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ON SOME MEAN SQUARE ESTIMATES IN THE RANKIN-SELBERG PROBLEM

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An overview of the classical Rankin-Selberg problem involving the asymptotic formula for sums of coefficients of holomorphic cusp forms is given. We also study the function $\Delta(x;\xi)$ $(0 \le \xi \le 1)$, the error term in the Rankin-Selberg problem weighted by ξ-th power of the logarithm. Mean square estimates for $\Delta(x;\xi)$ are proved.

1. THE RANKIN-SELBERG PROBLEM

The classical Rankin-Selberg problem consists of the estimation of the error term function

$$\Delta(x) := \sum_{n\le x} c_n - Cx, \tag{1.1}$$

where the notation is as follows. Let $\varphi(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL(2,\mathbb{Z})$, and denote by $a(n)$ the n-th Fourier coefficient of $\varphi(z)$ (see e.g., R. A. Rankin [**15**] for a comprehensive account). We suppose that $\varphi(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$, that is, $a(1) = 1$ and $T(n)\varphi = a(n)\varphi$ for every $n \in \mathbb{N}$. In (1.1) $C > 0$ is a suitable constant (see e.g., [**9**] for its explicit expression), and c_n is the convolution function defined by

$$c_n = n^{1-\kappa} \sum_{m^2|n} m^{2(\kappa-1)} \left|a\left(\frac{n}{m^2}\right)\right|^2.$$

The classical Rankin-Selberg bound of 1939 is

$$\Delta(x) = O(x^{3/5}), \tag{1.2}$$

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hitherto unimproved. In their works, done independently, R. A. Rankin [**14**] derives (1.2) from a general result of E. Landau [**11**], while A. Selberg [**17**] states the result with no proof. Although the exponent 3/5 in (1.2) represents one of the longest standing records in analytic number theory, recently there have been some developments in some other aspects of the Rankin-Selberg problem. In this paper we shall present an overview of some of these new results. In addition, we shall consider the weighted sum (the so-called Riesz logarithmic means of order ξ), namely

$$\frac{1}{\Gamma(\xi+1)}\sum_{n\le x} c_n \log^\xi\left(\frac{x}{n}\right) := Cx + \Delta(x;\xi) \qquad (\xi \ge 0), \tag{1.3}$$

where C is as in (1.1), so that $\Delta(x) \equiv \Delta(x;0)$. The effect of introducing weights such as the logarithmic weight in (1.3) is that the ensuing error term (in our case this is $\Delta(x;\xi)$) can be estimated better than the original error term (i.e., in our case $\Delta(x;0)$). This was shown by Matsumoto, Tanigawa and the author in [**9**], where it was proved that

$$\Delta(x;\xi) \ \ll_\varepsilon \ x^{(3-2\xi)/5+\varepsilon} \qquad (0 \le \xi \le 3/2). \tag{1.4}$$

Here and later ε denotes arbitrarily small constants, not necessarily the same ones at each occurrence, while $a \ll_\varepsilon b$ means that the constant implied by the $\ll$-symbol depends on ε. When $\xi = 0$ we recover (1.2) from (1.4), only with the extra 'ε' factor present. In this work we shall pursue the investigations concerning $\Delta(x;\xi)$, and deal with mean square bounds for this function.

2. THE FUNCTIONAL EQUATIONS

In view of (1.1) and (1.2) it follows that the generating Dirichlet series

$$Z(s) \ := \ \sum_{n=1}^\infty c_n n^{-s} \qquad (s = \sigma + it) \tag{2.1}$$

converges absolutely for $\sigma > 1$. The arithmetic function c_n is multiplicative and satisfies $c_n \ll_\varepsilon n^\varepsilon$. Moreover, it is well known (see e.g., R. A. Rankin [**14**], [**15**]) that $Z(s)$ satisfies for all s the functional equation

$$\Gamma(s+\kappa-1)\Gamma(s)Z(s) = (2\pi)^{4s-2}\Gamma(\kappa-s)\Gamma(1-s)Z(1-s), \tag{2.2}$$

which provides then the analytic continuation of $Z(s)$. In modern terminology $Z(s)$ belongs to the Selberg class $\mathcal{S}$ of L-functions of degree four (see A. Selberg [**18**] and the survey paper of Kaczorowski–Perelli [**10**]). An important feature, proved by G. Shimura [**19**] (see also A. Sankaranarayanan [**16**]) is

$$Z(s) = \zeta(s)\sum_{n=1}^\infty b_n n^{-s} = \zeta(s)B(s), \tag{2.3}$$

where $B(s)$ is holomorphic for $\sigma > 0$, $b_n \ll_\varepsilon n^\varepsilon$ (in fact $\sum_{n\le x} b_n^2 \le x\log^A x$ holds, too). It also satisfies the functional equation

$$\begin{aligned} B(s)\Delta_1(s) &= B(1-s)\Delta_1(1-s),\\ \Delta_1(s) &= \pi^{-3s/2}\Gamma\big(\tfrac12(s+\kappa-1)\big)\Gamma\big(\tfrac12(s+\kappa)\big)\Gamma\big(\tfrac12(s+\kappa+1)\big), \end{aligned}$$

and actually $B(s) \in \mathcal{S}$ with degree three. The decomposition (2.3) (the so-called 'Shimura lift') allows one to use, at least to some extent, results from the theory of $\zeta(s)$ in connection with $Z(s)$, and hence to derive results on $\Delta(x)$.

3. THE COMPLEX INTEGRATION APPROACH

A natural approach to the estimation of $\Delta(x)$, used by the author in [**8**], is to apply the classical complex integration technique. We shall briefly present this approach now. On using Perron's inversion formula (see e.g., the Appendix of [**3**]), the residue theorem and the convexity bound $Z(s) \ll_\varepsilon |t|^{2-2\sigma+\varepsilon}$ $(0\le\sigma\le1,\ |t|\ge1)$, it follows that

$$\Delta(x) = \frac{1}{2\pi i}\int_{\frac12-iT}^{\frac12+iT}\frac{Z(s)}{s}x^s\,\mathrm{d}s + O_\varepsilon\left(x^\varepsilon\left(x^{1/2}+\frac{x}{T}\right)\right) \quad (1\ll T\ll x). \tag{3.1}$$

If we suppose that

$$\int_X^{2X}\left|B\big(\tfrac12+it\big)\right|^2\mathrm{d}t \ll_\varepsilon X^{\theta+\varepsilon} \qquad (\theta\ge1), \tag{3.2}$$

and use the elementary fact $\big($see [**3**] for the results on the moments of $\left|\zeta\big(\tfrac12+it\big)\right|\big)$ that

$$\int_X^{2X}\left|\zeta\big(\tfrac12+it\big)\right|^2\mathrm{d}t \ll X\log X, \tag{3.3}$$

then from (2.3), (3.2), (3.3) and the Cauchy-Schwarz inequality for integrals we obtain

$$\int_X^{2X}\left|Z\big(\tfrac12+it\big)\right|\mathrm{d}t \ll_\varepsilon X^{(1+\theta)/2+\varepsilon}.$$

Therefore (3.1) gives

$$\Delta(x) \ll_\varepsilon x^\varepsilon(x^{1/2}T^{\theta/2-1/2}+xT^{-1}) \ll_\varepsilon x^{\frac{\theta}{\theta+1}+\varepsilon} \tag{3.4}$$

with $T = x^{1/(\theta+1)}$. This was formulated in [**8**] as

Theorem A. *If* θ *is given by* (3.2), *then*

$$\Delta(x) \ll_\varepsilon x^{\frac{\theta}{\theta+1}+\varepsilon}. \tag{3.5}$$

To obtain a value for θ, note that $B(s)$ belongs to the SELBERG class of degree three, hence $B\left(\frac{1}{2}+it\right)$ in (3.2) can be written as a sum of two DIRICHLET polynomials (e.g., by the reflection principle discussed in [**3**, Chapter 4]), each of length $\ll X^{3/2}$. Thus by the mean value theorem for DIRICHLET polynomials (op. cit.) we have $\theta \le 3/2$ in (3.2). Hence (3.5) gives (with unimportant ε) the RANKIN-SELBERG bound $\Delta(x) \ll_\varepsilon x^{3/5+\varepsilon}$. Clearly improvement will come from better values of θ. Note that the best possible value of θ in (3.2) is $\theta = 1$, which follows from general results on DIRICHLET series (see e.g., [**3**, Chapter 9]). It gives $1/2+\varepsilon$ as the exponent in the RANKIN-SELBERG problem, which is the limit of the method (the conjectural exponent $3/8+\varepsilon$, which is best possible, is out of reach; see the author's work [**4**]). To attain this improvement one faces ssentially the same problem as in proving the sixth moment for $\left|\zeta\left(\frac{1}{2}+it\right)\right|$, namely

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^6 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon},$$

only this problem is even more difficult, because the arithmetic properties of the coefficients b_n are even less known than the properties of the divisor coefficients

$$d_3(n) = \sum_{abc=n;a,b,c\in\mathbb{N}} 1,$$

generated by $\zeta^3(s)$. If we knew the analogue of the strongest sixth moment bound

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^6 \mathrm{d}t \ll T^{5/4}\log^C T \qquad (C>0),$$

namely the bound (3.2) with $\theta = 5/4$, then (3.1) would yield $\Delta(x) \ll_\varepsilon x^{5/9+\varepsilon}$, improving substantially (1.2).

The essential difficulty in this problem may be seen indirectly by comparing it with the estimation of $\Delta_4(x)$, the error term in the asymptotic formula for the summatory function of $d_4(n) = \sum_{abcd=n;a,b,c,d\in\mathbb{N}} 1$. The generating function in this case is $\zeta^4(s)$. The problem analogous to the estimation of $\Delta(x)$ is to estimate $\Delta_4(x)$, given the product representation

$$\sum_{n=1}^{\infty} d_4(n)n^{-s} = \zeta(s)G(s) = \zeta(s)\sum_{n=1}^{\infty} g(n)n^{-s} \quad (\sigma>1) \tag{3.6}$$

with $g(n) \ll_\varepsilon n^\varepsilon$ and $G(s)$ of degree three in the SELBERG class (with a pole of order three at $s=1$). By the complex integration method one gets $\Delta_4(x) \ll_\varepsilon x^{1/2+\varepsilon}$ (here 'ε' may be replaced by a log-factor) using the classical elementary bound $\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t \ll T\log^4 T$. Curiously, this bound for $\Delta_4(x)$ has never been improved; exponential sum techniques seem to give a poor result here. However, if one knows only (3.6), then the situation is quite analogous to the RANKIN–SELBERG problem, and nothing better than the exponent 3/5 seems obtainable. The bound $\Delta(x) \ll_\varepsilon x^{1/2+\varepsilon}$ follows also directly from (3.1) if the LINDELÖF hypothesis for $Z(s)$ (that $Z\left(\frac{1}{2}+it\right) \ll_\varepsilon |t|^\varepsilon$) is assumed.

4. MEAN SQUARE OF THE RANKIN–SELBERG ZETA–FUNCTION

Let, for a given $\sigma \in \mathbb{R}$,

$$\mu(\sigma) = \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log t} \tag{4.1}$$

denote the LINDELÖF function (the famous, hitherto unproved, LINDELÖF conjecture for $\zeta(s)$ is that $\mu(\sigma) = 0$ for $\sigma \ge \frac{1}{2}$, or equivalently that $\zeta\big(\frac{1}{2}+it\big) \ll_\varepsilon |t|^\varepsilon$). In [**8**] the author proved the following

Theorem B. *If $\beta = 2/\big(5-\mu\big(\frac{1}{2}\big)\big)$, then for fixed σ satisfying $\frac{1}{2} < \sigma \le 1$ we have*

$$\int_1^T |Z(\sigma+it)|^2\,\mathrm{d}t = T\sum_{n=1}^\infty c_n^2 n^{-2\sigma} + O_\varepsilon(T^{(2-2\sigma)/(1-\beta)+\varepsilon}). \tag{4.2}$$

This result is the sharpest one yet when σ is close to 1. For σ close to $\frac{1}{2}$ one cannot obtain an asymptotic formula, but only the upper bound (this is [**7**, eq. (9.27)])

$$\int_T^{2T} |Z(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon\ T^{2\mu(1/2)(1-\sigma)+\varepsilon}(T+T^{3(1-\sigma)}) \quad \big(\tfrac{1}{2} \le \sigma \le 1\big). \tag{4.3}$$

The upper bound in (4.3) follows easily from (2.3) and the fact that, as already mentioned, $B(s) \in \mathcal{S}$ with degree three, so that $B\big(\frac{1}{2}+it\big)$ can be approximated by DIRICHLET polynomials of length $\ll t^{3/2}$, and the mean value theorem for DIRICHLET polynomials yields

$$\int_T^{2T} |B(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon\ T^\varepsilon(T+T^{3(1-\sigma)}) \qquad \big(\tfrac{1}{2} \le \sigma \le 1\big).$$

Note that with the sharpest known result (see M. N. HUXLEY [**2**]) $\mu(1/2) \le 32/205$ we obtain $\beta = 410/961 = 0.426638917\ldots$. The limit is the value $\beta = 2/5$ if the LINDELÖF hypothesis $\big($that $\mu\big(\frac{1}{2}\big) = 0\big)$ is true. Thus (4.2) provides a true asymptotic formula for

$$\sigma\ >\ \frac{1+\beta}{2}\ =\ \frac{1371}{1922} = 0.7133194\ldots.$$

The proof of (4.2), given in [**8**], is based on the general method of the author's paper [**6**], which contains a historic discussion on the formulas for the left-hand side of (4.2) (see also K. MATSUMOTO [**12**]).

We are able to improve (4.2) in the case when $\sigma = 1$. The result is contained in

Theorem 1. *We have*

$$\int_1^T |Z(1+it)|^2\,\mathrm{d}t = T\sum_{n=1}^\infty c_n^2 n^{-2} + O_\varepsilon((\log T)^{2+\varepsilon}). \tag{4.4}$$

Proof. For $\sigma = \Re\mathrm{e}\, s > 1$ and $X \geq 2$ we have

$$Z(s) = \sum_{n\le X} c_n n^{-s} + \int_X^\infty x^{-s}\,\mathrm{d}\Big(\sum_{n\le x} c_n\Big) \tag{4.5}$$
$$= \sum_{n\le X} c_n n^{-s} + \frac{CX^{1-s}}{s-1} - \Delta(x)X^{-s} - s\int_X^\infty \Delta(x)x^{-s-1}\,\mathrm{d}x.$$

By using (1.2) it is seen that the last integral converges absolutely for $\sigma = \Re\mathrm{e}\, s > 3/5$, so that (4.5) provides the analytic continuation of $Z(s)$ to this region. Taking $s = 1+it, 1 \le t \le T, X = T^{10}$, it follows that

$$\int_1^T |Z(1+it)|^2\,\mathrm{d}t = \int_1^T \Big\{\Big|\sum_{n\le X} c_n n^{-1-it}\Big|^2 - 2C\Im\mathrm{m}\Big(\sum_{n\le X}\frac{c_n}{nt}\Big(\frac{X}{n}\Big)^{it}\Big)\Big\}\,\mathrm{d}t + O(1). \tag{4.6}$$

By the mean value theorem for Dirichlet polynomials we have

$$\int_1^T \Big|\sum_{n\le X} c_n n^{-1-it}\Big|^2\,\mathrm{d}t = T\sum_{n\le X} c_n^2 + O\Big(\sum_{n\le X} c_n^2 n^{-1}\Big) = T\sum_{n=1}^\infty c_n^2 + O_\varepsilon\Big((\log T)^{2+\varepsilon}\Big),$$

where we used the bound (see K. Matsumoto [**12**])

$$\sum_{n\le x} c_n^2 \ll_\varepsilon x(\log x)^{1+\varepsilon} \tag{4.7}$$

and partial summation. Finally we have

$$\sum_{n\le X}\frac{c_n}{n}\int_1^T \frac{1}{t}\left(\frac{X}{n}\right)^{it}\mathrm{d}t \ll \log\log T. \tag{4.8}$$

To see that (4.8) holds, note first that for $X - X/\log T \le n \le X$ the integral over t is trivially estimated as $\ll \log T$, and the total contribution of such n is

$$\ll \log T \sum_{X-X/\log T\le n\le X}\frac{c_n}{n}\,\mathrm{d}x \ll 1$$

on using (1.1)–(1.2). For the remaining n we note that the integral over t equals

$$\frac{\left(\frac{X}{n}\right)^{it}}{it\log(X/n)}\Bigg|_1^T + \frac{1}{i\log(X/n)}\int_1^T\left(\frac{X}{n}\right)^{it}\frac{\mathrm{d}t}{t^2}.$$

The contribution of those n is, using (1.1)–(1.2) again and making the change of variable $X/u = v$,

$$\begin{aligned}
&\ll \sum_{1\le n\le X-X/\log T}\frac{c_n}{n\log(X/n)} = \int_{1-0}^{X-X/\log T}\frac{1}{u\log(X/u)}\,\mathrm{d}(Cu+\Delta(u))\\
&= \int_1^{X-X/\log T}\frac{1}{u\log(X/u)}\left(C+\frac{\Delta(u)}{u}+\frac{\Delta(u)}{u\log(X/u)}\right)\mathrm{d}u + O(1)\\
&\ll \int_1^{X-X/\log T}\frac{\mathrm{d}u}{u\log(X/u)} + 1 = \int_{(1-1/\log T)^{-1}}^{X}\frac{\mathrm{d}v}{v\log v}+1\\
&= \log\log X - \log\log(1-1/\log T)^{-1} + 1 \ll \log\log T,
\end{aligned}$$

and (4.8) follows.

One can improve the error term in (4.4) to $O(\log^2 T)$, which is the limit of the method. I am very grateful to Prof. Alberto Perelli, who has kindly indicated this to me. The argument is very briefly as follows. Note that the coefficients c_n^2 are essentially the tensor product of the c_n's, and the c_n are essentially the tensor product of the $a(n)$'s; "essentially" means in this case that the corresponding L-functions differ at most by a "fudge factor", i.e., a Dirichlet series converging absolutely for $\sigma > 1/2$ and non-vanishing at $s = 1$. In terms of L-functions, the tensor product of the $a(n)$ (the coefficients of the tensor square L-function) corresponds to the product of $\zeta(s)$ and the L-function of Sym^2 (Shimura's lift). Moreover, Gelbart–Jacquet [**1**] have shown that Sym^2 is a cuspidal automorphic representation, so one can apply to the above product the general Rankin-Selberg theory to obtain "good properties" of the corresponding L-function. Since Sym^2 is irreducible, the L-function corresponding to c_n^2 has a double pole at $s = 1$ and a functional equation of Riemann type. It follows that the sum in (4.7) is asymptotic to $Dx\log x$ for some $D > 0$, and the assertion follows by following the preceding argument.

In concluding this section, let it be mentioned that, using (4.5), it easily follows that $Z(1+it) \ll \log|t|$ $(t \ge 2)$.

5. MEAN SQUARE OF $\Delta(x;\xi)$

In this section we shall consider mean square estimates for $\Delta(x;\xi)$, defined by (1.3). Although we could consider the range $\xi > 1$ as well, for technical reasons we shall restrict ourselves to the range $0 \le \xi \le 1$, which is the condition that will be assumed henceforth to hold. Let

$$\beta_\xi \;:=\; \inf\left\{\, \beta \ge 0 \;:\; \int_1^X \Delta^2(x;\xi)\,\mathrm{d}x \ll X^{1+2\beta} \,\right\}. \tag{5.1}$$

The definition of β_ξ is the natural analogue of the classical constants in mean square estimates for the generalized Dirichlet divisor problem (see [**3**, Chapter 13]). Our first result in this direction is

Theorem 2. *We have*

$$\frac{3-2\xi}{8} \;\le\; \beta_\xi \;\le\; \max\left(\frac{1-\xi}{2}, \frac{3-2\xi}{8}\right) \qquad (0\le\xi\le 1). \tag{5.2}$$

Proof. First of all, note that (5.2) implies that $\beta_\xi = (3-2\xi)/8$ for $\frac12 \le \xi \le 1$, so that in this interval the precise value of β_ξ is determined. The main tool in our investigations is the explicit Voronoï type formula for $\Delta(x;\xi)$. This is

$$\Delta(x;\xi) \;=\; V_\xi(x,N) + R_\xi(x,N), \tag{5.3}$$

where, for $N \gg 1$,
(5.4)

$$V_\xi(x,N) = (2\pi)^{-1-\xi}x^{(3-2\xi)/8}\sum_{n\le N} c_n n^{-(5+2\xi)/8}\cos\Bigl(8\pi(xn)^{1/4} + \tfrac12\bigl(\tfrac12-\xi\bigr)\pi\Bigr),$$
$$R_\xi(x,N) \ll_\varepsilon (xN)^\varepsilon\left(1 + x^{(3-\xi)/4}N^{-(1+\xi)/4} + (xN)^{(1-\xi)/4} + x^{(1-2\xi)/8}\right).$$

This follows from the work of U. Vorhauer [**20**] (for $\xi = 0$ this is also proved in [**9**]), specialized to the case when

$$A = \frac{1}{(2\pi)^2}, B = (2\pi)^4, M = L = 2, b_1 = b_2 = d_1 = d_2 = 1, \beta_1 = \kappa - \tfrac12, b_2 = \tfrac12,$$
$$\delta_1 = \kappa - \tfrac32, \delta_2 = -\tfrac12, \gamma = 1, p = B, q = 4, \lambda = 2, \Lambda = -1, C = (2\pi)^{-5/2}.$$

In (5.3)–(5.4) we take $N = x$, so that $R_\xi(x,N) \ll_\varepsilon x^{(1-\xi)/2+\varepsilon}$. Since $\frac{1-\xi}{2} \le \frac{3-2\xi}{8}$ for $\xi \ge \frac12$, the lower bound in (5.2) follows by the method of [4]. For the upper bound we use $c_n \ll_\varepsilon n^\varepsilon$ and note that ($\mathrm{e}(z) = \exp(2\pi i z)$)

$$\begin{aligned}
&\int_X^{2X}\Bigl|\sum_{K<k\le 2K} c_k k^{-(5+2\xi)/8}\mathrm{e}(4(xk)^{1/4})\Bigr|^2\,\mathrm{d}x\\
&\ll X + \sum_{k_1\ne k_2} c_{k_1}c_{k_2}(k_1k_2)^{-(5+2\xi)/8}\int_X^{2X}\mathrm{e}(4x^{1/4}(k_1^{1/4}-k_2^{1/4}))\,\mathrm{d}x\\
&\ll_\varepsilon X + X^{3/4+\varepsilon}K^{-(5+2\xi)/4}\sum_{k_1\ne k_2}\Bigl|k_1^{1/4}-k_2^{1/4}\Bigr|^{-1}\\
&\ll_\varepsilon X + X^{3/4+\varepsilon}K^{(1-\xi)/2},
\end{aligned}$$

where we used the first derivative test (cf. [**3**, Lemma 2.1]). Since $K \ll X$ and

$$\int_X^{2X}\Delta^2(x;\xi)\,\mathrm{d}x \ll \int_X^{2X}|V_\xi(x,N)|^2\,\mathrm{d}x + \int_X^{2X}|R(x,N)|^2\,\mathrm{d}x,$$

it follows that

$$\int_X^{2X}\Delta^2(x;\xi)\,\mathrm{d}x \ll_\varepsilon X^{(7-2\xi)/4+\varepsilon} + X^{2-\xi+\varepsilon},$$

which clearly proves the assertion.

Our last result is a bound for β_ξ, which improves on (5.2) when ξ is small. This is

Theorem 3. *We have*

$$\beta_\xi \le \frac{2-2\xi}{5-2\mu(\frac12)} \qquad \Big(0\le\xi\le\tfrac16\big(1+2\mu(\tfrac12)\big)\Big). \tag{5.5}$$

Proof. We start from

$$\Delta(x;\xi) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} Z(s)\frac{x^s}{s^{\xi+1}}\,\mathrm{d}s, \tag{5.6}$$

where $0 < c = c(\xi) < 1$ is a suitable constant (see K. Matsumoto [**13**] for a detailed derivation of formulas analogous to (5.6)). By the Mellin inversion formula we have (see e.g., the Appendix of [**3**])

$$Z(s)s^{-\xi-1} = \int_0^\infty \Delta(1/x;\xi)x^{s-1}\,\mathrm{d}x \qquad (\Re\mathrm{e}\, s = c).$$

Hence by Parseval's formula for Mellin tranforms (op. cit.) we obtain, for $\beta_\xi < \sigma < 1$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|Z(\sigma+it)|^2}{|\sigma+it|^{2\xi+2}}\,\mathrm{d}t &= \int_0^\infty \Delta^2(1/x;\xi)x^{2\sigma-1}\,\mathrm{d}x \\ &= \int_0^\infty \Delta^2(x;\xi)x^{-2\sigma-1}\,\mathrm{d}x \gg X^{-2\sigma-1}\int_X^{2X}\Delta^2(x;\xi)\,\mathrm{d}x.\end{aligned} \tag{5.7}$$

Therefore if the first integral converges for $\sigma = \sigma_0 + \varepsilon$, then (5.7) gives

$$\int_X^{2X} \Delta^2(x;\xi)\ \mathrm{d}x \ \ll X^{2\sigma+1},$$

namely $\beta_\xi \le \sigma_0$. The functional equation (2.2) and Stirling's formula in the form

$$|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-1/2}\mathrm{e}^{-\pi|t|/2}\big(1+O(|t|^{-1})\big) \qquad (|t|\ge t_0>0)$$

imply that

$$Z(s) \ = \ \mathcal{X}(s)Z(1-s), \quad \mathcal{X}(\sigma+it) \asymp |t|^{2-4\sigma} \quad (s=\sigma+it,\, 0\le\sigma\le1, |t|\ge2). \tag{5.8}$$

Thus it follows on using (4.3) that

$$\begin{aligned}\int_T^{2T}|Z(\sigma+it)|^2\,\mathrm{d}t &\ll T^{4-8\sigma}\int_T^{2T}|Z(1-\sigma+it)|^2\,\mathrm{d}t \\ &\ll_\varepsilon T^{4-8\sigma+2\mu(\frac12)\sigma+\max(1,3\sigma)+\varepsilon}.\end{aligned}$$

But we have $4 - 8\sigma + 2\mu(\frac12)\sigma + \max(1, 3\sigma) = 4 - 5\sigma + 2\mu\big(\frac12\big)\sigma < 2\xi + 2$ for

$$\sigma > \sigma_0 = \frac{2-2\xi}{5 - 2\mu\big(\frac12\big)}, \tag{5.9}$$

provided that $\sigma_0 \geq 1/3$, which occurs if $0 \leq \xi \leq \frac16\big(1 + 2\mu\big(\frac12\big)\big)$. Thus the first integral in (5.7) converges if (5.9) holds, and Theorem 3 is proved. Note that this result is a generalization of Theorem 7 in [**8**], which says that $\beta_0 \leq (2-2\xi)/\big(5 - 2\mu\big(\frac12\big)\big)$.

In the case when $\beta_\xi = (3-2\xi)/8$ we could actually derive an asymptotic formula for the integral of the mean square of $\Delta(x;\xi)$, much in the same way that this was done in [**9**] for the square of $\Delta_1(x) := \int_0^x \Delta(u)\,\mathrm{d}u$, where it was shown that

$$\int_1^X \Delta_1^2(x)\,\mathrm{d}x = DX^{13/4} + O_\varepsilon(X^{3+\varepsilon}) \tag{5.10}$$

with explicit $D > 0$ (in [**12**] the error term was improved to $O_\varepsilon(X^3(\log X)^{3+\varepsilon})$). In the case of $\Delta(x;1)$ the formula (5.10) may be used directly, since

$$\frac1x\Delta_1(x) = \frac1x\int_0^x \Delta(u)\,\mathrm{d}u = \Delta(x;1) + O_\varepsilon(x^\varepsilon). \tag{5.11}$$

To see that (5.11) holds, note that with $c = 1-\varepsilon$ we have

$$\begin{aligned}
\Delta(x;1) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z(s)\,\frac{x^s}{s^2}\,\mathrm{d}s\\
&= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z(s)\,\frac{x^s}{s(s+1)}\,\mathrm{d}s + \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} Z(s)\,\frac{x^s}{s^2(s+1)}\,\mathrm{d}s\\
&= \frac1x\int_0^x \Delta(u)\,\mathrm{d}u + \frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty} Z(s)\,\frac{x^s}{s^2(s+1)}\,\mathrm{d}s\\
&= \frac1x\Delta_1(x) + O_\varepsilon(x^\varepsilon),
\end{aligned}$$

on applying (5.8) to the last integral above.

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