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SHARP DOUBLE-EXPONENT TYPE BOUNDS FOR THE LEMNISCATE SINE FUNCTION

*Tie-Hong Zhao and Miao-Kun Wang**

In this paper, we will establish sharp inequalities of the lemniscate sine function and the so-called weighted (p, q)-exponential type function, of which the latter is an extension of the weighted Hölder mean. These results provide a systematic method for the previous obtained inequalities and give great improvements for bounds of the lemniscate sine function. As applications, several high accuracy approximations for the lemniscate sine function are also derived.

1. INTRODUCTION

Gauss' arc lemniscate sine is defined as follows

$$\operatorname{arcsl} x = \int_0^x \frac{dt}{\sqrt{1-t^4}} \tag{1}$$

for $|x| \le 1$, see [**3**, p. 259] and [**4**, (2.5)]. This function has a simple geometric interpretation, the arc length of which measured from the origin to a point with polar coordinates on the Bernoulli lemniscate $r^2 = \cos 2\theta$ is $\operatorname{arcsl} r$. Throughout this paper, the *first lemniscate constant* ω (c.f. [**3**, Theorem 1.7] and [**10**, 19.20.2]) is given by

$$\omega = \operatorname{arcsl}(1) = \frac{1}{\sqrt{2}}\mathcal{K}(1/\sqrt{2}) = \frac{[\Gamma(1/4)]^2}{4\sqrt{2\pi}} \approx 1.3110,$$

*Corresponding author. Miao-Kun Wang
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where $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt$ ($\mathrm{Re}\, x > 0$) is the classical Euler gamma function [**12, 29, 30, 33**] and

$$\mathcal{K}(r) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-r^2\sin^2\theta}} = \frac{\pi}{2} F\left(\tfrac{1}{2}, \tfrac{1}{2}; 1; r^2\right)$$

($0 < r < 1$) is the complete elliptic integral of the first kind [**13, 26, 31, 32**]. Here $F(a, b; c; x)$ is the Gaussian hypergeometric function defined, for $a, b, c \in \mathbb{R}$ with $c \neq 0, -1, -2, \cdots$, by

$$F(a, b; c; x) \equiv {}_2F_1(a, b; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{x^n}{n!}, \quad |x| < 1,$$

where $(a)_0 = 1$ for $a \neq 0$ and $(a)_n = a(a+1)(a+2)\cdots(a+n-1) = \Gamma(a+n)/\Gamma(a)$ is the *shifted factorial function* or the *Pochhammer symbol*. For more information on these functions and recently obtained related results, we refer the reader to [**6, 8, 11, 15, 16, 17, 18, 19, 35, 36, 37, 38, 39, 41, 42, 43**].

It has been proved in [**5**, Lemma 1] that the lemniscate function $\mathrm{arcsl}\, x$ can be expressed in temrs of Gaussian hypergeometric function, as follows

$$\mathrm{arcsl}\, x = \sum_{n=0}^{\infty} \frac{\Gamma(n+\frac{1}{2})}{\sqrt{\pi}(4n+1)\cdot n!} x^{4n+1} = xF\left(\tfrac{1}{4}, \tfrac{1}{2}; \tfrac{5}{4}; x^4\right). \tag{2}$$

The initial thread for this investigation begins with the following elegant inequality

$$\left(\frac{5}{3+2\sqrt{1-x^4}}\right)^{1/2} < \frac{\mathrm{arcsl}\, x}{x} < \frac{1}{(1-x^4)^{1/10}}$$

for all $0 < |x| < 1$, which is due to Neuman [**9**] in 2012, and namely the computable bounds for the lemniscate sine function are obtained.

Recently, in the study of Shafer-Fink's inequalities for the lemniscate functions, Wei, He and Wang [**20**] proved that the inequalities

$$\frac{\omega}{1+(\omega-1)\sqrt[4]{1-x^4}} < \frac{\mathrm{arcsl}\, x}{x} < \frac{5}{3+2\sqrt[4]{1-x^4}}, \tag{3}$$

$$\frac{5}{4+\sqrt{1-x^4}} < \frac{\mathrm{arcsl}\, x}{x} < \frac{\omega}{1+(\omega-1)\sqrt{1-x^4}}. \tag{4}$$

hold for $0 < |x| < 1$.

By studying simple bounds for the lemniscate type means, Zhao, Qian and Chu [**40**] established new bounds for the lemniscate sine function

$$\left(\frac{5}{3+2\sqrt{1-x^4}}\right)^{1/2} < \frac{\mathrm{arcsl}\, x}{x} < \left(\frac{\omega^2}{1+(\omega^2-1)\sqrt{1-x^4}}\right)^{1/2} \tag{5}$$

for all $0 < |x| < 1$.

For $(p, q) \in \mathbb{R}^2$ with $pq \neq 0$ and $\alpha \in (0, 1)$, we denote

$$H_\alpha(x; p, q) = \Big[(1-\alpha) + \alpha(1-x^4)^p\Big]^{-1/q} \tag{6}$$

for $0 < |x| < 1$. In the case of $p^2 + q^2 = 0$, if $\lim_{(p,q)\to(0,0)}(p/q) = \lambda \neq 0$, we can define

$$H_\alpha(x; \lambda) = (1-x^4)^{-\lambda\alpha}.$$

It is worth noting that $H_\alpha(x; p, q)$ or $H_\alpha(x; \lambda)$ degenerates to the infinite or a constant for other cases.

By applying (6), the inequalities (3)-(5) can be rewritten as

$$\begin{aligned} H_{1-\frac{1}{\omega}}(x; \tfrac{1}{4}, 1) &< \frac{\operatorname{arcsl} x}{x} < H_{\frac{2}{5}}(x; \frac{1}{4}, 1), \\ H_{\frac{1}{5}}(x; \tfrac{1}{2}, 1) &< \frac{\operatorname{arcsl} x}{x} < H_{1-\frac{1}{\omega}}(x; \tfrac{1}{2}, 1), \\ H_{\frac{2}{5}}(x; \tfrac{1}{2}, 2) &< \frac{\operatorname{arcsl} x}{x} < H_{1-\frac{1}{\omega^2}}(x; \tfrac{1}{2}, 2) \end{aligned}$$

for all $0 < |x| < 1$.

In light of these inequalities, it is natural to ask the following questions:

- Can these be extended to approximate $(\operatorname{arcsl} x)/x$ by the function $H_\alpha(x; p, q)$ defined in (6) for fixed $(p, q) \in \mathbb{R}^2$?
- Given $(p, q) \in \mathbb{R}^2$, what are the best possible parameters $\alpha, \beta \in (0, 1)$ such that
$$H_\alpha(x; p, q) < \frac{\operatorname{arcsl} x}{x} < H_\beta(x; p, q)$$
for all $0 < |x| < 1$?

The main goal of this paper is to answer the above questions.

2. PRELIMINARIES

2.1 Tools

To prove our results, we need two tools. The first tool is L'Hôpital Monotone Rule (see [**1**, Theorem 1.25]) which play an important role in dealing with the monotonicity of the ratio of two functions.

Proposition 1. Let $a, b \in \mathbb{R}$ with $a < b$, $f, g : [a, b] \mapsto \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If f'/g' is increasing (decreasing) on (a, b), then so are the functions

$$\frac{f(x) - f(a)}{g(x) - g(a)} \quad \text{and} \quad \frac{f(x) - f(b)}{g(x) - g(b)}.$$

If f'/g' is strict monotone, then the monotonicity in the conclusion is also strict.

Sometimes, f'/g' is not monotone but piecewise monotone. We now introduce a useful auxiliary function $\mathsf{H}_{f,g}$ (called H-funuction, see [**14, 34**]), which makes a bridge between the derivatives of the ratios f/g and f'/g'. For $-\infty \leq a < b \leq \infty$, let f and g be differentiable on (a, b) and $g' \neq 0$ on (a, b). Then the function $\mathsf{H}_{f,g}$ is defined by

$$\mathsf{H}_{f,g} := \frac{f'}{g'} g - f. \tag{7}$$

The function $\mathsf{H}_{f,g}$ has some well properties [**22**, Properties 1,2]. In particular, if f and g are twice differentiable on (a, b), then we have

$$\left(\frac{f}{g}\right)' = \frac{g'}{g^2}\left(\frac{f'}{g'} g - f\right) = \frac{g'}{g^2} \mathsf{H}_{f,g}, \tag{8}$$

$$\mathsf{H}'_{f,g} = \left(\frac{f'}{g'}\right)' g. \tag{9}$$

The second tool is a simple and useful criterion to determine the sign of a class of special series. The special series is named “Negative-Positive type power series” defined as follows.

Definition 2. [**32**, Definition 1] Let $m \geq 0$ be an integer. A power series $S(t)$ given by

$$S(t) = -\sum_{k=0}^{m} a_k t^k + \sum_{k=m+1}^{\infty} a_k t^k$$

is called a “Negative-Positive type power series”, or “NP type power series” for short, if its coefficients a_k for $k \geq 0$ satisfy

(i) $a_k \geq 0$ for all $k \geq 0$;

(ii) there exist at least two integers $0 \leq k_1 \leq m$ and $k_2 \geq m + 1$ such that $a_{k_1}, a_{k_2} > 0$.

Proposition 3. Let $S(t)$ be a Negative-Positive type power series converging on the interval $(0, r)$ $(r > 0)$. Then the following statements are true:

(1) If $S(r^-) \leq 0$, then $S(t) < 0$ for all $t \in (0, r)$;

(2) If $S(r^-) > 0$, then there exists $t_0 \in (0, r)$ such that $S(t) < 0$ for $t \in (0, t_0)$ and $S(t) > 0$ for $t \in (t_0, r)$.

Remark 4. In Definition 2, $-S(t)$ is called a "Positive-Negative type" (or "PN type" for short) power series. Further, if $a_k = 0$ for $k \geq k_2 + 1$, then $S(t)$ is called a "Negative-Positive type polynomial of degree k_2", or "NP type polynomial" for short. Likewise, $-S(t)$ is called a "Positive-Negative type polynomial of degree k_2", or "PN type polynomial" for short. A polynomial version of Proposition 3 appeared in [**24**], and another version of the series converging on $(0, \infty)$ can be found in [**23**].

Remark 5. Proposition 3 was proved in [**28**], which is a revised version of the electronic preprint [**27**], and proved differently in [**26**]. As a consequence of Proposition 3(2), for given $t_1 \in (0, r)$, if $S(t_1) > 0$, then $S(t) > 0$ for all $t \in (t_1, r)$, and if $S(t_1) < 0$, then $S(t) < 0$ for all $t \in (0, t_1)$.

2.2 Lemmas

In this section, we will present several lemmas which are used to prove the main results.

Lemma 6. *For $0 < |x| < 1$, we have*

$$\left(\frac{\operatorname{arcsl} x}{x}\right)^2 = \sum_{n=0}^{\infty} \frac{(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n} x^{4n}, \tag{10}$$

$$\frac{\operatorname{arcsl} x}{x\sqrt{1-x^4}} = \sum_{n=0}^{\infty} \frac{(\frac{3}{4})_n}{(\frac{5}{4})_n} x^{4n}. \tag{11}$$

Proof. For $c = a + b + 1/2$, it has been demonstrated in [**21**, Example 14.11] (see also [**7**]) that

$$\begin{aligned}\Big[F(a, b; c; x)\Big]^2 &= \frac{\Gamma(c)\Gamma(2c-1)}{\Gamma(2a)\Gamma(2b)\Gamma(a+b)} \sum_{n=0}^{\infty} \frac{\Gamma(2a+n)\Gamma(a+b+n)\Gamma(2b+n)}{n!\Gamma(c+n)\Gamma(2c-1+n)} x^n \\ &= \sum_{n=0}^{\infty} \frac{(2a)_n(2b)_n(a+b)_n}{n!(c)_n(2c-1)_n} x^n.\end{aligned} \tag{12}$$

According to (2), taking $a = 1/2$, $b = 1/4$ and $c = 5/4$ into (12) yields

$$\left(\frac{\operatorname{arcsl} x}{x}\right)^2 = \Big[F(\tfrac{1}{2}, \tfrac{1}{4}; \tfrac{5}{4}; x^4)\Big]^2 = \sum_{n=0}^{\infty} \frac{(1)_n(\frac{1}{2})_n(\frac{3}{4})_n}{n!(\frac{5}{4})_n(\frac{3}{2})_n} x^{4n} = \sum_{n=0}^{\infty} \frac{(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n} x^{4n},$$

which gives (10).

It is observed from (1) that the derivative of $\operatorname{arcsl} x$ is give by

$$\frac{d \operatorname{arcsl} x}{dx} = \frac{1}{\sqrt{1-x^4}}.$$

By this, it follows from (10) that

$$\frac{\operatorname{arcsl} x}{x\sqrt{1-x^4}} = \frac{1}{2x}\frac{d\operatorname{arcsl}^2 x}{dx} = \sum_{n=0}^{\infty}\frac{(\frac{3}{4})_n}{(\frac{5}{4})_n}x^{4n}.$$

This completes the proofs. □

Lemma 7. *Let*

$$\varphi_1(x) = 1 + (3-x^4)\frac{\operatorname{arcsl} x}{x\sqrt{1-x^4}} - 4\left(\frac{\operatorname{arcsl} x}{x}\right)^2,$$
$$\varphi_2(x) = x^4\left[\frac{\operatorname{arcsl} x}{x\sqrt{1-x^4}} - \left(\frac{\operatorname{arcsl} x}{x}\right)^2\right],$$
$$\varphi_3(x) = 1 - (1-x^4)\left[\frac{2\operatorname{arcsl} x}{x\sqrt{1-x^4}} - \left(\frac{\operatorname{arcsl} x}{x}\right)^2\right]$$

for $0 < x < 1$*. Then* $\varphi_i(x)$ *(*$i = 1, 2, 3$*) is absolutely monotonic on* $(0,1)$*. More precisely, we have the following series expansions*

$$\varphi_1(x) = \sum_{n=2}^{\infty}\frac{16n(n-1)(\frac{3}{4})_n}{(2n+1)(4n-1)(\frac{5}{4})_n}x^{4n}, \quad \varphi_2(x) = \sum_{n=2}^{\infty}\frac{2(n-1)(4n+1)(\frac{3}{4})_n}{(2n-1)(4n-1)(\frac{5}{4})_n}x^{4n},$$
$$\varphi_3(x) = \sum_{n=2}^{\infty}\frac{4(n-1)(4n+1)(\frac{3}{4})_n}{(4n^2-1)(4n-1)(\frac{5}{4})_n}x^{4n}.$$

Proof. From the formulas (10) and (11), it follows that

$$\begin{aligned}
\varphi_1(x) &= 1 + (3-x^4)\sum_{n=0}^{\infty}\frac{(\frac{3}{4})_n}{(\frac{5}{4})_n}x^{4n} - 4\sum_{n=0}^{\infty}\frac{(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n}x^{4n} \\
&= \sum_{n=2}^{\infty}\left(3 - \frac{4}{2n+1}\right)\frac{(\frac{3}{4})_n}{(\frac{5}{4})_n}x^{4n} - \sum_{n=1}^{\infty}\frac{(\frac{3}{4})_n}{(\frac{5}{4})_n}x^{4(n+1)} \\
&= \sum_{n=2}^{\infty}\frac{(6n-1)(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n}x^{4n} - \sum_{n=2}^{\infty}\frac{(\frac{3}{4})_{n-1}}{(\frac{5}{4})_{n-1}}x^{4n} \\
&= \sum_{n=2}^{\infty}\left(\frac{6n-1}{2n+1} - \frac{4n+1}{4n-1}\right)\frac{(\frac{3}{4})_n}{(\frac{5}{4})_n}x^{4n} = \sum_{n=2}^{\infty}\frac{16n(n-1)(\frac{3}{4})_n}{(2n+1)(4n-1)(\frac{5}{4})_n}x^{4n},
\end{aligned}$$

$$\begin{aligned}
\varphi_2(x) &= x^4\left[\sum_{n=0}^{\infty}\frac{(\frac{3}{4})_n}{(\frac{5}{4})_n}x^{4n} - \sum_{n=0}^{\infty}\frac{(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n}x^{4n}\right] = \sum_{n=1}^{\infty}\frac{2n(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n}x^{4(n+1)} \\
&= \sum_{n=2}^{\infty}\frac{2(n-1)(\frac{3}{4})_{n-1}}{(2n-1)(\frac{5}{4})_{n-1}}x^{4n} = \sum_{n=2}^{\infty}\frac{2(n-1)(4n+1)(\frac{3}{4})_n}{(2n-1)(4n-1)(\frac{5}{4})_n}x^{4n}
\end{aligned}$$

and

$$\begin{aligned}\varphi_3(x) &= 1-(1-x^4)\left[\sum_{n=0}^{\infty}\frac{2(\frac{3}{4})_n}{(\frac{5}{4})_n}x^{4n}-\sum_{n=0}^{\infty}\frac{(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n}x^{4n}\right]\\
&= \sum_{n=1}^{\infty}\frac{(4n+1)(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n}x^{4(n+1)}-\sum_{n=2}^{\infty}\frac{(4n+1)(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n}x^{4n}\\
&= \sum_{n=2}^{\infty}\frac{(4n-3)(4n+1)(\frac{3}{4})_n}{(2n-1)(4n-1)(\frac{5}{4})_n}x^{4n}-\sum_{n=2}^{\infty}\frac{(4n+1)(\frac{3}{4})_n}{(2n+1)(\frac{5}{4})_n}x^{4n}\\
&= \sum_{n=2}^{\infty}\frac{4(n-1)(4n+1)(\frac{3}{4})_n}{(4n^2-1)(4n-1)(\frac{5}{4})_n}x^{4n}.\end{aligned}$$

This completes the proofs. □

Lemma 8. *The function*

$$x\mapsto\frac{\log(x/\operatorname{arcsl}x)}{\log(1-x^4)}$$

is strictly decreasing from $(0,1)$ *onto* $(0,1/10)$.

Proof. Let $\phi_1(x)=\log(x/\operatorname{arcsl}x)$ and $\phi_2(x)=\log(1-x^4)$. Then differentiation yields

$$\frac{\phi_1'(x)}{\phi_2'(x)}=-\frac{\sqrt{1-x^4}(x-\sqrt{1-x^4}\operatorname{arcsl}x)}{4x^4\operatorname{arcsl}x},$$

$$\left[\frac{\phi_1'(x)}{\phi_2'(x)}\right]'=-\frac{x^2+x(3-x^4)\frac{\operatorname{arcsl}x}{\sqrt{1-x^4}}-4\operatorname{arcsl}^2x}{4x^5\operatorname{arcsl}^2x}=-\frac{\varphi_1(x)}{4x^3\operatorname{arcsl}^2x}, \tag{13}$$

where $\varphi_1(x)$ is defined as in Lemma 7.

It follows from Lemma 7 and (13) that $\varphi_1'(x)/\varphi_2'(x)$ is strictly decreasing on $(0,1)$.Using Proposition 1 gives that

$$\frac{\phi_1(x)}{\phi_2(x)}=\frac{\phi_1(x)-\phi_1(0^+)}{\phi_2(x)-\phi_2(0^+)}$$

is strictly decreasing on $(0,1)$. It is apparent to obtain two limiting values easily by l'Hôpital rule. □

Lemma 9. *Let* $(p,q)\in\mathbb{R}^2$ *and define the function* $x\mapsto\xi_{p,q}(x)$ *on* $(0,\infty)$ *by*

$$\xi_{p,q}(x)=8(2p-1)x^2+4(3p-q+1)x+2p-q.$$

Then we have the following conclusions:

(1) In the case of $p=1/2$,

- ⋄ $\xi_{p,q}(x)<0$ *for* $x\geq 2$ *if and only if* $q\geq 5/2$;

- ◇ *$\xi_{p,q}(x) \geq 0$ for $x \geq 2$ if and only if $q \leq 7/3$ with the equal only at $x = 2$;*
- ◇ *if $7/3 < q < 5/2$, then there exists $\tilde{x}_0 > 2$ such that $\xi_{p,q}(x) < 0$ for $x \in [2, \tilde{x}_0)$ and $\xi_{p,q}(x) > 0$ for $x \in (\tilde{x}_0, \infty)$.*

(2) In the case of $p > 1/2$,

- ◇ *$\xi_{p,q}(x) \geq 0$ for $x \geq 2$ if and only if $q \leq 10p - 8/3$ with the equal only at $x = 2$;*
- ◇ *if $q > 10p - 8/3$, then there exists $\tilde{x}_1 > 2$ such that $\xi_{p,q}(x) < 0$ for $x \in [2, \tilde{x}_1)$ and $\xi_{p,q}(x) > 0$ for $x \in (\tilde{x}_1, \infty)$.*

(3) In the case of $p < 1/2$,

- ◇ *$\xi_{p,q}(x) \leq 0$ for $x \geq 2$ if $q \geq \widehat{q}(p)$, where*

$$\widehat{q}(p) = \begin{cases} 10p - \frac{8}{3}, & p \leq \frac{13}{27}, \\ p + 2 - \sqrt{3(1-2p)}, & \frac{13}{27} < p < \frac{1}{2}. \end{cases}$$

  In particular, in the case of $q = 10p - 8/3$, namely, $\xi_{p,q}(2) = 0$, if $13/27 < p \leq 19/39$, then $\xi_{p,q}(x) \leq 0$ for $x \geq 3$; if $19/39 < p < 1/2$, then there exists $\tilde{x}_2 > 3$ such that $\xi_{p,q}(x) > 0$ for $x \in [3, \tilde{x}_2)$ and $\xi_{p,q}(x) < 0$ for $x \in (\tilde{x}_2, \infty)$.
- ◇ *If $q < 10p - 8/3$, then there exists $\tilde{x}_3 > 2$ such that $\xi_{p,q}(x) > 0$ for $x \in [2, \tilde{x}_3)$ and $\xi_{p,q}(x) < 0$ for $x \in (\tilde{x}_3, \infty)$.*

Proof. We now divide into three cases to complete the proof.

Case 1 $p = 1/2$.
In this case, $\xi_{p,q}(x)$ possibly reduces to a linear function.

- $\xi_{p,q}(x) \leq 0$ for $x \geq 2$ if and only if $3p - q + 1 \leq 0$ and $\xi_{p,q}(2) = 3(30p - 3q - 8) \leq 0$, equivalently, $q \geq 5/2$. At this point, $\xi_{1/2,q}(2) = 3(7 - 3q) \leq -3/2$, which gives $\xi_{p,q}(x) < 0$ for $x \geq 2$.
- $\xi_{p,q}(x) \geq 0$ for $x \geq 2$ if and only if $3p - q + 1 \geq 0$ and $\xi_{p,q}(2) \geq 0$, equivalently, $q \leq 7/3$. It was observed that $\xi_{1/2,q}(x) = 1 + 10x - q(1 + 4x) \geq 1 + 10x - 7(1 + 4x)/3 = 2(x - 2)/3 \geq 0$, which implies that $\xi_{p,q}(x) = 0$ only holds for $x = 2$.
- If $7/3 < q < 5/2$, then $\xi_{1/2,q}(x)$ is a linear and increasing function. This together with $\xi_{1/2,q}(2) = 3(7 - 3q) < 0$ implies that there exists $\tilde{x}_0 > 2$ such that $\xi_{p,q}(x) < 0$ for $x \in (2, \tilde{x}_0)$ and $\xi_{p,q}(x) > 0$ for $x \in (\tilde{x}_0, \infty)$.

Case 2 $p > 1/2$.
In this case, $\xi_{p,q}(x)$ is a quadratic function whose graph is a upward opening parabola. Let us denote the symmetric axes of $\xi_{p,q}(x)$ by

$$x_s = \frac{1 + 3p - q}{4(1 - 2p)}.$$

We divide the proof into two subcases.

- If $q \le 10p - 8/3$, then $\xi_{p,q}(2) \ge 0$ and

$$x_s \le \frac{1+3p-(10p-8/3)}{4(1-2p)} = 2 - \frac{27p-13}{2p-1} < 2.$$

  which demonstrates that $\xi_{p,q}(x)$ is strictly increasing on $(2,\infty)$. So $\xi_{p,q}(x) \ge \xi_{p,q}(2) \ge 0$ for $x \ge 2$. In particular, $\xi_{p,q}(x) = 0$ holds only at $x = 2$.
- If $q > 10p - 8/3$, namely, $\xi_{p,q}(2) < 0$, then the graph property of $\xi_{p,q}(x)$ enables us to know that there exists $\tilde{x}_1 > 2$ such that $\xi_{p,q}(x) < 0$ for $x \in (2, \tilde{x}_1)$ and $\xi_{p,q}(x) > 0$ for $x \in (\tilde{x}_1, \infty)$.

Case 3 $p < 1/2$.
Now the graph of $\xi_{p,q}(x)$ is a downward opening parabola. As in Case 2, we divide the proof into two subcases.

- If $\xi_{p,q}(2) \le 0$, then $q \ge 10p - 8/3$. We will discuss in two subsubcases.
  - $x_s \le 2$. Then we have $q \ge 19p - 7$. Hence $\xi_{p,q}(x) \le \xi_{p,q}(2) = 0$ for $x \ge 2$ due to the monotonicity of $\xi_{p,q}(x)$. In this subcase, $q \ge \max\{10p - 8/3, 19p - 7\}$ which is equivalent to

$$\text{(14)}\quad p \le \frac{13}{27} \text{ and } q \ge 10p - \frac{8}{3} \quad \text{or} \quad \frac{13}{27} < p < \frac{1}{2} \text{ and } q \ge 19p - 7.$$

  - $x_s > 2$. Then $13/27 < p < 1/2$ and $10p - 8/3 \le q < 19p - 7$. In this subsubcase, from the property of quadratic function, there is still a situation such that $\xi_{p,q}(x) \le 0$ for $x \ge 2$. That is, the determination of $\xi_{p,q}(x)$ is non-positive, more precisely,

$$\Delta(p,q) = 16[q^2 - 2(p+2)q + p^2 + 10p + 1] \le 0,$$

    which can be solved as

$$\text{(15)}\qquad p + 2 - \sqrt{3(1-2p)} \le q \le p + 2 + \sqrt{3(1-2p)}.$$

    It was observed that

$$\Delta(p, 10p - 8/3) = \frac{(27p-13)^2}{9} > 0$$

    and

$$\Delta(p, 19p-7) = -6(1-2p)(27p-13) < 0.$$

    This together with the graphic property of $q \mapsto \Delta(p,q)$ and (15) leads to the conclusion that (p,q) needs to satisfy

$$\text{(16)}\qquad \frac{13}{27} < p < \frac{1}{2} \quad \text{and} \quad p + 2 - \sqrt{3(1-2p)} \le q < 19p - 7.$$

In summary, in this subsubcase, $\xi_{p,q}(x) \le 0$ for $x \ge 2$ if $q \ge \widehat{q}(p)$, where $\widehat{q}(p)$ can be defined by (14) and (16). In particular, in the subsubcase of $q = 10p - 8/3$, we clearly see that $\xi_{p,q}(2) = 0$. If $13 < 27 < p \le 19/39$, then it follows that

$$\xi_{p,q}(x) = -8(1-2p)(x-2)\left[x - 3 + \frac{19-39p}{6(1-2p)}\right] \le 0 \tag{17}$$

for $x \ge 3$. If $19/39 < p < 1/2$, then it follows from (17) that $\xi_{p,q}(3) > 0$, which in conjunction with the graph of $\xi_{p,q}$ implies that there exists $\tilde{x}_2 > 3$ such that $\xi_{p,q}(x) > 0$ for $x \in [3, \tilde{x}_2)$ and $\xi_{p,q}(x) < 0$ for $x \in (\tilde{x}_2, \infty)$.

- If $\xi_{p,q}(2) > 0$, that is, $q < 10p - 8/3$, then by the property of graph, it shows that there exists $\tilde{x}_3 > 2$ such that $\xi_{p,q}(x) > 0$ for $x \in (2, \tilde{x}_3)$ and $\xi_{p,q}(x) < 0$ for $x \in (\tilde{x}_3, \infty)$.

□

Remark 10. Due to Lemma 9, it remains to discuss the sign of $\xi_{p,q}(x)$ for $x \ge 2$ in the case of $(p,q) \in \mathcal{R}_0$, where

$$\mathcal{R}_0 = \left\{(p,q) \,\middle|\, \tfrac{13}{27} < p < \tfrac{1}{2}, \quad 10p - \tfrac{8}{3} < q < p + 2 - \sqrt{3(1-2p)}\right\}.$$

As a matter of fact, in this case, the sign of $\xi_{p,q}(x)$ will change alternately with x, which can not be easily dealt with in the sequel. For the sake of clarity, the regions (p,q) of Lemma 9 will be illustrated in Figure 1.

3. MAIN RESULTS

Before proving the main results, we will show the following monotonicity theorem. Lemma 9 will be used to prove Theorem 11. To facilitate the presentation, we need to divide the regions of (p,q) in Lemma 9.

Since p and q are required to be non-zero in Theorem 11, the regions of (p,q) in three cases of Lemma 9 should minus two axes, which are expressed by

$$\mathcal{R}_{\frac{1}{2}} = \left\{(p,q) \in \mathbb{R}_0^2 \,\middle|\, p = \frac{1}{2}\right\}, \quad \mathcal{R}_{\frac{1}{2}}^{+} = \left\{(p,q) \in \mathbb{R}_0^2 \,\middle|\, p > \frac{1}{2}\right\},$$
$$\mathcal{R}_{\frac{1}{2}}^{-} = \left\{(p,q) \in \mathbb{R}_0^2 \,\middle|\, \begin{array}{c} p < \frac{1}{2}, \\ q < 10p - \frac{8}{3} \text{ or } q \ge \widehat{q}(p) \end{array}\right\},$$

where $\mathbb{R}_0$ denotes the set of non-zero real numbers. Further, we define several regions as follows:

$$\mathcal{R}_{1,1} = \left\{\begin{array}{c} p < 0, \\ q > 0 \end{array}\right\}, \quad \mathcal{R}_{1,2} = \left\{\begin{array}{c} 0 < p < \frac{1}{2}, \\ 10p - \frac{8}{3} \le q < 0 \end{array}\right\}, \quad \mathcal{R}_{1,3} = \left\{\begin{array}{c} p = \frac{1}{2}, \\ 0 < q \le \frac{7}{3} \end{array}\right\},$$

$$\mathcal{R}_{1,4}=\left\{\begin{array}{c}p>\frac{1}{2},\\ 0<q\le 10p-\frac{8}{3}\end{array}\right\},\quad \mathcal{R}_{2,1}=\left\{\begin{array}{c}p>\frac{1}{2},\\ q<0\end{array}\right\},\quad \mathcal{R}_{2,2}=\left\{\begin{array}{c}p=\frac{1}{2},\\ q<0\end{array}\right\},$$

$$\mathcal{R}_{2,3}=\left\{\begin{array}{c}p=\frac{1}{2},\\ q\ge q_0\end{array}\right\},\quad \mathcal{R}_{2,4}=\left\{\begin{array}{c}0<p<\frac{1}{2},\\ q\ge \widehat{q}(p),\ q>0\end{array}\right\},\quad \mathcal{R}_{2,5}=\left\{\begin{array}{c}\frac{13}{27}<p\le\frac{19}{39},\\ q=10p-\frac{8}{3}\end{array}\right\},$$

$$\mathcal{R}_{2,6}=\left\{\begin{array}{c}p<0,\\ 10p-\frac{8}{3}\le q<0\end{array}\right\},\quad \mathcal{R}_{3,1}=\left\{\begin{array}{c}p>\frac{1}{2},\\ q>10p-\frac{8}{3}\end{array}\right\},\quad \mathcal{R}_{3,2}=\left\{\begin{array}{c}p=\frac{1}{2},\\ \frac{7}{3}\le q<q_0\end{array}\right\},$$

$$\mathcal{R}_{3,3}=\left\{\begin{array}{c}0<p<\frac{1}{2},\\ q<\min\{10p-\frac{8}{3},0\}\end{array}\right\},\quad \mathcal{R}_{4,1}=\left\{\begin{array}{c}0<p<\frac{1}{2},\\ 0<q<10p-\frac{8}{3}\end{array}\right\},$$

$$\mathcal{R}_{4,2}=\left\{\begin{array}{c}\frac{19}{39}<p<\frac{1}{2},\\ q=10p-\frac{8}{3}\end{array}\right\},\quad \mathcal{R}_{4,3}=\left\{\begin{array}{c}p<0,\\ q<10p-\frac{8}{3}\end{array}\right\},$$

where $q_0=2.39904\cdots$ is the unique zero point of $\varrho(q)=q+2\omega-2\omega^{q+1}$ on $(\frac{7}{3},\frac{5}{2})$, since $\varrho'(q)=1-2\omega^{q+1}\ln\omega<0$ for $q\in(\frac{7}{3},\frac{5}{2})$, and $\varrho(2.399039)=4.50078\times10^{-7}$ and $\varrho(2.399041)=-2.69471\times10^{-7}$ by numerical experiments. Moreover, it can be easily seen that

$$\mathcal{R}_{\frac{1}{2}}=\mathcal{R}_{1,3}\cup\mathcal{R}_{2,2}\cup\mathcal{R}_{2,3}\cup\mathcal{R}_{3,2},\qquad \mathcal{R}_{\frac{1}{2}}^{+}=\mathcal{R}_{1,4}\cup\mathcal{R}_{2,1}\cup\mathcal{R}_{3,1},$$
$$\mathcal{R}_{\frac{1}{2}}^{-}=\mathcal{R}_{1,1}\cup\mathcal{R}_{1,2}\cup\mathcal{R}_{2,4}\cup\mathcal{R}_{2,5}\cup\mathcal{R}_{2,6}\cup\mathcal{R}_{3,3}\cup\mathcal{R}_{4,1}\cup\mathcal{R}_{4,2}\cup\mathcal{R}_{4,3},$$

which gives a complete division of $\mathcal{R}_{\frac{1}{2}},\mathcal{R}_{\frac{1}{2}}^{+}$ and $\mathcal{R}_{\frac{1}{2}}^{-}$.

Let us define $\mathcal{R}_1=\bigcup_{j=1}^{4}\mathcal{R}_{1,j}$, $\mathcal{R}_2=\bigcup_{j=1}^{6}\mathcal{R}_{2,j}$, $\mathcal{R}_3=\bigcup_{j=1}^{3}\mathcal{R}_{3,j}$ and $\mathcal{R}_4=\bigcup_{j=1}^{3}\mathcal{R}_{4,j}$, more precisely, which are given by (see Figure 2)

$$\mathcal{R}_1=\left\{(p,q)\,\middle|\,\begin{array}{c}p<0\\ q>0\end{array}\ \text{or}\ \begin{array}{c}0<p<\frac{4}{15}\\ 10p-\frac{8}{3}\le q<0\end{array}\ \text{or}\ \begin{array}{c}p\ge\frac{1}{2}\\ 0<q\le 10p-\frac{8}{3}\end{array}\right\},$$
$$\mathcal{R}_2=\left\{(p,q)\,\middle|\,\begin{array}{c}p\ge\frac{1}{2}\\ q<0\end{array}\ \text{or}\ \begin{array}{c}p=\frac{1}{2}\\ q\ge q_0\end{array}\ \text{or}\ \begin{array}{c}0<p<\frac{1}{2}\\ q\ge\widehat{q}(p),\ q>0\end{array}\ \text{or}\ \begin{array}{c}\frac{13}{27}<p\le\frac{19}{39}\\ q=10p-\frac{8}{3}\end{array}\right.$$
$$\left.\text{or}\ \begin{array}{c}p<0\\ 10p-\frac{8}{3}\le q<0\end{array}\right\},$$
$$\mathcal{R}_3=\left\{(p,q)\,\middle|\,\begin{array}{c}p>\frac{1}{2}\\ q>10p-\frac{8}{3}\end{array}\ \text{or}\ \begin{array}{c}p=\frac{1}{2}\\ \frac{7}{3}<q<q_0\end{array}\ \text{or}\ \begin{array}{c}0<p<\frac{1}{2}\\ q<\min\{10p-\frac{8}{3},0\}\end{array}\right\},$$
$$\mathcal{R}_4=\left\{(p,q)\,\middle|\,\begin{array}{c}\frac{4}{15}<p<\frac{1}{2}\\ 0<q<10p-\frac{8}{3}\end{array}\ \text{or}\ \begin{array}{c}\frac{19}{39}<p<\frac{1}{2}\\ q=10p-\frac{8}{3}\end{array}\ \text{or}\ \begin{array}{c}p<0\\ q<10p-\frac{8}{3}\end{array}\right\}.$$

For the sake of convenience, we denote by $\mathcal{R}=\bigcup_{j=1}^{4}\mathcal{R}_j$ and the region of (p,q) in Remark 10 by

$$\mathcal{R}_0=\left\{(p,q)\,\middle|\,\frac{13}{27}<p<\frac{1}{2},\quad 10p-\frac{8}{3}<q<p+2-\sqrt{3(1-2p)}\right\}.$$

Apparently, $\mathbb{R}_0^2=\mathcal{R}\cup\mathcal{R}_0$. In the sequel, we mainly focus on the parameters $(p,q)\in\mathcal{R}$.

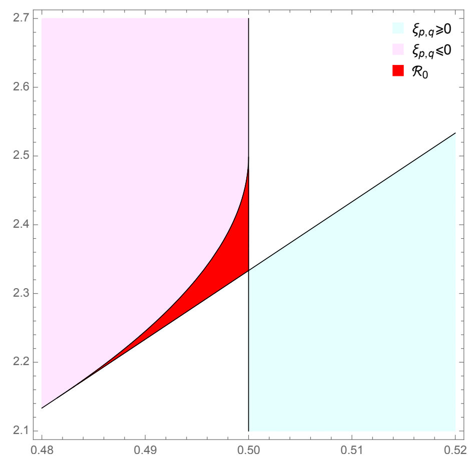

Figure 1: The sign of $\xi_{p,q}$.

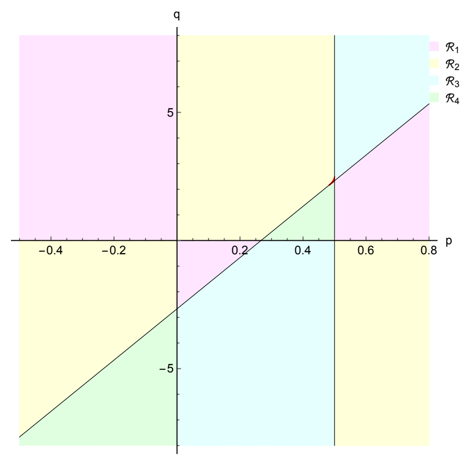

Figure 2: Visualized regions of $\mathcal{R}_j$.

Theorem 11. *Let $(p,q)\in\mathcal{R}$ and define the function $x\mapsto F_{p,q}(x)$ on $(0,1)$ by*

$$F_{p,q}(x)=\frac{1-(x/\operatorname{arcsl}x)^q}{1-(1-x^4)^p}$$

Then we have the following statements conclusions:

(1) $F_{p,q}$ is strictly increasing on $(0,1)$ if and only if $(p,q)\in\mathcal{R}_1$.

(2) $F_{p,q}$ is strictly decreasing on $(0,1)$ if $(p,q)\in\mathcal{R}_2$.

(3) If $(p,q)\in\mathcal{R}_3$, then there exists $x_1=x_1(p,q)\in(0,1)$ such that $F_{p,q}$ is strictly decreasing on $(0,x_1)$ and strictly increasing on $(x_1,1)$.

(4) If $(p,q)\in\mathcal{R}_4$, then there exists $x_2=x_2(p,q)\in(0,1)$ such that $F_{p,q}$ is strictly increasing on $(0,x_2)$ and strictly decreasing on $(x_2,1)$.

Proof. Let $f_1(x)=1-(x/\operatorname{arcsl}x)^q$ and $f_2(x)=1-(1-x^4)^p$. Then it is clear that $F_{p,q}(x)=f_1(x)/f_2(x)$ and $f_1(0^+)=f_2(0)=0$.

Differentiating $f_1(x)$ and $f_2(x)$ leads to

$$\frac{f_1'(x)}{f_2'(x)}=\frac{q}{4p}\left(\frac{x}{\operatorname{arcsl}x}\right)^{q-1}\frac{x-\sqrt{1-x^4}\operatorname{arcsl}x}{x^3(1-x^4)^{p-1/2}\operatorname{arcsl}^2x}, \tag{18}$$

$$\begin{aligned}\left[\frac{f_1'(x)}{f_2'(x)}\right]'&=\frac{q}{4px^5(1-x^4)^p}\left(\frac{x}{\operatorname{arcsl}x}\right)^{q+2}\Big[4p\varphi_2(x)-\varphi_1(x)-q\varphi_3(x)\Big]\\&\equiv\frac{q}{4px^5(1-x^4)^p}\left(\frac{x}{\operatorname{arcsl}x}\right)^{q+2}\Phi_{p,q}(x),\end{aligned} \tag{19}$$

where $\varphi_1(x),\varphi_2(x)$ and $\varphi_3(x)$ are defined as in Lemma 7.

By Lemma 7, it follows that

$$\Phi_{p,q}(x) = 4p\sum_{n=2}^{\infty}\frac{2(n-1)(4n+1)(\frac{3}{4})_n}{(2n-1)(4n-1)(\frac{5}{4})_n}x^{4n} - \sum_{n=2}^{\infty}\frac{16n(n-1)(\frac{3}{4})_n}{(2n+1)(4n-1)(\frac{5}{4})_n}x^{4n} - q\sum_{n=2}^{\infty}\frac{4(n-1)(4n+1)(\frac{3}{4})_n}{(4n^2-1)(4n-1)(\frac{5}{4})_n}x^{4n} = \sum_{n=2}^{\infty}\frac{4(n-1)\xi_{p,q}(n)}{(4n^2-1)(4n-1)}\frac{(\frac{3}{4})_n}{(\frac{5}{4})_n}x^{4n}, \tag{20}$$

where $\xi_{p,q}(x)$ is defined in Lemma 9.

By (7), (11) and (18), it is observed that

$$H_{f_1,f_2}(0^+) = \lim_{x\to 0^+}\left[\frac{f_1'(x)}{f_2'(x)}f_2(x) - f_1(x)\right] = \lim_{x\to 0^+}\left[\frac{q(x-\sqrt{1-x^4}\operatorname{arcsl} x)}{4px^5}\cdot f_2(x)\right] = \lim_{x\to 0^+}\frac{q\operatorname{arcsl} x}{10px\sqrt{1-x^4}}\cdot\lim_{x\to 0^+}f_2(x) = 0, \tag{21}$$

$$H_{f_1,f_2}(1^-) = \lim_{x\to 1^-}\left[\frac{f_1'(x)}{f_2'(x)}f_2(x) - f_1(x)\right] = \begin{cases}\infty, & p > \frac{1}{2},\\ \frac{\varrho(q)}{2\omega^{q+1}}, & p = \frac{1}{2},\\ \frac{1}{\omega^q} - 1, & p < \frac{1}{2}.\end{cases} \tag{22}$$

To obtain the monotonicity of $F_{p,q}$, it suffices to verify the sign of $\left[\frac{f_1'(x)}{f_2'(x)}\right]'$. It follows easily from (19) that

$$\operatorname{sgn}\left\{\left[\frac{f_1'(x)}{f_2'(x)}\right]'\right\} = \operatorname{sgn}(pq)\cdot\operatorname{sgn}\left[\Phi_{p,q}(x)\right]. \tag{23}$$

We divide into four cases to complete the proof.

Case I $(p,q)\in\mathcal{R}_1$.

- If $p<0$ and $q>0$ or $0<p<4/15$ and $10p-8/3\le q<0$, then it is apparent that $q\ge\max\{0,10p-8/3\}\ge\widehat{q}(p)$. This together with Lemma 9(3) gives $\xi_{p,q}(n)\le 0$ for $n\ge 2$. According to this with (20), it follows from $pq<0$ and (23) that $f_1'(x)/f_2'(x)$ is strictly increasing on $(0,1)$ and so is $f_1(x)/f_2(x)$ by Proposition 1.
- If $p\ge 1/2$ and $0<q\le 10p-8/3$, equivalently, $p=1/2$ and $0<q\le 7/3$ or $p>1/2$ and $0<q\le 10p-8/3$, then it follows from Lemma 9(1),(2) that $\xi_{p,q}(n)\ge 0$ for $n\ge 2$. Combining this with (20), the condition $pq>0$, (23) and Proposition 1 lead to the conclusion that $f_1(x)/f_2(x)$ is strictly increasing on $(0,1)$.

Case II $(p,q) \in \mathcal{R}_2$.

- If $p \geq 1/2$ and $q < 0$, then $q \leq 7/3 \leq 10p - 8/3$, and it follows from Lemma 9(1) and (2) that $\xi_{p,q}(n) \geq 0$ for $n \geq 2$. Due to $pq < 0$, by (20) and (23), we conclude that $f_1'(x)/f_2'(x)$ is strictly decreasing on $(0,1)$ and so is $f_1(x)/f_2(x)$ by Proposition 1.
- If $0 < p < 1/2$ and $q \geq \widehat{q}(p)$, $q > 0$ or $p < 0$ and $10p - 8/3 \leq q < 0$, then in either case, by the expression of $\widehat{q}(p)$, it can be easily seen that $p < 1/2$ and $q \geq \widehat{q}(p)$. This, by Lemma 9(3), implies that $\xi_{p,q}(n) \leq 0$ for $n \geq 2$. According to this, it follows from $pq > 0$, (20), (23) and Proposition 1 that $f_1(x)/f_2(x)$ is strictly decreasing on $(0,1)$.
- If $13/27 < p \leq 19/39$ and $q = 10p - 8/3$, then we clearly see from Lemma 9(3) that $\xi_{p,q}(n) \leq 0$ for $n \geq 2$, which in conjunction with $pq > 0$, (20), (23) and Proposition 1 that $f_1(x)/f_2(x)$ is strictly decreasing on $(0,1)$.
- In the case of $p = 1/2$ and $q \geq q_0 \approx 2.39904$, we divide into two subcases.
  - $p = 1/2$ and $q \geq 5/2$. In this case, Lemma 9(1) and (20), (23) together with $pq > 0$ and Proposition 1 lead to the conclusion that $f_1(x)/f_2(x)$ is strictly decreasing on $(0,1)$.
  - $p = 1/2$ and $q_0 \leq q < 5/2$. By Lemma 9(1), it follows that there exists $n_0 \geq 2$ such that $\xi_{p,q}(n) < 0$ for $2 \leq n \leq n_0$ and $\xi_{p,q}(n) > 0$ for $n \geq n_0$. This together with (20) implies that $\Phi_0(x)$ is an *NP*-type power series. Hence, by Proposition 3 and (23), $f_1'(x)/f_2'(x)$ is strictly decreasing on $(0,1)$ if $\Phi_0(1^-) \leq 0$ or there exists $\delta_0 \in (0,1)$ such that $f_1'(x)/f_2'(x)$ is strictly decreasing on $(0,\delta_0)$ and strictly increasing on $(\delta_0,1)$ if $\Phi_0(1^-) > 0$. So is $\mathsf{H}'_{f_1,f_2}(x)$ by (9) and $f_2(x) > 0$. In either case, we obtain
    (24)
    $$\mathsf{H}_{f_1,f_2}(x) < \max\left\{\mathsf{H}_{f_1,f_2}(0^+), \mathsf{H}_{f_1,f_2}(1^-)\right\} = \max_{q_0 \leq q < 5/2}\left\{0, \frac{\varrho(q)}{2\omega^{q+1}}\right\} = 0$$
    for $x \in (0,1)$ by (21) and (22), where the last equal follows from $\varrho(q) \leq 0$ for $q_0 \leq q < 5/2$ due to the monotonicity of $\varrho(q)$. Hence $f_1(x)/f_2(x)$ is strictly decreasing on $(0,1)$ following from (8), (24) and $f_2'(x) > 0$.

Case III $(p,q) \in \mathcal{R}_3$.

- If $p > 1/2$ and $q > 10p - 8/3$ or $p = 1/2$ and $7/3 < q < q_0$, then it follows from Lemma 9(1)(2) that there exists $n_1 \geq 2$ such that $\xi_{p,q}(n) < 0$ for $2 \leq n \leq n_1$ and $\xi_{p,q}(n) > 0$ for $n \geq n_1$. Moreover, by Lemma 7, we obtain

  $$\begin{aligned}(25) \qquad \Phi_{p,q}(1^-) &= \lim_{x\to 1^-}\left[4p\varphi_2(x) - \varphi_1(x) - q\varphi_3(x)\right] \\ &= 4(1-p)\omega^2 - 1 - q + 2(2p-1)\lim_{x\to 1^-}\frac{\operatorname{arcsl} x}{x\sqrt{1-x^4}}.\end{aligned}$$

This gives $\Phi_{p,q}(1^-) = \infty$ if $p > 1/2$ or $\Phi_{p,q}(1^-) = 2\omega^2 - 1 - q > 2\omega^2 - 1 - q_0 > 0.03$ if $p = 1/2$ and $7/3 < q < q_0$. According to this with the sign of $\xi_{p,q}(n)$, Proposition 3 and (23) together with $pq > 0$ and $f_2(x) > 0$ make us to know that there exists $\delta_1 \in (0,1)$ such that $H_{f_1,f_2}(x)$ is strictly decreasing on $(0,\delta_1)$ and strictly increasing on $(\delta_1,1)$. So is $f_1(x)/f_2(x)$ due to (21) and (22) together with (8) and $f_2'(x) > 0$.

- If $0 < p < 1/2$ and $q < \min\{10p - 8/3, 0\}$, then by (25) we clearly see that $\Phi_{p,q}(1^-) = -\infty$. This together with $pq < 0$, Proposition 3 and Lemma 9(3) implies that there exists $\delta_2 \in (0,1)$ such that $[f_1'(x)/f_2'(x)]'$ is strictly decreasing on $(0,\delta_2)$ and strictly increasing on $(\delta_2,1)$. So is $H_{f_1,f_2}(x)$ by (9) and $f_2(x) > 0$. In this case, it follows from (21) and (22) that $H_{f_1,f_2}(0^+) = 0$ and $H_{f_1,f_2}(1^-) = 1/\omega^q - 1 > 0$. According to this with (8), $f_2'(x) > 0$ and the piecewise monotonicity of $H_{f_1,f_2}(x)$, we conclude that the exists $x_1 \in (0,1)$ such that $f_1(x)/f_2(x)$ is strictly decreasing on $(0,x_1)$ and strictly increasing on $(x_1,1)$.

Case IV $(p,q) \in \mathcal{R}_4$.
In this case, it follows from (25) and Lemma 9(3) that there exists $\delta_3 \in (0,1)$ such that $\Phi_{p,q}(x) > 0$ for $x \in (0,\delta_3)$ and $\Phi_{p,q}(x) < 0$ for $x \in (\delta_3,1)$. This together with $pq > 0$ and (23) implies that $(f_1'/f_2')'$ is strictly increasing on $(0,\delta_3)$ and strictly decreasing on $(\delta_3,1)$.

- If $4/15 < p < 1/2$ and $0 < q < 10p - 8/3$ or $19/39 < p < 1/2$ and $q = 10p - 8/3$, then due to $f_2(x) > 0$ and (9), $H_{f_1,f_2}(x)$ has the same piecewise monotonicity property of $(f_1'/f_2')'$. According to this with (21) and (22), it can be seen that there exists $x_2 \in (0,1)$ such that $H_{f_1,f_2}(x) > 0$ for $x \in (0,x_2)$ and $H_{f_1,f_2}(x) < 0$ for $x \in (x_2,1)$. This proves Theorem 11(4) by (8) and $f_2'(x) > 0$.
- If $p < 0$ and $q < 10p - 8/3$, then it follows from (9) and $f_2(x) < 0$ that $H_{f_1,f_2}(x)$ has the opposite piecewise monotonicity property of $(f_1'/f_2')'$. At this point, $H_{f_1,f_2}(1^-) > 0$ by (22). This together with (8), (21) and $f_2'(x) < 0$ shows that there exists $x_2 \in (0,1)$ that f_1/f_2 is strictly increasing on $(0,x_2)$ and strictly decreasing on $(x_2,1)$.

To obtain the necessity of Theorem 11(1), by the above analysis, it suffices to show that $F_{p,q}$ is not strictly increasing on $(0,1)$ when $(p,q) \in \mathcal{R}_0$. For $(p,q) \in \mathcal{R}_0$, it is easy to see that $\xi_{p,q}(2) = 3(30p - 3q - 8) < 0$, which in conjunction with (20), (23) and $pq > 0$ implies that there exists a sufficiently small $\varepsilon \in (0,1)$ such that $f_1'(x)/f_2'(x)$ is strictly decreasing on $(0,\varepsilon)$. Combining this with $f_1(0^+) = f_2(0) = 0$, Proposition 1 enables us to know that $F_{p,q}$ is strictly decreasing on $(0,\varepsilon)$.

□

Before stating the main result, we need to divide $\mathcal{R}$ into several new regions.

Let us define the function $q \mapsto \iota(q)$ on $(-\infty, \infty)$ by

$$\iota(q) = \frac{q\omega^q}{10(\omega^q - 1)}, \quad q \neq 0 \quad \text{and} \quad \iota(0) = \frac{1}{10\log\omega}.$$

By Proposition 1, it is easy to see that $\iota(q)$ is continuous at the origin and strictly increasing from $(-\infty, \infty)$ onto $(0, \infty)$. Moreover, for $p > 0$,

$$\frac{q}{10p} - \left(1 - \frac{1}{\omega^q}\right) = \frac{(\omega^q - 1)[\iota(q) - p]}{p\omega^q} \begin{cases} \geq 0, & if \quad q[\iota(q) - p] \geq 0, \\ \leq 0, & if \quad q[\iota(q) - p] \leq 0. \end{cases} \tag{26}$$

For later use, the symbol $A^{\pm}$ stands for the intersection of A and $\{(p,q) |\ p > 0,\ q[\iota(q) - p] \geq (\leq) 0\}$. Now we are able to give the following regions.

- The case $pq > 0$:

$$D_1 = \mathcal{R}_{1,3} \cup \mathcal{R}_{1,4} \cup \mathcal{R}_{4,1}^- \cup \mathcal{R}_{4,2}^-, \quad D_2 = \mathcal{R}_{2,6} \cup \mathcal{R}_{4,3}, \quad D_3 = \mathcal{R}_{3,1} \cup \mathcal{R}_{3,2},$$
$$D_4 = \mathcal{R}_{2,3} \cup \mathcal{R}_{2,4} \cup \mathcal{R}_{2,5} \cup \mathcal{R}_{4,1}^+ \cup \mathcal{R}_{4,2}^+,$$
$$E_1 = \mathcal{R}_{1,3} \cup \mathcal{R}_{1,4} \cup \mathcal{R}_{3,1}^- \cup \mathcal{R}_{3,2}^-, \quad E_2 = \mathcal{R}_{2,3} \cup \mathcal{R}_{2,4} \cup \mathcal{R}_{2,5} \cup \mathcal{R}_{2,6} \cup \mathcal{R}_{3,1}^+ \cup \mathcal{R}_{3,2}^+,$$
$$E_3 = \mathcal{R}_4\,.$$

- The case $pq < 0$:

$$\widehat{D}_1 = \mathcal{R}_{1,2} \cup \mathcal{R}_{3,3}^-, \quad \widehat{D}_2 = \mathcal{R}_{1,1}, \quad \widehat{D}_3 = \mathcal{R}_{2,1} \cup \mathcal{R}_{2,2} \cup \mathcal{R}_{3,3}^+,$$
$$\widehat{E}_1 = \mathcal{R}_{1,1} \cup \mathcal{R}_{1,2}, \quad \widehat{E}_2 = \mathcal{R}_{3,3}, \quad \widehat{E}_3 = \mathcal{R}_{2,1} \cup \mathcal{R}_{2,2}\,.$$

Observe that $\mathcal{R} = \left(\bigcup_{j=1}^4 D_j\right) \cup \left(\bigcup_{j=1}^3 \widehat{D}_j\right) = \left(\bigcup_{j=1}^3 E_j\right) \cup \left(\bigcup_{j=1}^3 \widehat{E}_j\right)$, we refer to see Figures 3 and 4 for clarity.

We are now in a position to prove the main result.

Theorem 12. *Let* $m_F = \min\limits_{x\in(0,1)}\{F_{p,q}(x)\}$ *and* $M_F = \max\limits_{x\in(0,1)}\{F_{p,q}(x)\}$. *Then for any fixed* $(p,q) \in \mathcal{R}$, *the double inequality*

$$H_\alpha(x; p, q) \leq \frac{\operatorname{arcsl} x}{x} \leq H_\beta(x; p, q) \tag{27}$$

holds for all $0 < |x| < 1$ *with the equal only for certain point* x *if and only if*

$$\alpha \leq \alpha(p,q) \quad \textit{and} \quad \beta \geq \beta(p,q) \quad \textit{for } pq > 0,$$
$$\alpha \geq \widehat{\alpha}(p,q) \quad \textit{and} \quad \beta \leq \widehat{\beta}(p,q) \quad \textit{for } pq < 0,$$

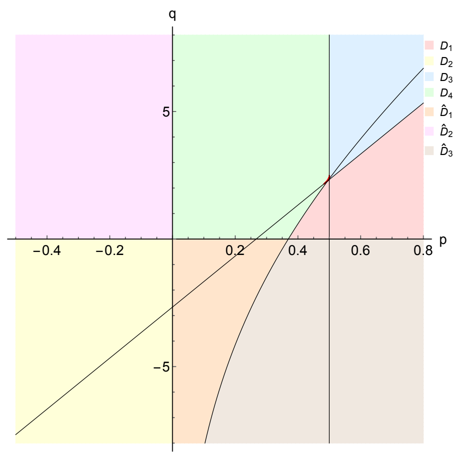

Figure 3: The regions D_j and $\widehat{D}_j$.

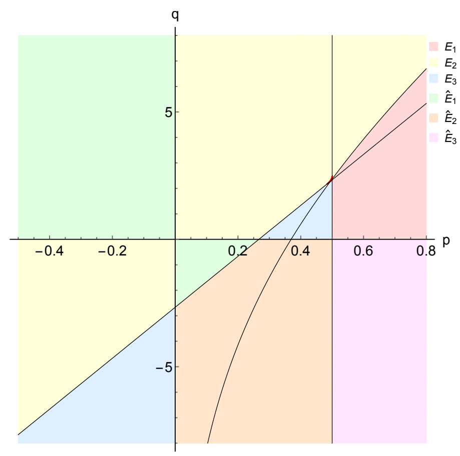

Figure 4: The regions E_j and $\widehat{E}_j$.

where

$$\alpha(p,q)=\begin{cases}\frac{q}{10p}, & (p,q)\in D_1,\\ 0, & (p,q)\in D_2,\\ m_F, & (p,q)\in D_3,\\ 1-\frac{1}{\omega^q}, & (p,q)\in D_4,\end{cases}\qquad \beta(p,q)=\begin{cases}1-\frac{1}{\omega^q}, & (p,q)\in E_1,\\ \frac{q}{10p}, & (p,q)\in E_2,\\ M_F, & (p,q)\in E_3,\end{cases}$$

$$\widehat{\alpha}(p,q)=\begin{cases}1-\frac{1}{\omega^q}, & (p,q)\in \widehat{D}_1,\\ 0, & (p,q)\in \widehat{D}_2,\\ \frac{q}{10p}, & (p,q)\in \widehat{D}_3,\end{cases}\qquad \widehat{\beta}(p,q)=\begin{cases}\frac{q}{10p}, & (p,q)\in \widehat{E}_1,\\ m_F, & (p,q)\in \widehat{E}_2,\\ 1-\frac{1}{\omega^q}, & (p,q)\in \widehat{E}_3.\end{cases}$$

In particular, if $\lim_{(p,q)\to(0,0)}(p/q)=\lambda\neq 0$, *the double inequality*

$$H_\alpha(x;\lambda)<\frac{\operatorname{arcsl} x}{x}<H_\beta(x;\lambda) \tag{28}$$

holds for all $0<|x|<1$ *if and only if*

$$\begin{cases}\alpha\le 0 \quad \textit{and} \quad \beta\ge\frac{1}{10\lambda}, & \lambda>0,\\ \alpha\ge 0 \quad \textit{and} \quad \beta\le\frac{1}{10\lambda}, & \lambda<0.\end{cases}$$

Proof. Since the functions $\operatorname{arcsl} x$ and x are odd on $(-1,1)$, we may assume that $x\in(0,1)$. For $(p,q)\in\mathbb{R}_0^2$, it is a matter of simple transformation to verify that the inequality (27) is equivalent to

$$\alpha\le F_{p,q}(x)\le\beta, \quad \text{if} \quad pq>0, \tag{29}$$

$$\beta\le F_{p,q}(x)\le\alpha, \quad \text{if} \quad pq<0, \tag{30}$$

Table 1: Description of the range of $F_{p,q}$ for $(p,q) \in \mathcal{R}$

(a) The case of $pq > 0$

Regions		The range of $F_{p,q}$
$\mathcal{R}_{1,3}$		$\left(\frac{q}{10p}, 1-\frac{1}{\omega^q}\right)$
$\mathcal{R}_{1,4}$		$\left(\frac{q}{10p}, 1-\frac{1}{\omega^q}\right)$
$\mathcal{R}_{2,3}$		$\left(1-\frac{1}{\omega^q}, \frac{q}{10p}\right)$
$\mathcal{R}_{2,4}$		$\left(1-\frac{1}{\omega^q}, \frac{q}{10p}\right)$
$\mathcal{R}_{2,5}$		$\left(1-\frac{1}{\omega^q}, \frac{q}{10p}\right)$
$\mathcal{R}_{2,6}$		$\left(0, \frac{q}{10p}\right)$
$\mathcal{R}_{3,1}$	$\mathcal{R}_{3,1}^{+}$	$\left[m_F, \frac{q}{10p}\right)$
	$\mathcal{R}_{3,1}^{-}$	$\left[m_F, 1-\frac{1}{\omega^q}\right)$
$\mathcal{R}_{3,2}$	$\mathcal{R}_{3,2}^{+}$	$\left[m_F, \frac{q}{10p}\right)$
	$\mathcal{R}_{3,2}^{-}$	$\left[m_F, 1-\frac{1}{\omega^q}\right)$
$\mathcal{R}_{4,1}$	$\mathcal{R}_{4,1}^{+}$	$\left(1-\frac{1}{\omega^q}, M_F\right]$
	$\mathcal{R}_{4,1}^{-}$	$\left(\frac{q}{10p}, M_F\right]$
$\mathcal{R}_{4,2}$	$\mathcal{R}_{4,2}^{+}$	$\left(1-\frac{1}{\omega^q}, M_F\right]$
	$\mathcal{R}_{4,2}^{-}$	$\left(\frac{q}{10p}, M_F\right]$
$\mathcal{R}_{4,3}$		$\left(0, M_F\right]$

(b) The case of $pq < 0$

Regions		The range of $F_{p,q}$
$\mathcal{R}_{1,1}$		$\left(\frac{q}{10p}, 0\right)$
$\mathcal{R}_{1,2}$		$\left(\frac{q}{10p}, 1-\frac{1}{\omega^q}\right)$
$\mathcal{R}_{2,1}$		$\left(1-\frac{1}{\omega^q}, \frac{q}{10p}\right)$
$\mathcal{R}_{2,2}$		$\left(1-\frac{1}{\omega^q}, \frac{q}{10p}\right)$
$\mathcal{R}_{3,3}$	$\mathcal{R}_{3,3}^{+}$	$\left[m_F, \frac{q}{10p}\right)$
	$\mathcal{R}_{3,3}^{-}$	$\left[m_F, 1-\frac{1}{\omega^q}\right)$

where $F_{p,q}$ is defined as in Theorem 11. So the inequalities (29) and (30) hold for all $x \in (0,1)$ if and only if

$$\alpha \le \alpha(p,q) = \inf_{x\in(0,1)}\{F_{p,q}(x)\} \quad \text{and} \quad \beta \ge \beta(p,q) = \sup_{x\in(0,1)}\{F_{p,q}(x)\} \quad \text{for } pq>0,$$

$$\alpha \ge \widehat{\alpha}(p,q) = \sup_{x\in(0,1)}\{F_{p,q}(x)\} \quad \text{and} \quad \beta \le \widehat{\beta}(p,q) = \inf_{x\in(0,1)}\{F_{p,q}(x)\} \quad \text{for } pq<0.$$

Note that, by (18),

$$F_{p,q}(0^+) = \lim_{x\to 0^+}\frac{f_1'(x)}{f_2'(x)} = \frac{q}{10p}, \quad F_{p,q}(1^-) = \begin{cases} 0, & p<0, \\ 1-\frac{1}{\omega^q}, & p>0. \end{cases} \tag{31}$$

Moreover, it has been proved in Theorem 11 the monotonicity of $F_{p,q}(x)$ on $(0,1)$ for $(p,q) \in \mathcal{R}$. This together with (26) and (31) gives the range of $F_{p,q}$ for each $(p,q) \in \mathcal{R}$, which is illustrated in Table 1. Hence, the specific expression of $\alpha(p,q), \beta(p,q)$, $\widehat{\alpha}(p,q)$ and $\widehat{\beta}(p,q)$ can be obtained from Table 1.

In particular, in the case of $(p,q) = (0,0)$, if $\lim_{(p,q)\to(0,0)}(q/p) = \lambda \ne 0$, it is considered to approximate $(\operatorname{arcsl} x)/x$ by $H_\alpha(x;\lambda)$, which is the limiting expression of $H_\alpha(x;p,q)$. It can be easily verified that the double inequality (28) is equivalent to

$$\lambda\alpha < \frac{\log(x/\operatorname{arcsl} x)}{\log(1-x^4)} < \lambda\beta \iff \begin{cases} \alpha < \frac{\log(x/\operatorname{arcsl} x)}{\lambda\log(1-x^4)} < \beta, & \lambda > 0, \\ \beta < \frac{\log(x/\operatorname{arcsl} x)}{\lambda\log(1-x^4)} < \alpha, & \lambda < 0 \end{cases} \tag{32}$$

for all $x \in (0,1)$. So the sufficient and necessary conditions which make the inequality (28) hold can be easily obtained from (32), the sign of λ and Lemma 8. □

Remark 13. In Theorem 12, the best constants $\alpha(p,q)$ (resp. $\beta(p,q)$, $\widehat{\beta}(p,q)$) is not computable simply if $(p,q) \in D_3$ (resp. E_3, $\widehat{E}_2$). In this case, Theorem 11(3) (resp. (4)) enables us to know that $F_{p,q}$ is strictly convex (or concave) on $(0,1)$, that is, $F'_{p,q}$ is strictly increasing (or decreasing) on $(0,1)$, one can find an approximation of the extreme point with arbitrary precision by numerical experiments.

4. APPLICATIONS

In this section, Theorem 12 will be applied to obtain some new inequalities involving the arc lemniscate sine function.

Recall that a real-valued function $A(x)$ is said to be an n-order approximation for $f(x)$ as $x \to 0^+$ if

$$\lim_{x\to 0^+} \frac{|f(x)-A(x)|}{x^n} = c > 0.$$

Moreover, if the inequality $f(x) \geq (\leq) A(x)$ holds for all $x \in (0,1)$, then $A(x)$ is called an n-order lower (upper) approximation for $f(x)$. It is well known that the order n is one of the important measure of accuracy of the given approximation $A(x)$ for $f(x)$, see [**2, 25**].

As $x \to 0^+$, by Taylor series, we obtain

$$\frac{\operatorname{arcsl} x}{x} - H_\alpha(x;p,q) = \left(\frac{1}{10} - \frac{\alpha p}{q}\right)x^4 + \frac{12\alpha pq(p-1) + q^2 - 12\alpha^2p^2(q+1)}{24q^2}x^8$$
$$+ \left[\frac{5}{208} - \frac{\alpha p(p-1)(p-2)}{6q} + \frac{\alpha^2p^2(p-1)(q+1)}{2q^2} - \frac{\alpha^3p^3(q+1)(2q+1)}{6q^3}\right]x^{12} + o(x^{12}).$$

We can find the parameters α, p and q such that all the coefficient for x^4, x^8 and x^{12} are zero, explicitly,

$$\alpha = \frac{43}{95}, \quad p = \frac{19}{39}, \quad q = \frac{86}{39}.$$

It can be easily seen that $q = 10p - 8/3$ and so $(19/39, 86/39) \in \mathcal{R}_{2,5} \subset D_4 \cap E_2$. By Theorem 12, we obtain the following corollary.

Corollary 1. *The inequality*

$$H_{1-\omega^{-\frac{86}{39}}}\left(x;\frac{19}{39},\frac{86}{39}\right)<\frac{\operatorname{arcsl}x}{x}<H_{\frac{43}{95}}\left(x;\frac{19}{39},\frac{86}{39}\right)$$

holds for all $0<|x|<1$*, where* $H_{\frac{43}{95}}\left(x;\frac{19}{39},\frac{86}{39}\right)$ *is an 16-order upper approximation.*

A special case of our interest is that (p,q) passing through the origin. Assuming that $q=kp$ with $k\neq 0$, which is required not to intersect with $\mathcal{R}_0$. It is observed that both of the points $(13/27,58/27)$ and $(1/2,5/2)$ belong to $\mathcal{R}_2$ and also the boundary points of $\mathcal{R}_0$. If two lines $q=k_1p$ and $q=k_2p$ pass through these two points, then it can be easily obtained that $k_1=\frac{58}{13}$ and $k_2=5$. That is to say, $\{(p,q)|\ q=kp,\ k\neq 0\}\cap\mathcal{R}_0=\emptyset$ if $k\in(-\infty,0)\cup(0,\frac{58}{13}]\cup[5,\infty)$.

As a consequence of Theorem 12, we pose the following propositions.

Proposition 14. *(1) For* $k\in(-\infty,\frac{58}{13}]$ *with* $k\neq 0$*, the inequality*

$$H_{k/10}(x;p_1,kp_1)<\frac{\operatorname{arcsl}x}{x}<H_{k/10}(x;p_2,kp_2) \tag{33}$$

holds for all $0<|x|<1$ *with the best possible constants* $p_1=\sigma$ *and* $p_2=\tau$*, where*

$$\sigma:=\sigma(k)=\frac{\log\left[10/(10-k)\right]}{k\log\omega}\quad\textit{and}\quad\tau:=\tau(k)=\frac{8}{3(10-k)}.$$

(2) For $k\in[5,10)$*, the inequality* (33) *holds for all* $0<|x|<1$ *with the best possible constants* $p_1=\tau$ *and* $p_2=\sigma$*.*

Proof. Inequality (33) can be directly derived from Theorem 11 if we can show that

$$\begin{cases}(\sigma,k\sigma)\in D_1\quad\text{and}\quad(\tau,k\tau)\in E_2,\quad\text{if}\quad k\in(0,\frac{58}{13}],\\(\tau,k\tau)\in D_1\quad\text{and}\quad(\sigma,k\sigma)\in E_2,\quad\text{if}\quad k\in[5,10),\\(\sigma,k\sigma)\in\widehat{D}_3\quad\text{and}\quad(\tau,k\tau)\in\widehat{E}_1,\quad\text{if}\quad k\in(-\infty,0).\end{cases}$$

Firstly, it can be easily verified, by Proposition 1, that $k\mapsto\sigma(k)$ and $k\mapsto\tau(k)/\sigma(k)$ is strictly increasing on $(-\infty,10)$. Secondly, by the expression of σ and τ, it follows from $\omega^{k\sigma}=10/(10-k)$ that $\iota(k\sigma)=(k\sigma\omega^{k\sigma})/[10(\omega^{k\sigma}-1)]=\sigma$ and $k\tau=10\tau-8/3$. That is to say, $(\sigma,k\sigma)$ satisfies the curve $p=\iota(q)$ and $(\tau,k\tau)$ is on the straight line $q=10p-8/3$.

- $k\in(0,\frac{58}{13}]$.
  In this case, $\sigma>\sigma(0^+)=1/[10\log\omega]>0$ and $\sigma\leq\sigma(\frac{58}{13})=0.489\cdots<1/2$. Moreover,

$$\frac{8}{3(10-k)\sigma}=\frac{\tau(k)}{\sigma(k)}\leq\frac{\tau(\frac{58}{13})}{\sigma(\frac{58}{13})}=0.984559\cdots<1\quad\Longleftrightarrow\quad 0<k\sigma<10\sigma-\frac{8}{3}. \tag{34}$$

This gives $(\sigma, k\sigma) \in \mathcal{R}_{4,1}$ and So $(\sigma, k\sigma) \in \mathcal{R}_{4,1}^- \subset D_1$. On the other hand,

$$\frac{4}{15} = \tau(0) < \tau < \tau(\tfrac{58}{13}) = \frac{13}{27} \quad \text{and} \quad k\tau = 10\tau - \frac{8}{3} = \widehat{q}(\tau).$$

That is to say, $(\tau, k\tau) \in \mathcal{R}_{2,4} \subset E_2$.

- $k \in [5, 10)$.
  Due to $\tau > \tau(5) = 8/15 > 1/2$, $(\tau, k\tau) \in \mathcal{R}_{1,4} \subset D_1$. On the other hand, $\sigma \geq \sigma(5) = 0.5119\cdots > 1/2$ and

  $$\frac{8}{3(10-k)\sigma} = \frac{\tau(k)}{\sigma(k)} \geq \frac{\tau(5)}{\sigma(5)} = 1.0418\cdots > 1 \quad \Longleftrightarrow \quad k\sigma > 10\sigma - \frac{8}{3},$$

  which gives $(\sigma, k\sigma) \in \mathcal{R}_{3,1}$ and $(\sigma, k\sigma) \in \mathcal{R}_{3,1}^+ \in E_2$.

- $k \in (-\infty, 0)$.
  In this case, we obtain $0 = \sigma(-\infty) < \sigma < \sigma(0^-) = 1/(10\log\omega) = 0.3692\cdots < 1/2$. As in (34), it follows from $\tau(0)/\sigma(0) < 1$ that $k\sigma < 10\sigma - 8/3$, which gives $(\sigma, k\sigma) \in \mathcal{R}_{3,3}$ and so $(\sigma, k\sigma) \in \mathcal{R}_{3,3}^+ \subset \widehat{D}_3$. On the other hand, $(\tau, k\tau) \in \mathcal{R}_{1,2} \subset \widehat{E}_1$ follows from $0 < \tau < \tau(0) = 4/15$.

In the following, we shall prove that the constants are the sharp for each case. We only give the proof for the case $k \in (0, \frac{58}{13}]$ and others are similar.

For $k \in (0, \frac{58}{13}]$, as in (34), one has $\tau < \sigma$. If $p \in (\tau, \sigma)$, namely,

$$\frac{4}{15} < \frac{8}{3(10-k)} < p < \frac{\log[10/(10-k)]}{k\log\omega} < \frac{1}{2},$$

which can be simplified to

$$0 < kp < 10p - \frac{8}{3} \quad \text{and} \quad 0 < p < \frac{kp\omega^{kp}}{10(\omega^{kp}-1)} = \iota(kp),$$

That is to say, $(p, kp) \in \mathcal{R}_{4,1}^+ \subset D_4 \cap E_3$. By Theorem 12, it follows that

$$H_{1-\omega^{-kp}}(x; p, kp) < \frac{\operatorname{arcsl} x}{x} < H_{M_F}(x; p, kp)$$

holds for all $0 < |x| < 1$ with the optimal parameters $1 - \omega^{-kp}$ and M_F. For $(p, kp) \in \mathcal{R}_{4,1}^+$, it follows from Theorem 11(4) and (26) that $1 - \omega^{-kp} < k/10$ and $M_F > k/10$. Due to the monotonicity of $\alpha \mapsto H_\alpha(x; p, kp)$, the optimality of parameters enables us to know that there exist two numbers $\hat{x}_1, \hat{x}_2 \in (0, 1)$ such that

$$H_{k/10}(\hat{x}_1; p, kp) > \frac{\operatorname{arcsl}\hat{x}_1}{\hat{x}_1} \quad \text{and} \quad H_{k/10}(\hat{x}_2; p, kp) < \frac{\operatorname{arcsl}\hat{x}_2}{\hat{x}_2}.$$

It is observed that $p \mapsto H_{k/10}(x; p, kp)$ is strictly decreasing (increasing) on $(-\infty, \infty)$ for $k > (<)0$. This gives the sharpness of τ and σ. $\square$

By applying Proposition 14, we give the following corollaries.

Corollary 2. *The inequality*

$$H_{29/65}\big(x;p_3,58p_3/13\big)<\frac{\operatorname{arcsl}x}{x}<H_{29/65}\big(x;p_4,58p_4/13\big)$$

holds for all $0<|x|<1$ *if and only if* $p_3\geq[13\log(65/36)]/(58\log\omega)=0.48903\cdots$ *and* $p_4\leq 13/27=0.48148\cdots$.

Corollary 3. *The inequality*

$$H_{1/2}\big(x;p_5,5p_5\big)<\frac{\operatorname{arcsl}x}{x}<H_{1/2}\big(x;p_6,5p_6\big)$$

holds for all $0<|x|<1$ *if and only if* $p_5\geq 8/15=0.5333\cdots$ *and* $p_6\leq(\log 2)/(5\log\omega)=0.5119\cdots$.

Remark 15. By numerical experiments, we can show that $k\mapsto H_{k/10}\big(x;\sigma(k);k\sigma(k)\big)$ is strictly increasing on $(-\infty,10)$ and $k\mapsto H_{k/10}\big(x;\tau(k);k\tau(k)\big)$ is strictly decreasing on $(-\infty,10)$. Moreover, $p\mapsto H_{k/10}(x;p,kp)$ is strictly decreasing on $(-\infty,\infty)$ for $k>0$.

Based on these monotonicity properties, it follows from Corollaries 2 and 3 that

$$\begin{aligned}
&H_{1-1/\omega}(x;\tfrac14,1)<H_{1-1/\omega}(x;\tfrac{1}{10(1-1/\omega)},1)<H_{29/65}\left(x;\sigma(\tfrac{58}{13}),\tfrac{58}{13}\sigma(\tfrac{58}{13})\right)\\
&\quad<\frac{\operatorname{arcsl}x}{x}<H_{29/65}\big(x;\tau(\tfrac{58}{13}),\tfrac{58}{13}\tau(\tfrac{58}{13})\big)<H_{2/5}(x;\tfrac49,\tfrac{16}{9})<H_{2/5}(x;\tfrac14,1),
\end{aligned}$$

$$\begin{aligned}
&H_{1/5}(x;\tfrac12,1)<H_{1/5}\big(x;\sigma(2),2\sigma(2)\big)<H_{29/65}\left(x;\sigma(\tfrac{58}{13}),\tfrac{58}{13}\sigma(\tfrac{58}{13})\right)\\
&\quad<\frac{\operatorname{arcsl}x}{x}<H_{1/2}\big(x;\sigma(5),5\sigma(5)\big)<H_{1-1/\omega}(x;\tfrac12;1),
\end{aligned}\tag{35}$$

$$\begin{aligned}
&H_{2/5}(x;\tfrac12,2)<H_{2/5}\big(x;\sigma(4),4\sigma(4)\big)<H_{29/65}\big(x;\sigma(\tfrac{58}{13}),\tfrac{58}{13}\sigma(\tfrac{58}{13})\big)\\
&\quad<\frac{\operatorname{arcsl}x}{x}<H_{1/2}\big(x;\sigma(5),5\sigma(5)\big)<H_{1-1/\omega^2}(x;\tfrac12;2),
\end{aligned}\tag{36}$$

where the last inequal signs in (35) and (36) follow from numerical experiments.

Hence, Corollaries 2 and 3 give the great improvements for inequalities (3)-(5).

Remark 16. As $x\to 0^+$, by Taylor series, we have

$$\begin{aligned}
\frac{\operatorname{arcsl}x}{x}-H_{29/65}\left(x;\frac{13}{27},\frac{58}{27}\right)&=-\frac{2x^{12}}{78975}+o(x^{16}),\\
\frac{\operatorname{arcsl}x}{x}-H_{1/2}\left(x;\frac{8}{15},\frac{8}{3}\right)&=\frac{x^{12}}{4875}+o(x^{16}),
\end{aligned}$$

which together with Corollaries 2 and 3 implies that $H_{29/65}\left(x;\frac{13}{27},\frac{58}{27}\right)$ and $H_{1/2}\left(x;\frac{8}{15},\frac{8}{3}\right)$ give the 12-order upper and lower bounds for $(\operatorname{arcsl}x)/x$, respectively.

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Tie-Hong Zhao
School of Mathematics,
Hangzhou Normal University,
Hangzhou 311121, China
E-mail: *tiehong.zhao@hznu.edu.cn*
Orcid: *0000-0002-6394-1094*

Miao-kun Wang
Department of Mathematics,
Huzhou University,
Huzhou 313000, China
E-mail: *wangmiaokun@zjhu.edu.cn*
Orcid: *0000-0002-0895-7128*