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WHAT IS HADAMARD'S INEQUALITY?

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The Hadamard inequality usually stated as a result valid for convex functions only, actually holds for many other functions. We argue that an attempt ought to be made to close this gap by either changing the inequality or considering the measures in the integrals as a "second variable."

Hadamard's Inequality [**1**] is usually stated as

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f\,\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{1}$$

or in Fejer's version (left hand side only)(and change the interval)

$$\int_{-1}^{1} p(x)\,\mathrm{d}x\ f(0) \le \int_{-1}^{1} p(x)f(x)\,\mathrm{d}x \tag{2}$$

$$\text{provided } p \ge 0 \text{ and } p(x) = p(-x). \tag{3}$$

In both cases, f is presumed to be convex and continuous and p integrable. It is the convexity of f that we wish to challenge. Let us look at the proof on $[-1,1]$. Since *the graph of f lies above any supporting line at* $(0, f(0))$ *and below the chord joining* $(-1, f(-1))$ *and* $(1, f(1))$, we have for some c

$$f(0) + cx \le f(x) \le \frac{f(1)+f(-1)}{2} + \frac{f(1)-f(-1)}{2}\,x. \tag{4}$$

Integrating with respect to Lebesgue measure yields (1) and (2) by integrating with respect to $p(x)\,\mathrm{d}x$.

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Although any convex function satisfies (4), there are a host of other non-convex functions that do also. Merely the emphasized assumption is required. But even that is not necessary. Take any function that is odd about the center of the interval, then both (1) and (2) are satisfied trivially. But more. Since the inequalities are linear, the class of all functions satisfying (1) or (2) is a cone, i.e. closed under sums and positive multiples. For example, $1+\sin 1000\pi x$ satisfies these inequalities. From our point of view, citing (1) or (2) as a basic inequality for convex function is unsatisfactory.

There are several remedies available. The first is suggested by passing from (1) to (2). That is, we should consider bounding

$$\frac{1}{P_0}\int_{-1}^{1} f(x)\,\mathrm{d}\mu(x)$$

where μ is any (say Borel) measure and $P_0=\int_{-1}^{1}\mathrm{d}\mu>0$, and μ is any signed measure. That is, we now have 2 “variables,” the class of functions, and the class of measures. Typically, we would want to prove a bound for the “largest” class of functions and the “largest” class of measures.

In [**3**] one of us solved this dilemma by changing the inequality, fixing the class of functions as the convex ones, and finding an appropriate class of measures. We gave a definition of the “largest” class referred to above. Specifically, let all functions f be continuous and

$$\widehat{M_0}=\{\mu|\mu(-1,1)\ge 0,\ \int_{t}^{1}(x-1)\,\mathrm{d}\mu(x)\ge 0,\ \int_{-1}^{t}(t-x)\,\mathrm{d}\mu(x)\ge 0\ \text{ for }\ t\in[-1,1]\}$$

and $M_0=\{f|\quad f \text{ is a continuous convex function}\}$. Then we proved in [**3**] that

$$P_0 f\left(\frac{P_1}{P_0}\right)\le\int_{-1}^{1} f\,\mathrm{d}\mu,\qquad P_1=\int_{-1}^{1}x\,\mathrm{d}\mu,\quad P_0=\int_{-1}^{1}\mathrm{d}\mu \tag{6}$$

holds for all $f\in M_0$ if and only if $\mu\in\widehat{M_0}$ and (6) holds for all $\mu\in\widehat{M_0}$ if and only if $f\in M_0$.

We call instances where we can prove both these statements a “best possible inequality.” That is, the inequality for all convex f characterizes $\widehat{M_0}$ and the inequality for all measures in $\widehat{M_0}$ characterizes M_0. The reason this works, is that (6) uses the fact that the graph lies above all supporting lines by varying the measure. As a comment, we proved in [**3**] a version of the right hand inequality of (4) which at the time we did not know that it is also a best possible inequality. But we can give a proof that it is.

For the right hand inequality

$$\int_{-1}^{1} f(x)\,\mathrm{d}\mu(x)\le P_0\,\frac{f(-1)+f(1)}{2}+P_1\,\frac{f(1)-f(-1)}{2}\,, \tag{6'}$$

the result was that this hold for all convex f and measures μ such that the boundary value problem

$$y''\mathrm{d}t = \mathrm{d}\mu(t); \quad y(-1) = y(1) = 0 \quad \text{has a solution}$$

$y(x) \le 0$ on $[-1, 1]$. This means that if $G(x,t)$ is the GREEN's function for the problem $Ly = y''$, $y(\pm 1) = 0$ then $y(x) = \int_{-1}^{1} G(x,t)\,\mathrm{d}\mu(t)$. In [**3**] it is stated that it is unlikely that $(6')$ is a best possible inequality. In fact it is. We state this as our first new theorem.

Theorem 1. *Let* $M_0^1 = \{\mu|$ *if* $y''(t)\,\mathrm{d}t = d\mu(t) \quad y(\pm 1) = 0$ *then* $y(x) \le 0$ *on* $[-1, 1]\}$. *Then*

$$(6') \quad \textit{holds for all convex } f \ \textit{ if and only if } \ \mu \in M_0^1;$$

and

$$(6') \quad \textit{holds for all} \quad \mu \in M_0^1 \ \textit{ if and only if } \quad f \ \textit{ is convex.}$$

Proof. The sufficiency in both cases is Theorem 5 of [**3**].

The GREEN's function is given by

$$G(x,t) = \begin{cases} (t-1)x; & -1 \le x \le t \le 1 \\ (x-1)t; & -1 \le t \le x \le 1 \end{cases}.$$

If we take $\mu = \delta_{\frac{a+b}{2}} - \frac{1}{2}\,\delta_a - \frac{1}{2}\,\delta_b$ when δ_t is the unit mass at t and $-1 \le a \le b \le 1$ then $\int_{-1}^{1} G(x,t)\,\mathrm{d}\mu(t) = G\left(x, \frac{a+b}{2}\right) - \frac{1}{2}\,G(x,a) - G(x,b) \le 0$ by the convexity of G. So $\mu \in M_0^1$ and $(6')$ becomes

$$f\left(\frac{a+b}{2}\right) - \frac{1}{2}\,f(a) - \frac{1}{2}\,f(b) \le 0 \quad \text{since} \quad P_0 = P_1 = 0.$$

Since f is always assumed to be continuous, this make f convex, proving the second statement of the theorem. For the other part observe that $f(t) = G(x,t)$ for fixed x is convex and $f(\pm 1) = 0$ so $(6')$ becomes $\int_{-1}^{1} G(x,t)\,\mathrm{d}\mu(t) \le 0$ and μ is necessarily in M_0^1. $\square$

But suppose we do not want to change the inequality (1) and retain some best possible inequality. Two preliminary comments are in order for the inequality

$$f(0)P_0 \le \int_{-1}^{1} f\,\mathrm{d}\mu. \tag{7}$$

First, this inequality does not respect $\mu\{0\}$. For if we replace μ for $\mu + c\delta_0$, δ_0 the unit mass at 0, this just adds $cf(0)$ to both sides of the inequality. Therefore we can never deduce anything about $\mu\{0\}$. Secondly, if we add a constant to f, that constant times P_0 appears on both sides. In particular, we may require $f(0) = 0$. We may consider the inequality

$$(8) \qquad 0 \leq \int_{-1}^{1} f \,\mathrm{d}\mu \quad \text{with the proviso that} \quad f(0) = 0.$$

As a test case, let us look at FEJER's version. Let $\widehat{M_1} = \{\mu| \quad \text{for} \quad A \subset (0,1], \mu(A) = \mu(-A) \geq 0\}$.

Theorem 2. *Let* $M_1 = \{f|\ f$ *is continuous,* $f(0) = 0$ *and* $f(t) + f(-t) \geq 0$ *for* $t \in (0,1]\}$

then (8) *holds for all* $\mu \in \widehat{M_1}$ *if and only if* $f \in M_1$, *and*
(8) *holds for all* $f \in M_1$ *if and only if* $\mu \in \widehat{M_1}$.

Proof. For $\mu \in \widehat{M_1}$, $\int_{-1}^{1} f \,\mathrm{d}\mu = \int_{0}^{1} \big(f(t) + f(-t)\big) \,\mathrm{d}\mu$ so if $f \in M_1$ (8) holds. For the converse take $\mu = \delta_t + \delta_{-t}, 0 < t \leq 1$, then $0 \leq \int_{-1}^{1} f \,\mathrm{d}\mu = f(t) + f(-t)$ so $f \in M_1$. This proves the first if only if and the "if" part of the second statement. To prove that $\mu \in \widehat{M_1}$ if (8) holds for all $f \in M_1$, we take $f = \chi_A - \chi_{-A}$ for a measurable $A \subset (0,1]$. Then f is odd so $f(0) = 0$ and $f \in M_1$. Now (8) leads to $\mu(A) - \mu(-A) \geq 0$. Taking $-f$ we get the reverse inequality. Finally we take $f = \chi_A + \chi_{-A}$ to get $\mu(A) \geq 0$. These f are not continuous but can be approximated in L_1.

For the original inequality (7) the class M_1 now becomes those functions f for which

$$(9) \qquad \frac{f(t) + f(-t)}{2} \geq f(0)$$

i.e. the even part has its minimum at 0. □

It is a triviality to prove

Theorem 3. *The inequality* (8) *holds for all* $\mu \geq 0$ *if and only if* $f \geq 0$ *and the inequality* (8) *holds for all* $f \geq 0$ *if and only if* $\mu \geq 0$. *Consequently,* (7) *is a best possible inequality for the classes*
$M^* = \{f|\ f$ *has a minimum at* $0\}$ *and*
$M^* = \{\mu|\mu \geq 0\}$.

The reduction of HADAMARD's inequality (7) to (8) allows us to prove a variety of best possible inequalities. We always work inside this class of functions which are continuous with $f(0) = 0$, but state the results in terms of the inequality (7) with the measures always in $\widehat{U}$, the class of regular BOREL measures.

Thus HADAMARD's (and FEJER's) inequality (1) and (2) are in fact about functions which have their minimum at the center of the interval. As a second simple example, let $M_2 = \{f|f$ is non-decreasing on $[-1,1]\}$ and $\widehat{M_2} = \{\mu| \int_{-1}^{t} \mathrm{d}\mu \leq 0$ for $-1 \leq t < 0$ and $\int_{t}^{1} \mathrm{d}\mu \geq 0$ for $0 < t \leq 1\}$.

Theorem 4. *The Hadamard inequality* (7) *is best possible with the pairs* M_2 *and* $\widehat{M_2}$.

Proof. If f is increasing then there are measures λ, σ such that $\lambda \leq 0$ $f(x) = \int_{x}^{0} d\lambda$ for $-1 \leq x < 0$ and $f(x) = \int_{0}^{x} \mathrm{d}\sigma$ for $0 < x \leq 1$ and $\sigma \geq 0$. (Recall $f(0) = 0$). Then

$$\int_{-1}^{1} f(x)\,\mathrm{d}\mu(x) = \int_{-1}^{0}\int_{x}^{0} \mathrm{d}\lambda(t)\mathrm{d}\mu(x) + \int_{0}^{1}\int_{0}^{x} \mathrm{d}\sigma(t)\mathrm{d}\mu(x) = \int_{-1}^{0}\int_{-1}^{t} \mathrm{d}\mu(x)\mathrm{d}\lambda(t) + \int_{0}^{1}\int_{t}^{1} \mathrm{d}\mu(x)\mathrm{d}\sigma(t).$$

Consequently, if $\mu \in \widehat{M_2}$ and $f \in M_2$,(8) is satisfied. On the other hand if we take $f = \chi_{[t.1]}$ for $0 < t \leq 1$ then (8) becomes $\int_{t}^{1} \mathrm{d}\mu \geq 0$ and if $f = -\chi_{[-1,t]}$ for $-1 \leq t < 0$ then $\int_{-1}^{t} \mathrm{d}\mu \leq 0$. These f are not in M_2 but can be approximated in L_1 by elements of M_2. So $\mu \in \widehat{M_2}$. If we take $\mu = \delta_t - \delta_s$ for $0 < s < t < 1$ then $\mu \in \widehat{M_2}$ and (8) becomes $f(s) \leq f(t)$ and if $-1 < s < t < 0$ we take $\mu = \delta_g - \delta_s$ then $\mu \in \widehat{M_2}$ and $f(s) \leq f(t)$. So $f \in M_2$. □

To add one multiplicity suppose

$$C = \{f|f(0) = 0,\ f \text{ is convex on } [-1,1] \subset U\}$$

and we seek the measures for which (8) holds. For example $f(x) \equiv \pm|x|$ is in C. So that $\int_{-1}^{1} x\,\mathrm{d}\mu = 0$ is necessary. We may subtract a supporting line at 0 since $\int_{-1}^{1} x\,\mathrm{d}\mu = 0$ and we do not change the inequality. That is f may be assumed to be increasing and convex on [0.1]. In particular for any such f we have $f = f_1 + f_2$ where

$$f_1 = \begin{cases} f & 0 \leq x \leq 1 \\ 0 & -1 \leq x \leq 0 \end{cases} \quad \text{and} \quad f_2 = \begin{cases} 0 & 0 \leq x \leq 1 \\ f & -1 \leq 1 \leq 0 \end{cases}.$$

Inequality (7) holds for all f if and only if it holds for all such f_1 and f_2 since all these are convex. We have (10)

$$f_1(x) = \int_{0}^{x} (x-t)\,\mathrm{d}\sigma(t) \text{ for some } \sigma \geq 0. \tag{10}$$

Then

$$\int_0^1 f_1(x)\,\mathrm{d}\mu(x) = \int_0^1 \int_0^x (x-t)\,\mathrm{d}\sigma(t)\mathrm{d}\mu(x) = \int_0^1 \int_t^1 (x-t)\,\mathrm{d}\mu(x)\mathrm{d}\sigma(t) \geq 0 \tag{11}$$

if $\int_t^1 (x-t)\,\mathrm{d}\mu(x) = \int_t^1 \int_t^x \mathrm{d}s\mathrm{d}\mu(x) = \int_t^1 \int_s^1 \mathrm{d}\mu ds \geq 0$. This condition is

$$\int_t^1 \mu[s,1]\,\mathrm{d}s \geq 0, \quad 0 < t \leq 1, \tag{12}$$

and is necessary and sufficient for (8) to hold. The necessity is obtained by taking σ to be a point mass in (10) and (11), i.e. f is an angle. In a similar way the condition on $[-1,0]$ is $\int_{-1}^t \mu(-1,s)\,\mathrm{d}s \geq 0$, $-1 \leq t < 0$. $\big(f_2(x) = \int_x^0 (t-x)\,\mathrm{d}\sigma(t).\big)$

To complete the best possible statements, one has to prove the convexity of f. Take $\mu = \frac{1}{2}\,\delta_x + \frac{1}{2}\,\delta_y - \delta_{\frac{x+y}{2}}$ for $-1 \leq x < y < 1$. Then

$$\mu(s,1] = \begin{cases} 0 & s \leq x,\ s \geq y \\ \frac{1}{2} & \frac{x+y}{2} \leq s < y \\ -\frac{1}{2} & x < s < \frac{x+y}{2} \end{cases};$$

so that $\int_t^1 \mu(s,1]\,\mathrm{d}s = \begin{cases} 0 & t \geq y,\ t \leq x \\ \frac{1}{2}\,(y-t) & \frac{x+y}{2} < t < y \\ \frac{1}{2}\,(t-x) & x < t < \frac{x+y}{2} \end{cases}$. This is ≥ 0 for $t \in [0,1]$ regardless of where x, $\frac{x+y}{2}$, and y are.

Consequently μ satisfies (10) and (8) become $\frac{1}{2}\,f(x) + \frac{1}{2}\,f(y) \geq f\left(\frac{x+y}{2}\right)$. Note that the function $\int_t^1 \mu[s,1]\,\mathrm{d}s$ is a spline with support $[x,y]$. So taking

$$\widehat{C} = \left\{\mu \middle| \int_{-1}^1 x\,\mathrm{d}\mu = 0;\ \int_{-1}^t \mu[-1,s]\,\mathrm{d}s \geq 0 \text{ for } -1 \leq t \leq 0;\ \int_t^1 \mu[s,1]\,\mathrm{d}s \geq 0 \text{ for } 0 \leq t \leq 1\right\}$$

we have the best possible inequality (9).

We give one other simple example. Suppose we consider the class $M = \{f|f$ for even and $f \geq 0$ on $[-1,1]$, $f(0) = 0\}$. What is the appropriate class of measures for (8) to hold?

Theorem 5. *Let M be as above and $M^* = \{\mu|\mu(A)+\mu(-A) \geq 0$ for all $A \subset (0,1]\}$. Then* (8) *is a best possible inequality.*

Proof. Since $\int_{-1}^{1} f(x)\,\mathrm{d}\mu(x) = \int_{0}^{1} f(x)\big(\mathrm{d}\mu(x) + \mathrm{d}\mu(-x)\big)$ for $f \in M$, (8) hold for $f \in M$ and $\mu \in M^*$. To get the necessity first pick $f(x) = \chi_A(x)$ for any $A \subset (0,1]$. Evidently f is even and $f(0) = 0$. The inequality (8) yield $\mu(A) + \mu(-A) \geq 0$ so $\mu \in M^*$. On the other hand if $\mu = \delta_x - \delta_{-x}$ then clearly $\mu \in M^*$ and (8) yields $f(x) + f(-x) \geq 0$. If we take $\mu = \delta_x - \delta_{-x}$ then $\mu(A) + \mu(-A) = 0$ so $\mu \in M^*$ and (8) yields $f(x) - f(-x) \geq 0$. Combining the two inequalities give $f(x) \geq 0$. Finally taking $\mu = \delta_{-x} - \delta_x$ we also get $f(-x) - f(x) \geq 0$, so that $f(x) = f(-x)$. □

For a modification of Hadamard's inequality that adds a term, see Fink [**5**]. For other views of the inequality, see [**6**] or [**7**].

We have argued that at minimum, Hadamard's inequality should be stated for a class of measures. Then it is about functions with minimums at the center of the interval (Theorem 3) or about functions whose even part about the center has a minimum there (Theorem 4).

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