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SOME RESULTS ON STARLIKE AND CONVEX FUNCTIONS

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Let $\mathcal{A}$ be the class of analytic functions in the unit disk that are normalized with $f(0) = f'(0) - 1 = 0$. In this paper we give sharp sufficient conditions on the expression

$$\frac{1 - \alpha + \alpha z f''(z)/f'(z)}{zf'(z)/f(z)}$$

that implies starlikeness and convexity of function f.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denotes the class of functions $f(z)$ that are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

Further, let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and we write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disk $\mathcal{U}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. Specially, if $g(z)$ is univalent in $\mathcal{U}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

If $-1 \le B < A \le 1$ then an important class is defined by

$$S^*[A, B] = \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\}.$$

Geometrically, this means that the image of $\mathcal{U}$ by $zf'(z)/f(z)$ is inside the open disk centered on the real axis with diameter end points $(1 - A)/(1 - B)$ and $(1 + A)/(1 + B)$. Special selection of A and B lead us to the following classes:

- $S^*[1, -1] \equiv S^*$ is the *class of starlike functions*;

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- $S^*[1-2\alpha,-1] \equiv S^*(\alpha)$, $0 \le \alpha < 1$, is the *class of starlike functions of order* α.

Also, $K^*(\alpha)$, $0 \le \alpha < 1$, is the class of *convex functions of order* α, defined by $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, i.e., $\mathrm{Re}\left(1+\frac{zf''(z)}{f'(z)}\right) > \alpha$, $z \in \mathcal{U}$.

In this paper we will study the class

$$G_{\lambda,\alpha} = \left\{ f \in \mathcal{A} : \left| \frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)} - (1-\alpha) \right| < \lambda, z \in \mathcal{U} \right\},$$

$0 < \alpha \le 1$, $\lambda > 0$, and give sufficient conditions that embed it into the classes $S^*[A,B]$ and $K(\delta)$, $0 \le \delta < 1$. Comparison with previous known results will be done.

In that purpose from the theory of first-order differential subordinations we will make use of the following lemma.

Lema 1 ([**1**]). *Let* $q(z)$ *be univalent in the unit disk* $\mathcal{U}$, *and let* $\theta(\omega)$ *and* $\phi(\omega)$ *be analytic in a domain* D *containing* $q(\mathcal{U})$, *with* $\phi(\omega) \ne 0$ *when* $\omega \in q(\mathcal{U})$. *Set* $Q(z) = zq'(z)\phi\big(q(z)\big)$, $h(z) = \theta(q(z)) + Q(z)$, *and suppose that*

i) $Q(z) \in S^*$; *and*

ii) $\mathrm{Re}\,\frac{zh'(z)}{Q(z)} = \mathrm{Re}\left\{\frac{\theta'(q(z))}{\phi(q(z))} + \frac{zQ'(z)}{Q(z)}\right\} > 0$, $z \in \mathcal{U}$.

If $p(z)$ *is analytic in* $\mathcal{U}$, *with* $p(0) = q(0)$, $p(\mathcal{U}) \subseteq D$ *and*

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{1}$$

then $p(z) \prec q(z)$, *and* $q(z)$ *is the best dominant of* (1)

2. MAIN RESULTS AND CONSEQUENCES

In the beginning, using Lemma 1 we will prove the following result.

Theorem 1. *Let* $f \in \mathcal{A}$, $-1 \le B < A \le 1$ *and* $\frac{1+|A|}{3+|A|} \le \alpha \le 1$. *If*

$$\frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + (1-2\alpha)\frac{1+Bz}{1+Az} + \frac{\alpha z(A-B)}{(1+Az)^2} \equiv h(z)$$

then $f \in S^*[A,B]$. *This result is sharp.*

Proof. We choose $p(z) = \frac{f(z)}{zf'(z)}$, $q(z) = \frac{1+Bz}{1+Az}$, $\theta(\omega) = (1-2\alpha)\omega + \alpha$ and $\phi(\omega) = -\alpha$. Then $q(z)$ is convex, thus univalent, because $1 + zq''(z)/q'(z) =$

$(1-Az)/(1+Az)$; $\theta(\omega)$ and $\phi(\omega)$ are analytic in the domain $D=\mathcal{C}$ which contains $q(\mathcal{U})$ and $\phi(\omega)$ when $\omega\in q(\mathcal{U})$. Further,

$$Q(z)=zq'(z)\phi(q(z))=\frac{\alpha(A-B)z}{(1+Az)^2}$$

is starlike because $\dfrac{zQ'(z)}{Q(z)}=\dfrac{1-Az}{1+Az}$. Further,

$$h(z)=\theta(q(z))+Q(z)=\alpha+(1-2\alpha)\frac{1+Bz}{1+Az}+\frac{\alpha z(A-B)}{(1+Az)^2}$$

and

$$\operatorname{Re}\frac{zh'(z)}{Q(z)}=\operatorname{Re}\left(1-\frac{1}{\alpha}+\frac{2}{1+Az}\right)>1-\frac{1}{\alpha}+\frac{2}{1+|A|},$$

$z\in\mathcal{U}$, which is greater or equal to zero if and only if $\alpha\geq\dfrac{1+|A|}{3+|A|}$. Therefore from Lemma 1 follows that $p(z)\prec q(z)$, i.e., $f\in S^*[A,B]$.

The result is sharp as the functions ze^{Az} and $z(1+Bz)^{A/B}$ show in the cases $B=0$ and $B\neq 0$, respectively. □

Remark 1. According to the definition of subordination, the sharpness of the result of Theorem 1 means that $h(\mathcal{U})$ is the greatest region in the complex plane with the property that if

$$\frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)}\in h(\mathcal{U})$$

for all $z\in\mathcal{U}$ then $f(z)\in S^*[A,B]$.

The following corollary embeds $G_{\lambda,\alpha}$ into $S^*[A,B]$.

Corollary. *$G_{\lambda,\alpha}\subseteq S^*[A,B]$ when $\dfrac{1+|A|}{3+|A|}\leq\alpha\leq 1$ and*

$$\lambda=(A-B)\cdot\frac{(1-2\alpha)|A|-(1-3\alpha)}{(1+|A|)^2}.$$

This result is sharp, i.e., given λ is the greatest so that inclusion holds.

Proof. In order to prove this corollary, due to Theorem 1 it is enough to show that $\lambda=\min\{|h(z)-(1-\alpha)|:|z|=1\}\equiv\widehat{\lambda}$, where $h(z)$ is defined as in the statement of the theorem and

$$h(z)-(1-\alpha)=-z(A-B)\cdot\frac{A(1-2\alpha)z+1-3\alpha}{(1+Az)^2}.$$

Further, let

$$\begin{aligned}\psi(t)&\equiv\left|h(e^{i\gamma\pi/2})-(1-\alpha)\right|^2\\&=(A-B)^2\cdot\frac{\left((1-2\alpha)^2A^2+2(1-3\alpha)(1-2\alpha)At+(1-3\alpha)^2\right)}{(1+2At+A^2)^2},\end{aligned}$$

$t = \cos(\gamma\pi/2) \in [-1,1]$. Thus $\widehat{\lambda} = \min\{\sqrt{\psi(t)} : -1 \le t \le 1\}$.

If $\alpha \le 1/2$ then $1-2\alpha \ge 0$ and having in mind that $1-3\alpha \le -\frac{2|A|}{3+|A|} \le 0$ we receive that $\psi(t)$ is a monotone function and

$$\widehat{\lambda} = \min\{\sqrt{\psi(-1)}, \sqrt{\psi(1)}\} = \min\{|h(-1)-(1-\alpha)|, |h(1)-(1-\alpha)|\} = \lambda.$$

The last equality holds because $1-3\alpha+A(1-2\alpha)z \ge 0$ is equivalent to $\alpha \ge \frac{1+|A|}{3+|A|} \ge \frac{1-|A|}{3-2|A|}$.

If $\alpha > 1/2$ we have the following analysis. Equation $\psi_t'(t) = 0$ has unique solution

$$t_* = -\frac{A^2(1-\alpha)(1-2\alpha)+(1-3\alpha)(1-4\alpha)}{2A(1-2\alpha)(1-3\alpha)}.$$

It can be verified that $|t_*| > 1$ is equivalent to

$$\varphi(A,\alpha) \equiv A^2(1-\alpha)(1-2\alpha) - 2|A|(1-2\alpha)(1-3\alpha) + (1-3\alpha)(1-4\alpha) > 0.$$

Now, $\varphi(A,\alpha)$ is decreasing function of $|A| \in [0,1]$ which implies $\varphi(A,\alpha) \ge \varphi(1,\alpha) = 2\alpha^2 > 0$. Thus, $|t_*| > 1$ which implies that $\psi(t)$ is a monotone function on $[-1,1]$ leading to $\widehat{\lambda} = \min\{\sqrt{\psi(t)} : -1 \le t \le 1\} = \min\{\sqrt{\psi(-1)}, \sqrt{\psi(1)}\} = \min\{|h(-1)-(1-\alpha)|, |h(1)-(1-\alpha)|\}$. At the end, the function

$$\eta(A,\alpha) \equiv |h(1)-(1-\alpha)| - |h(-1)-(1-\alpha)| = 2A \cdot \frac{1-A^2-2\alpha(2-A^2)}{(1+A)^2(1-A)^2}$$

has the opposite sign of the sign of coefficient A. Therefore,

$$\widehat{\lambda} = \left\{ \begin{array}{ll} |h(1)-(1-\alpha)|, & A \ge 0 \\ |h(-1)-(1-\alpha)|, & A < 0 \end{array} \right\} = \lambda.$$

Sharpness of the result follows from the sharpness of Theorem **??** (see Remark 1) and the fact that obtained λ is the greatest which embeds the disc $|\omega-(1-\alpha)| < \lambda$ in $h(\mathcal{U})$. □

The following example exhibits some concrete conclusions that can be obtained from the results of the previous section by specifying the values α, A, B.

Example 1. Let $-1 \le B < A \le 1$.

i) $G_{\lambda,1/2} \subseteq S^*[A,B]$ when $\lambda = \frac{A-B}{2(1+|A|)^2}$.

ii) $G_{\lambda,1} \subseteq S^*[A,B]$ when $\lambda = (A-B)\cdot\frac{2-|A|}{2(1+|A|)^2}$.

iii) $G_{\lambda,1/(2-\gamma)} \subseteq S^*[A,B]$ when $\gamma \ge -\frac{1-|A|}{1+|A|}$ and $\lambda = (A-B)\cdot\frac{1+\gamma-\gamma|A|}{2(1+|A|)^2}$.

iv) $G_{\lambda,\alpha} \subseteq S^*$ when $1/2 \le \alpha \le 1$ and $\lambda = \alpha/2$.

v) $G_{\lambda,\alpha} \subseteq S^*[0,B] \subset S^*(1/(1-B))$ when $1/3 \leq \alpha \leq 1$, $-1 \leq B < 0$ and $\lambda = B(1-3\alpha)$.

Given λ is the greatest so that inclusions hold.

Remark 2. The result from Example 1 (i) is the same as in Corollary 2.6 in [**5**]. Also, for $\alpha = 1/2$ in Example 1 (v) we receive the same result as in Theorem 1 from [**2**]. Finally, for $\alpha = 1$ and $B = -1$ in Example 1(v) we receive the same result as in Corollary 2 from [**3**].

Next theorem studies connection between $G_{\lambda,\alpha}$ and the class of convex functions of some order.

Theorem 2. *$G_{\lambda,\alpha} \subseteq K\left(2-\frac{1}{\alpha}\right)$ when $\frac{1}{2} \leq \alpha < 1$ and $\lambda = \dfrac{(1-\alpha)(3\alpha-1)}{\sqrt{2(5\alpha^2-4\alpha+1)}}$.*

Proof. Let $f \in G_{\lambda,\alpha}$ and $B = \dfrac{\lambda}{(1-3\alpha)}$. Then, by Example 1 (v) we have $f \in S^*[0,B]$, i.e., $\left|\dfrac{f(z)}{zf'(z)} - 1\right| < B$, $z \in \mathcal{U}$. Further,

$$1 + \frac{zf''(z)}{f'(z)} - \left(2 - \frac{1}{\alpha}\right) = \frac{zf'(z)}{\alpha f(z)} \cdot \frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)}.$$

and for all $z \in \mathcal{U}$ we obtain

$$\begin{aligned}
\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - 2 + \frac{1}{\alpha}\right)\right| &\leq \left|\arg\frac{zf'(z)}{f(z)}\right| + \left|\arg\frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)}\right| \\
&\leq \arcsin|B| + \arcsin\frac{\lambda}{1-\alpha} \\
&= \arcsin\left(\frac{\lambda}{1-\alpha} \cdot \sqrt{1-B^2} + |B| \cdot \sqrt{1 - \frac{\lambda^2}{(1-\alpha)^2}}\right) \\
&= \arcsin 1 = \frac{\pi}{2},
\end{aligned}$$

i.e., $f \in K\left(2 - \frac{1}{\alpha}\right)$. □

Example 2. For $\alpha = 1/2$ and $\alpha = 1/(2-\gamma)$ in the previous theorem we receive

i) $G_{\lambda,1/2} \subseteq K$ when $\lambda = \sqrt{2}/4$.

ii) $G_{\lambda,1/(2-\gamma)} \subseteq K(\gamma)$ when $0 \leq \gamma < 1$ and $\lambda = \dfrac{1-\gamma^2}{(2-\gamma)\sqrt{2(1+\gamma^2)}}$.

Remark 3. By putting $\alpha = \dfrac{1}{2-\gamma}$, $0 \leq \gamma < 1$, we receive the result from Theorem 2 in [**4**].

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