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METHOD OF FACTORIZATION OF ORDINARY DIFFERENTIAL OPERATORS AND SOME OF ITS APPLICATIONS

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The paper is dedicated to analytical and algebraic approaches to the problem of the integration of ordinary differential equations. The first part is devoted to linear ordinary differential equations of the second and nth orders, while the second deals with nonlinear ordinary differential equations. Factorization of nonlinear equations of the second and the third orders both through commutative and noncommutative nonlinear differential operators are considered. The method of the exact linearization for nonlinear equations is explained. Some applications are also considered.

1. INTRODUCTION

The contents of this paper are closely connected to the problem of the integration of ordinary differential equations. Factorization of differential operators is a very effective method for analyzing both linear and nonlinear ordinary differential equations. It uses analogies between differential operators and algebraic polynomials.

The prehistory of this method goes back to investigations of G. Frobenius [**29**], E. Landau, [**43**] and G. Mammana [**47**].

The most efficacious is simultaneously using factorization method and variables transformation.

A great contribution to the problem of integrating ordinary differential equations was made by mathematicians of Serbia and the former Yugoslavia: M. Petrović, T. Pejović, D. S. Mitrinović, B. Popov, I. Šapkarev, I. Bandić, P.

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Vasić, J. Kečkić, V. Kocić and others. The journal "Publications of the Faculty of Electrical Engineering - Series Mathematics" (1956–2006), which was founded by Professor D. S. Mitrinović, have played significant role in the regeneration of interest in the problem of the solution of ordinary differential equations iin closed form.

At the present time the importance of this problem has increased considerably. Closed-form solutions are necessary both for new mathematical models in the natural sciences and for the testiing of numerical and analytic algorithms.

In Section 2 we consider differential algebras of differential operators. We place the main emphasis on their factorization.

In Section 3 it is shown how to use the method of LODE-2 and LODE-n transformation. The Kummer-Liouville transformation, that is applied in this work, is the most general transformation of variables that preserves the order and the linearity of the given equation.

The solutions of the classical Kummer's and Halphen's problems of LODE-2 and LODE-n equivalence are given.

The criteria of LODE-n reducibility to equations with constant coefficients are pointed out.

In Section 4 we consider the method of autonomization for nonlinear differential equations. It is applicable for equations that can be representad as a sum of linear and nonlinear parts. The test for autonomization is also adduced. The generalized Emden-Fowler's equation and generalized Ermakov's equation, which frequently appear in different applications, are considered. The very important idea of a nonlinear superposition principle for nonlinear differential equations is given.

In Section 5 the method of linearization of nonlinear differential equations (see Berkovich [**18, 22**]) is applied to the equations of the second and third orders. A nonlinear oscillator and the Euler-Poinsot case in the problem of the gyroscope are good examples of the effectiveness of this method.

In Section 6 we simultaneously apply the method of transformation of variables and factorization of nonlinear differential operators to the generalized Emden-Fowler's equation of the third order, to Lienard's equation and to the equation of the anharmonic oscillator.

2. DIFFERENTIAL ALGEBRA OF DIFFERENTIAL OPERATORS

Definitions of the main concepts can be found in the following books: Kaplansky [**32**], Magid [**46**], Singer [**56**] and Berkovich [**13, 22**].

2.1. Differential field

Definition 1. *A differential field is a pair* (F, δ), *where* F *is a functional field and* δ *is a derivation. Let* K *be a number field of characteristic 0* (*i.e. constant field* F). *It may be algebraically closed, or it may be not.*

$$a' := \delta(a), \quad a \in F,$$

$$a \in F_0 \Leftrightarrow a' \in F_0, \quad c \in K \Leftrightarrow c' = 0.$$

EXAMPLE 1. Field (F, δ), where $\delta = \dfrac{\mathrm{d}}{\mathrm{d}x} = D$, $\delta = x\,\dfrac{\mathrm{d}}{\mathrm{d}x}$. Further let δ be D.

EXAMPLE 2. Field $(\mathbb{C}(x), D)$, where $\mathbb{C}(x)$ is the field of rational functions over the field of complex numbers $\mathbb{C}$.

2.2. Ring of differential operators

Consider the set of differential operators of arbitrary order

$$L = a_n D^n + \cdots + a_1 D + a_0,$$

where $n \in \mathbb{N}$, $a_i \in F_0$, $\forall i$. Multiplication in F_0 is determined by the rule:

$$Da = aD + D(a) = aD + a'. \tag{2.1}$$

From (2.1) LEIBNITZ' formula follows:

$$D^i b = \sum_{k=0}^{i} \binom{i}{k} b^{(i-k)} D^k.$$

$F_0[D]$ is an associative but not a commutative ring.

2.3. Factorization of differential operators

Definition 2. *An operator, L, is factorizable in F_0 if it can be represented as the product of differential operators of lower order. The latter operators have coefficients in F_0. Under factorization the source number field may be extended to the algebraically closed field $\overline{K}$.*

Equivalent definition:

Definition 3. *The equation, $Ly = 0$, of order n is factorizable in F_0 if both this equation and the equation, $My = 0$, of order less than n have a common nontrivial integral.*

Otherwise L is said to be not factorizable in F_0.

2.4. Right differential analogue of Bezout's theorem

Theorem 1. *Dividing L by $D - \alpha$ from the right we get*

$$f(x) = \exp\Big(-\int \alpha\,\mathrm{d}x\Big) L \exp\Big(\int \alpha\,\mathrm{d}x\Big).$$

In the ring $F_0[D]$ HORNER-type schemes take place by analogy with algebraic polynomials.

Using the right differential analogue of HORNER's scheme one can make an expansion

$$L = \sum_{s=0}^{n-1} \beta_s D^s (D - \alpha), \quad \beta_{n-1} = 1.$$

Using the left differential analogue of HORNER's scheme one can make an expansion:

$$L = (D-\alpha)\sum_{s=0}^{n-1}\beta_s D^s, \quad \beta_{n-1} = 1.$$

2.5. Conjugation operator and its properties

Definition 4. *Transformation of conjugation, τ, is linear operator that acts on the Linear Ordinary Differential Operator (LODO) as it pointed out below:*

$$\tau\big(p(x)D^n\big) = (-1)^n D^n p(x) = (-1)^n \sum_{k=0}^{n}\binom{n}{k}p^{(k)}D^{n-k},$$
$$\tau\Big(\sum_{s=0}^{n} C_s p_s(x) D^s\Big) = \sum_{s=0}^{r} C_s \tau\big(p_s D^s\big), \quad C_s = \text{const}.$$

Let L^* be the operator, τL, that is formally conjugated to L

$$L^* \equiv \tau\Big(\sum_{k=0}^{n} a_k D^k\Big) = \sum_{k=0}^{n}(-1)^k D^k a_k = \sum_{k=0}^{n}\sum_{s=0}^{k}(-1)^k\binom{k}{s}a_k^{(s)}D^{k-s}.$$

Let L and M be LODOes. Then

$$\tau(LM) = \tau(M)\tau(L) = M^* L^*.$$

2.6. Left differential analogue of Bezout's theorem

Theorem 2. *Dividing L by $D-\alpha$ from the left we get*

$$g(x) = \exp\Big(\int \alpha\, \mathrm{d}x\Big) L^* \exp\Big(-\int \alpha\, \mathrm{d}x\Big).$$

2.7. Selfconjugated and antiselfconjugated operators

Theorem 3. *A selfconjugated operator, L_{2n}, can be represented as*

$$L_{2n} \equiv \prod_{k=2n}^{1}(\beta_k D - \alpha_k) = \prod_{k=1}^{n}(\beta_k D + \beta_k' + \alpha_k)\prod_{k=n}^{1}(\beta_k D - \alpha_k).$$

Theorem 4. *An antiselfconjugated operator, L_{2n+1}, can be represented as*

$$\begin{aligned} L_{2n+1} &\equiv \prod_{s=2n+1}^{1}(\beta_s D - \alpha_s) \\ &= \prod_{k=1}^{n}(\beta_k D + \beta_k' + \alpha_k)\Big(-2\int \alpha_{n+1}\,\mathrm{d}x D - \alpha_{n+1}\Big)\prod_{k=n}^{1}(\beta_k D - \alpha_k). \end{aligned}$$

2.8. Reducible selfconjugated and antiselfconjugated operators

Theorem 5 (see Berkovich, Rozov and Eishinsky [**4**]). *A selfconjugated operator that admits the factorization,*

$$L_{2n} = \prod_{k=1}^{n} \Big(D + \frac{2n+1-2k}{2n-1}\,\alpha\Big) \prod_{k=n}^{1} \Big(D - \frac{2n+1-2k}{2n-1}\,\alpha\Big), \tag{2.2}$$

can be represented as

$$\exp\Big(\frac{4n}{2n-1}\int \alpha \,\mathrm{d}x\Big) L_{2n} = \Big(\exp\Big(\frac{2}{2n-1}\int \alpha\,\mathrm{d}x\Big)(D-\alpha)\Big)^{2n}.$$

Theorem 6 [**4**]. *An antiselfconjugated operator that admits the factorization,*

$$L_{2n+1} = \prod_{k=1}^{n} \Big(D + \frac{n+1-k}{n}\,\alpha\Big) D \prod_{k=n}^{1} \Big(D - \frac{n+1-k}{n}\alpha\Big), \tag{2.3}$$

can be represented as

$$\exp\Big(\frac{2n+1}{n}\int \alpha \,\mathrm{d}x\Big) L_{2n+1} = \Big(\exp\Big(\frac{1}{n}\int \alpha\,\mathrm{d}x\Big)(D-\alpha)\Big)^{2n+1}.$$

The operator, (2.2), is called a *reducible selfconjugated operator.*

The operator, (2.3), is called a *reducible antiselfconjugated operator.*

2.9. Liouvillian and Euler expansions

A set Λ is a generalized Liouvillian (Euler) expansion of the field F_0 if there is a tower of fields,

$$F_0 \subset F_1 \subset \ldots \subset F_n = \Lambda,$$

such that one of the following conditions is fulfilled

- a. $F_i = F_{i-1}(\alpha)$, where $F_{i-1}(\alpha)$ is the field of rational functions of α with coefficients from F_{i-1} and $\alpha' \in F_{i-1}$.
- b. $F_i = F_{i-1}(\alpha)$, $\alpha \neq 0$, $\alpha'/\alpha \in F_{i-1}$.
- c. $F_i = F_{i-1}(\alpha)$, where α satisfies an algebraic equation of order $n \geq 2$.
- d. $F_i = F_{i-1}(y_1, y_2)$, where y_1 and y_2 constitute a basis of the equation

$$y'' + a_1 y' + a_0 y = 0, \quad a_1, a_0 \in F_{i-1}. \tag{2.4}$$

If (a), (b) or (c) is satisfied, then we get a Liouvillian expansion Λ_0. If in addition condition (d) is satisfied, then we have a generalized Liouvillian (Euler) expansion Λ of the field F_0.

2.10. Picard-Vessiot expansion

Definition 5. *The Picard-Vessiot expansion for the equation*

$$Ly \equiv \sum_{s=0}^{n} a_s y^{(s)} = 0, \quad a_s \in F_0 \tag{2.5}$$

is the differential field $F_0(y_1, \dots, y_n)$, where $y_1, y_2, \dots, y_n$ is a basis of equation (2.5).

Definition 6. *Equation* (2.5) *can be integrated in quadratures if $PV \subset \Lambda_0$.*

Definition 7 *Equation* (2.5) *has an Euler solution if $PV \subset \Lambda$.*

2.11. Mammana's theorems

Theorem 7. *It is always possible to factorize the equation $Ly = 0$ by an infinite number of ways through operators of the first order*

$$Ly \equiv \prod_{k=n}^{1} (D - \alpha_k) y = 0, \tag{2.6}$$

where α_k are complex-valued functions of x.

Example 3.

$$D^2 + 1 \equiv \left(D + \frac{i(c_1 e^{ix} - c_2 e^{-ix})}{c_1 e^{ix} + c_2 e^{-ix}} \right) \left(D - \frac{i(c_1 e^{ix} - c_2 e^{-ix})}{c_1 e^{ix} + c_2 e^{-ix}} \right).$$

Theorem 8. *Suppose that we have an equation $Ly = 0$, $a_s \in \mathbb{C}^s(I)$, $I = \{x | a < x < b\}$. Let α_k be real-valued functions in I.*

Factorization of (2.6) *in I exists if and only if any solution $y(x)$ of the equation $Ly = 0$ is nonoscillating, i.e. it has no more than $n - 1$ zeroes* (*counted according to their multiplicity*) *in I.*

Example 4.

$$D^2 + 1 \equiv \left(D + \frac{-c_1 \sin x + c_2 \cos x}{c_1 \cos x + c_2 \sin x} \right) \left(D - \frac{-c_1 \sin x + c_2 \cos x}{c_1 \cos x + c_2 \sin x} \right).$$

2.12. Factorization in ground differential field

The equation

$$y'' + a_0 y = 0 \tag{2.7}$$

admits the factorization

$$(D + \alpha)(D - \alpha) y = 0, \tag{2.8}$$

where $\alpha(x)$ satisfies the Riccati equation

$$\alpha' + \alpha^2 + a_0 = 0, \quad a_0 \in \mathbb{C}(x).$$

They also have the form

$$\alpha = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{c_{ij}}{(x - r_i)^j} + p(x), \quad \alpha \in \mathbb{C}(x),$$

where $p(x)$ is a polynomial.

Example 5 (Kovacic [**41**]). Equation

$$y'' = \Big(x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}\Big) y$$

admits the factorization

$$\Big(D + \frac{1}{x+1} + \frac{1}{x-1} - \frac{3}{2x} + \frac{1}{x^2} + x - 1\Big)\Big(D - \frac{1}{x+1} - \frac{1}{x-1} + \frac{3}{2x} - \frac{1}{x^2} - x + 1\Big) y = 0$$

and has the particular solution

$$y = (x^2 - 1)\, x^{-3/2} \exp\Big(-\frac{1}{x} + \frac{1}{2}x^2 - x\Big).$$

The factorization of differential operators of order n

$$L = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0,$$

namely representation as

$$L = (\beta_n D - \alpha_n)(\beta_{n-1} D - \alpha_{n-1}) \cdots (\beta_2 D - \alpha_2)(\beta_1 D - \alpha_1),$$

was considered in works by (Mitrinović [**48**], Popov [**54**], Berkovich [**13, 21, 22, 25**] and others).

2.13. Factorization in the quadratic expansion of the field F_0

Lemma 1 (see, for e.g., Kaplansky [**32**]). *The factorization,* (2.8)*, takes place in the quadratic expansion of the field F_0, or in other words the condition*

$$\alpha^2 - p(x)\alpha + q(x) = 0, \quad p, q \in F_0, \quad p \neq 0$$

is fullfiled if and only if the following relations are satisfied:

$$p'' + 3pp' + p^3 + 2a' + 4ap = 0, \qquad 2q = p' + p^2 + 2a.$$

EXAMPLE 6. The equation (see [**22, 25**])

$$Ly \equiv y'' + \left(\frac{3}{16}x^{-2} - bx^{-1}\right) y = 0$$

admits the factorization

$$Ly \equiv \left(D + \frac{1}{4}\,x^{-1} \pm \sqrt{b}\,x^{-1/2}\right)\left(D - \frac{1}{4}\,x^{-1} \mp \sqrt{b}\,x^{-1/2}\right) y = 0,\ b > 0,$$

and has solutions

$$y = x^{1/4}\big(c_1 \exp(2\sqrt{bx}\,) + c_2 \exp(-2\sqrt{bx}\,)\big).$$

2.14. Analogues of Vieta's formulæ and LODE-2 solutions

Suppose we have linear ordinary differential equation of the second order (LODE-2)

$$Ly \equiv y'' + a_1 y' + a_0 y = 0, \tag{2.9}$$

where the operator L admits the factorization

$$L \equiv (D - \alpha_2)(D - \alpha_1). \tag{2.10}$$

From formulæ (2.9) and (2.10) the analogues of VIÈTE's formulæ follow

$$a_1 = -(\alpha_1 + \alpha_2), \qquad a_0 = \alpha_2\alpha_1 - \alpha_1',$$

where α_1 and α_2 satisfy the RICCATI equations [1]

$$\alpha_1' + {\alpha_1}^2 + a_1\alpha_1 + a_0 = 0, \quad \alpha_2' - {\alpha_2}^2 - a_1\alpha_2 - a_0 = 0.$$

Linearly independent solutions of equations (2.9) and (2.10) have the form

$$y_1 = e^{\int \alpha_1 \,\mathrm{d}x}, \quad y_2 = e^{\int \alpha_1 \,\mathrm{d}x} \!\int e^{\int (\alpha_2 - \alpha_1)\,\mathrm{d}x}\mathrm{d}x.$$

The linear nonhomogeneous equation, $Ly = f(x)$, where L admits the factorization (2.10), has the particular solution

$$\overline{y} = e^{\int \alpha_1 \,\mathrm{d}x}\int \left(e^{\int (\alpha_2 - \alpha_1)\,\mathrm{d}x}\!\int e^{-\int \alpha_2 \,\mathrm{d}x} f(x)\,\mathrm{d}x\right)\mathrm{d}x.$$

2.15. Factorization of Lamé's operator

Suppose we have LAMÉ's equation,

$$Ly \equiv y'' - (2\wp(x) + \lambda)y = 0, \quad \lambda = \wp(\varepsilon), \tag{2.11}$$

[1] We remark that criteria of an integrability of RICCATI's equation were considered, in particular, in the papers (MITRINOVIĆ and VASIĆ [**49, 50**]).

where $\wp(x)$ is the WEIERSTRASS elliptic function. LAMÉ's operator admits the factorization

$$L = \big(D + \zeta(x \pm \varepsilon) - \zeta(X) \mp \zeta(\varepsilon)\big)\big(D - \zeta(x \pm \varepsilon) + \zeta(x) \pm \zeta(\varepsilon)\big)$$

and equation (2.11) the has general solution

$$y(x) = c_1 \frac{\sigma(x+\varepsilon)}{\sigma(x)}\, e^{-\zeta(\varepsilon)x} + c_2 \frac{\sigma(x-\varepsilon)}{\sigma(x)}\, e^{\zeta(\varepsilon)x},$$

where the WEIERSTRASS functions $\wp(x)$, $\sigma(x)$ and $\zeta(x)$ are connected by the relations

$$\wp(x) = -\zeta'(x),\ \zeta(x) = \frac{\sigma'(x)}{\sigma(x)}\,, \quad \zeta(x+\varepsilon) - \zeta(x) - \zeta(\varepsilon) = \frac{\wp'(x) - \wp'(\varepsilon)}{\wp(x) - \wp(\varepsilon)}\,.$$

EXAMPLE 7. The degenerate case: $\wp(x) = \dfrac{1}{x^2}\,.$

Equation (2.11) takes the form

$$Ly \equiv y'' - \left(\frac{2}{x^2} + \frac{1}{\alpha^2}\right) y = 0,$$

where L admits the factorization

$$L = \left(D + \frac{1}{x+\alpha} - \frac{1}{x} \mp \frac{1}{\alpha}\right)\left(D - \frac{1}{x+\alpha} + \frac{1}{x} \pm \frac{1}{\alpha}\right),$$

has the general solution

$$y = c_1 \frac{x+\alpha}{x}\, e^{-x/\alpha} + c_2 \frac{x-\alpha}{x}\, e^{x/\alpha}.$$

EXAMPLE 8. Degenerate case. Suppose that

$$\wp(x) = \frac{1}{\sin^2 x} - \frac{1}{3}\,.$$

LAMÉ's equation has the form

$$Ly \equiv y'' - \left(\frac{2}{\sin^2 x} + \operatorname{ctg}^2 \varepsilon\right) y = 0, \tag{2.12}$$

where the operator L admits the factorization

$$L = \big(D + \operatorname{ctg}(x \pm \varepsilon) - \operatorname{ctg} x \pm \operatorname{ctg}\varepsilon\big)\big(D - \operatorname{ctg}(x \pm \varepsilon) + \operatorname{ctg} x \mp \operatorname{ctg}\varepsilon\big).$$

Equation (2.12) has general solution

$$y = c_1 \frac{\sin(x+\varepsilon)}{\sin x}\, e^{-x \operatorname{ctg}\varepsilon} + c_2 \frac{\sin(x-\varepsilon)}{\sin x}\, e^{x \operatorname{ctg}\varepsilon}.$$

2.16. Factorization of the third-order Halphen's 0perator

HALPHEN's equation of the third order is

$$Ly \equiv y''' - 3\wp(x)y' - \left(\frac{3}{2}\wp'(x) + \frac{1}{2}\wp'(\alpha)\right) y = 0, \tag{2.13}$$

where the operator L admits the factorization

$$\begin{aligned} L = &\big(D + \zeta(x+\alpha+\beta) - \zeta(x) - \zeta(\alpha) - \zeta(\beta)\big) \times \\ &\times\big(D - \zeta(x+\alpha+\beta) + \zeta(x+\alpha) + \zeta(\beta)\big)\big(D - \zeta(x+\alpha) + \zeta(x) + \zeta(\alpha)\big). \end{aligned}$$

The general solution of HALPHEN's equation, (2.13), has the form

$$y(x) = c_1 \frac{\sigma(x+\alpha)}{\sigma(x)}\, e^{-x\zeta(\alpha)} + c_2 \frac{\sigma(x+\beta)}{\sigma(x)}\, e^{-x\zeta(\beta)} + c_3 \frac{\sigma(x+\gamma)}{\sigma(x)}\, e^{-x\zeta(\gamma)},$$

where

$$\wp'^2(x) - \wp'^2(\alpha) = 0, \quad \wp(\alpha) + \wp(\beta) + \wp(\gamma) = 0.$$

EXAMPLE 9. The degenerate case: $\wp = \dfrac{1}{x^2}$.

The equation

$$Ly \equiv y''' - \frac{3}{x^2}y' + \left(\frac{3}{x^3} + \frac{1}{\alpha^3}\right) y = 0,$$

where the operator L admits the factorization

$$\begin{aligned} L = &\left(D + \frac{1}{x+\alpha+\beta} + \frac{1}{x} - \frac{1}{\alpha} - \frac{1}{\beta}\right) \times \\ &\times \left(D - \frac{1}{x+\alpha+\beta} + \frac{1}{x+\alpha} + \frac{1}{\beta}\right)\left(D - \frac{1}{x+\alpha} + \frac{1}{x} + \frac{1}{\alpha}\right), \end{aligned}$$

has the general solution

$$y = c_1 \frac{x+\alpha}{x}\, e^{-x/\alpha} + c_2 \frac{x+\beta}{x} e^{-x/\beta} + c_3 \frac{x+\gamma}{x}\, e^{-x/\gamma}.$$

Note. LAMÉ's operator and HALPHEN's operator are commutative. The KORTEWEG-DE VRIES's equation, well-known in the theory of solitons, $u_t = 6uu_x + u_{xxx}$ is generated by commutative condition of the corresponding pair of operators of the second and third orders.

2.17. Operational identities

Differential operators of higher orders may admits a factorization not only through operators of the first order but also operators of other orders. Operational identities in this case are useful. In the paper, (BERKOVICH, KVAL'WASSER [**3**]), such identities are constructed, for example

$$(xD^2 + aD)^m = \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(a+m)}{\Gamma(a+m-k)} x^{m-k} D^{2m-k}, \tag{2.14}$$

where $\Gamma(a+m) = (a+m-1)\cdots(a+1)a\Gamma(a)$;

$$\left(xD^2 + \left(m - \frac{2n+1}{2}\right)D\right)^{\frac{2n+1}{2}}$$
$$= \pm \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{\Gamma\left(\frac{2n+1}{2}+1\right)}{\Gamma\left(\frac{2n+1}{2}-k+1\right)} x^{\frac{2n+1}{2}-k} D^{2n+1-k}.$$

The identity (2.14) was generalized in the paper of (KLAMKIN and NEWMAN [**37**]).

3. TRANSFORMATION OF LODE

3.1. Statement of Kummer's problem

Suppose that we have the equations

$$y'' + a_1(x)y' + a_0(x)y = 0, \quad a_k \in \mathbb{C}^k(I), \quad I = \{x|a < x < b\}, \quad k = 0, 1, \tag{3.1}$$

$$\ddot{z} + b_1(t)\,\dot{z} + b_0\,(t)z = 0, \quad b_k \in \mathbb{C}^k(J), \quad J = \{t|\alpha < t < \beta\}, \tag{3.2}$$

and the KUMMER-LIOUVILLE transformation

$$y = v(x)z, \quad \mathrm{d}t = u(x)\mathrm{d}x, \quad v, u \in \mathbb{C}^2(I), uv \neq 0. \tag{3.3}$$

It is an invertible transformation, that is, the Jacobian

$$J = \left(\frac{\partial(y, \quad x)}{\partial(z, \quad t)}\right) = \begin{vmatrix} \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial z} \end{vmatrix} \neq 0.$$

Is it possible to transform (3.1) to (3.2) with the help of KL-transformation (3.3)?

3.2. Solution of Kummer's problem

Theorem 9 (see BERKOVICH [**10**], BERKOVICH and ROZOV [**15**]). *Equation* (3.1) *can be transformed to* (3.2) *with transformation* (3.3) *if and only if the following conditions for the KL-transformation are satisfied*:

$$v(x) = |u(x)|^{-1/2} \exp\left(-\frac{1}{2}\int a_1\,\mathrm{d}x + \frac{1}{2}\int b_1(t)\,\mathrm{d}t\right),$$

$$\frac{1}{2}\,\frac{t'''}{t'} - \frac{3}{4}\left(\frac{t''}{t'}\right)^2 + B_0(t)\,t'^2 = A_0(x), \tag{3.4}$$

where (3.4) *is the Kummer-Schwartz equation of the third order* (*KS*-3), *and*

$$A_0(x) = a_0 - \frac{1}{4}\,{a_1}^2 - \frac{1}{2}\,a_1', \qquad B_0(t) = b_0 - \frac{1}{4}\,{b_1}^2 - \frac{1}{2}\,\dot{b}_1$$

are semiinvariants of equations (3.1) *and* (3.2) *respectively* (see PEJOVIĆ [**51**]), *and* v *and* u *also satisfy the equation*

$$v'' + a_1 v' + a_0 v - b_0 u^2 v = 0. \tag{3.5}$$

EXAMPLE 10 (see ŠAPKAREV [**55**], VASIĆ [**57**]):

$$y'' + \left(\frac{f f'}{f^2 + b^2} - \frac{f''}{f'}\right) y' - \frac{a^2 f'^2}{f^2 + b^2}\, y = 0,\ f = f(x).$$

By the transformation

$$\mathrm{d}t = \frac{f'}{\sqrt{f^2 + b^2}}\,\mathrm{d}x$$

this equation is reduced to the equation $\ddot{y} - \dfrac{a^2}{b^2}\, y = 0$.

3.3. Kummer-Schwartz and Ermakov equations

Suppose $a_1 = b_1 = 0$. The equation (ERMAKOV [**26**], see BERKOVICH and ROZOV [**8**])

$$v'' + a_0 v - b_0 v^{-3} = 0 \tag{3.5$'$}$$

has the general solution (see also PINNEY [**53**])

$$v(x) = \sqrt{A{Y_2}^2 + BY_1Y_2 + C{Y_1}^2}, \quad B^2 - 4AC = -4b_0, \tag{3.6}$$

where $Y_1, Y_2 = Y_1 \int Y_1^{-2}\,\mathrm{d}x$ forms a basis of the second-order equation

$$Y'' + a_0 Y = 0. \tag{3.7}$$

The KUMMER-SCHWARZ equation of the second order (KS-2),

$$\frac{1}{2}\,\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + b_0 u^2 = a_0, \tag{3.8}$$

has general solution of the form

$$u(x) = (A{Y_2}^2 + BY_1Y_2 + C{Y_1}^2)^{-1}, \quad B^2 - 4AC = -4b_0. \tag{3.9}$$

3.4. LODE-2 Related by KL transformation

Equation (2.5) with a “carrier”, a_0, generates the next sequence of related equations [**12, 25**]

$$y_k'' + a_k y_k = 0,$$

where

$$a_k = a_0 - \sum_{s=1}^{k} b_{0s} u_s^2, \quad a_k = a_{k-1} - b_{0k} u_k^2,$$
$$\frac{1}{2}\frac{u_s''}{u_s} - \frac{3}{4}\left(\frac{u_s'}{u_s}\right)^2 - \frac{1}{4}\delta_s u_s^2 = a_{s-1},\ \delta_s = b_{1s}^2 - 4b_{0s},$$
$$y_k^{(1,2)} = |u_k|^{-1/2}\exp\left(\pm\frac{1}{2}\, b_{1k}\int u_k\,\mathrm{d}x\right),\ b_{1k} \neq 0.$$

3.5. Examples of related equations

The following equations are related to the equation $y'' = 0$ [**12, 25**].

Example 11. $y'' - \big(m(m+1)x^{-2} + T^4\big)y = 0, \quad T = \alpha x^{-m} + \beta x^{m+1},\ m \neq \dfrac{1}{2}$.

General solution: $y(x) = T\left(M\,\mathrm{ch}\left(\dfrac{x^{-m}}{\gamma T}\right) + N\,\mathrm{sh}\left(\dfrac{x^{-m}}{\gamma T}\right)\right), \quad \gamma = (2m+1)\beta.$

Example 12. $y'' + \left(\dfrac{1}{4x^2} + \dfrac{1}{x^2 S^4}\right)y = 0, \quad S = \alpha\log x + \beta.$

General solution: $y = \sqrt{x}\,S\Big(M\cos\dfrac{1}{\alpha S} + N\sin\dfrac{1}{\alpha S}\Big).$

3.6. Halphen’s problem for LODE-n

Suppose the equations (Halphen [**30**], Berkovich [**11, 22**])

$$L_n y \equiv y^{(n)} + \sum_{k=0}^{n}\binom{n}{k} a_k y^{(n-k)} = 0, \quad a_k \in \mathbb{C}^{n-k}(I), \tag{3.10}$$

$$M_n Z \equiv z^{(n)}(t) + \sum_{k=1}\binom{n}{k} b_k z^{(n-k)}(t) \quad b_k \in \mathbb{C}^{n-k}(J), \tag{3.11}$$

and the KL-transformation

$$y = v(x)z,\ \ \mathrm{d}t = u(x)\,\mathrm{d}x,\ \ vu \neq 0,\ v, u \in \mathbb{C}^n(I). \tag{3.12}$$

Problem 1: Find necessary and sufficient conditions of equivalence of (3.10) and (3.11) under the KL transformation (3.12).

Problem 2: Classify equations (3.10) with the help of canonical forms.

3.7. Lemmas of LODE-n equivalence

Lemma 2. *Equations* (3.10) *and* (3.11) *are equivalent if and only if the following system is compatible*

$$\{t,x\} + \frac{3}{n+1}\,B_2 t'^2 = \frac{3}{n+1}\,A_2(x),$$
$$\frac{t^{IV}}{t'} - 6\,\frac{t'''\,t''}{t'^2} + 6\left(\frac{t''}{t'}\right)^3 + \frac{12}{n+1}\,A_2\,\frac{t''}{t'} + \frac{4}{n+1}\,B_3 t'^3 = \frac{4}{n+1}A_3, \dots,$$
$$\left((t')^{\frac{1-n}{2}}\right)^{(n)} + \sum_{k=2}^{n}\binom{n}{k}A_k\left((t')^{\frac{1-n}{2}}\right)^{(n-k)} - B_n(t')^{\frac{n+1}{2}} = 0,$$

where A_k, B_k are semiinvariants of equations (3.8) *and* (3.9) *respectively*:

$$A_2 = a_2 - a_1^2 - a_1', \quad A_3 = a_3 + 2a_1^3 - 3a_1a_2 - a_1'', \dots$$

Lemma 3. *Equations* (3.10) *and* (3.11) *are equivalent if the following conditions are satisfied*

$$v'' - \frac{n-2}{n-1}\,\frac{v'^2}{v} + 3\,\frac{n-1}{n+1}\,A_2 v - 3\,\frac{n-1}{n+1}\,B_2 v^{\frac{n-5}{n-1}} = 0,$$
$$v''' - 3\,\frac{n-3}{n-1}\,\frac{v'v''}{v} + 2\,\frac{(n-2)(n-3)}{(n-1)^2}\,\frac{v'^3}{v^2}\,\frac{12}{n+1}\,A_2 v' + 2\,\frac{n-1}{n+1}\,A_3 v - 2\,\frac{n-1}{n+1}\,B_3 v^{\frac{n-7}{n-1}} = 0,$$
$$\vdots$$
$$v^{(n)} + \sum_{k=2}^{n}\binom{n}{k}A_k v^{(n-k)} - B_n v^{-\frac{n+1}{n-1}} = 0.$$

3.8. Theorem of LODE-n equivalence

Theorem 10. *Equations* (3.10) *and* (3.11) *are equivallent if and only if the following relations between their invariants are satisfied*: $I_0(A) = u^3 I_0(B)$, $J_{n,1}(A) = u^4 J_{n,1}(B)$, $J_{n,2}(A) = u^5 J_{n,2}(B), \dots, J_{n,n-3}(A) = u^n J_{n,n-3}(B)$, *where* $\int u(x)\,\mathrm{d}x = t(x)$ *satisfies the equation (KS-3)*

$$\{t,x\} + \frac{3}{n+1}B_2 t'^2 = \frac{3}{n+1}A_2, \{t,x\} = \frac{1}{2}\frac{t'''}{t'} - \frac{3}{4}\left(\frac{t''}{t'}\right)^2,$$

and $I_0(A)$ is Laguerre's invariant (Laguerre [**42**])

$$I_0(A) = A_3 - \frac{3}{2}\,A_2' = a_3 - 3a_1a_2 + 2a_1^3 + 3a_1a_1' + \frac{1}{2}\,a_1'' - \frac{3}{2}\,a_2',$$

and

$$J_{n,1}(A) = A_1 - 2A_3 + \frac{6}{5}\,A_2'' - \frac{3(5n+7)}{5(n+1)}\,A_2^2,$$
$$J_{n,2}(A), \dots, J_{n,n-3}(A)$$

are Halphen's invariants.

3.9. Halphen's canonical forms

Class	Invariants	Transformation $y = u_k^{-\frac{n-1}{2}} z,\ \mathrm{d}t = u_k\,\mathrm{d}x$	Halphen's Canonical forms
Y_0	$I_0 \neq 0$	$u_0 = \sqrt[3]{I_0}$	Principal (H_{n0}), depends on $n-2$ parameters
Y_k, $k = \overline{1, n-3}$	$I_0 = I_{n,1} = \cdots = I_{n,k-1} = 0,\ I_{n,k} = J_{n,k} \neq 0$	$u_k = \sqrt[k+3]{I_{n,k}}$	Degenerate (H_{nk}), depends of $n-k-2$ parameters
Y_{n-2}	$I_0 = I_{n,1} = \cdots = I_{n,n-3} = 0$	$\frac{1}{2}\frac{u''_{n-2}}{u_{n-2}} - \frac{3}{4}\left(\frac{u'_{n-2}}{u_{n-2}}\right)^2 = \frac{3}{n+1}A_2$	Elementary denegerate $(H_{nn-2}) : z^{(n)}(t) = 0$

3.10. Forsythe's canonical forms

Class	Invariants	Transformation $y = u^{-\frac{n-1}{2}} z,\ \mathrm{d}t = u\,\mathrm{d}x$	Forsythe's canonical forms
Y_0	$I_0 \neq 0$	$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 = \frac{3}{n+1}A_2$	Principal (F_{n0}), depends on $n-2$ parameters
Y_k, $k = \overline{1, n-3}$	$I_0 = I_{n,1} = \cdots = I_{n,k-1}$ $=0,\ I_{n,k} = J_{n,k} \neq 0$		Degenerate (F_{nk}), depends on $n-k-2$ parameters
Y_{n-2}	$I_0 = I_{n,1} = \cdots = I_{n,n-3} = 0$		Elementary degenerate $(F_{nn-2}) : z^{(n)}(t) = 0$

Note. Halphen [**30**] found canonical forms for the equations of orders $n = 3$ and $n = 4$.

Forsyth [**28**] found the canonical form F_{n0}.

3.11. Criteria of LODE-n reducibility

Equation (3.10) is locally reducible (by Halphen) if it can be transformed to the following form

$$M_n z \equiv z^{(z)} + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(t) = 0, \quad b_k = \text{const} \tag{3.13}$$

by the KL-transformation, (3.12).

Theorem 11 [**13, 22**]. *The followiing conditions are equivalent*:

1. *Equation* (3.10) *is reducible*;
2. *The operator* L_n *admits noncommutative factorization*

$$L_n = \prod_{k=n}^{1} \left(D - \frac{v'}{v} - (k-1)\,\frac{u'}{u} - r_k u \right),$$

where

$$v = |u|^{-\frac{n-1}{2}} \exp\left(-\smallint \alpha_1 \,\mathrm{d}x + b_1 \smallint u \,\mathrm{d}x\right),$$
$$\frac{1}{2}\,\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + \frac{3}{n+1}\,B_2 u^2 = \frac{3}{n+1}\,A_2,$$

r_k *are roots of the characteristic equation*

$$M_n(r) \equiv r^n + \sum_{k=1}^{n} \binom{n}{k} b_k\, r^{n-k} = 0; \tag{3.14}$$

3. *The operator* $u^{-n} L_n$ *admits the commutative factorization*

$$u^{-n} L_n = \prod_{k=1}^{n} \left(\frac{1}{u}\,D - \frac{v'}{vu} - r_k \right);$$

4. *there exist four functions*, ω, w, λ *and* μ, *namely*

$\omega = v^{-1}u^{1-n},\ w = v^{-1}u^{-n},\ \lambda = u^{-1},\ \mu = -v'v^{-1}u^{-1}$, (see Fayet [**27**])

such that

$$\omega L_n(\lambda D + \mu) y = D\,(\omega L_n y)\,;$$

5. $Y(x)$ *is solution of* (3.10) *if* $y(x)$ *is solution of* (3.10) : (see Kakeya [**31**])

$$Y(x) = \frac{1}{u}\,y' - \frac{v'}{vu}\,y;$$

6. *I_0 and $J_{n,k}$ are connected in a special way, namely in Theorem* 10 $I_0(B)$, $J_{n,1}(B), \dots, J_{n,n-3}(B)$ *are constants;*

7. *Absolute Halphen's invariants* $h_k = \text{const}$ (Halphen [**30**]);

8. *Equation* (3.10) *admits a point symmetry with a generator*

$$X = \frac{1}{u}\frac{\partial}{\partial x} + \frac{v'}{uv}\, y\, \frac{\partial}{\partial y}.$$

4. AUTONOMIZATION OF NODE

We consider nonautonomous nonlinear ordinary differential equations (NODE) [**5,6**].

4.1. Nonlinear equations with reducible linear part

$$y^{(n)} + \sum_{k=1}^{n} \binom{n}{k} a_k\, y^{(n-k)} + F(x, y, y', \dots, y^{(m)}) = 0. \tag{4.1}$$

Theorem 12. *Equation* (4.1) *can be reduced to an autonomous form*

$$z^{(n)}(t) + \sum_{k=1}^{n} \binom{n}{k} b_k\, z^{(n-k)}(t) + a\, \Phi\big(z, z'(t), \dots, z^{m}(t)\big) = 0$$

by the KL-transformation (3.12) *if and only if the nonlinear part F can be represented as*

$$F = au^n v \Phi \left(\frac{y}{v}, \frac{1}{v}\left(\frac{1}{u}D - \frac{v'}{vu}\right) y, \dots, \frac{1}{v}\left(\frac{1}{u}D - \frac{v'}{vu}\right)^{m} y \right).$$

Bandić (see for example [**2**]) transformed nonlinear equations by applying the so-called relative derivatives $\Delta_k = y^{(k)}/y$ (Petrovich [**52**]).

4.2. Test for autonomization

1. Using the criteria for reducibility, verify whether $L_n y = 0$ is reducible.

2. If $L_n y = 0$ is reducible (it always is for $n = 2$), represent the general solution in the form:

$$y = v \sum_{k=1}^{n} c_k \exp(r_k U), \quad U = \textstyle\int u \, dx,$$

where r_k are distinct roots of the characteristic equation (3.14), or in the form

$$y = v \sum_{k=1}^{m} \sum_{s=1}^{\ell_k} \frac{1}{(s-1)!} U^{s-1} \exp(r_k U), \quad \sum_{k=1}^{m} \ell_k = n,$$

where r_k are multiple roots of characteristic equation (3.14),

$$u(x) = (AY_2{}^2 + BY_2Y_1 + CY_1{}^2)^{-1}, \quad B^2 - 4AC = -\frac{12}{n+1}\,B_2,$$

and Y_1, $Y_2 = Y_1 \int \mathrm{d}x/Y_1{}^2$ are linearly independent solutions of

$$Y'' + \frac{3}{n+1}\,A_2 Y = 0. \tag{4.2}$$

4.3. Principles of nonlinear superposition

Let there be given the equation

$$f(x, y, y', \dots, y^{(n)}) = 0. \tag{4.3}$$

A system of functions

$$\{Y_1(x), \dots, Y_m(x)\} \tag{4.4}$$

(see LIE [**45**]) forms a fundamental system of solutions (FSS) of equation (4.3) if its general solution can be represented in the form

$$y = F(Y_1, Y_2, \dots, Y_m; c_1, \dots, c_n), \tag{4.5}$$

where (4.4) are particular solutions of (4.3), particular solutions of the adjoint nonlinear equation

$$\varphi(X, Y, Y', \dots, Y^{(m)}) = 0$$

or they (4.4) are FSS of the adjoint linear equation

$$Y^{(m)} + \sum_{k=1}^{m} \binom{m}{k} a_k(x)\, Y^{(m-k)} = 0.$$

Function (4.5) is called a *nonlinear superposition principle* for equation (4.3) (see WINTERNITZ [**58**], BERKOVICH [**13**]).

Note. Formulas (3.6) and (3.9) are nonlinear superposition principles for the ERMAKOV equation, (3.5'), and for the KUMMER-SCHWARTZ equation (KS-2), (3.8), respectively.

4.4. Generalized Emden-Fowler equation of the second order

Theorem 13. *In order that the equation*

$$y'' + f(x)\, y^n = 0, \quad n \neq 0, \quad n \neq 1,$$

lead to

$$\ddot{z} \pm b_1 \dot{z} + b_0 z + c z^n = 0,$$

it is necessary and sufficient that

$$
\begin{aligned}
&f_1(x) = (\alpha_1 x + \beta_1)^{-\frac{3+n}{2} \pm \frac{b_1(1-n)}{2\sqrt{\delta_1}}} (\alpha_2 x + \beta_2)^{-\frac{3+n}{2} \mp \frac{b_1(1-n)}{2\sqrt{\delta_1}}} \ \delta_1 > 0,\\
&f_2(x) = (Ax^2 + Bx + C)^{-\frac{3+n}{2}} \exp\left(\pm \frac{(1-n)\, b_1}{\sqrt{\delta_2}} \arctan \frac{2AX + B}{\sqrt{-\delta_2}}\right) \ \delta_2 < 0,\\
&f_3(x) = (\alpha x + \beta)^{-(n+3)} \exp\left(\pm \frac{(1-n) b_1}{2\alpha(\alpha x + \beta)}\right), \quad \delta_3 = 0,\ \alpha \neq 0,\\
&f_4(x) = (\alpha x + \beta)^{-\frac{n+3}{2} + b_1 \frac{1-n}{2\alpha}}, \quad \delta_4 = \alpha^2 > 0,\\
&f_5(x) = C \exp\left(\pm \frac{1-n}{2}\, b_1 x\right), \quad \delta_5 = 0.
\end{aligned}
$$

(see also Kečkić [**35**], Kocić [**38**], Berkovich [**7, 16**], Leach [**44**]).

4.5. Ermakov systems

The system (Ermakov [**26**])

$$
\begin{cases} \ddot{x} + a_0(t) x = 0 \\ \ddot{y} + a_0 y = b_0 y^{-3} \end{cases}
$$

havs the integral (invariant):

$$
\frac{1}{2}(\dot{x} y - \dot{y} x)^2 + \frac{1}{2} b_0 \left(\frac{x}{y}\right)^2 = C.
$$

The generalized Ermakov system (Berkovich [**22**]) is

$$
\begin{cases} \ddot{x} + a_1(t)\dot{x} + a_0(t) x = a f(t) x^m y^n F(x, y) \\ \ddot{y} + a_1(t)\dot{y} + a_0(t) y = b f(t) x^n y^m G(x, y), \end{cases} \qquad a, b = \text{const.} \tag{4.5}
$$

If the left part of system (4.5) is reduced to constant coefficients by the KL-transformation

$$
x = v(t) X,\ y = v(t) Y,\ \mathrm{d}T = u(t)\,\mathrm{d}t
$$

and thus

$$
F = F(y/x),\ G = G(x/y),\ f(t) = v^{1-m-n} u^2,\ m = -(n+3)
$$

, system (4.5) possesses the first integral (invariant)

$$
\begin{aligned}
&I = \frac{1}{2}\varphi^2 (x\dot{y} - y\dot{x})^2 + a \int^{x/y} u^{n+1} F(u)\,\mathrm{d}u + b \int^{y/x} u^{n+1} G(u)\,\mathrm{d}u,\\
&\varphi = \exp\left(\int^{x} a_1(t)\,\mathrm{d}t\right).
\end{aligned}
$$

4.6. Generalized Ermakov's equation of the n-order

Theorem 14. *Equation*

$$y^{(n)} + \sum_{k=2}^{n} \binom{n}{k} a_n y^{(n-k)} + b_n y^{\frac{1+n}{2-n}} = 0,$$

where $L_n y = 0$ *is reducible, has the two-parameter solution*

$$y = p\left(AY_1{}^2 + BY_1Y_2 + CY_2{}^2\right)^{\frac{n-1}{2}}, \quad B^2 - 4AC = q,$$

where Y_1 *and* Y_2 *are linearly independent solutions of equation*

$$Y'' + \frac{3}{n+1}\, a_2 Y = 0$$

and admits three-dimensional LIE *algebra with generators*

$$\begin{aligned}
X_1 &= Y_1{}^2 \frac{\partial}{\partial x} + (n-1)Y_1Y_1'\, y \frac{\partial}{\partial y},\\
X_2 &= Y_1Y_2 \frac{\partial}{\partial x} + \frac{n-1}{2}\,(Y_1Y_2' + Y_2Y_1')\, y \frac{\partial}{\partial y},\\
X_3 &= Y_2{}^2 \frac{\partial}{\partial x} + (n-1)Y_2Y_2'\, y \frac{\partial}{\partial y}
\end{aligned}$$

and commutators

$$[X_1, X_2] = X_1,\ [X_2, X_3] = X_3,\ [X_3, X_1] = -2X_2.$$

5. LINEARIZATION OF NODE

In the papers [**17-21, 23**] and in the book [**22**] we have already investigated autonomous nonlinear ordinary differential equations (NODE)

$$y^{(n)} = F(y, y', \dots, y^{(n-1)}). \tag{5.1}$$

Lemma 4. *In order that equation* (5.1) *can be linearized by the nonlinear transformation*

$$y = v(y)z, \quad \mathrm{d}t = u(y)\,\mathrm{d}x \tag{5.2}$$

to equation (3.13), *it is necessary and sufficient that equation* (5.1) *admit the noncommutative factorization*

$$\prod_{k=n}^{1}\left(D - \left(\frac{v^*}{v} - (k-1)\frac{u^*}{u}\right)y' - r_k u\right) y = 0$$

or the commutative factorization

$$\prod_{k=1}^{n}\left(\frac{1}{u}D-\frac{v^*}{uv}y'-r_k\right)y=0,$$

where r_k are roots of the characteristic equation (3.14).

5.1. Linearization of second-order equations

The equation

$$y''+f(y)y'^2+b_1\varphi(y)y'+\psi(y)=0,\quad b_1=\mathrm{const} \tag{5.3}$$

can be linearized by the transformation (5.2) to the equation

$$\ddot{z}+b_1\dot{z}+b_0z+c=0,\ a,\ b,\ c=\mathrm{const}, \tag{5.4}$$

if and only if

$$\psi(y)=\varphi\exp\left(-\textstyle\int f\,\mathrm{d}y\right)\left(b_0\textstyle\int\varphi e^{\int f\,\mathrm{d}y}\,\mathrm{d}y+\dfrac{c}{\beta}\right). \tag{5.5}$$

Here the transformation (5.2) is

$$z=\beta\int\varphi\exp\left(\textstyle\int f\,\mathrm{d}y\right)\mathrm{d}y,\quad \mathrm{d}t=\varphi(y)\,\mathrm{d}x,$$

where $\beta=$const is a normalizing factor. One-parameter solutions of the equations (5.3) and (5.5), where $c=0$, are

$$r_kx+C_k=\int\frac{\exp\left(\int f\,\mathrm{d}y\right)\mathrm{d}y}{\int\varphi\exp\left(\int f\,\mathrm{d}y\right)\mathrm{d}y},$$

where distinct r_k, $(k=1,2)$, satisfy the equation

$$r^2+b_1r+b_0=0.$$

5.2. Nonlinear Oscillator

The equation

$$y''+f(y)y^2\pm a^2\psi(y)=0 \tag{5.6}$$

by transformation

$$z=\sqrt{2\int\psi\exp\left(2\textstyle\int f\,\mathrm{d}y\right)\mathrm{d}y},\quad \mathrm{d}t=z^{-1}\psi\exp\left(\textstyle\int f\,\mathrm{d}y\right)\mathrm{d}x$$

is reduced to the form

$$\ddot{z}\pm a^2z=0.$$

Equation (5.6) has the first integrals:

$$y'^2 = a^2 \Big(C \mp 2\!\int \psi \exp\big(2\!\int f\,\mathrm{d}y\big)\,\mathrm{d}y\Big) \exp\big(-2\!\int f\,\mathrm{d}y\big)$$

and also the one-parameter solutions:

$$\int \frac{\exp\big(2\!\int f\,\mathrm{d}y\big)\,\mathrm{d}y}{z} = \pm\sqrt{\mp\, a^2}\,x + C.$$

5.3. Linearization of third-order equations

We find conditions for linearization of the equation

$$y''' + f_5(y)y'y'' + f_4(y)y'' + f_3(y)y'^3 + f_2(y)y'^2 + f_1(y)y' + f_0(y) = 0 \tag{5.7}$$

to the equation

$$\dddot{z} + b_2\ddot{z} + b_1\dot{z} + b_0 z + c = 0 \tag{5.8}$$

by a transformation of the form (5.2).

Theorem 15. *Equation* (5.7) *can be linearized if and only if it can be represented in the form*

$$\begin{aligned} y''' + f(y)y'y'' + \frac{1}{9}\left(3\,\frac{\varphi^{**}}{\varphi} - 5\,\frac{\varphi^{*2}}{\varphi^2} - f\,\frac{\varphi^*}{\varphi} + f^2 + 3f^*\right) y'^3 \\ +b_2\varphi y'' + \frac{1}{3}\,b_2\varphi\left(f + \frac{\varphi^*}{\varphi}\right) y'^2 + b_1\varphi^2 y' \\ +\varphi^{5/3}\left(b_0\!\int \varphi^{4/3}\exp\left(\frac{1}{3}\!\int f\,\mathrm{d}y\right)\mathrm{d}y + \frac{c}{\beta}\right)\exp\left(-\frac{1}{3}\!\int f\,\mathrm{d}y\right) = 0. \end{aligned} \tag{5.9}$$

Equation (5.9) by the transformation

$$z = \beta\varphi^{4/3}\exp\left(\frac{1}{3}\!\int f\,\mathrm{d}y\right)\mathrm{d}y, \quad \mathrm{d}t = \varphi(y)\,\mathrm{d}x$$

is reduced to the linear form (5.8) and, if $c = 0$, has the distinct one-parameter solutions

$$r_k x + c_k = \int \frac{\varphi^{4/3}\exp\left(\frac{1}{3}\int f\,\mathrm{d}y\right)\mathrm{d}y}{\int \varphi^{4/3}\exp\left(\frac{1}{3}\int f\,\mathrm{d}y\right)\mathrm{d}y}\,,$$

where r_k satisfy the equation

$$r^3 + b_2 r^2 + b_1 r + b_0 = 0.$$

We remark that KEČKIĆ [**34–36**] and KOCIĆ [**38, 39**] investigated nonlinear equations of the second and third orders in another way.

5.4. Euler-Poinsot case in the problem of the gyroscope

Suppose we have the coupled system

$$\left\{\begin{aligned} &A\dot p-(B-C)qr=0,\\ &B\dot q-(C-A)rp=0,\\ &C\dot r-(A-B)pq=0,\end{aligned}\right. \tag{5.10}$$

where p, q and r are the components of the angular velocity in the directions of its principal axes of inertia, A, B and C are its principal moments of inertia. Eliminating the variables we get a noncoupled system of nonlinear third-order equations:

$$y_i''' - \frac{1}{y_i}\,y_i'y_i'' + b_i y_i' {y_i}^2 = 0, \quad (') = \mathrm{d}/\mathrm{d}x_i, \tag{5.11}$$

where b_i is expressed through A, B and C. By the transformations

$$z_i = {y_i}^2,\ \mathrm{d}s_i = y_i\,\mathrm{d}x_i.$$

equations (5.11) are reduced to the linear equations

$$z_i'''(s_i) + b_i z(s_i)' = 0.$$

As a result equations (5.11) have the parametrical solutions:

$$y_i = \Big(2\Big(A_{1i}\cos\big(\sqrt{b_i}s_i+\theta\big)+A_{2i}\Big)\Big)^{1/2},\ x_i = \int\frac{\mathrm{d}s_i}{\Big(2\big(A_{1i}\cos(\sqrt{b_i}s_i+\theta)+A_{2i}\big)\Big)^{1/2}}.$$

6. SIMULTANEOUS USING OF DIFFERENT METHODS

6.1. Generalized Emden-Fowler equations of the third order

The equation

$$y''' + bx^s y^n = 0,\ n \neq 0, n \neq 1$$

can be reduced by the transformation

$$y = v_1(x)v_2\big(y/v_1(x)\big)z, \quad \mathrm{d}t = u_1(x)u_2\big(y/v_1(x)\big)\,\mathrm{d}x$$

to a linear equarion if and only if $n=-5/2$, $s=1$ or $n=-7/2$, $s=3$ respectively. The equations

$$y''' + bxy^{-5/2} = 0,$$
$$y''' + bx^3y^{-7/2} = 0$$

by the transformation

$$z = x^2y^{-1}, \quad \mathrm{d}t = xy^{-3/2}\,\mathrm{d}x$$

are reduced to the linear forms

$$z_t''' - b = 0,$$
$$z_t''' - bz = 0$$

respectively.

6.2. Factorization of Lienard's equation

The equation

$$y'' + a_1(y)y' + a_0(y)y = 0 \tag{6.1}$$

admits factorization of the form

$$\big(D - \alpha_2(y)\big)\big(D - \alpha_1(y)\big)y = 0, D = \mathrm{d}/\mathrm{d}x,$$
$$a_1 = -(\alpha_1 + \alpha_2 + \alpha_1^* y),\ \alpha_0 = \alpha_1\alpha_2,\ (*) = \mathrm{d}/\mathrm{d}y,$$

where α_1 satisfies the Abel equation of the second kind

$$y\alpha_1 \frac{d\alpha_1}{dy} + {\alpha_1}^2 + a_1\alpha_1 + a_0 = 0,$$

and α_2 satisfies the Abel equation of the first kind

$$a_0 y \frac{\mathrm{d}\alpha_2}{\mathrm{d}y} = \alpha_2^3 + a_1\alpha_2^2 + \alpha_2(a_0 + a_0^* y).$$

Equation (6.1) was considered by Bandić [**1**] in a different way.

6.3. Anharmonic oscillator

If

$$(n+3)^2 b_0 = 2(n+1)b_1^2,$$

the equation

$$y'' + b_1 y' + b_0 y + by^n = 0$$

admits the factorization

$$\left(D - r_2 - k_2 y^{\frac{n-1}{2}}\right)\left(D - r_2 - k_2 y^{\frac{n-1}{2}}\right) y = 0, \quad D = \mathrm{d}/\mathrm{d}x,$$

where

$$r_1 = -\frac{2b_1}{n+3}\,,\ r_2 = -\frac{n+1}{n+3}b_1\,,\ k_1 = \pm\sqrt{-\frac{2b}{n+1}}\,,\ k_2 = \mp\sqrt{-\frac{b(n+1)}{2}}\,,$$

and has the one-parameter system of solutions

$$y = \left(\pm\frac{n+3}{b_1}\sqrt{-\frac{b}{2(n+1)}} + C\exp\left(\frac{b_1(n+1)}{n+3}\,x\right)\right)^{\frac{2}{1-n}}.$$

7. CONCLUSION

The methods, discussed in the present work, do not minimize the importance of the other analytical methods, nor the methods of numerical analysis, nor the qualitative theory of differential equations. Only by simultaneously using all of them shall we get the best effect, but the construction of algorithms for solving ordinary differential equations in closed form is the most important goal for any effective theory of ordinary differential equations. Explicit formulas concentrate all the information about the given ordinary differential equation. In this connection we mention the following works: L. Berkovich and F. Berkovich [**14**], Berkovich and Evlakhov [**24**], in which some algorithms of LODE-2 factorization and variables transformation were implemented in REDUCE. Further implementation of such algorithms for nonlinear equations and linear high-order equations is an actual problem. It is the author's opinion that further elaboration of factorization and variable transformation can cast new light on many solved and unsolved questions of natural science.

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