_{a,b}^k(x_{P_m})\right)\\ + \frac{1}{(n-1)!(b-a)}\int_a^b \left(\sum_{i=1}^{m} p_i R_n(x_i, s) - P_m R_n(x_{P_m}, s)\right) f^{(n)}(s)ds,\end{aligned}$$

where $P_m = \sum_{i=1}^m p_i$, $x_{P_m} = \frac{1}{P_m}\sum_{i=1}^m p_i x_i$, and where $\Phi_{a,b}^k(\cdot)$ is defined by (6).

Proof. Substituting x_i, $i = 1, 2, \dots, m$, in the Fink identity (5), multiplying the corresponding identity by p_i and summing all identities, we have that

$$\begin{aligned}\sum_{i=1}^{m} p_i f(x_i) = \frac{nP_m}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{m} p_i \Phi_{a,b}^k(x_i)\right) \\ + \frac{1}{(n-1)!(b-a)}\int_a^b \left(\sum_{i=1}^{m} p_i R_n(x_i,s)\right) f^{(n)}(s)ds,\end{aligned} \tag{8}$$

after changing the order of summation. Similarly, putting $x_{P_m} = \frac{1}{P_m}\sum_{i=1}^m p_i x_i$ in (5), we obtain the identity

$$\begin{aligned}P_m f(x_{P_m}) = \frac{nP_m}{b-a}\int_a^b f(t)dt + \frac{1}{b-a}\sum_{k=1}^{n-1} P_m \Phi_{a,b}^k(x_{P_m}) \\ + \frac{P_m}{(n-1)!(b-a)}\int_a^b R_n(x_{P_m},s) f^{(n)}(s)ds.\end{aligned}$$

Hence, (7) follows by subtracting the previous two relations. □

Remark 2. In [**3**], the authors have proved that the function $R_n(\cdot,s)$ is convex, with respect to the first variable and for every fixed s, when $n \ge 4$ is even integer. Clearly, this follows by taking its second partial derivative (with respect to the first variable):

$$\frac{\partial^2 R_n}{\partial x^2}(x,s) = \begin{cases} (n-1)(n-2)(x-s)^{n-3}(s-a), & a \le s \le x \le b, \\ (n-1)(n-2)(x-s)^{n-3}(s-b), & a \le x < s \le b. \end{cases}$$

However, if $n = 2$, then

$$R_2(x,s) = \begin{cases} (x-s)(s-a), & a \le s \le x \le b \\ (x-s)(s-b), & a \le x < s \le b. \end{cases}$$

It should be noticed here that $R_2(\cdot,s)$ is a piecewise linear function which is obviously convex on $[a,b]$ for every fixed $s \in [a,b]$. In conclusion, $R_n(\cdot,s)$ is convex for all positive even integers n.

Due to the previous remark, the integral on the right-hand side of identity (7) is non-negative whenever the function f is n-convex, where n is non-negative even integer. Of course, this fact provides a slightly extended Jensen-type inequality which has been established in [**3**]. We give it here in a more suitable form.

Theorem 3 (see also [**3**]). *Let n be positive even integer and let $f : [a,b] \to \mathbb{R}$ be n-convex function such that $f^{(n-1)}$ is absolutely continuous. If $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a,b]^m$ and $\mathbf{p} = (p_1, p_2, \ldots, p_m) \in \mathbb{R}_+^m$, then holds the inequality*

$$\mathcal{J}_m(f,\mathbf{x},\mathbf{p}) \ge \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{m} p_i \Phi_{a,b}^k(x_i) - P_m \Phi_{a,b}^k(x_{P_m})\right), \tag{9}$$

where $P_m = \sum_{i=1}^m p_i$, $x_{P_m} = \frac{1}{P_m}\sum_{i=1}^m p_i x_i$, and where $\Phi_{a,b}^k(\cdot)$ is defined by (6)*. Furthermore, if f is n-concave then the sign of inequality* (9) *is reversed.*

Let f be a function that is both n-convex and convex in a classical sense, for example $f(x) = x^n$, $x \in [a, b]$, where n is even. Generally speaking, if the right-hand side of inequality (9) is non-negative, then it represents the improvement of the Jensen inequality (1). Otherwise, inequality (9) is weaker than the usual Jensen inequality. However, in the case of $n = 2$, things become much clearer.

Remark 4. If $n = 2$, inequality (9) reduces to

$$\mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) \geq \frac{1}{b-a}\left(\sum_{i=1}^{m} p_i \Phi_{a,b}^1(x_i) - P_m \Phi_{a,b}^1(x_{P_m})\right), \tag{10}$$

where $\Phi_{a,b}^1(x) = f(b)(x-b) - f(a)(x-a)$ is the linear function. Since the Jensen inequality becomes equality in the case of linear function, the right-hand side of (10) is equal to zero, which yields positivity of the Jensen functional. This fact can also be derived in another way. Namely, if $n = 2$, identity (7) becomes

$$\begin{aligned} \mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) &= \frac{1}{b-a}\int_a^b \left(\sum_{i=1}^{m} p_i R_2(x_i, s) - P_m R_2(x_{P_m}, s)\right) f''(s)ds \\ &= \sum_{i=1}^{m} p_i F(x_i) - P_m F(x_{P_m}), \end{aligned} \tag{11}$$

since $\sum_{i=1}^{m} p_i \Phi_{a,b}^1(x_i) - P_m \Phi_{a,b}^1(x_{P_m}) = 0$. Here, F stands for the function

$$F(x) = \frac{1}{b-a}\int_a^b R_2(x, s) f''(s)ds.$$

It is not hard to show that F is a convex function. Obviously, F is differentiable and we have that

$$\begin{aligned} F'(x) &= \frac{1}{b-a}\left[\int_a^x \frac{\partial R_2}{\partial x}(x, s) f''(s)ds + \int_x^b \frac{\partial R_2}{\partial x}(x, s) f''(s)ds\right] \\ &= \frac{1}{b-a}\left[\int_a^x (s-a) f''(s)ds + \int_x^b (s-b) f''(s)ds\right] = f'(x), \end{aligned}$$

that is, $F''(x) = f''(x)$. Hence, F is convex on $[a, b]$, which again yields positivity of the Jensen functional. In each case, the derived result is meaningful, so inequality (9) can be regarded as an extension of the classical Jensen inequality.

Remark 5. Let's return to identity (11) once more. This formula represents integral form of the Jensen functional for twice differentiable functions. In fact, this expression measures the difference between the right-hand side and left-hand side of the Jensen inequality using an integral. That is why identity (11) can be understood as an improvement of the classical Jensen inequality. Moreover, applying mutual bounds in (4) to the convex function $R_2(\cdot, s)$, identity (11) again provides usual bounds for the Jensen functional.

Let us conclude this section with a short discussion about one of the most interesting reverses of the Jensen inequality, the so called Lah-Ribarič inequality. The Lah-Ribarič inequality asserts that if $f : [a,b] \to \mathbb{R}$ is a convex function, $\mathbf{x} = (x_1, x_2, \dots, x_m) \subseteq [a,b]^m$, $\mathbf{p} = (p_1, p_2, \dots, p_m) \subset \mathbb{R}_+^m$, then

$$\frac{1}{P_m}\sum_{i=1}^{m} p_i f(x_i) \le \frac{b - x_{P_m}}{b-a} f(a) + \frac{x_{P_m} - a}{b-a} f(b), \tag{12}$$

where $P_m = \sum_{i=1}^m p_i$ and $x_{P_m} = \frac{1}{P_m}\sum_{i=1}^m p_i x_i$ (for more details, see [**10**] and [**12**, p. 98]).

Following the Jensen inequality equipped with the higher order convexity, Butt et al. [**3**], also derived a version of the Lah-Ribarič inequality that leans on higher order convexity, by rewriting (12) as the difference between the left-hand side and the right-hand side. On the other hand, it is much more natural to define the Lah-Ribarič functional as the difference between the right-hand side and the left-hand side of (12) multiplied by P_m:

$$\mathcal{L}_m(f, \mathbf{x}, \mathbf{p}) = P_m\left(\frac{b - x_{P_m}}{b-a} f(a) + \frac{x_{P_m} - a}{b-a} f(b)\right) - \sum_{i=1}^{m} p_i f(x_i). \tag{13}$$

Now, the main goal is to transform the above functional using the Fink identity. More precisely, considering (5) with $x = a$ and $x = b$, we arrive at the following identity:

$$\begin{aligned}
&P_m\left(\frac{b - x_{P_m}}{b-a} f(a) + \frac{x_{P_m} - a}{b-a} f(b)\right) \\
&= \frac{nP_m}{b-a}\int_a^b f(t)dt + \frac{P_m}{(b-a)^2}\sum_{k=1}^{n-1}\left((b - x_{P_m})\Phi_{a,b}^k(a) + (x_{P_m} - a)\Phi_{a,b}^k(b)\right) \\
&\quad + \frac{P_m}{(n-1)!(b-a)^2}\int_a^b \left((b - x_{P_m})R_n(a,s) + (x_{P_m} - a)R_n(b,s)\right) f^{(n)}(s)ds.
\end{aligned}$$

Furthermore, subtracting (8) from the above identity, we obtain the following form of the Lah-Ribarič functional:

$$\begin{aligned}
&\mathcal{L}_m(f, \mathbf{x}, \mathbf{p}) \\
&= \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\frac{P_m}{b-a}\left((b - x_{P_m})\Phi_{a,b}^k(a) + (x_{P_m} - a)\Phi_{a,b}^k(b)\right) - \sum_{i=1}^{m} p_i \Phi_{a,b}^k(x_i)\right) \\
&\quad + \frac{1}{(n-1)!(b-a)}\left(\frac{P_m}{b-a}\int_a^b \left((b - x_{P_m})R_n(a,s) + (x_{P_m} - a)R_n(b,s)\right) f^{(n)}(s)ds \right.\\
&\qquad\qquad \left. - \int_a^b \left(\sum_{i=1}^{m} p_i R_n(x_i, s)\right) f^{(n)}(s)ds\right).
\end{aligned} \tag{14}$$

Let's note once again that in [**3**] functional (14) has the opposite sign. Now, following the ideas as in Theorem 3, Butt et al. [**3**], established the Lah-Ribarič inequality for n-convex functions, where n is even integer such that $n \geq 4$. However, taking into account our Remark 2, the corresponding result can also be extended for the case of $n = 2$.

Corollary 6 (see also [**3**]). *Let n be positive even integer and let $f : I \to \mathbb{R}$ be n-convex function such that $f^{(n-1)}$ is absolutely continuous. Then holds the inequality*

$$
\begin{aligned}
&\mathcal{L}_m(f, \mathbf{x}, \mathbf{p}) \\
&\geq \frac{1}{b-a} \sum_{k=1}^{n-1} \left(\frac{P_m}{b-a} \Big((b - x_{P_m}) \Phi_{a,b}^k(a) + (x_{P_m} - a) \Phi_{a,b}^k(b) \Big) - \sum_{i=1}^{m} p_i \Phi_{a,b}^k(x_i) \right).
\end{aligned} \tag{15}
$$

Proof. Since the function $R_n(\cdot, s)$ is convex on $[a, b]$ for even n (see Remark 2), it follows that

$$
P_m \left(\frac{b - x_{P_m}}{b-a} R_n(a, s) + \frac{x_{P_m} - a}{b-a} R_n(b, s) \right) \geq \sum_{i=1}^{m} p_i R_n(x_i, s),
$$

by the Lah-Ribarič inequality (12). Therefore, the integral on the right-hand side of (14) is non-negative which implies (15). □

Remark 7. It should be noticed here that if f is n-concave then the sign of inequality (15) is reversed due to the fact that $f^{(n)}(s) \leq 0$, $s \in [a, b]$, for n-concave function.

3. SUPERADDITIVITY AND MONOTONICITY OF THE JENSEN FUNCTIONAL IN COMPANION WITH THE HIGHER ORDER CONVEXITY

The main objective of this section is to establish the corresponding superadditivity and monotonicity relations of the Jensen functional for n-convex functions, where n is even. Clearly, the key role will be played by identity (7), established via the Fink identity. Our first result is an extension of (2), i.e. superadditivity form of the Jensen functional in the described setting.

Theorem 8. *Let n be positive even integer and let $f : I \to \mathbb{R}$ be n-convex function such that $f^{(n-1)}$ is absolutely continuous. Then holds the inequality*

$$
\begin{aligned}
&\mathcal{J}_m(f, \mathbf{x}, \mathbf{p} + \mathbf{q}) \\
&\geq \mathcal{J}_m(f, \mathbf{x}, \mathbf{p}) + \mathcal{J}_m(f, \mathbf{x}, \mathbf{q}) \\
&\quad + \frac{1}{b-a} \sum_{k=1}^{n-1} \Big(P_m \Phi_{a,b}^k(x_{P_m}) + Q_m \Phi_{a,b}^k(x_{Q_m}) - (P_m + Q_m) \Phi_{a,b}^k(x_{P_m + Q_m}) \Big),
\end{aligned} \tag{16}
$$

where $\mathbf{x} \in [a,b]^m \subset I^m$, $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^m$, $P_m = \sum_{i=1}^m p_i$, $x_{P_m} = \frac{1}{P_m}\sum_{i=1}^m p_i x_i$, *and where* $\Phi_{a,b}^k(\cdot)$ *is defined by* (6).

Proof. The first step we have to do is rewrite the Jensen functional (7) with m-tuple $\mathbf{p}+\mathbf{q}$ instead of $\mathbf{p}$, that is,

$$
\begin{aligned}
&\mathcal{J}_m(f,\mathbf{x},\mathbf{p}+\mathbf{q})\\
&= \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\sum_{i=1}^m (p_i+q_i)\Phi_{a,b}^k(x_i) - (P_m+Q_m)\Phi_{a,b}^k(x_{P_m+Q_m})\right)\\
&\quad + \frac{1}{(n-1)!(b-a)}\int_a^b \left(\sum_{i=1}^m (p_i+q_i)R_n(x_i,s) - (P_m+Q_m)R_n(x_{P_m+Q_m},s)\right) f^{(n)}(s)ds.
\end{aligned}
$$

Therefore, we arrive at the following identity

$$
\begin{aligned}
&\mathcal{J}_m(f,\mathbf{x},\mathbf{p}+\mathbf{q}) - \mathcal{J}_m(f,\mathbf{x},\mathbf{p}) - \mathcal{J}_m(f,\mathbf{x},\mathbf{q})\\
&= \frac{1}{b-a}\sum_{k=1}^{n-1}\left(P_m\Phi_{a,b}^k(x_{P_m}) + Q_m\Phi_{a,b}^k(x_{Q_m}) - (P_m+Q_m)\Phi_{a,b}^k(x_{P_m+Q_m})\right)\\
&\quad + \frac{1}{(n-1)!(b-a)}\int_a^b \Big(P_mR_n(x_{P_m},s) + Q_mR_n(x_{Q_m},s)\\
&\qquad\qquad - (P_m+Q_m)R_n(x_{P_m+Q_m},s)\Big) f^{(n)}(s)ds.
\end{aligned}
\tag{17}
$$

Furthermore, since the function $R_n(\cdot,s)$ is convex on $[a,b]$ for every fixed value s and for even $n \geq 2$, we have that

$$
\begin{aligned}
&P_mR_n(x_{P_m},s) + Q_mR_n(x_{Q_m},s)\\
&= (P_m+Q_m)\left(\frac{P_m}{P_m+Q_m}R_n(x_{P_m},s) + \frac{Q_m}{P_m+Q_m}R_n(x_{Q_m},s)\right)\\
&\geq (P_m+Q_m)\,R_n\left(\frac{P_mx_{P_m}+Q_mx_{Q_m}}{P_m+Q_m},s\right)\\
&= (P_m+Q_m)R_n(x_{P_m+Q_m},s),
\end{aligned}
$$

which together with $f^{(n)}(s) \geq 0$ yields positivity of the integral on the right-hand side of (17). This means that (16) holds, as claimed. □

If $n=2$, then the sum on the right-hand side of inequality (16) vanishes, so we obtain the starting superadditivity relation (2).

Remark 9. If the function $f : I \to \mathbb{R}$ from Theorem 8 is n-concave, then the sign of inequality (16) is reversed. Namely, the case of the reversed inequality sign in (16) holds due to the fact that $f^{(n)}(s) \leq 0$, $s \in [a,b]$, for n-concave function. Of course, this fact will be valid for the results throughout this section, so it will be omitted.

Now, Theorem 8 implies some kind of monotonicity of the Jensen functional which can be regarded as an extension of the classical relation (3).

Corollary 10. *Let n be positive even integer and let $f : I \to \mathbb{R}$ be n-convex function such that $f^{(n-1)}$ is absolutely continuous. Further, let $\mathbf{x} \in [a,b]^m \subset I^m$ and let $\mathbf{p}, \mathbf{q} \in \mathbb{R}_+^m$ be such that $\mathbf{p} \geq \mathbf{q}$. Then holds the inequality*

$$\begin{aligned} &\mathcal{J}_m(f,\mathbf{x},\mathbf{p}) - \mathcal{J}_m(f,\mathbf{x},\mathbf{q}) \\ &\geq \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\sum_{i=1}^m (p_i - q_i)\Phi_{a,b}^k(x_i) - P_m\Phi_{a,b}^k(x_{P_m}) + Q_m\Phi_{a,b}^k(x_{Q_m})\right), \end{aligned} \tag{18}$$

where $P_m = \sum_{i=1}^m p_i$, $x_{P_m} = \frac{1}{P_m}\sum_{i=1}^m p_i x_i$, and where $\Phi_{a,b}^k(\cdot)$ is defined by (6).

Proof. Considering (16) with $\mathbf{p}-\mathbf{q}$ instead of $\mathbf{p}$, we have that

$$\begin{aligned} &\mathcal{J}_m(f,\mathbf{x},\mathbf{p}) - \mathcal{J}_m(f,\mathbf{x},\mathbf{p}-\mathbf{q}) - \mathcal{J}_m(f,\mathbf{x},\mathbf{q}) \\ &\geq \frac{1}{b-a}\sum_{k=1}^{n-1}\Big((P_m - Q_m)\Phi_{a,b}^k(x_{P_m-Q_m}) + Q_m\Phi_{a,b}^k(x_{Q_m}) - P_m\Phi_{a,b}^k(x_{P_m})\Big). \end{aligned}$$

In addition, inequality (9) yields

$$\mathcal{J}_m(f,\mathbf{x},\mathbf{p}-\mathbf{q}) \geq \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\sum_{i=1}^m (p_i-q_i)\Phi_{a,b}^k(x_i) - (P_m - Q_m)\Phi_{a,b}^k(x_{P_m-Q_m})\right),$$

which together with the previous relation implies (18), as claimed. □

If $n = 2$, then the right-hand side of (18) vanishes, which provides usual monotonicity of the Jensen functional for a convex function in the classical sense.

Remark 11. Inequality (18) can be derived directly from identity (7), without using Theorem 8. For this purpose, let's start with the identity:

$$\begin{aligned} &\mathcal{J}_m(f,\mathbf{x},\mathbf{p}) - \mathcal{J}_m(f,\mathbf{x},\mathbf{q}) \\ &= \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\sum_{i=1}^m (p_i - q_i)\Phi_{a,b}^k(x_i) - P_m\Phi_{a,b}^k(x_{P_m}) + Q_m\Phi_{a,b}^k(x_{Q_m})\right) \\ &+ \frac{1}{(n-1)!(b-a)}\int_a^b \left(\sum_{i=1}^m p_i R_n(x_i,s) - P_m R_n(x_{P_m},s)\right) f^{(n)}(s)ds \\ &- \frac{1}{(n-1)!(b-a)}\int_a^b \left(\sum_{i=1}^m q_i R_n(x_i,s) - Q_m R_n(x_{Q_m},s)\right) f^{(n)}(s)ds. \end{aligned} \tag{19}$$

Now, since $R_n(\cdot,s)$ is convex function in a usual sense, the classical monotonicity principle (3) implies that the relation

$$\sum_{i=1}^m p_i R_n(x_i,s) - P_m R_n(x_{P_m},s) \geq \sum_{i=1}^m q_i R_n(x_i,s) - Q_m R_n(x_{Q_m},s)$$

holds for every fixed value $s \in [a, b]$. Consequently, the difference between two integrals in (19) is non-negative, so (18) holds.

In order to conclude this section, we give mutual bounds for the Jensen functional (7) in terms of the adjoint non-weighted functional. The corresponding bounds that hold for n-convex functions, where n is even, can be regarded as an extension of bounds in (4).

Corollary 12. *Let n be positive even integer and let $f : I \to \mathbb{R}$ be n-convex function such that $f^{(n-1)}$ is absolutely continuous. Then hold the inequalities*

$$\begin{aligned} &\mathcal{J}_m(f,\mathbf{x},\mathbf{p}) - mp_{\min}\mathcal{I}_m(f,\mathbf{x}) \\ &\geq \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{m}(p_i - p_{\min})\Phi_{a,b}^k(x_i) - P_m\Phi_{a,b}^k(x_{P_m}) + mp_{\min}\Phi_{a,b}^k(\overline{x}_m)\right) \end{aligned} \tag{20}$$

and

$$\begin{aligned} &mp_{\max}\mathcal{I}_m(f,\mathbf{x}) - \mathcal{J}_m(f,\mathbf{x},\mathbf{p}) \\ &\geq \frac{1}{b-a}\sum_{k=1}^{n-1}\left(\sum_{i=1}^{m}(p_{\max} - p_i)\Phi_{a,b}^k(x_i) - mp_{\max}\Phi_{a,b}^k(\overline{x}_m) + P_m\Phi_{a,b}^k(x_{P_m})\right), \end{aligned} \tag{21}$$

where $\mathbf{x} \in [a,b]^m \subset I^m$, $\mathbf{p} \in \mathbb{R}_+^m$, $p_{\min} = \min_{1\leq i\leq m} p_i$, $p_{\max} = \max_{1\leq i\leq m} p_i$, $P_m = \sum_{i=1}^m p_i$, $x_{P_m} = \frac{1}{P_m}\sum_{i=1}^m p_i x_i$, $\overline{x}_m = \frac{1}{m}\sum_{i=1}^m x_i$, and where $\Phi_{a,b}^k(\cdot)$ is defined by (6).

Proof. Since $(p_{\min}, p_{\min}, \ldots, p_{\min}) \leq \mathbf{p} \leq (p_{\max}, p_{\max}, \ldots, p_{\max})$, the corresponding result follows from (18). □

Note also that the right-hand sides of (20) and (21) vanish for $n = 2$, so in this setting we obtain already known bounds (4).

In the sequel, we will establish some new estimates for power means. In particular, we obtain new arithmetic-geometric and arithmetic-harmonic mean inequalities in both quotient and difference forms. Similar results, in companion with the Montgomery identity, have been established in our earlier paper [**2**].

4. APPLICATION TO POWER MEANS

A quasi-arithmetic mean M_ψ of $\mathbf{x} = (x_1, x_2, \ldots, x_m) \in [a,b]^m$ with weights $\mathbf{p} = (p_1, p_2, \ldots, p_m) \subset \mathbb{R}_+^m$ is defined by

$$M_\psi(\mathbf{x},\mathbf{p}) = \psi^{-1}\left(\frac{1}{P_m}\sum_{i=1}^m p_i\psi(x_i)\right), \tag{22}$$

where $\psi : [a, b] \to \mathbb{R}$ is a continuous strictly monotone function and $P_m = \sum_{i=1}^{m} p_i$. If $p_1 = p_2 = \cdots = p_m$, we obtain the non-weighted mean, denoted by

$$m_\psi(\mathbf{x}) = \psi^{-1}\left(\frac{1}{m}\sum_{i=1}^{m}\psi(x_i)\right).$$

Throughout this section we denote $\psi_a = \min\{\psi(a), \psi(b)\}$, $\psi_b = \max\{\psi(a), \psi(b)\}$, where χ and ψ are strictly monotone functions. In this setting we consider the Jensen functional

$$\mathcal{J}_m(\chi \circ \psi^{-1}, \psi(\mathbf{x}), \mathbf{p}) = P_m\left(\chi(M_\chi(\mathbf{x}, \mathbf{p})) - \chi(M_\psi(\mathbf{x}, \mathbf{p}))\right),$$

that is, the inequality

$$\chi(M_\chi(\mathbf{x}, \mathbf{p})) \geq \chi(M_\psi(\mathbf{x}, \mathbf{p})), \tag{23}$$

provided that $\chi \circ \psi^{-1}$ is a convex function (for more details, see [**11, 12**]). Of course, inequalities in (4) provide the following refinement and reverse of (23) (for more details, see [**8, 9**]):

$$\begin{aligned} &mp_{\max}\left(\chi(m_\chi(\mathbf{x})) - \chi(m_\psi(\mathbf{x}))\right)\\ &\geq P_m\left(\chi(M_\chi(\mathbf{x}, \mathbf{p})) - \chi(M_\psi(\mathbf{x}, \mathbf{p}))\right)\\ &\geq mp_{\min}\left(\chi(m_\chi(\mathbf{x})) - \chi(m_\psi(\mathbf{x}))\right). \end{aligned} \tag{24}$$

The most typical example of quasi-arithmetic mean (22) is a power mean defined by

$$M_r(\mathbf{x}, \mathbf{p}) = \begin{cases} \left(\frac{1}{P_m}\sum_{i=1}^{m} p_i {x_i}^r\right)^{\frac{1}{r}}, & r \neq 0,\\ \left(\prod_{i=1}^{m} {x_i}^{p_i}\right)^{\frac{1}{P_m}}, & r = 0. \end{cases} \tag{25}$$

Clearly, (25) is a special case of (22) when $\psi(t) = t^r$, for $r \neq 0$, and $\psi(t) = \log t$, for $r = 0$. Recall that $r = 1, 0, -1$ provide the arithmetic, geometric and harmonic mean, respectively. The most important power mean inequality describes monotonicity property of means and asserts that if $r < s$, then

$$M_r(\mathbf{x}, \mathbf{p}) \leq M_s(\mathbf{x}, \mathbf{p}). \tag{26}$$

If $p_1 = p_2 = \cdots = p_m$, we obtain the non-weighted power mean

$$m_r(\mathbf{x}) = \begin{cases} \left(\frac{1}{m}\sum_{i=1}^{m} {x_i}^r\right)^{\frac{1}{r}}, & r \neq 0,\\ \left(\prod_{i=1}^{m} x_i\right)^{\frac{1}{m}}, & r = 0. \end{cases}$$

Of course, by putting $\chi(t) = t^s$ and $\psi(t) = t^r$, $s, r \neq 0$, in (24), one obtains refinement and reverse of (26), i.e.

$$\frac{mp_{\max}}{P_m}(m_s^s(\mathbf{x}) - m_r^s(\mathbf{x})) \geq M_s^s(\mathbf{x}, \mathbf{p}) - M_r^s(\mathbf{x}, \mathbf{p}) \geq \frac{mp_{\min}}{P_m}(m_s^s(\mathbf{x}) - m_r^s(\mathbf{x})), \tag{27}$$

provided that $s \leq 0 < r$, $r < 0 \leq s$, $0 < r \leq s$ or $s \leq r < 0$ (see also [**6**]). For more details about quasi-arithmetic and power means, the reader is referred to monographs [**11, 12**], as well as to papers [**6, 8, 9**] and the references cited therein.

Now, our aim is to derive power mean inequalities relying on higher order convexity. The starting point in this direction is to find a suitable conditions under which the function $f = \chi \circ \psi^{-1}$ is n-convex or n-concave.

Since the power means are special cases of quasi-arithmetic means for particular choices of χ and ψ, let us first set $\chi(t) = t^s$ and $\psi(t) = t^r$, $t > 0$, where $r, s \neq 0$. In this case the function $f(t) = (\chi \circ \psi^{-1})(t) = t^{\frac{s}{r}}$ is n times differentiable, for every $n \in \mathbb{N}$, and we have that

$$f^{(n)}(t) = \prod_{k=1}^{n} \left(\frac{s}{r} - k + 1\right) t^{\frac{s}{r}-n} = \left(\frac{s}{r}\right)^{\underline{n}} t^{\frac{s}{r}-n},$$

where $\underline{n}$ stands for the falling factorial, i.e. $x^{\underline{n}} = x(x-1)\cdots(x-n+1)$. Now, we need to discuss the sign of the above n-th derivation. Of course, we are interested in the case when n is even. Then, $f^{(n)}(t) \geq 0$ if

$$\frac{s}{r} \in (-\infty, 0] \cup [1,2] \cup [3,4] \cup \ldots \cup [n-3, n-2] \cup [n-1, \infty), \tag{28}$$

while $f^{(n)}(t) < 0$ if

$$\frac{s}{r} \in (0,1) \cup (2,3) \cup (4,5) \cup \ldots \cup (n-2, n-1). \tag{29}$$

We proceed with cases when one of the parameters r and s is equal to zero. If $s = 0$, then by putting $\chi(t) = \log t$ and $\psi(t) = t^r$, we have that $f(t) = (\chi \circ \psi^{-1})(t) = \frac{1}{r}\log t$ is n times differentiable for every $n \in \mathbb{N}$, and so

$$f^{(n)}(t) = \frac{1}{r}(-1)^{n-1}(n-1)!\; t^{-n}.$$

Consequently, it follows that f is n-convex if $r > 0$ and n is odd, or if $r < 0$ and n is even. On the other hand, f is n-concave if $r > 0$ and n is even, or if $r < 0$ and n is odd. In the case when $r > 0$ the function $\psi(t) = t^r$ is strictly increasing, so $\psi_a = a^r$ and $\psi_b = b^r$, while for $r < 0$ we have $\psi_a = b^r$ and $\psi_b = a^r$.

Finally, if $r = 0$, then by putting $\chi(t) = t^s$ and $\psi(t) = \log t$, we obtain that the function $f(t) = (\chi \circ \psi^{-1})(t) = e^{st}$ is also n times differentiable, for every $n \in \mathbb{N}$, so we have $f^{(n)}(t) = s^n\, e^{st}$. Hence, if $s > 0$, then f is n-convex for every $n \in \mathbb{N}$. Otherwise, if $s < 0$, then f is n-convex for every even n, and n-concave for every odd n. In addition, the function $\psi(t) = \log t$ is strictly increasing, so in this case we have $\psi_a = \log a$ and $\psi_b = \log b$. For more detailed discussion about the n-th derivative of the above functions, the reader is referred to our earlier paper [**2**].

Now, by virtue of the Jensen-type inequality (9) and the above discussion, we obtain a whole series of power mean inequalities that correspond to monotonicity relation (26). Our first result refers to parameters r and s not equal to zero.

Corollary 13. *Let n be positive even integer, let $r, s \neq 0$ be real numbers, and let $\mathbf{x} \in [a,b]^m \subset \mathbb{R}_+^m$, $\mathbf{p} \in \mathbb{R}_+^m$. If (28) holds, then*

$$\begin{aligned} & M_s^s(\mathbf{x},\mathbf{p}) - M_r^s(\mathbf{x},\mathbf{p}) \\ & \geq \frac{1}{b^r - a^r} \sum_{k=1}^{n-1} \frac{n-k}{k!} \overline{\left(\frac{s}{r}\right)}^{k-1} \left(\frac{1}{P_m} \sum_{i=1}^{m} p_i h_{a,b}^{k,r,s}(x_i) - h_{a,b}^{k,r,s}(M_r(\mathbf{x},\mathbf{p})) \right), \end{aligned} \tag{30}$$

where

$$h_{a,b}^{k,r,s}(x) = b^{s-r(k-1)}(x^r - b^r)^k - a^{s-r(k-1)}(x^r - a^r)^k. \tag{31}$$

Conversely, if (29) *holds, then the sign of inequality* (30) *is reversed.*

Proof. Let's consider inequality (9) with $f(t) = t^{\frac{s}{r}}$, with m-tuple $\mathbf{x}^r = (x_1^r, x_2^r, \ldots, x_m^r)$ instead of $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, and with $\min\{a^r, b^r\}$, $\max\{a^r, b^r\}$ instead of a, b, respectively. Then (30) holds due to an obvious formula

$$\Phi_{a^r,b^r}^k(x^r) = \frac{n-k}{k!} \overline{\left(\frac{s}{r}\right)}^{k-1} h_{a,b}^{k,r,s}(x).$$

□

Remark 14. In particular, let $s = 1$ and $r = -1$. Then, $\frac{s}{r} = -1$, which means that inequality (30) is valid in this case since (28) holds. It should be noticed here that this setting provides a variant of the arithmetic-harmonic inequality. Namely, by putting $q_i = \frac{P_m}{p_i}$, $i = 1, 2, \ldots, m$, we obtain the inequality

$$\begin{aligned} & \sum_{i=1}^{m} \frac{x_i}{q_i} - \frac{1}{\sum_{i=1}^{m} \frac{1}{q_i x_i}} \\ & \geq \frac{1}{b^{-1} - a^{-1}} \sum_{k=1}^{n-1} \frac{(-1)^{k-1}(n-k)}{k} \left(\sum_{i=1}^{m} \frac{h_{a,b}^{k,-1,1}(x_i)}{q_i} - h_{a,b}^{k,-1,1}\left(\frac{1}{\sum_{i=1}^{m} \frac{1}{q_i x_i}} \right) \right). \end{aligned}$$

Our next two consequences of inequality (9) refer to the cases when one of parameters r and s is equal to zero. Of course, in these cases we deal with a comparison with the geometric mean. In particular, we obtain variants of arithmetic-geometric mean inequality in both quotient and difference forms.

Corollary 15. *Let n be positive even integer, let $r < 0$, and let $\mathbf{x} \in [a,b]^m \subset \mathbb{R}_+^m$, $\mathbf{p} \in \mathbb{R}_+^m$. Then holds the inequality*

$$\begin{aligned} & \log M_0(\mathbf{x},\mathbf{p}) - \log M_r(\mathbf{x},\mathbf{p}) \\ & \geq \frac{1}{r(b^r - a^r)} \sum_{k=1}^{n-1} \frac{(-1)^{k-2}(n-k)}{(k-1)k} \left(\frac{1}{P_m} \sum_{i=1}^{m} p_i h_{a,b}^{k,r,0}(x_i) - h_{a,b}^{k,r,0}(M_r(\mathbf{x},\mathbf{p})) \right), \end{aligned} \tag{32}$$

where $h_{a,b}^{k,r,0}(\cdot)$ is defined by (31). *Otherwise, if $r > 0$, then the sign of* (32) *is reversed.*

Proof. It is similar to the proof of Corollary 13, except that we consider $f(t) = \frac{1}{r}\log t$ instead of $f(t) = t^{\frac{s}{r}}$. Then, (32) holds after utilizing the formula

$$\Phi^k_{a^r,b^r}(x^r) = \frac{(-1)^{k-2}(n-k)}{r(k-1)k} h^{k,r,0}_{a,b}(x).$$

□

Remark 16. In the case of $r = 1$, the reverse of (32) yields a new version of the arithmetic-geometric mean inequality in a quotient form. More precisely, by putting $r = 1$ and $q_i = \frac{P_m}{p_i}$, $i = 1, 2, \ldots, m$, in the reverse of (32), we arrive at the inequality

$$\left(\frac{\prod_{i=1}^m x_i^{\frac{1}{q_i}}}{\sum_{i=1}^m \frac{x_i}{q_i}}\right)^{b-a} \leq \exp\left(\sum_{k=1}^{n-1} \frac{(-1)^{k-2}(n-k)}{(k-1)k}\left(\sum_{i=1}^m \frac{h^{k,1,0}_{a,b}(x_i)}{q_i} - h^{k,1,0}_{a,b}\left(\sum_{i=1}^m \frac{x_i}{q_i}\right)\right)\right),$$

which holds for a positive even integer n.

Corollary 17. *Let n be positive even integer, let s be real number, and let $\mathbf{x} \in [a,b]^m \subset \mathbb{R}^m_+$, $\mathbf{p} \in \mathbb{R}^m_+$. Then holds the inequality*

$$\begin{aligned} & M_s^s(\mathbf{x},\mathbf{p}) - M_0^s(\mathbf{x},\mathbf{p}) \\ & \geq \frac{1}{\log\frac{b}{a}} \sum_{k=1}^{n-1} \frac{(n-k)s^{k-1}}{k!}\left(\frac{1}{P_m}\sum_{i=1}^m p_i l^{k,s}_{a,b}(x_i) - l^{k,s}_{a,b}(M_0(\mathbf{x},\mathbf{p}))\right), \end{aligned} \tag{33}$$

where

$$l^{k,s}_{a,b}(x) = b^s \log^k \frac{x}{b} - a^s \log^k \frac{x}{a}. \tag{34}$$

Proof. We consider (9) with $f(t) = e^{st}$ and with $\log \mathbf{x} = (\log x_1, \log x_2, \ldots, \log x_m)$, $\log a$, $\log b$ instead of $\mathbf{x} = (x_1, x_2, \ldots, x_m)$, a, b, respectively. Then, (33) holds due to

$$\Phi^k_{\log a,\log b}(\log x) = \frac{(n-k)s^{k-1}}{k!} l^{k,s}_{a,b}(x).$$

□

Remark 18. By putting $s = 1$ and $q_i = \frac{P_m}{p_i}$, $i = 1, 2, \ldots, m$, in (33), we arrive at the following version of the arithmetic-geometric mean inequality in a difference form:

$$\begin{aligned} & \sum_{i=1}^m \frac{x_i}{q_i} - \prod_{i=1}^m x_i^{\frac{1}{q_i}} \\ & \geq \frac{1}{\log\frac{b}{a}} \sum_{k=1}^{n-1} \frac{n-k}{k!}\left(\sum_{i=1}^m \frac{l^{k,1}_{a,b}(x_i)}{q_i} - l^{k,1}_{a,b}\left(\prod_{i=1}^m x_i^{\frac{1}{q_i}}\right)\right). \end{aligned} \tag{35}$$

Let's note once again that this inequality is valid for a positive even integer n. In this setting, we have that $\sum_{i=1}^m \frac{1}{q_i} = 1$. This form of the arithmetic-geometric mean inequality is usually referred to as the Young-type inequality.

If $n = 2$, then, according to our discussion in Remark 4, inequalities (30), (32) and (33) reduce to the basic mean inequality (26), since their right-hand sides vanish. Generally speaking, if $n > 2$ is even, then any of relations (30), (32) or (33) can represent refinement of the basic monotonicity relation (26) if its right-hand side is non-negative. Otherwise, it is weaker than (26).

Now, our goal is to derive versions of inequalities (30), (32) and (33) that rely on Corollary 12 from the previous section. In other words, we give mutual bounds for the differences of power means in terms of the corresponding non-weighted means (for all possible cases of parameters r and s). Of course, the established relations are related to the inequalities in (27).

Corollary 19. *Let n be positive even integer, let $r, s \neq 0$ be real numbers, and let $\mathbf{x} \in [a,b]^m \subset \mathbb{R}_+^m$, $\mathbf{p} \in \mathbb{R}_+^m$. If condition* (28) *holds, then hold the inequalities*

$$\begin{aligned}
&M_s^s(\mathbf{x},\mathbf{p}) - M_r^s(\mathbf{x},\mathbf{p}) - \frac{mp_{\min}}{P_m}\left(m_s^s(\mathbf{x}) - m_r^s(\mathbf{x})\right)\\
&\geq \frac{1}{b^r-a^r}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left(\frac{s}{r}\right)^{\underline{k-1}}\\
&\qquad\times\left(\frac{1}{P_m}\sum_{i=1}^{m}(p_i-p_{\min})h_{a,b}^{k,r,s}(x_i) - h_{a,b}^{k,r,s}(M_r(\mathbf{x},\mathbf{p})) + \frac{mp_{\min}}{P_m}h_{a,b}^{k,r,s}(m_r(\mathbf{x}))\right)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{mp_{\max}}{P_m}\left(m_s^s(\mathbf{x}) - m_r^s(\mathbf{x})\right) - \left(M_s^s(\mathbf{x},\mathbf{p}) - M_r^s(\mathbf{x},\mathbf{p})\right)\\
&\geq \frac{1}{b^r-a^r}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left(\frac{s}{r}\right)^{\underline{k-1}}\\
&\qquad\times\left(\frac{1}{P_m}\sum_{i=1}^{m}(p_{\max}-p_i)h_{a,b}^{k,r,s}(x_i) - \frac{mp_{\max}}{P_m}h_{a,b}^{k,r,s}(m_r(\mathbf{x})) + h_{a,b}^{k,r,s}(M_r(\mathbf{x},\mathbf{p}))\right),
\end{aligned}$$

where $h_{a,b}^{k,r,s}(\cdot)$ is defined by (31) *and $p_{\min} = \min_{1\leq i\leq m} p_i$, $p_{\max} = \max_{1\leq i\leq m} p_i$. Otherwise, if* (29) *holds, then the signs of both inequalities are reversed.*

Corollary 20. *Let n be positive even integer, let $r < 0$, and let $\mathbf{x} \in [a,b]^m \subset \mathbb{R}_+^m$, $\mathbf{p} \in \mathbb{R}_+^m$. Then hold the inequalities*

$$\begin{aligned}
&\log\frac{M_0(\mathbf{x},\mathbf{p})}{M_r(\mathbf{x},\mathbf{p})} - \frac{mp_{\min}}{P_m}\log\frac{m_0(\mathbf{x})}{m_r(\mathbf{x})}\\
&\geq \frac{1}{r(b^r-a^r)}\sum_{k=1}^{n-1}\frac{(-1)^{(k-2)}(n-k)}{(k-1)k}\\
&\qquad\times\left(\frac{1}{P_m}\sum_{i=1}^{m}(p_i-p_{\min})h_{a,b}^{k,r,0}(x_i) - h_{a,b}^{k,r,0}(M_r(\mathbf{x},\mathbf{p})) + \frac{mp_{\min}}{P_m}h_{a,b}^{k,r,0}(m_r(\mathbf{x}))\right)
\end{aligned}$$

and

$$\frac{mp_{\max}}{P_m}\log\frac{m_0(\mathbf{x})}{m_r(\mathbf{x})}-\log\frac{M_0(\mathbf{x},\mathbf{p})}{M_r(\mathbf{x},\mathbf{p})})$$
$$\geq\ \frac{1}{r(b^r-a^r)}\sum_{k=1}^{n-1}\frac{(-1)^{k-2}(n-k)}{(k-1)k}$$
$$\times\left(\frac{1}{P_m}\sum_{i=1}^{m}(p_{\max}-p_i)h_{a,b}^{k,r,0}(x_i)-\frac{mp_{\max}}{P_m}h_{a,b}^{k,r,0}(m_r(\mathbf{x}))+h_{a,b}^{k,r,0}(M_r(\mathbf{x},\mathbf{p}))\right),$$

where $h_{a,b}^{k,r,0}(\cdot)$ *is defined by* (31) *and* $p_{\min}=\min_{1\leq i\leq m}p_i$, $p_{\max}=\max_{1\leq i\leq m}p_i$. *Conversely, if* $r>0$*, then the signs of both inequalities are reversed.*

Corollary 21. *Let* n *be positive even integer, let* s *be real number and let* $\mathbf{x}\in[a,b]^m\subset\mathbb{R}_+^m$, $\mathbf{p}\in\mathbb{R}_+^m$. *Then,*

$$M_s^s(\mathbf{x},\mathbf{p})-M_0^s(\mathbf{x},\mathbf{p})-\frac{mp_{\min}}{P_m}\left(m_s^s(\mathbf{x})-m_0^s(\mathbf{x})\right)$$
$$\geq\ \frac{1}{\log\frac{b}{a}}\sum_{k=1}^{n-1}\frac{(n-k)s^{k-1}}{k!}$$
$$\times\left(\frac{1}{P_m}\sum_{i=1}^{m}(p_i-p_{\min})l_{a,b}^{k,s}(x_i)-l_{a,b}^{k,s}(M_0(\mathbf{x},\mathbf{p}))+\frac{mp_{\min}}{P_m}l_{a,b}^{k,s}(m_0(\mathbf{x}))\right)$$

and

$$\frac{mp_{\max}}{P_m}\left(m_s^s(\mathbf{x})-m_0^s(\mathbf{x})\right)-\left(M_s^s(\mathbf{x},\mathbf{p})-M_0^s(\mathbf{x},\mathbf{p})\right)$$
$$\geq\ \frac{1}{\log\frac{b}{a}}\sum_{k=1}^{n-1}\frac{(n-k)s^{k-1}}{k!}$$
$$\times\left(\frac{1}{P_m}\sum_{i=1}^{m}(p_{\max}-p_i)l_{a,b}^{k,s}(x_i)-\frac{mp_{\max}}{P_m}l_{a,b}^{k,s}(m_0(\mathbf{x}))+l_{a,b}^{k,s}(M_0(\mathbf{x},\mathbf{p}))\right),$$

where $l_{a,b}^{k,s}(\cdot)$ *is defined by* (34) *and* $p_{\min}=\min_{1\leq i\leq m}p_i$, $p_{\max}=\max_{1\leq i\leq m}p_i$.

In should be noticed here that any of inequalities in Corollaries 19, 20 and 21 can represent an improvement of (27), provided that its right-hand side is non-negative.

Remark 22. The previous three corollaries can also be exploited in establishing some particular mean inequalities as in Remarks 14, 16 and 18. For an illustration, we give here only mutual bounds for the difference between the arithmetic and geometric mean expressed in terms of the corresponding non-weighted means. More precisely, by putting $s=1$ and $q_i=\frac{P_m}{p_i}$, $i=1,2,\ldots,m$, in Corollary 21, we arrive at the inequalities

$$\begin{aligned}
&\sum_{i=1}^m \frac{x_i}{q_i}-\prod_{i=1}^m x_i^{\frac{1}{q_i}}-\frac{m}{q_{\max}}\left(\frac{1}{m}\sum_{i=1}^m x_i-\left(\prod_{i=1}^m x_i\right)^{\frac{1}{m}}\right)\\
&\geq \frac{1}{\log\frac{b}{a}}\sum_{k=1}^{n-1}\frac{(n-k)}{k!}\\
&\qquad\times\left(\sum_{i=1}^m\left(\frac{1}{q_i}-\frac{1}{q_{\max}}\right)l_{a,b}^{k,1}(x_i)-l_{a,b}^{k,1}\left(\prod_{i=1}^m x_i^{\frac{1}{q_i}}\right)+\frac{m}{q_{\max}}l_{a,b}^{k,1}\left(\left(\prod_{i=1}^m x_i\right)^{\frac{1}{m}}\right)\right)
\end{aligned}$$

and

$$\begin{aligned}
&\frac{m}{q_{\min}}\left(\frac{1}{m}\sum_{i=1}^m x_i-\left(\prod_{i=1}^m x_i\right)^{\frac{1}{m}}\right)-\left(\sum_{i=1}^m \frac{x_i}{q_i}-\prod_{i=1}^m x_i^{\frac{1}{q_i}}\right)\\
&\geq \frac{1}{\log\frac{b}{a}}\sum_{k=1}^{n-1}\frac{(n-k)}{k!}\\
&\qquad\times\left(\sum_{i=1}^m\left(\frac{1}{q_{\min}}-\frac{1}{q_i}\right)l_{a,b}^{k,1}(x_i)+l_{a,b}^{k,1}\left(\prod_{i=1}^m x_i^{\frac{1}{q_i}}\right)-\frac{m}{q_{\min}}l_{a,b}^{k,1}\left(\left(\prod_{i=1}^m x_i\right)^{\frac{1}{m}}\right)\right),
\end{aligned}$$

where $q_{\min}=\min_{1\leq i\leq m} q_i$ and $q_{\max}=\max_{1\leq i\leq m} q_i$.

5. HÖLDER-TYPE INEQUALITIES BASED ON THE FINK IDENTITY

In this section we give another significant consequence of the established Jensen-type inequalities. Among many applications of the Jensen inequality, the Hölder inequality certainly stands out. Let (Ω,Σ,μ) be σ-finite measure space and let $\sum_{i=1}^m\frac{1}{q_i}=1$, $q_i>1$. Recall that if $f_i\in L^{q_i}(\Omega)$, $i=1,2,\ldots,m$, are non-negative measurable functions, then holds the inequality

$$\int_\Omega\prod_{i=1}^m f_i(x)d\mu(x)\leq\prod_{i=1}^m\|f_i\|_{q_i}. \tag{36}$$

It is well known that the Hölder inequality can be derived in several ways, among others via the arithmetic-geometric mean inequality, i.e. the Young inequality (for more details, see [**11, 12**]). Using this fact, the Young-type relation (35) can be exploited in establishing a new Hölder-type inequality based on the Fink identity.

It should be noticed here that since the corresponding result leans on identity (5), we have to require some additional conditions on non-negative measurable functions $f_i\in L^{q_i}(\Omega)$, $i=1,2,\ldots,m$.

Theorem 23. *Let* (Ω, Σ, μ) *be* σ*-finite measure space and let* $\sum_{i=1}^{m} \frac{1}{q_i} = 1$, $q_i > 1$, $i = 1, 2, \ldots, m$. *Further, suppose that* $f_i \in L^{q_i}(\Omega)$, $i = 1, 2, \ldots, m$, *are non-negative measurable functions such that*

$$a^{\frac{1}{q_i}} \|f_i\|_{q_i} \leq f_i(x) \leq b^{\frac{1}{q_i}} \|f_i\|_{q_i}, \; x \in \Omega, \; i = 1, 2, \ldots, m, \tag{37}$$

where $0 < a < b$. *Then holds the inequality*

$$\begin{aligned} &\prod_{i=1}^{m} \|f_i\|_{q_i} - \int_{\Omega} \prod_{i=1}^{m} f_i(x) d\mu(x) \\ &\geq \frac{\prod_{i=1}^{m} \|f_i\|_{q_i}}{\log \frac{b}{a}} \\ &\times \sum_{k=1}^{n-1} \frac{n-k}{k!} \left(\sum_{i=1}^{m} \frac{1}{q_i} \int_{\Omega} l_{a,b}^{k,1} \left(\frac{f_i^{q_i}(x)}{\|f_i\|_{q_i}^{q_i}} \right) d\mu(x) - \int_{\Omega} l_{a,b}^{k,1} \left(\prod_{i=1}^{m} \frac{f_i(x)}{\|f_i\|_{q_i}} \right) d\mu(x) \right), \end{aligned} \tag{38}$$

where $l_{a,b}^{k,s}(\cdot)$ *is defined by* (34).

Proof. The key idea is to put $f_i^{q_i}(x)/\|f_i\|_{q_i}^{q_i}$, $x \in \Omega$, instead of x_i, $i = 1, 2, \ldots, m$, in (35). Of course, this can be done due to conditions in (37). Therefore, we arrive at the inequality

$$\begin{aligned} &\sum_{i=1}^{m} \frac{f_i^{q_i}(x)}{q_i \|f_i\|_{q_i}^{q_i}} - \prod_{i=1}^{m} \frac{f_i(x)}{\|f_i\|_{q_i}} \\ &\geq \frac{1}{\log \frac{b}{a}} \sum_{k=1}^{n-1} \frac{n-k}{k!} \left(\sum_{i=1}^{m} \frac{1}{q_i} l_{a,b}^{k,1} \left(\frac{f_i^{q_i}(x)}{\|f_i\|_{q_i}^{q_i}} \right) - l_{a,b}^{k,1} \left(\prod_{i=1}^{m} \frac{f_i(x)}{\|f_i\|_{q_i}} \right) \right). \end{aligned}$$

Now, integrating the above inequality over Ω with respect to the measure μ, we get

$$\begin{aligned} &\sum_{i=1}^{m} \frac{1}{q_i} - \frac{\int_{\Omega} \prod_{i=1}^{m} f_i(x) d\mu(x)}{\prod_{i=1}^{m} \|f_i\|_{q_i}} \\ &\geq \frac{1}{\log \frac{b}{a}} \sum_{k=1}^{n-1} \frac{n-k}{k!} \left(\sum_{i=1}^{m} \frac{1}{q_i} \int_{\Omega} l_{a,b}^{k,1} \left(\frac{f_i^{q_i}(x)}{\|f_i\|_{q_i}^{q_i}} \right) d\mu(x) - \int_{\Omega} l_{a,b}^{k,1} \left(\prod_{i=1}^{m} \frac{f_i(x)}{\|f_i\|_{q_i}} \right) d\mu(x) \right). \end{aligned}$$

Finally, since $\sum_{i=1}^{m} \frac{1}{q_i} = 1$, multiplying the last inequality by $\prod_{i=1}^{m} \|f_i\|_{q_i}$, we obtain (38), as asserted. □

Based on Remark 22, below we give two more Hölder-type inequalities. In fact, we obtain mutual bounds for the Hölder inequality in a difference form, with which we conclude this paper.

Theorem 24. *Suppose that the assumptions as in Theorem 23 are satisfied. Then hold the inequalities*

$$
\begin{aligned}
&1-\frac{\int_\Omega \prod_{i=1}^m f_i(x)d\mu(x)}{\prod_{i=1}^m \|f_i\|_{q_i}}-\frac{m}{q_{\max}}\left(1-\frac{\int_\Omega \prod_{i=1}^m f_i^{\frac{q_i}{m}}(x)d\mu(x)}{\prod_{i=1}^m \|f_i\|_{q_i}^{\frac{q_i}{m}}}\right)\\
&\geq \frac{1}{\log\frac{b}{a}}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left(\sum_{i=1}^m\Big(\frac{1}{q_i}-\frac{1}{q_{\max}}\Big)\int_\Omega l_{a,b}^{k,1}\Big(\frac{f_i^{q_i}(x)}{\|f_i\|_{q_i}^{q_i}}\Big)d\mu(x)\right.\\
&\qquad\left.-\int_\Omega l_{a,b}^{k,1}\left(\prod_{i=1}^m\frac{f_i(x)}{\|f_i\|_{q_i}}\right)d\mu(x)+\frac{m}{q_{\max}}\int_\Omega l_{a,b}^{k,1}\left(\prod_{i=1}^m\frac{f_i^{\frac{q_i}{m}}(x)}{\|f_i\|_{q_i}^{\frac{q_i}{m}}}\right)d\mu(x)\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{m}{q_{\min}}\left(1-\frac{\int_\Omega \prod_{i=1}^m f_i^{\frac{q_i}{m}}(x)d\mu(x)}{\prod_{i=1}^m \|f_i\|_{q_i}^{\frac{q_i}{m}}}\right)-\left(1-\frac{\int_\Omega \prod_{i=1}^m f_i(x)d\mu(x)}{\prod_{i=1}^m \|f_i\|_{q_i}}\right)\\
&\geq \frac{1}{\log\frac{b}{a}}\sum_{k=1}^{n-1}\frac{n-k}{k!}\left(\sum_{i=1}^m\Big(\frac{1}{q_{\min}}-\frac{1}{q_i}\Big)\int_\Omega l_{a,b}^{k,1}\Big(\frac{f_i^{q_i}(x)}{\|f_i\|_{q_i}^{q_i}}\Big)d\mu(x)\right.\\
&\qquad\left.+\int_\Omega l_{a,b}^{k,1}\left(\prod_{i=1}^m\frac{f_i(x)}{\|f_i\|_{q_i}}\right)d\mu(x)-\frac{m}{q_{\min}}\int_\Omega l_{a,b}^{k,1}\left(\prod_{i=1}^m\frac{f_i^{\frac{q_i}{m}}(x)}{\|f_i\|_{q_i}^{\frac{q_i}{m}}}\right)d\mu(x)\right),
\end{aligned}
$$

where $q_{\min}=\min_{1\leq i\leq m} q_i$ *and* $q_{\max}=\max_{1\leq i\leq m} q_i$.

Proof. The proof is similar to the proof of Theorem 23 except that we use relations from Remark 22 instead of inequality (35). □

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