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COLOR-BOUNDED HYPERGRAPHS, III: MODEL COMPARISON

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Generalizing previous models of hypergraph coloring — due to Voloshin, Drgas-Burchardt and Łazuka, and the present authors – in this paper we introduce and study the structure class that we call *stably bounded hypergraphs*. In this model, a hypergraph is viewed as a six-tuple $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b} : \mathcal{E} \to \mathbb{N}$ are given integer-valued functions on the edge set. A mapping $\varphi : X \to \mathbb{N}$ is a *proper vertex coloring* if it satisfies the following conditions for each edge $E \in \mathcal{E}$: the number of colors in E is at least $\boldsymbol{s}(E)$ and at most $\boldsymbol{t}(E)$, while the largest number of vertices having the same color inside E is at least $\boldsymbol{a}(E)$ and at most $\boldsymbol{b}(E)$.

Taking different subsets of $\{\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}\}$ (as combinations of nontrivial conditions on colorability) result in a hierarchy of structure classes with respect to vertex coloring. The main issue of this paper is to carry out a detailed analysis of how those classes are related. This includes the study of possible chromatic polynomials and 'feasible sets' — that is, the set $\Phi(\mathcal{H})$ of integers k such that $\mathcal{H}$ has a proper vertex coloring with exactly k colors — with or without assuming that the number of vertices is the same under the different combinations of color-bound conditions, or restricting the edge sizes. Furthermore, substantial change is observed concerning the algorithmic complexity of recognizing hypergraphs that are uniquely colorable and $\Phi(\mathcal{H}) = \{\,|X| - 1\}$.

1. INTRODUCTION

The main goal of this paper is to describe a unified framework for various concepts in the coloring theory of hypergraphs, and to study how some of its naturally arising subclasses are interrelated. The model presented here includes, as

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particular cases, the proper (vertex) colorings in the classical sense, moreover the class of mixed hypergraphs, and also the color-bounded hypergraphs that have been introduced recently.

We assume throughout that $X = \{x_1, \ldots, x_n\}$ is a finite vertex set and $\mathcal{E} = \{E_1, \ldots, E_m\}$ is the edge set, where each E_i is a nonempty subset of X. Under a given mapping $\varphi : X \to \mathbb{N}$ – for which we shall use the term *vertex coloring*, or simply *coloring* – a set $Y \subseteq X$ is *monochromatic* if $\varphi(y) = \varphi(y')$ for all $y, y' \in Y$; and Y is said to be *polychromatic* if $\varphi(y) \neq \varphi(y')$ for any two distinct $y, y' \in Y$. Introducing the notation $\varphi(Y)$ for the image of Y under φ (i.e., the set of colors appearing in Y), 'monochromatic' and 'polychromatic' mean $|\varphi(Y)| = 1$ and $|\varphi(Y)| = |Y|$, respectively.

The classical theory of graph and hypergraph coloring proceeds by *excluding monochromatic edges*. Around the mid-1990's, Voloshin [**18, 19**] introduced the more general structures of *mixed hypergraphs*, in which some edges – that are called *$\mathcal{D}$-edges* – are not allowed to be monochromatic, while some edges – the *$\mathcal{C}$-edges* – are not allowed to be polychromatic. A *bi-edge* is a vertex subset (edge) which is a $\mathcal{C}$-edge and a $\mathcal{D}$-edge at the same time. In the past decade, the theory of mixed hypergraphs developed rapidly. For a concise survey on the subject, we refer to [**16**]; see also the research monograph [**20**] and the regularly updated web site [**21**].

In this paper we view hypergraphs as six-tuples $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$, where

$$\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b} : \mathcal{E} \to \mathbb{N}$$

are given integer-valued functions on the edge set; they will play important role concerning colorings. To simplify notation, we define

$$s_i := \boldsymbol{s}(E_i), \qquad t_i := \boldsymbol{t}(E_i), \qquad a_i := \boldsymbol{a}(E_i), \qquad b_i := \boldsymbol{b}(E_i)$$

and assume throughout that the inequalities

$$1 \le s_i \le t_i \le |E_i|, \qquad 1 \le a_i \le b_i \le |E_i|$$

are valid for all edges E_i.

Given a coloring $\varphi : X \to \mathbb{N}$ and any $Y \subseteq X$, we denote by $\mu(Y)$ (by $\pi(Y)$, respectively) the largest cardinality of a monochromatic (resp. polychromatic) subset of Y. A *stable coloring* of $\mathcal{H}$ is a mapping such that

$$s_i \le \pi(E_i) \le t_i \qquad \text{and} \qquad a_i \le \mu(E_i) \le b_i \qquad \text{for all} \qquad E_i \in \mathcal{E}$$

If these conditions are met, we may also call φ a *proper (vertex) coloring* of $\mathcal{H}$. Hence, the bounds s_i and t_i force that the largest *polychromatic* subset of each E_i has at least s_i and at most t_i vertices, whereas the largest *monochromatic* subset of E_i must have at least a_i and at most b_i colors.

We shall refer to $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$ as *color-bound functions*, and to s_i, t_i, a_i, b_i as *color-bounds* on edge E_i. In this setting the pair $(\boldsymbol{s}, \boldsymbol{t})$ restricts the *maximum number* of colors inside an edge, while $(\boldsymbol{a}, \boldsymbol{b})$ is responsible for the *maximum multiplicity* of colors on the edge. We introduce the terminology *stably bounded hypergraph* for $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$; this phrase may be viewed as an alternative rewritten form of '$(\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$-ly bounded'.

Our paper is a continuation of the study of *color-bounded hypergraphs*, introduced and investigated in [**2**–**4**], in which only the functions $\boldsymbol{s}, \boldsymbol{t}$ were considered as restrictions. In those works, strongly motivated by the paper of Drgas-Burchardt and Łazuka [**6**] (concerning the function $\boldsymbol{s}$ itself) and the concept of mixed hypergraphs, substantial differences have been pointed out between the classes of mixed and color-bounded hypergraphs. Though more general than all those, our present model with its four color-bound functions is still a subclass of the 'pattern hypergraphs' introduced by Dvořák et al. in [**7**], since in the latter the collection of feasible coloring patterns may be specified for each edge separately. Compared to that, however, our more restrictive conditions allow us to prove stronger results.

In the present paper we give a detailed analysis of the relations among the four color-bound functions. The subsets of $\{\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}\}$, as combinations of nontrivial conditions on colorability, form a hierarchy with respect to the strength of models concerning vertex coloring. In a way, the pair $(\boldsymbol{s}, \boldsymbol{a})$ is universal; but, interestingly enough, the partial order among the classes is not always the same, as it may depend on the aspect under which the allowed colorings are compared. Our results indicate that concerning the possible numbers of colors on a given number of vertices, the more restrictive function is the *monochromatic* upper bound $\boldsymbol{b}$ (cf. Theorems 1 and 2), while with respect to the number of color partitions in general the stronger restriction is the *polychromatic* upper bound $\boldsymbol{t}$ (see Section 2.3).

Although the decision problem whether a hypergraph admits any proper coloring is NP-complete for all nontrivial combinations of the conditions, nevertheless some algorithmic questions exhibit further substantial differences among the color-bound types. This fact is demonstrated concerning unique colorability in Section 3. On the other hand, there are subclasses of stably bounded hypergraphs that admit efficient coloring algorithms. Some of them will be discussed in the forthcoming paper [**5**].

While revising the manuscript, we have learned that the subclass of hypergraphs with bound $\boldsymbol{b}$ was studied previously in [**14**], [**11**] and [**1**], especially concerning approximation algorithms for the minimum number of colors in a proper coloring.

Further notation and terminology. Conditions of the types $s_i = 1$, $t_i = |E_i|$, $a_i = 1$, and $b_i = |E_i|$ have no effect on the colorability properties of $\mathcal{H}$, because they are trivially satisfied in every coloring. For this reason, we may restrict our attention to the subset of $\{\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}\}$ that really means some conditions on at least one edge. We shall use Capital letters to indicate them. For instance, by an (S, T)-hypergraph we mean one where $a_i = 1$ and $b_i = |E_i|$ hold for all edges. In such hypergraphs it is usually the case – though not required by definition – that there is at least one edge $E_{i'}$ with $s_{i'} > 1$ and at least one edge $E_{i''}$ with $t_{i''} < |E_{i''}|$.

Otherwise, e.g. if $s_i = 1$ also holds for all i, we may simply call it a T-hypergraph.

Colorability, feasible sets, chromatic spectrum. A hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ may admit no colorings at all; this is the case already with the mixed hypergraphs. We call $\mathcal{H}$ *colorable* if it has at least one proper coloring; and otherwise we say that it is *uncolorable*.

Assume that $\mathcal{H}$ is colorable. By a *k-coloring* we mean a proper vertex coloring with *exactly* k colors; that is, a coloring $\varphi : X \to \mathbb{N}$ with $|\varphi(X)| = k$. (Observe that this is a slight deviation from standard terminology.) The set

$$\Phi(\mathcal{H}) := \{k \mid \mathcal{H} \text{ has a } k\text{-coloring}\}$$

is termed the *feasible set* of $\mathcal{H}$. Assuming that $\mathcal{H}$ is colorable, the largest and smallest possible numbers of colors in a feasible coloring are termed the *upper chromatic number* and *lower chromatic number* of $\mathcal{H}$, respectively. In notation,

$$\chi(\mathcal{H}) = \min \Phi(\mathcal{H}), \qquad \overline{\chi}(\mathcal{H}) = \max \Phi(\mathcal{H}).$$

Each k-coloring φ of $\mathcal{H}$ induces a *color partition*, $X = X_1 \cup \cdots \cup X_k$, where the X_i are the inclusionwise maximal monochromatic subsets of X in the coloring φ. We shall denote by r_k the number of proper color partitions with precisely k nonempty classes. By the *chromatic spectrum* of $\mathcal{H}$ we mean the $\overline{\chi}$-tuple $(r_1, r_2, \ldots, r_{\overline{\chi}})$. We should mention, however, that in other parts of the literature (see [**20**]) the chromatic spectrum is defined as the n-tuple $(r_1, r_2, \ldots, r_n)$. Nevertheless, it is clear that the two representations are equivalent whenever the number of vertices is irrelevant; therefore we prefer to keep the shorter notation in the present context.

2. SMALL VALUES AND REDUCTIONS

Here we point out some simple relations among the color-bound functions $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$. It will turn out that on 3-uniform hypergraphs without further restrictions, four different models are equivalent. On the other hand, for hypergraphs with arbitrary edge sizes, one of them is universal.

Proposition 1. *Let E_i be an edge in a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$. If $|E_i| \le 3$, then $\pi(E_i) + \mu(E_i) = |E_i| + 1$.*

Proof. It suffices to observe that any E_i has a unique partition into 1 or $|E_i|$ classes, verifying $\pi(E_i) + \mu(E_i) = |E_i| + 1$ for such trivial partitions; moreover, if $|E_i| = 3$, then the size distribution in precisely two nonempty partition classes is uniquely determined as $(2, 1)$, so that $\pi(E_i) = \mu(E_i) = 2$ in this case. □

Corollary 1. *Let $E_i \in \mathcal{E}$ be an edge with at most three vertices.*

1. *If $|E_i| = 1$, then $s_i = t_i = a_i = b_i = 1$ necessarily holds, and the edge may be deleted without changing the coloring properties of $\mathcal{H}$.*

2. *If* $|E_i| = 2$, *then between the local conditions the following equivalences are valid for* $k = 1, 2$.

   (i) $s_i = k \Leftrightarrow b_i = 3 - k$,

   (ii) $a_i = k \Leftrightarrow t_i = 3 - k$.

3. *If* $|E_i| = 3$, *then between the local conditions the following equivalences are valid for* $k = 1, 2, 3$.

   (i) $s_i = k \Leftrightarrow b_i = 4 - k$,

   (ii) $a_i = k \Leftrightarrow t_i = 4 - k$.

An important consequence is that, in the restricted class of 3-uniform hypergraphs, each pair in $(\boldsymbol{s}, \boldsymbol{b}) \times (\boldsymbol{t}, \boldsymbol{a})$ represents any nontrivial combination of $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$ in full generality:

Corollary 2. *If each edge of* $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ *has at most three vertices, then* $\mathcal{H}$ *has an equivalent description as an*

- (S, T)*-hypergraph,*
- (S, A)*-hypergraph,*
- (T, B)*-hypergraph,*
- (A, B)*-hypergraph.*

Proof. Based on Corollary 1, every s-condition and a-condition can be transcribed to an equivalent b-condition and t-condition, respectively; and vice versa. □

The coincidences of conditions above do not carry over for edges with $|E_i| > 3$. Indeed, a 4-element set admits 2-partitions of both types $2 + 2$ and $3 + 1$ (and the situation is even worse for larger edges), hence there is no strict relation between $\pi(E_i)$ and $\mu(E_i)$ in either direction. Nevertheless, the following implications remain valid for edges of any size, by the pigeon-hole principle.

Proposition 2. *Let* E_i *be any edge in a hypergraph* $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$. *Then, between the conditions the following equivalences are valid*:

(i) $s_i = 2 \Leftrightarrow b_i = |E_i| - 1$,

(ii) $a_i = 2 \Leftrightarrow t_i = |E_i| - 1$. □

In particular, for mixed hypergraphs we obtain

Corollary 3. *Every mixed hypergraph is a member of all the four classes of* (S, T)*-,* (S, A)*-,* (T, B)*-, and* (A, B)*-hypergraphs at the same time.*

Proof. Every $\mathcal{C}$-edge E_i can be interpreted as $(s_i, a_i) = (1, 2)$, a $\mathcal{D}$-edge E_i corresponds to the bounds $(s_i, a_i) = (2, 1)$, whereas a bi-edge E_i is equivalent to the bounds $(s_i, a_i) = (2, 2)$. This yields membership in (S, A). Transcription to the other three models can be done via Proposition 2. □

REMARK. Contrary to mixed hypergraphs, in the general model the edges E_i of cardinality 2 with $t_i = 1$ or $a_i = 2$ usually cannot be contracted, despite their two vertices must get the same color in every proper coloring. The reason is that in $(\boldsymbol{a}, \boldsymbol{b})$ the multiplicities of colors are of essence. To keep track of them, one would need to introduce weighted vertices and interpret $(\boldsymbol{a}, \boldsymbol{b})$ as weighted conditions. We do not study weighted hypergraphs in the present paper.

2.1. CLASS REDUCTIONS AND COLORABILITY

Some combinations between color-bound conditions can be done; moreover, some of their combinations always admit a proper coloring. We summarize these facts as follows.

Table 1

1. *Colorable pairs*:
   - (S, B)-hypergraphs allow every edge to be polychromatic, therefore the upper chromatic number equals the number of vertices.
   - (T, A)-hypergraphs allow every edge to be monochromatic, therefore the lower chromatic number equals 1.
2. *Combinations admitting uncolorability*:

   These are the sets of color-bound functions intersecting both (S, B) and (T, A); i.e., the minimal such sets are the pairs (S, T), (S, A), (A, B), and (T, B). The following operations show that all of them can be reduced to (S, A).
   - $b_i < |E_i|$: insert all $(b_i + 1)$-subsets of E_i with lower color-bound $s = 2$, and omit the condition b_i from E_i. This eliminates the function $\boldsymbol{b}$.
   - $t_i < |E_i|$: insert all $(t_i + 1)$-subsets of E_i with lower color-bound $a = 2$, and omit the condition t_i from E_i. This eliminates the function $\boldsymbol{t}$.
3. *Universal classes for colorability problems*:
   - S-hypergraphs [**6**] are universal models for n-colorable stably bounded structures where the question is to determine χ.
   - A-hypergraphs are universal models for 1-colorable stably bounded structures where the question is to determine $\overline{\chi}$.
   - (S, A)-hypergraphs are universal models for stably bounded structures where both χ and $\overline{\chi}$ are of interest.

Concerning feasible sets, the following assertions are valid.

Proposition 3. *If a hypergraph* $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ *has* $\chi(\mathcal{H}) = 1$ *or* $\overline{\chi}(\mathcal{H}) = |X|$, *then its chromatic spectrum is continuous. Moreover, every interval of positive integers can be realized as the feasible set of hypergraphs with just one edge in each of the four types* (S,T), (S,A), (T,B), *and* (A,B); *and such a realization is possible even with an* S*-hypergraph and with a* B*-hypergraph.*

Proof. If $\chi(\mathcal{H}) = 1$, then we have non-restrictive bounds $s_i = 1$ and $b_i = |E_i|$ for all edges E_i. Let φ be a k-coloring of $\mathcal{H}$ with $k > 1$. Taking the union of two arbitrarily chosen color classes of φ as just one new color class, no edge E_i will have smaller $\mu(E_i)$ or larger $\pi(E_i)$, hence a proper $(k-1)$-coloring is constructed. Starting from $k = \overline{\chi}(\mathcal{H})$, we obtain that all numbers of colors between 1 and $\overline{\chi}$ admit a proper coloring.

Similarly, for $\overline{\chi}(\mathcal{H}) = |X|$ we have $t_i = |E_i|$ and $a_i = 1$ for all E_i. If φ is a k-coloring of $\mathcal{H}$ with $k < |X|$, then some color class has more than one vertex, and splitting it into two nonempty classes in an arbitrary way we cannot violate the conditions s_i and b_i, so that a proper $(k+1)$-coloring is obtained. Starting from $k = \chi(\mathcal{H})$, all numbers of colors between χ and $|X|$ admit a proper coloring.

In order to construct a hypergraph $\mathcal{H} = (X, \mathcal{E})$ with feasible set $\Phi(\mathcal{H}) = \{\ell \mid p \le \ell \le q\}$ for any given $q \ge p \ge 1$, we let $|X| = q$ that will ensure $\overline{\chi} = q$ for both S- and B-hypergraphs. To satisfy the equation $\chi = p$, we may simply assign $s = p$ to an edge whose cardinality is between p and q. In type B, this edge should have cardinality exactly p, assigned with the color-bound $b = 1$ that makes it polychromatic. □

Remark 2. Conditions involving S are more flexible than those with B. Namely, for the types (S,T) and (S,A) we may take $\boldsymbol{s}(X) = p$ with <u>any</u> number n of vertices, because either of the conditions $\boldsymbol{t}(X) = q$ and $\boldsymbol{a}(X) = n - q + 1$ yields then $\overline{\chi} = q$, as the total number of colors cannot be larger than $n - \boldsymbol{a}(X) + 1$. On the other hand, concerning the hypergraph $\mathcal{H} = (X, \{X\})$ the only possible choice for $\boldsymbol{b}(X)$ is $\lceil n/p \rceil$, which guarantees $\chi = p$ if and only if $(p-1)\lceil n/p \rceil \le n-1$. Thus, for (T,B) and (A,B) the set of feasible orders n is precisely $\bigcup_{b \ge 1} \{n \in \mathbb{N} \mid pb - b + 1 \le n \le pb\}$.

Proposition 4. *For every finite sequence* $(r_2, \dots, r_k)$ *of nonnegative integers with* $r_k > 0$, *there exist* (S,T)-, (S,A)-, $(A,B)-$, *and* (T,B)*-hypergraphs whose upper chromatic number is* k *and chromatic spectrum is* $(r_1 = 0, r_2, \dots, r_k)$.

Proof. As proved by Kráľ in [**9**], every spectrum $(r_2, \dots, r_k)$ occurs in non-1-colorable (i.e., with $r_1 = 0$) mixed hypergraphs. Since every mixed hypergraph belongs to all of the four types by Corollary 3, the assertion follows. □

Hence, in hypergraphs $\mathcal{H}$ belonging to class types other than the trivially colorable ones which are subsets of $\{S,B\}$ and $\{T,A\}$, it remains a substantial question to determine the feasible set $\Phi(\mathcal{H})$. On the other hand, chromatic spectra and chromatic polynomials are of interest for trivially colorable classes, too.

2.2. LARGE GAPS IN THE CHROMATIC SPECTRUM

Jiang et al. constructed in [**8**] a *mixed* hypergraph on $2k+4$ vertices and with a gap of size k in the chromatic spectrum, for all $k \geq 1$. This $2k+4$ is the smallest possible order, what follows from another result of the same paper (though this consequence is not formulated there explicitly).

In this subsection we extend this result by pointing out that the minimality of $2k+4$ for a gap of size k remains valid in the more general class of (T, A, B)-hypergraphs, too. In contrast to this, in [**3**] the exact minimum for (S, T)-hypergraphs has been proved to be $k+5$, and at the end of this subsection we show that the same is valid for (S, A)-hypergraphs as well.

First, we prove an assertion that we shall use as a lemma but it can be of interest in itself, too. It contains, as subcases, all the types of (T, B)-, (A, B)-, and mixed hypergraphs.

Proposition 5. *If a (T, A, B)-hypergraph on n vertices has a gap at g, then its lower chromatic number χ is at least $2g - n + 2$.*

Proof. By definition, there exists an integer $j \geq g+1$ such that the hypergraph has a coloring φ with exactly j colors but there is no proper $(j-1)$-coloring.

Suppose first that $2j - 2 \geq n$. Then there occur at least $2j - n \geq 2$ singleton color classes in φ. Considering two of them, say $\{x\}$ and $\{y\}$, their union yields a non-feasible $(j-1)$-coloring. After the identification of $\varphi(x)$ and $\varphi(y)$, however, all the bounds t_i and a_i remain fulfilled. Consequently, the obtained $(j-1)$-coloring can be non-feasible only because of an edge E_i containing both vertices x and y and having bound $b_i = 1$. Thus, x and y must have different colors in every feasible coloring. For the same reason, any two vertices from the at least $2j - n$ singletons are differently colored in any χ-coloring, too. This implies $\chi \geq 2j - n$. Due to the condition $j \geq g+1$, the inequality $\chi \geq 2g - n + 2$ follows.

On the other hand, if $2j - 2 < n$, considering the gap at $g \leq j - 1$ we obtain the upper bound $2g - n + 2 \leq 2j - n \leq 1$, thus the inequality $\chi \geq 2g - n + 2$ automatically holds. □

We mention the following consequence that was proved for *mixed* hypergraphs in [**8**]. Tightness follows from a construction of the same paper.

Corollary 4. *If a (T, A, B)-hypergraph is ℓ-colorable and has a gap at g, then it has at least $2g + 2 - \ell$ vertices.*

Theorem 1. *If a (T, A, B)-hypergraph has a gap of size $k \geq 1$ in its chromatic spectrum, then it has at least $2k+4$ vertices. Moreover, this bound is sharp; that is, for every positive integer k there exist mixed, (T, B)- and (A, B)-hypergraphs on $|X| = 2k+4$ vertices, whose chromatic spectrum has a gap of size k.*

Proof. Suppose that a (T, A, B)-hypergraph has an ℓ-coloring and an $(\ell + k + 1)$-coloring, but all integers in between are gaps. Then we can apply Proposition 5 with $g = \ell + k$, so that $2\ell + 2k - n + 2 \leq \chi \leq \ell$ is obtained. Moreover, every

1-colorable hypergraph has continuous chromatic spectrum, hence $\ell \geq 2$ holds and the above facts imply that the lower bound $n \geq \ell + 2k + 2 \geq 2k + 4$ is valid.

To show that the bound is sharp, we consider the construction from [**8**]. The hypergraph $\mathcal{H}_{2,k+3}$ is defined on the $(2k+4)$-element vertex set $\{x_1, x_2, a_1, a_2, \ldots, a_{k+1}, b_1, b_2, \ldots, b_{k+1}\}$, with the following edges:

- Triples of the form $\{x_i, a_j, b_j\}$ for $i = 1, 2$ and for all $1 \leq j \leq k+1$. They are bi-edges in the mixed hypergraph and have bounds $(t, b) = (2, 2)$ or $(a, b) = (2, 2)$ in the other models.
- Quadruples of the form $\{a_i, a_j, b_i, b_j\}$ for all $1 \leq i < j \leq k+1$, as $\mathcal{D}$-edges, with bounds $(t, b) = (4, 3)$ or $(a, b) = (1, 3)$.
- Triples of the form $\{a_i, a_j, b_i\}$ and $\{a_i, b_i, b_j\}$ for any two distinct indices $i, j \in \{1, 2, \ldots, k+1\}$. They are $\mathcal{C}$-edges or equivalently have bounds $(t, b) = (2, 3)$ and $(a, b) = (2, 3)$, respectively.
- The pair $\{x_1, x_2\}$ as a $\mathcal{D}$-edge, with bounds $(t, b) = (2, 1)$ or $(a, b) = (1, 1)$.

The feasible set of this hypergraph is $\{2, k+3\}$, as it was proved in [**8**]. This fact remains valid in all of the three models considered, thus the assertion follows. $\square$

In [**3**] we proved that (S, T)-hypergraphs can have a gap of size k only if the number of vertices is at least $k+5$, and this bound is tight. Now, we extend the lower bound of this result to all stably bounded hypergraphs, and show that it is tight already for (S, A)-hypergraphs.

Theorem 2. *If a stably bounded hypergraph has a gap of size $k \geq 1$ in its chromatic spectrum, then it has at least $k+5$ vertices. Moreover, this estimate is sharp, already for the type (S, A); that is, for every positive integer k there exists an (S, A)-hypergraph on $|X| = k+5$ vertices, whose chromatic spectrum has a gap of size k.*

Proof. We have proved in Proposition 3 that if a stably bounded hypergraph has a 1-coloring or a totally polychromatic n-coloring, then its chromatic spectrum is continuous. Hence, the only possibility for having a gap of size $k \geq 1$ on fewer than $k+5$ vertices would be with $n = k+4$ and with the feasible set $\{2, k+3\}$.

Assume for a contradiction that this is the case. Because of 2-colorability, the inequality $s_i \leq 2$ is valid for every edge E_i. Let now φ be a coloring with precisely $k+3 = n-1$ colors. Then $k+2$ of the color classes (i.e., all but one) in φ are singletons. Taking the union of two arbitrarily chosen 1-element classes, say $\{x\}$ and $\{y\}$, we get a non-feasible color partition. This change can never decrease the size of monochromatic subsets or increase the number of distinct colors occurring inside any edge, therefore all of the bounds a_i and t_i are kept satisfied.

Hence, there exists an edge E_i for which either the bound $s_i \leq 2$ gets violated, or its monochromatic subset becomes larger than b_i. The former means, however, that E_i becomes monochromatic. That is, we have $s_i = 2$ and $E_i = \{x, y\}$, hence

x and y are colored differently in every feasible coloring. On the other hand, since $\varphi(x) \neq \varphi(y)$, in the modified coloring the color of $\{x, y\}$ does not occur on any other vertex. Hence, if b_i gets violated, then $b_i = 1$ must hold, and again we can conclude that x and y are colored differently in every feasible coloring. This property is valid for any two of the $k + 2 \geq 3$ singletons, what contradicts to the assumption that the hypergraph is 2-colorable. Hence, there cannot exist any color-bounded hypergraphs with a gap of size k on fewer than $k + 5$ vertices.

To show that a gap of size k is realizable on $k + 5$ vertices, we refer to the corresponding construction from [**3**], that has feasible set $\{3, k + 4\}$. The (S, T)-hypertree described there has edges only of size 4, with bounds $(s, t) = (3, 3)$, what can be interpreted with bounds $(s, a) = (3, 2)$ in an equivalent way. Hence, an (S, A)-hypergraph with the required properties is obtained. □

Theorems 1 and 2 together characterize the minimum order of a hypergraph of any nontrivial type for a gap of size k: the minimum is $k + 5$ if and only if the type contains (S, T) or (S, A), and it is $2k + 4$ if and only if it does not contain S but contains B and at least one of T and A. In any other case, the spectrum is gap-free.

2.3. COMPARISON OF THE SETS OF CHROMATIC POLYNOMIALS

We have already seen that any type of nontrivial combinations of $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{a}$, $\boldsymbol{b}$ can be expressed with $(\boldsymbol{s}, \boldsymbol{a})$ on applying part 2 of Table 1, if no structural conditions are imposed; and, furthermore, for 3-uniform hypergraphs each pair in $(\boldsymbol{s}, \boldsymbol{b}) \times (\boldsymbol{t}, \boldsymbol{a})$ would work equally nicely. Here we prove that this latter equivalence is not valid in general.

To formulate observations providing a more detailed information, let us denote by $\mathcal{P}_{X,Y}$ and $\mathcal{P}_X$ the sets of chromatic polynomials belonging to the classes of hypergraphs of type (X, Y) and of type X, respectively, for any $X, Y \in \{S, T, A, B\}$. Similarly, the set of chromatic polynomials appearing in the case of mixed hypergraphs will be denoted by $\mathcal{P}_m$.

Theorem 3. *For the sets of chromatic polynomials belonging to* (S, A)-, (A, B)-, (S, T)-, (T, B)-, *and mixed hypergraphs, the relations* $\mathcal{P}_{S,A} = \mathcal{P}_{A,B} \supsetneqq \mathcal{P}_{S,T} = \mathcal{P}_{T,B} = \mathcal{P}_m$ *hold.*

Proof.

1. According to Corollary 3, each mixed hypergraph has a chromatic equivalent in each of those four stably bounded subclasses. Thus, the set $\mathcal{P}_m$ is contained in each of $\mathcal{P}_{S,A}$, $\mathcal{P}_{A,B}$, $\mathcal{P}_{S,T}$, and $\mathcal{P}_{T,B}$.

2. On the other hand, as it has been shown, the bound $b_i < |E_i|$ can be replaced by some (b_i+1)-element $\mathcal{D}$-edges, whilst the elimination of the bound $t_i < |E_i|$ can be done by inserting some $(t_i + 1)$-element $\mathcal{C}$-edges. Therefore, every (T, B)-hypergraph has a chromatically equivalent mixed hypergraph (on the

same vertex set). Taking into consideration the observation 1, the equality of $\mathcal{P}_{T,B}$ and $\mathcal{P}_m$ is obtained.

3. It was proved in [**3**] that $\mathcal{P}_{S,T} = \mathcal{P}_m$. It worth noting that there exists an (S,T)-hypergraph with no chromatic equivalent mixed hypergraph on the same number of vertices. For instance, due to [**3**], there exists an (S,T)-hypergraph with a gap of size 2 on seven vertices, whilst in the case of mixed hypergraphs it needs at least eight vertices, by a result of [**8**].

4. By the elimination of $\boldsymbol{t}$, the (S,T)-hypergraphs can be modeled in (S,A), hence $\mathcal{P}_{S,A} \supseteq \mathcal{P}_{S,T}$. We are going to show that the sets of chromatic spectra, and consequently also the chromatic polynomials, of (S,A)- and (S,T)-hypergraphs are not equal.

   Let $\mathcal{H}_{s,a}$ have four vertices and just one 4-element edge with bounds $a = 3$ and $s = 1$. Obviously, $r_1 = 1$ and $r_2 = 4$. On the other hand, it was proved in [**3**] that in 1-colorable (S,T)-hypergraphs the value of r_2 always is of the form $2^{n-1} - 1$. Since this property is not valid for $\mathcal{H}_{s,a}$, it cannot have a chromatically equivalent (S,T)-hypergraph.

5. Since any chromatic spectrum with $r_1 = 0$ belongs to some mixed hypergraphs, the same holds for (S,A)- and (A,B)-hypergraphs, too. Thus, a difference between $\mathcal{P}_{S,A}$ and $\mathcal{P}_{A,B}$ might occur only on hypergraphs with $r_1 = 1$. The assumption of 1-colorability in an (S,A)-hypergraph implies that every edge E_i has bounds $(s_i, a_i) = (1, a_i)$, whereas in an (A,B)-hypergraph it implies $(a_i, b_i) = (a_i, |E_i|)$ for every edge. These two color-bound conditions clearly are equivalent on each edge. Hence, the possible chromatic spectra and consequently the chromatic polynomials are the same: $\mathcal{P}_{S,A} = \mathcal{P}_{A,B}$.

   Nevertheless, there exist some (S,A)-hypergraphs not having chromatic equivalent (A,B)-hypergraphs on the same number of vertices. Similarly to the example in step 4 of the proof, one can see that there exists an (S,A)-hypergraph on seven vertices with feasible set $\{3,6\}$, but to generate this feasible set in (A,B)-hypergraphs needs at least eight (in fact, at least nine) vertices. $\square$

As regards modeling with the same number of vertices, the previous proof yields the following observation.

Remark 3. Every mixed hypergraph has a chromatically equivalent (T,B)-hypergraph such that their vertex sets are of the same cardinality, and vice versa. This stronger condition does not hold for any other pairs of the models listed above.

We close this subsection with supplements of Theorem 3 regarding other types of stably bounded hypergraphs.

Proposition 6. *Concerning the possible chromatic polynomials of S-, T-, A-, B-, $\mathcal{C}$- ('mixed', without $\mathcal{D}$-edges) and $\mathcal{D}$- (classical) hypergraphs the following relations hold*:

1. $\mathcal{P}_m \supsetneqq \mathcal{P}_S = \mathcal{P}_{S,B} \supsetneqq \mathcal{P}_{\mathcal{D}} = \mathcal{P}_B$,
2. $\mathcal{P}_{S,A} \supsetneqq \mathcal{P}_A = \mathcal{P}_{A,T} \supsetneqq \mathcal{P}_{\mathcal{C}} = \mathcal{P}_T$,
3. *$\mathcal{P}_m$ and $\mathcal{P}_A$ are incomparable.*

Proof.

1. Every $\mathcal{D}$-edge E_i can be interpreted equivalently with bound $b_i = |E_i| - 1$, whilst a bound $b_i < |E_i|$ can be replaced by some $(b_i + 1)$-element $\mathcal{D}$-edges, therefore $\mathcal{P}_{\mathcal{D}} = \mathcal{P}_B$ holds.

   Every $\mathcal{D}$-edge evidently means an edge with bound $s = 2$, hence $\mathcal{P}_{\mathcal{D}} \subseteq \mathcal{P}_S$ is clear. On the other hand, let us consider the S-hypergraph $\mathcal{H} = (X, \{X\}, \boldsymbol{s})$ with $|X| = 5$ vertices and with color-bound $\boldsymbol{s}(X) = 3$. Its chromatic spectrum is $(0, 0, 25, 10, 1)$. Assuming a $\mathcal{D}$-hypergraph with this spectrum, it should have five vertices and each of its 3-partitions should yield a proper coloring. In particular, for any three vertices there should exist a coloring where they get the same color, implying that there can occur $\mathcal{D}$-edges only of sizes 4 and 5. Consequently, the 2-partitions with color classes of size 2 and 3 are not forbidden, what contradicts $r_2 = 0$. Therefore, this S-hypergraph has no equivalent $\mathcal{D}$-hypergraph, implying $\mathcal{P}_S \supsetneqq \mathcal{P}_{\mathcal{D}}$.

   By the elimination of $\boldsymbol{b}$, we can transform the structures of type (S, B) to type S, hence $\mathcal{P}_S = \mathcal{P}_{S,B}$. It is also clear that $\mathcal{P}_S \subseteq \mathcal{P}_{S,T} = \mathcal{P}_m$, and that mixed hypergraphs having gaps in their chromatic spectra cannot be modeled in S-hypergraphs. That is, $\mathcal{P}_S \subsetneqq \mathcal{P}_m$ is obtained.

2. Any $\mathcal{C}$-edge E_i can be considered as an edge with bound $t_i = |E_i| - 1$, whilst any bound $t_i < |E_i|$ can be expressed by $\mathcal{C}$-edges, hence $\mathcal{P}_{\mathcal{C}} = \mathcal{P}_T$.

   By eliminating $\boldsymbol{t}$, every (A, T)-hypergraph can be rewritten only with the bound $\boldsymbol{a}$, thus $\mathcal{P}_A = \mathcal{P}_{A,T}$. Moreover $\mathcal{P}_{S,A} \supseteq \mathcal{P}_A$ trivially holds.

   To show that there exist A-hypergraphs having no chromatically equivalent $\mathcal{C}$-hypergraphs, we recall the example from step 4 in the proof of Theorem 3. This (S, A)-hypergraph can be considered as just an A-hypergraph, and since it has no equivalent of type (S, T), the same is true for mixed- and $\mathcal{C}$-hypergraphs, too. Consequently, $\mathcal{P}_A \supsetneqq \mathcal{P}_{\mathcal{C}}$, and because of the 1-colorability of every A-hypergraph, $\mathcal{P}_{S,A} \neq \mathcal{P}_A$ is valid as well.

3. By the previous example there exist A-hypergraphs that have no equivalent mixed hypergraphs whereas mixed hypergraphs admitting no 1-coloring cannot be equivalent to any A-hypergraphs. □

Proposition 7. *Concerning the possible chromatic polynomials of stably bounded hypergraphs involving at least three types of conditions, the following equations hold:*

1. $\mathcal{P}_{S,T,A,B} = \mathcal{P}_{S,A,B} = \mathcal{P}_{S,A,T} = \mathcal{P}_{S,A}$,
2. $\mathcal{P}_{A,B,T} = \mathcal{P}_{A,B} = \mathcal{P}_{S,A}$,

3. $\mathcal{P}_{S,T,B} = \mathcal{P}_{S,T}$.

Proof. The reductions described in part 2 of Table 1 yield:

- The color-bound function $\boldsymbol{t}$ can be expressed by the function $\boldsymbol{a}$. Consequently, if a type contains T and A together, then omitting T the set of possible chromatic polynomials does not change.

- Similarly, the function $\boldsymbol{b}$ can be reduced to $\boldsymbol{s}$, therefore in the presence of S the cancelation of B cannot make a change in the set of possible chromatic polynomials.

These observations immediately imply the statements listed above, except for the last equation in part 2, what has been proved in Theorem 3. □

3. COMPLEXITY OF TESTING COLORABILITY

In this section we investigate the time complexity of the following two algorithmic problems.

COLORABILITY

Instance: A hypergraph $\mathcal{H}$ of a given type.

Question: Is $\mathcal{H}$ colorable?

UNIQUE k-COLORABILITY

Instance: A hypergraph $\mathcal{H}$ of a given type, together with a proper k-coloring φ.

Question: Does $\mathcal{H}$ admit any proper coloring other than φ?

For the former, we simply extend the NP-hardness result of [**4**] from (S,T)-hypergraphs to all nontrivial combinations of the color-bound functions. On the other hand, the situation with the latter problem is more interesting. We choose the value $k = n - 1$ and prove that two of the non-trivial pairs, namely those containing S, lead to intractability; but the other two, containing B, admit a good characterization and polynomial-time algorithms.

In general, it should be noted that COLORABILITY clearly belongs to NP, whereas UNIQUE k-COLORABILITY is in co-NP. Moreover, since a hypergraph on n vertices cannot have more than $\binom{n}{2}$ proper $(n-1)$-colorings, we can see that for $k = n - 1$ (and also if k is as large as n minus a constant) it does not change the complexity status of the problem if a k-coloring is not given in the input.

3.1. COLORABILITY OF 3-UNIFORM HYPERGRAPHS

It was first observed in [**17**] that the recognition problem of colorable mixed hypergraphs is NP-complete in general, and also when restricted to 3-uniform mixed hypergraphs. There are some important classes with a nice structure, however, that admit efficient algorithms.

A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is called a *hypertree* if there exists a tree graph T on the same vertex set X as $\mathcal{H}$, such that each edge $E_i \in \mathcal{E}$ induces a subtree in T. In [**15**] a simple necessary and sufficient condition was given for the colorability of *mixed hypertrees*, from which an efficient algorithm is obtained, too.

On the other hand, we have shown in [**4**] that the colorability of 3-uniform (S, T)-hypertrees is NP-complete. We have also seen in Corollary 2 that every 3-uniform stably bounded hypergraph has equivalent representations with all the types of (S, T)-, (S, A)-, (T, B)-, and (A, B)-hypergraphs, and those can be constructed in linear time. In this way, an input of any of these types can efficiently be transformed to an (S, T)-hypergraph. Consequently, the result of [**4**] can be extended as follows.

Theorem 4. *The* Colorability *problem is* NP*-complete on each of the following classes of hypergraphs*:

- 3*-uniform* (S, T)*-hypertrees*,
- 3*-uniform* (S, A)*-hypertrees*,
- 3*-uniform* (T, B)*-hypertrees*,
- 3*-uniform* (A, B)*-hypertrees.*

It is worth comparing Theorem 4 with the following results: there are linear-time algorithms for deciding whether a *mixed* hypertree is colorable, and also for finding a proper coloring if there exists one [**15**], whereas determining the upper chromatic number of a mixed hypertree without edges larger than three is NP-complete [**10**].

Let us note further that NP-completeness remains valid if we assume that the host tree is a star. On the other hand, it will be proved in the forthcoming paper [**5**] that 3-uniform stably bounded *interval* hypergraphs admit a linear-time colorability test and a linear-time coloring algorithm, too.

3.2. UNIQUELY $(n-1)$-COLORABLE (S, T)- AND (S, A)-HYPERGRAPHS

Although it is hard to test whether an unrestricted mixed hypergraph is uniquely colorable [**17**], this is not the case if $\overline{\chi}$ is very large. For the latter case, Niculitsa and Voss [**12**] described a characterization of uniquely $(n-1)$-colorable, and also of uniquely $(n-2)$-colorable mixed hypergraphs.

In sharp contrast to this, we have proved in [**3**] that the recognition of uniquely $(n-1)$-colorable (S,T)-hypergraphs is hard. Here we show how the construction can be extended to (S,A)-hypergraphs.

Theorem 5. *The* UNIQUE $(n-1)$-COLORABILITY *problem is co-*NP*-complete on* (S,A)*-hypergraphs.*

Proof. As we have already mentioned, membership in co-NP is clear. To prove hardness, let us recall from [**3**] the reduction for (S,T)-hypergraphs, from the problem of determining the *chromatic number of Steiner triple systems*.

PHELPS and RÖDL proved in [**13**] that it is NP-complete to decide whether a STEINER triple system – viewed as a 3-uniform $\mathcal{D}$- (classical) hypergraph – is colorable with 14 colors. Given an input STEINER triple system $\mathcal{S} = STS(n-2) = (X, \mathcal{B})$ of order $n-2$ with vertex set $X = \{x_1, \dots, x_{n-2}\}$ and edge set $\mathcal{B}$, an (S,T)-hypergraph $\mathcal{H} = (X', \mathcal{E}, \boldsymbol{s}, \boldsymbol{t})$ is constructed as follows. We set $X' = X \cup \{z_1, z_2\}$, where z_1, z_2 are two new vertices, and consider the following edges with respective color-bounds:

- $B' = B \cup \{z_1, z_2\}$ with $\boldsymbol{s}(B') = 4$ and $\boldsymbol{t}(B') = 5$, for all blocks $B \in \mathcal{B}$;
- $W' = W \cup \{z_1, z_2\}$ with $\boldsymbol{s}(W') = 1$ and $\boldsymbol{t}(W') = 16$, for all 15-element subsets of X;
- $e_{i,j} = \{x_i, z_j\}$ with $\boldsymbol{s}(e_{i,j}) = \boldsymbol{t}(e_{i,j}) = 2$, for all $1 \le i \le n-2$ and $j = 1, 2$.

In this (S,T)-hypergraph, every t_i is either $|E_i|$ or $|E_i| - 1$. Hence, it is easy to eliminate $\boldsymbol{t}$ along the lines of Proposition 2 and obtain an equivalent (S,A)-hypergraph: we simply define

$$\boldsymbol{a}(B') = 1, \qquad \boldsymbol{a}(W') = 2, \qquad \boldsymbol{a}(e_{i,j}) = 1$$

for all edges $B', W', e_{i,j} \in \mathcal{E}$. From the argument in [**3**] it follows that $\mathcal{H}$ is *not* uniquely $(n-1)$-colorable if and only if $\mathcal{S}$ has a proper coloring with at most 14 colors; and certainly the same holds for the derived (S,A)-hypergraph, too. Thus, co-NP-hardness follows. □

3.3. UNIQUELY $(n-1)$-COLORABLE (T,B)- AND (A,B)-HYPERGRAPHS

In this subsection we characterize the uniquely $(n-1)$-colorable (T,B)- and (A,B)-hypergraphs. In the models (S,A) and (S,T) studied in the previous subsection, the decision problem of unique $(n-1)$-colorability was proved to be co-NP-complete. In contrast to this, the characterization presented below yields polynomial-time algorithms for (T,B)- and (A,B)-hypergraphs.

Before the characterization, let some terminology be introduced:

- $\{x,y\}$ is called $\mathcal{B}_1$-edge if x and y are contained in a common hyperedge E_i having bound $b_i = 1$. (This corresponds to a graph-edge in the usual sense.)

- E_i is called $\mathcal{B}_2$-edge if $b_i = 2$.
- E_i is called $\mathcal{C}$-edge if $t_i = |E_i| - 1$.

Concerning a given (T, B)-hypergraph, the set of $\mathcal{C}$-, $\mathcal{B}_1$- and $\mathcal{B}_2$-edges will be denoted by $\mathcal{C}$, $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. As a side-product of the characterization theorem, it will turn out that if an edge has bound $b_i \geq 3$, then the exact value of b_i has no influence on unique $(n-1)$-colorability.

Theorem 6. *A (T, B)-hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{t}, \boldsymbol{b})$ on $|X| = n$ vertices is uniquely $(n-1)$-colorable if and only if the following conditions hold*:

(α) $\max\limits_{E_i \in \mathcal{E}} (|E_i| - t_i) = 1$.

(β) *The set $C^* := \bigcap\limits_{C \in \mathcal{C}} C$ contains at least two vertices and induces a complete $\mathcal{B}_1$-graph minus one $\mathcal{B}_1$-edge.*

Moreover, denoting by y_1 and y_2 the vertices from the missing $\mathcal{B}_1$-edge,

(γ) *$X \setminus \{y_1, y_2\}$ is a complete $\mathcal{B}_1$-graph.*

(δ) *For each vertex $x \in X \setminus C^*$, at least one of the relations $\{x, y_1\} \in \mathcal{B}_1$, $\{x, y_2\} \in \mathcal{B}_1$ and $\{x, y_1, y_2\} \subseteq E_i \in \mathcal{B}_2$ holds.*

(ϵ) *For each pair of vertices $x_j, x_k \in X \setminus C^*$, if $\{x_j, x_k\}$ intersects every $\mathcal{C}$-edge, then either there exist $\mathcal{B}_1$-edges $\{z, x_j\}$ and $\{z, x_k\}$ for a $z \in \{y_1, y_2\}$, or there exist $\mathcal{B}_1$-edges $\{z, y_1\}$ and $\{z, y_2\}$ for a $z \in \{x_j, x_k\}$.*

Proof. Consider a uniquely $(n-1)$-colorable (T, B)-hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{t}, \boldsymbol{b})$. Since it admits an $(n-1)$-coloring, where each edge has at least $|E_i| - 1$ colors, $t_i \geq |E_i| - 1$ holds. On the other hand, since the n-coloring is not feasible, there is some hyperedge with bound $t_i = |E_i| - 1$. Consequently, we have $\max\limits_{E_i \in \mathcal{E}} (|E_i| - t_i) = 1$, according to ($\alpha$).

Let φ be a proper $(n-1)$-coloring, and assume without loss of generality that its color classes are $\{x_1\}, \{x_2\}, \ldots, \{x_{n-2}\}$, and $\{y_1, y_2\}$. Every $\mathcal{C}$-edge has to involve vertices with a common color by φ, moreover the color class $\{y_1, y_2\}$ cannot be a $\mathcal{B}_1$-edge, therefore:

(β_1) $\{y_1, y_2\} \subseteq C^*$ and $\{y_1, y_2\} \notin \mathcal{B}_1$.

Taking the union of any two color classes from φ, the obtained $(n-2)$-coloring is not feasible, what can be caused only by breaking some bound b_i. We are going to analyze the various vertex partitions with $n-2$ classes.

- The contraction of any two singletons $\{x_j\}$ and $\{x_k\}$ is forbidden, hence there exists an edge $E_i \supseteq \{x_j, x_k\}$ with bound $b_i = 1$. That is, $\{x_j, x_k\} \in \mathcal{B}_1$ for every $1 \leq j < k \leq n-2$, so that (γ) holds.

- The contraction of any singleton $\{x_j\}$ and $\{y_1, y_2\}$ is also forbidden by some bound b_i, consequently at least one of the alternatives from (δ) holds.
- Any $(n-2)$-partition containing the two non-singleton color classes $\{x_j, y_1\}$ and $\{x_k, y_2\}$ is non-feasible, hence either a $\mathcal{C}$-edge omits both x_j and x_k (since it includes both y_1 and y_2), or there occur $\mathcal{B}_1$ edges in both sets $\{\{x_j, y_1\}, \{x_k, y_2\}\}$ and $\{\{x_j, y_2\}, \{x_k, y_1\}\}$. This means, the implication of (ϵ) is valid.

By assumption, φ is the unique coloring of $\mathcal{H}$; thus, the coloring with singletons and the only two-element color-class $\{x_j, y_k\}$ is non-feasible for all $1 \le j \le n-2$ and $1 \le k \le 2$. If x_j belongs to each $\mathcal{C}$-edge, the bounds t_i are fulfilled, hence in this case there surely occurs $\{x_j, y_k\}$ as a $\mathcal{B}_1$-edge:

(β_2) If $x_j \in C^*$, then $\{x_j, y_1\} \in \mathcal{B}_1$ and $\{x_j, y_2\} \in \mathcal{B}_1$ hold.

The properties (β_1), (β_2) and (γ) together ensure the existence of a complete $\mathcal{B}_1$-graph minus one $\mathcal{B}_1$-edge on the intersection of $\mathcal{C}$-edges, implying that (β) is fulfilled, too.

Now, assume a hypergraph $\mathcal{H}$ satisfying the conditions $(\alpha) - (\epsilon)$ of the theorem. Unique $(n-1)$-colorability is verified as follows:

- By the requirement (α), the hypergraph admits no n-coloring.
- Consider the $(n-1)$-coloring φ, where the only monochromatic vertex pair is $\{y_1, y_2\}$. According to (β), both y_1 and y_2 are contained in each $\mathcal{C}$-edge, and hence, due to (α) all the bounds from $\boldsymbol{t}$ are satisfied. Since $\{y_1, y_2\} \notin \mathcal{B}_1$, every hyperedge E_i containing both y_1 and y_2, has bound $b_i \ge 2$, whilst each of the remaining hyperedges involves no monochromatic vertex pair. Therefore, all bounds from $\boldsymbol{b}$ are fulfilled, the color partition $\{x_1\}, \{x_2\}, \dots, \{x_{n-2}\}, \{y_1, y_2\}$ is feasible.
- According to (γ):

  ($\star$) There is no feasible partition with a color class containing both x_j and x_k (for all $1 \le j < k \le n-2$).

  Thus, the only possibility for a second $(n-1)$-coloring would be a partition with 2-element color class $\{x_j, y_k\}$ (for some $1 \le j \le n-2$ and $1 \le k \le 2$). But if $x_j \in C^*$, there is contained a forbidden $\mathcal{B}_1$-edge due to (β), whilst if $x_j \notin C^*$, then some forbidden polychromatic $\mathcal{C}$-edge would arise. Consequently, no $(n-1)$-coloring different from φ can be feasible.
- To prove that no $(n-2)$-colorings exist:

  ($\star\star$) There is no feasible partition containing $\{x_j, y_1, y_2\}$ as a color class. If $x_j \in C^*$, the class contains two forbidden $\mathcal{B}_1$-edges, due to (β). And if $x_j \notin C^*$, the property (δ) ensures that there occurs either a forbidden $\mathcal{B}_1$-edge in the 3-element color class, or this class is involved in a $\mathcal{B}_2$-edge. All these cases are impossible.

($\star\star\star$) The pairs $\{x_j, y_1\}$ and $\{x_k, y_2\}$ cannot be color classes simultaneously. Such a coloring is trivially non-feasible if there exists a $\mathcal{C}$-edge containing neither x_j nor x_k. Also, if at least one of the vertices x_j and x_k is contained in C^*, the partition is forbidden by a $\mathcal{B}_1$-edge according to (β). In the third case, when all $\mathcal{C}$-edges meet $\{x_j, x_k\}$ but their intersection doesn't, the conditions of (ϵ) are satisfied, hence its conclusion excludes the feasibility of this partition.

The claims ($\star$), ($\star\star$) and ($\star\star\star$) together imply that the hypergraph $\mathcal{H}$ admits no $(n-2)$-coloring.

- Because of ($\star$), the vertices $x_1, x_2, \ldots, x_{n-2}$ have mutually distinct colors in every feasible coloring, therefore $\mathcal{H}$ admits no coloring with fewer than $n-2$ colors.

Thereupon, the hypergraph is uniquely $(n-1)$-colorable, and this completes the proof. □

There is no restriction for the exact value of bounds $b_i \geq 3$ in the characterization, therefore we immediately get the following corollary:

Corollary 5. *Let $\mathcal{H}$ and $\mathcal{H}'$ be (T, B)-hypergraphs on n vertices, and suppose that $\mathcal{H}'$ can be obtained from $\mathcal{H}$ by replacing each bound $b_i \geq 3$ with some bound $3 \leq b_i' \leq |E_i|$. Then $\mathcal{H}$ is uniquely $(n-1)$-colorable if and only if so is $\mathcal{H}'$. In particular, concerning unique $(n-1)$-colorability, $\mathcal{H}$ can be reduced to a T-hypergraph supplemented with some $\mathcal{B}_1$- edges and 3-element $\mathcal{B}_2$-edges, that is, with a classical ($\mathcal{D}$-) hypergraph of rank at most three.*

Except for the first property (α), the above characterization gives conditions only for the color-bound function $\boldsymbol{b}$ and for the edges having bound $t_i = |E_i| - 1$. Since the restriction $\max_{E_i \in \mathcal{E}} (|E_i| - t_i) = 1$ can be equivalently expressed with the bound $\boldsymbol{a}$, we get an analogous characterization for uniquely $(n-1)$-colorable (A, B)-hypergraphs, too. The terms $\mathcal{B}_1$ and $\mathcal{B}_2$ are used as above; $\mathcal{C}$-edge means a hyperedge E_i with bound $a_i = 2$.

Theorem 7. *An (A, B)-hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{a}, \boldsymbol{b})$ with $|X| = n$ vertices is unique $(n-1)$-colorable if and only if the following conditions hold:*

(α') $\max\limits_{E_i \in \mathcal{E}} a_i = 2.$

(β) *The set $C^* := \bigcap\limits_{C \in \mathcal{C}} C$ contains at least two vertices and induces a complete $\mathcal{B}_1$-graph minus one $\mathcal{B}_1$-edge.*

Moreover, denoting by y_1 and y_2 the vertices of the omitted $\mathcal{B}_1$-edge,

(γ) *$X \setminus \{y_1, y_2\}$ is a complete $\mathcal{B}_1$-graph.*

(δ) *For each vertex $x \in X \setminus C^*$, at least one of the relations $\{x, y_1\} \in \mathcal{B}_1$, $\{x, y_2\} \in \mathcal{B}_1$ and $\{x, y_1, y_2\} \subseteq E_i \in \mathcal{B}_2$ holds.*

(ϵ) *For each pair of vertices $x_j, x_k \in X \setminus C^*$, if $\{x_j, x_k\}$ intersects every $\mathcal{C}$-edge, then either there exist $\mathcal{B}_1$-edges $\{z, x_j\}$ and $\{z, x_k\}$ for some $z \in \{y_1, y_2\}$, or there exist $\mathcal{B}_1$-edges $\{z, y_1\}$ and $\{z, y_2\}$ for some $z \in \{x_j, x_k\}$.*

Proof. If the condition (α') is valid for a given (A, B)-hypergraph, we can replace each bound $a_i = 2$ by $t_i = |E_i| - 1$, whilst the non-restricting $a_i = 1$ can be rewritten as $t_i = |E_i|$, and we get a chromatically equivalent (T, B)-hypergraph on the same vertex set. The obtained (T, B)-hypergraph is uniquely $(n-1)$-colorable if and only if so is the original (A, B)-hypergraph. Also, the $\mathcal{B}_1$-, $\mathcal{B}_2$- and $\mathcal{C}$-edges are the same, hence in this case the conditions (α')–(ϵ) give an exact characterization for the (A, B)-hypergraph.

On the other hand, if the condition (α') does not hold, then either $\overline{\chi} < n - 1$ or $\overline{\chi} = n$ or the hypergraph is uncolorable, so it is not uniquely $(n-1)$-colorable in either case. Hence, (α') is indeed necessary for unique $(n-1)$-colorability. □

As it was our purpose, the characterization theorems make it possible to design polynomial-time algorithms for testing unique $(n-1)$-colorability in the two hypergraph classes in question. As a matter of fact, on the one hand it is obvious that the condition $a_i \leq 2$ is necessary for unique $(n-1)$-colorability in *every* stably bounded hypergraph; while, on the other hand, the proof of Theorem 7 shows that if this condition holds, then the color-bound function $\boldsymbol{a}$ can completely be replaced with a suitably chosen $\boldsymbol{t}$. Thus, the following more general result is obtained.

Theorem 8. *The decision problem* UNIQUE $(n-1)$-COLORABILITY *can be solved in polynomial time for (T, A, B)-hypergraphs.*

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