APPLICABLE ANALYSIS AND DISCRETE MATHEMATICS, **1** (2007), 1–2.
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FROM *PEF* TO *AADM*, VIA *MAGT*

Milan Merkle

This journal, *Applicable Analysis and Discrete Mathematics (AADM)*, continues the former *Publikacije Elektrotehničkog Fakulteta - serija Matematika (PEF)*, or with the English title, *Publications of the Faculty of Electrical Engineering - series Mathematics*. Founded in the year 1956 as *Serija Matematika i Fizika*, the journal will continue its life under the new name, with two issues per year, appearing in April and October. This first issue of *AADM* is published simultaneously with the last issue of *PEF*.

The new name reflects the scope of the journal, which remains a general mathematical journal, concentrated around areas such as classical mathematical analysis (including convexity and inequalities with applications), functional analysis, differential and difference equations, special functions, combinatorics and graph theory, applications in probability and statistics, numerical analysis and computer science.

This first issue features twenty four papers presented at the conference *Topics in Mathematical Analysis and Graph Theory (MAGT)* that was held in Belgrade, September 1–4, 2006, as a satellite meeting to the International Congress of Mathematicians in Madrid. The conference was organized as a part of the celebration of the fiftieth anniversary of *PEF*, and the new name was given to the journal in the course of the conference.

The activities related to the conference were initiated in summer of the year 2005. We started with the project of preserving all past issues of *PEF* in a digital form and posting them on the Web site of the journal; another version was made on compact discs to be distributed to participants of the *MAGT*. Then, a sequence of tasks was in order to be done. Some of them were annoying, boring, not wanted, but, generally it was a challenge to try to find best solutions.

Mathematicians are typically lonely workers, and this enterprise was calling for a team work. For me, it was an extraordinary experience to coordinate several teams with many interconnected tasks. None of us had any previous involvement in organizing any small or big conference, and our ideas were coming from recollections of what we liked or disliked in meetings that we had attended in the past. In the course of preparing the conference, we learned a variety of new things, from non-mathematical world, bits about banking, tourism, diplomacy, politics, food, law, public relations, etc. There were many people involved in the organization of the *MAGT*. The complete list of names can be found at the Web site http://magt.etf.bg.ac.yu, together with other details and photos from the conference.

The selection of topics for the *MAGT* was made according to (past and present) scientific interests of members of the Department of Applied Mathematics at the Faculty of Electrical Engineering. The selection of invited speakers was made with an idea to meet old friends and to make new friendships. The talks were interesting enough to keep most of participants inside the conference rooms, for all four days of the scientific program. It's not that they did not have where else to go. It was a nice and sunny late summer, and in the conference bags they could find a plenty of exciting hints to spend time in outdoor cafés, to visit *places of interest*, or to stroll along the banks of two big rivers. But they opted to attend the talks, which was a clear sign that the scientific purpose of the *MAGT* was met.

Unfortunately, the conference was definitively the last meeting with our two dear friends and distinguished colleagues. Francis K. Bell passed away on December 19, 2006. Lev M. Berkovich passed away on March 14, 2007. Their contributions to Mathematics will continue to live in journals and books, and memories to them will remain in our hearts.

With this special issue, the work related to the *MAGT* is finally over. While expressing my gratitude to all who supported and participated in the organizing of the conference, I am anxiously looking forward to receiving new submissions of quality, for future issues of the *AADM*.

Applicable Analysis and Discrete Mathematics, **1** (2007), 3–4.
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A CHANGE OF ERAS: OBSERVATIONS OF A STRANGER

PGL Leach

The invitation, doubtlessly engineered by the distinguished Russian mathematician Lev Berkovich, to participate in the celebration of the fiftieth anniversary of the *Publications of the Faculty of Electrical Engineering, University of Belgrade, Series: Mathematics and Physics* could not be refused.

The conference was set to take place in Belgrade. My knowledge of the area was limited. I knew a bit about the history of the Serbs clawing their homeland back from the Turks, the incident in Sarajevo, the ascendency of Tito and the more recent process of the disintegration of Yugoslavia. For me as a youth in post-WWII Australia there were pictures of the partisans operating in rather rugged country and people who left the country seeking another life.

I have spent most of my life in the Southern Hemisphere (Australia and South Africa), but have strong connections with some parts of Europe. Serbia was not one of those parts. I accepted the invitation for reasons combining my relationship with some of the attendees and an interest in that part of the world. I 'persuaded' one of my Greek collaborators to share the adventure with me and the beginning of September found us in Belgrade.

Belgrade and its surrounds came as a bit of a surprise. The rugged country of the partisans was the flood plain of the rivers Danube and Sava. The city was generally attractive, apart from some bombed buildings which left me quite ill – I have experienced some minor bombings by terrorists, but the bombing in Belgrade was on an international scale – and angry at the perpetrators of such assaults on everyman.

The conference was to celebrate the fiftieth anniversary of the establishment of the *Publications of the Faculty of Electrical Engineering, University of Belgrade, Series: Mathematics and Physics* by D. S. Mitrinović, a distinguished Serbian mathematician. In addition to a solid Serbian contingent there were participants from twenty-three other countries covering every continent apart from Antarctica. It was evident that some of the delegates were associates of Serbian mathematics from a long time past even though from distant countries.

The themes of the meeting were unusual in that they covered differential equations, analysis and discrete mathematics with an emphasis on graph theory. Nevertheless the participants managed to coexist peacefully! One obvious feature of the lectures was their high mathematical content. Generally the papers were presented in the current international language. Some were in Serbian and, being one of the World's more accomplished nonlinguists, I have a great sympathy for those speakers. As it happened, the accompanying visual material made the context intelligible. That is one of the consolations of Mathematics.

In many respects the meeting was a learning experience for me. Talk after talk referred to the work of Serbian mathematicians and scientists in general. Of this I was ignorant and so doubly thankful that I had been invited to join the celebration of the anniversary.

Historically journals have been published by learned societies and universities. The *Publications of the Faculty of Electrical Engineering, University of Belgrade, Series: Mathematics and Physics* is one of many which reflect this tradition to be found not only in Europe but in other places, India for example. For reasons which are not immediately obvious the proceedings/reports of a faculty no longer seem to attract the respect they formerly had. In the case of the *Publications of the Faculty of Electrical Engineering, University of Belgrade, Series: Mathematics and Physics* the present meeting was asked to suggest a new title for the journal. A 'secret ballot' was held and the new title *Applicable Analysis and Discrete Mathematics* adopted. I have to confess that this was not my idea, but that of Themistocles Rassias. I could only congratulate him on a better name than that of which I thought.

This conference was unusual in that we had the question of the name of the journal to be decided. In other respects it was very much mainstream in that we enjoyed a civic reception by the Deputy Mayor of the City at one of the palaces in the centre of the city, a guided tour of the old town, a concert and a fashion show in a gallery of frescoes from ancient churches and an excursion to mountain Fruška Gora, town Sremski Karlovci and the monastery Krušedol, which looked completely unlike my idea of Serbia. Milan Merkle contrived an excellent combination of scientific and cultural activities. As a personal aside my Greek colleague and I visited Topola to see the mausoleum of the former Royal Family. We are both into mosaics and were suitably impressed.

The beauty of the Serbian people and their constant smile on their faces, even though history has treated them badly, made a strong impact. I hope that Milan Merkle and the rest of the organizers do think towards a sequel to this productive meeting in order for all of us who took part in this one and the many others who were not fortunate enough to be present have the opportunity to return to Serbia once again.

Slike 1 strana

Slike 2 strana

Slike 3 strana

Slike 4 strana

Applicable Analysis and Discrete Mathematics, **1** (2007), 9–17.
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TOPICS IN MATHEMATICAL ANALYSIS AND GRAPH THEORY (MAGT 2006): REALIZED PROGRAM

Editorial note. In four days of scientific program of MAGT 2006, the total of 98 participants from 25 countries presented 20 invited and 67 contributed papers. Scientific activities were running in the building of Faculty of Mechanical Engineering, University of Belgrade, while the welcome cocktail, on-site registration and the opening were held in the building of the Faculty of Electrical Engineering. Abstract of presented talks can be found in [**1**]. The abstract book [**2**] was printed prior to the conference, and contains abstracts of all submitted and accepted talks. Twenty four peer-refereed full papers are published in this issue (pages 18–323).

Thursday, August 31

18:00 – 20:00 *Welcome cocktail and on-site registration*

Friday, September 1

8:00 – 9:30 On-site registration (continuation)

9:30 – 10:30 Opening and transfer to Mechanical Engineering

10:30 – 11:00 Coffee break

Room A: Mathematical Analysis (chair: Hari M. Srivastava)

11:00 – 11:45 Themistocles M. Rassias, Dep. of Math., Nation. Tech. Univ. of Athens, Greece: *On the Mazur-Ulam theorem and the Aleksandrov problem for isometric mappings*

11:45 – 12:30 A. M. Fink, Mathematics Department, Iowa State University, USA: *What really is Hadamard's inequality*

12:30 – 12:45 Coffee break

12:45 – 13:30 Gradimir V. Milovanović, Faculty of Electronic Engineering, University of Niš, Serbia: *Formal orthogonal polynomials with respect to a moment functional and applications*

Room B: Graph Theory (chair: Domingos Moreira Cardoso)

11:00 – 11:45 Peter Rowlinson, Department of Computing Science and Mathematics, University of Stirling, Scotland: *Star complements and extremal graphs*

11:45 – 12:30 Zsolt Tuza, Computer and Automation Research Institute, Hungarian Academy of Sciences, Hungary: *Colorings of hypergraphs with local conditions*

13:30 – 14:30 Lunch

Room A: Topics in Analysis (chair: Kostadin Trenčevski)

14:30 – 14:50 G. S. Srivastava: *Spaces of entire functions of two complex variables*

14:50 – 15:10 Kostadin Trenčevski: *On the conjecture concerning the complex manifolds with odd complex dimension*

15:10 – 15:30 Monica Moulin Ribeiro Merkle: *A free boundary problem between two parallel planes*

15:30 – 15:50 Coffee break

15:50 – 16:10 Slobodanka Janković: *The property of good decomposition for slowly varying functions*

Room B: Tournaments (chair: Zsolt Tuza)

14:50 – 15:10 Masaya Takahashi, Takahiro Watanabe, Takeshi Yoshimura: *A consideration of the score sequence pair problems of* (r_{11}, r_{12}, r_{22})*-tournaments*

15:10 – 15:30 Jea-Hyun Park, Soon-Yeong Chung: *The chip-firing game and the Dirichlet game on weighted graphs*

15:30 – 15:50 Hiroaki Mohri: *Fixed charged network flow problem and its cooperative game*

15:50 – 16:10 Ljiljana Branković, Yuqing Lin; *Graceful labeling of trees with maximum degree* 3

Room C: Topics in Analysis (chair: Zagorka Lozanov-Crvenković)

14:30 – 14:50 Dennis Nemzer: *Poisson's summation formula for boehmians*

16:40 – 17:00 Dušanka Perišić, Zagorka Lozanov-Crvenković: *Hermite expansion of ultradistibution*

16:10 – 16:40 Coffee break

Room A: Topics in Analysis (chair: Pietro Cerone)

16:40 – 17:00 Milan Jovanović, Djura Paunić: *Convex functions - some historical notes*

17:00 – 17:20 Dragan Doder: *On the set of inequalities*

17:20 – 17:40 Branko Malešević, Ratko Obradović: *A method of proving a class of inequalities via Pade approximations*

17:40 – 18:00 Ćemal Doličanin, Milisav Stefanović, Sead Rešić: *On pseudoscalar product of unisotropic vectors*

Room B: Topics in Graph Theory (chair: Carlos M. da Fonseca)

16:40 – 17:00 Vadim E. Levit, Eugen Mandrescu: *Partial unimodality for independence polynomials of some compound graphs*

17:00 – 17:20 Mehdi Alaeiyan, Mohsen Ghasemi: *Normal Cayley digraphs with valency* 2 *on groups*

17:20 – 17:40 Joel Ratsaby: *Density of smooth boolean function*

Room C: Topics in Analysis (chair: Zagorka Lozanov-Crvenković)

16:40 – 17:00 Dušanka Perišić, Zagorka Lozanov-Crvenković: Hermite expansion of ultradistibution

19:00 *Reception in the City Hall*

Saturday, September 2

Room A: Mathematical Analysis (chair: Gradimir V. Milovanović)

9:00 – 9:45 Ingram Olkin, School of Education, Stanford University, USA: *Inequalities: some probabilistic, some matric, and some both*

9:45 – 10:30 Hari M. Srivastava, Department of Mathematics and Statistics, University of Victoria, Canada: *Some Fox-Wright generalized hypergeometric functions and associated families of convolution operators*

Room B: Graph Theory (chair: Slobodan K. Simić)

9:00 – 9:45 Domingos Moreira Cardoso, Universidade de Aveiro, Departamento de Matematica, Aveiro - Portugal: *The class of graphs with convex-qp stability number*

9:45 – 10:30 Tomaž Pisanski, IMFM, University of Ljubljana and University of Primorska, Slovenia: *Morse matchnigs of colored graphs*

10:30 – 11:00 Coffee break

Room A: Mathematical Analysis (chair: Themistocles M. Rassias)

11:00 – 11:45 Pietro Cerone, School of Computer Science and Mathematics, Faculty of Health, Engineering and Science, Victoria, Australia: *Special Functions: Their approximation and bounds*

11:45 – 12:30 Sever Dragomir, Faculty of Engeneering and Sciences, Victoria University of Technology, Australia: *New inequalities of the Kantorovich type for bounded linear operators in Hilbert spaces*

Room B: Graph Theory (chair: Slobodan K. Simić)

11:00 – 11:45 Dragoš Cvetković, Faculty of Electrical Engineering, University of Belgrade, Serbia: *Some properties of signless Laplacian eigenvalues of graphs*

12:30 – 14:30 Lunch

Room A: Fixed point theory (chair: Ljiljana Gajić)

14:30 – 14:50 Siniša Ješić, Milan Tasković, Nataša Babačev: *Transversal spaces and fixed point theorems*

14:50 – 15:10 Ljiljana Gajić: *On fixed point in D-metric spaces*

15:10 – 15:30 Ivan D. Arandjelović: *Note on asymptotic contractions*

15:30 – 15:50 Siniša Ješić, Rale Nikolić: *Common fixed point theorems for R-weakly commuting mappings defined on fuzzy metric spaces*

Room B: Spectra of Graphs (chair: Kristina Vušković)

14:30 – 14:50 Francesco Belardo, Enzo Maria Li Marzi, Slobodan K. Simić: *Ordering graphs with the index in the interval* $(2, \sqrt{2+\sqrt{5}}\,)$

14:50 – 15:10 Francis K. Bell, Dragoš Cvetković, Peter Rowlinson, Slobodan K. Simić: *Graphs for which the least eigenvalue in minimal*

15:10 – 15:30 Mirko Lepović: *Some results on conjugate integral graphs*

15:30 – 15:50 Carlos M. da Fonseca: *Location of eigenvalues of acyclic matrices*

15:50 – 16:10 Miroslav Petrović, Bojana Borovićanin: *On the graphs with maximal index*

Room C: Special Functions and Numerical Analysis (chair: Ljiljana Petković)

14:30 – 14:50 Nenad Cakić: *An unified explicit formula for Stirling numbers*

14:50 – 15:10 Tibor K. Pogany, Živorad Tomovski: *On Mathieu-type series which terms contain generalized hypergeometric function qFp and Meijer's G-function*

15:10 – 15:30 Purshottam Narain Agrawal, Asha Ram Gairola: *On Miccelli combination of modified Bernstein polynomials*

15:30 – 15:50 Aleksandra Erić: *Polynomial interpolation problem for skew polynomials*

15:50 – 16:10 Ljiljana D. Petković, Miodrag S. Petković; *On the Newton-like method for the inclusion of polynomial zeros*

16:10 – 16:40 Coffee break

Room A: Topics in Analysis (chair: Ljiljana Gajić)

17:00 – 17:20 Pratulananda Das, B. K. Lahiri: *I and I* - convergence in topological spaces*

Room B: Topics in Graph Theory (chair: Tomaž Pisanski)

16:40 – 17:00 Zoran Radosavljević: *On uncyclic reflexive graphs*

17:00 – 17:20 Marija Rašajski: *On a class of maximal reflexive θ graphs generated by Smith graphs*

17:20 – 17:40 Kristina Vušković, F. Maffray, N. Trotignon; *A combinatorial algorithm for maximum weighted clique for a subclass of perfect graphs*

17:40 – 18:00 Silvana Petruševa; *Solving a shortest path problem in an environment which has Eulerian graph representation with a fractal structure, with an emotional agent*

18:00 – 18:20 Dejan Tošić, Slobodan K. Simić, Milka Potrebić: *Analysis of electric circuits with MATHEMATICA*

Room C: Numerical Analysis (chair: Nenad Cakić)

16:40 – 17:00 Germain E. Randriambelosoa: *Approximate solution of beam differential equation*

20:00 *Conference concert in Galerija fresaka (museum of frescoes): Pavle Aksentijević and the group Zapis; fashion show by Verica Planić*

Sunday, September 3

Room A: Mathematical Analysis (chair: Milan Merkle)

9:00 – 9:45 Soon-Yeong Chung, Department of Mathematics, Sogang University, Korea: *Inverse conductivity problems in the electrical networks*

9:45 – 10:30 Stevan Pilipović, Faculty of Sciences and Mathematics, University of Novi Sad, Serbia: *Algebra of generalized functions, generalized hyperfunctions and algebra of megafunctions*

Room B: Graph theory (chair: Peter Rowlinson)

9:45 – 10:30 Nataša Pržulj, Department of Computer Science, University of California, Irvine, USA: *Protein-Protein interaction networks: Issues, models, and comparisons*

10:30 – 11:00 Coffee break

Room B: Graph Theory (chair: Milan Merkle)

11:00 – 11:15 Zoran Obradović, Information Science and Technology Center, Temple University, USA: *Using gene ontology graphs for biomarker selection from integrated microarray, proteomics and clinical data* (joint work with Hongbox Hie and Slobodan Vučetić)

Room B: Publications - Past and Present (chair: Milan Merkle)

11:15 – 13:00 Speakers:

- Dragoš Cvetković
- Slaviša Prešić
- Dobrilo Tošić
- Hari M. Srivastava
- A. M. Fink
- Ingram Olkin
- Lev M. Berkovich
- Themistocles M. Rassias
- Peter Rowlinson
- Dragoš Cvetković
- Zsolt Tuza
- Sever Dragomir
- Slobodan K. Simić

15:00 – 17:00 *Walking tour: sightseeing Belgrade*

20:00 *Conference dinner, restaurant Ima dana*

Monday, September 4

Room A: Mathematical Analysis (chair: Gradimir V. Milovanović)

9:00 – 9:45 Aleksandar Ivić, Faculty of Mining and Geology, University of Belgrade, Serbia: *The Rankin - Selberg problem*

9:45 – 10:30 Wayne Hayes, School of Information and Computer Science, Uni. of California, USA: *Outer solar system on the edge of chaos*

Room B: Mathematical Analysis (chair: Milan Merkle)

9:00 – 9:45 Lev M. Berkovich, Samara State University, Russia: *Method of a factorization of ordinary differential operators and its applications*

9:45 – 10:30 Peter G. L. Leach, University of KwaZulu-Natal, Durban, South Africa: *Nonlocal symmetries: past, present and future*

10:30 – 11:00 Coffee break

Room A: Ordinary differential equations (chair: Lev M. Berkovich)

11:00 – 11:20 Dragan Dimitrovski, Vladimir Rajović, Aleksandar Dimitrovski (In memory of Professor Mitrinović): *On the need for and importance of a thematic monograph exclusively on periodical solutions of differential equations*

11:20 – 11:40 Dragan Dimitrovski, Vladimir Rajović, Aleksandar Dimitrovski: *Global aspect of non-homogenous differential equation of the second order*

11:40 – 12:00 Miloje Rajović, Rade Stojiljkovi*ć* *On types, form and supremum of solutions of ordinary homogenous linear differential equations of the second order*

12:00 – 12:20 Boro M. Piperevski: *On a correlation between the nature of a solution of a class of differential equation of n-th order and the solutions of its adequate characteristic algebraic equation of n-th degree*

12:20 – 12:40 Stana Cvejić, Milena Lekić: *Sturm's theorems through iterations*

Room B: Computer Science (chair: Tatjana Lutovac)

11:00 – 11:20 Žarko Mijajlović, Miloš Milošević, Aleksandar Perović: *Some properies of posynomial rings*

11:20 – 11:40 Milica Andjelić: *On the matrix equation* $XA - AX = \tau(X)$

11:40 – 12:00 Dejan Živković: *Non-polynomial lower bound for monotone depth-*3 *circuits computing an* NC^1*-complete function*

12:00 – 12:20 Nenad Krdžavac: *Implementation tableau algorithm for a description logic using a model transformation*

12:20 – 12:40 Miodrag Rašković, Zoran Marković, Zoran Ognjanović: *A logic with imprecise conditional probabilities*

12:40 – 14:30 Lunch

Room A: Differential equations (chair: Soon-Yeong Chung)

14:50 – 15:10 Biljana Jolevska-Tuneska: *On a differential equation with non-standard coefficients*

15:10 – 15:30 Jong-Ho Kim, Soon-Yeong Chung: (p, w)*-harmonic functions and inverse problems on nonlinear network*

15:30 – 15:50 Yun-Sung Chung, Soon-Yeong Chung: *Boundary value problems for diffusion and elastic operators on networks*

15:50 – 16:10 Diana Dolićanin, Valery G. Romanovski, Tatjana Mirković: *Linearizability conditions of a polynomial system of degree five*

Room B: Stochastic (chair: Slobodanka Janković)

14:30 – 14:50 Takashi Matsuhisa: *Bayesian communication leading to Nash equilibrium*

14:50 – 15:10 Tzuu-Shuh Chiang, Yunshyong Chow: *Optimal Ventcel graphs, minimal cost spanning trees and asymptotic probabilities*

15:10 – 15:30 Djordje V. Vukomanović: Probability on a universe of discourse

15:30 – 15:50 Miroslav M. Ristić, Biljana C. Popović, Aleksandar Nastić, Miodrag Djordjević: *A bivariate Marshall and Olkin exponential minification process*

15:50 – 16:10 Slobodanka S. Mitrović: *Stochastic modeling of the growth process*

Room C: Various Topics (chair: Monica Moulin Ribeiro Merkle)

14:30 – 14:50 Maximiliano Pinto Damas, Lilian Markenzon, Nair Maria Maia de Abreu: *The concept of tuner set for graphs*

14:50 – 15:10 Nebojša Nikolić, Rade Lazović: *The size of some antichains for multisets*

15:10 – 15:30 Mirjana Stojanović, R. Gorenflo: *Diffusion-wave problem*

15:30 – 15:50 Tatjana Lutovac, James Harland: *An algebraic approach to redundancy analysis of sequent proofs*

15:50 – 16:10 Nikola Tuneski: *Starlikeness and convexity of a class of analytic functions*

16:10 – 16:40 Coffee break

Room A: Differential equations (chair: Soon-Yeong Chung)

16:40 – 17:00 Young-Su Lee, Soon-Yeong Chung: Stability for cubic functional equation in the spaces of generalized functions

17:00 – 17:20 A. Carmona, E. Bendito, A. M. Encinas, J. M. Gesto: *Potential theory for BVP's in finite networks*

Room C: Various Topics (chair: MONICA MOULIN RIBEIRO MERKLE)

16:40 – 17:00 OLIVERA DJORDJEVIĆ: *On a Littlewood-Paley type inequality*

17:00 – 17:20 MIOMIR STANKOVIĆ, PREDRAG M. RAJKOVIĆ, SLADJANA D. MARINKOVIĆ *On the fractional integrals and derivatives in quantum calculus*

17:20 – 17:40 VLADIMIR BALTIĆ *On the number of certain types of restricted permutatuions*

17:40 *Closing ceremony*

Tuesday, September 5

Excursion to Sremski Karlovci and surroundings

REFERENCES

1. *Abstracts of talks presented at The International Mathematical Conference: Topics in Mathematical Analysis and Graph Theory.* Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat., **18** (2007), 94–138.
2. Program and Abstracts Book of The International Mathematical Conference: *Topics in Mathematical Analysis and Graph Theory.* Faculty of Electrical Engineering, University of Belgrade, 2006.

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ON THE ALEKSANDROV PROBLEM FOR ISOMETRIC MAPPINGS

Themistocles M. Rassias

In this paper some relations between linearity and isometry are investigated for mappings which preserve some distance. Several open problems are discussed.

1. INTRODUCTION

Let X, Y be two metric spaces, d_1, d_2 the distances on X and Y, respectively. A mapping $f : X \to Y$, of X onto Y, is defined to be an **isometry** if

$$d_2\big(f(x), f(y)\big) = d_1(x, y)$$

for all elements x, y of X.

S. Mazur and S. Ulam [**14**] have proved that every isometry of a normed real vector space onto a normed real vector space is a linear mapping up to translation. Consider then the following condition (**distance one preserving property**) for the mapping $f : X \to Y$.

(DOPP) *Given $x, y \in X$ with $d_1(x, y) = 1$. Then $d_2\big(f(x), f(y)\big) = 1$.*

A. D. Aleksandrov [**1**] posed the following problem:

> *Under what conditions is a mapping of a metric space into itself preserving unit distance an isometry?*

2000 Mathematics Subject Classification. 51K05.

Key Words and Phrases. Aleksandrov problem, Mazur-Ulam theorem, isometry, Banach space, metric, distance, sphere.

The basic "**problem of conservative distances**" is whether the existence of a single conservative distance for f implies that f is an isometry of X into Y (cf. [**6**, 17]).

F. S. BECKMAN and D. A. QUARLES [**2**] proved that if $f : E^n \to E^n$ for $2 \le n < \infty$ satisfies condition (DOPP), then f is an isometry, where E^n is a finite-dimensional real Euclidean space. Independently from BECKMAN and QUARLES, R. L. BISHOP [**5**], P. ZVENGROWSKI [**23**], D. GREENWELL and P. D. JOHNSON [**7**] have obtained different proofs of the same result. For non-Euclidean spaces the BECKMAN-QUARLES result has been obtained by the Russian school, notably by A. GUC [**8**], A. V. KUZ'MINYH [**13**].

This property does not hold for E^1, the Euclidean line. A simple counterexample is the following:

Let $f : E^1 \to E^1$ be defined by

$$f(x) = \begin{cases} x+1 & \text{if } x \text{ is an integer point,} \\ x & \text{otherwise.} \end{cases}$$

Nevertheless, one may ask about a solution with additional assumptions (for instance continuity or differentiability of f). The answer is still negative:

EXAMPLE 1.1. Define $f : E^1 \to E^1$ by

$$f(x) = x + \frac{1}{7}\sin(2\pi x).$$

The function f is an analytic diffeomorphism satisfying the (DOPP), but is not an isometry.

Also this property does not hold for E^∞, a HILBERT space. A counterexample can be made in the following way: Let $\{y_i\}$ be a countable everywhere dense set of points. Define $g : E^\infty \to \{y_i\}$ such that $d(x, g(x)) < 1/2$. Define $h : \{y_i\} \to \{a_i\}$ such that $h(y_i) = a_i$, where a_i is the point in E^∞ with coordinates $(a_{i1}, a_{i2}, \ldots)$ such that $a_{ij} = \delta_{ij}/\sqrt{2}$, where δ_{ij} is the KRONECKER delta. Then

$$f = gh : E^\infty \to E^\infty$$

satisfies condition (DOPP). If $d(x, y) = 1$, then $g(x) \ne g(y)$ and hence $f(x) \ne f(y)$, but f is not an isometry.

It is not yet known what does it happen in E^∞ even with the additional condition of continuity of the mapping.

Conjecture 1.2. *A continuous mapping $f : E^\infty \to E^\infty$ satisfying condition* (DOPP) *must be an isometry.*

In this paper, we will survey recent developments on the ALEKSANDROV problem and the MAZUR-ULAM theorem for mappings which preserve some distances.

2. RESULTS AND OPEN PROBLEMS

B. Mielnik and Th. M. Rassias [**15**] have proved the following

Theorem 2.1. *Every homeomorphism $f : E^n \to E^n$ $(2 < n \leq \infty)$ with a non-trivial conservative distance $\ell > 0$ is an isometry.*

The case of mapping $f : E^n \to E^m$ $(2 \leq n < m < +\infty)$

In the following we outline a method to show how to construct examples to prove that for each positive integer n there exists a positive integer m and a unit distance preserving mapping $f : E^n \to E^m$ that is not an isometry. The following example illustrates the case of a mapping $f : E^2 \to E^8$. For this consider partitioning the plane into squares of unit diagonal as follows:

Each square contains the bottom edge, the left edge and the bottom left corner but none of the other corners. Now label the nine vertices of the unit 8-simplex in E^8 and map each square labeled i to the i-th vertex. This mapping satisfies condition (DOPP) but is not an isometry.

Remark. Using hexagons instead of squares one can construct such a mapping from $E^2 \to E^6$. This idea extends easily to higher dimensions.

Th. M. Rassias [**16**] has proved the following

Theorem 2.2. *For any integer $n \geq 1$, there exists an integer n_m such that for $N \geq n_m$ it follows that there exists a mapping $f : E^n \to E^N$ which is distance one preserving but is not an isometry.*

It is not yet known whether there is a distance 1-preserving mapping $f : E^2 \to E^3$ which is not an isometry. It is also an open problem whether there is a continuous mapping $f : E^n \to E^m$ for $m > n$ which satisfies the (DOPP) but is not an isometry.

Combining continuity and distance preserving properties for the mapping we can formulate the following

Conjecture 2.3. *If M is a locally Euclidean manifold of finite dimension greater or equal to two, then there is a distance a such that for any $b < a$, every mapping $f : M \to M$ preserving distance b is an isometry.*

In E^n three classical metrics induce the same topology:

$$d_m(x,y) = \max\{|x_i - y_i| : i = 1, 2, \ldots, n\},$$
$$d_\Sigma(x,y) = \sum_{i=1}^{n} |x_i - y_i|,$$

and the Euclidean metric d_E, where $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$.

In the following we consider the isometry problem with respect to these metrics (see [**6**]).

Problem. Does the condition (DOPP) suffice for a mapping $f : E^n \to E^k$ with respect to these metrics to be an isometry if $2 \le n < k < +\infty$?

It is obvious that for $n = 1$ all three metrics are the same.

Consider the space E^2 with the metric d_m. In this case the mapping may satisfy (DOPP) and not be an isometry. For this consider the following

Example 2.4. Let $f : E^2 \to E^2$ be defined by

$$f(x, y) = ([x], [y])$$

(in Cartesian coordinates, $[x]$ denotes the integer part of x). This mapping, which corresponds every point to the left-bottom corner of a suitable square with sides of length equal to one, with range equal to $\mathbb{Z}^2$ ($\mathbb{Z}$ denotes the set of integers) is not an isometry but it preserves distance one.

Let us consider now the metric d_Σ.

Example 2.5. Consider the mapping g defined by

$$g = \left(\sqrt{2} \cdot R_{\pi/4}\right) \circ f \circ \left(\frac{1}{\sqrt{2}} \cdot R_{\pi/4}^{-1}\right),$$

where f is as in Example 2.4 and $R_{\pi/4}$ is the rotation:

$$(x, y) \mapsto \left(\frac{x+y}{\sqrt{2}}, \ \frac{y-x}{\sqrt{2}}\right).$$

The rotation maps unit balls in metric d_m to balls of radius $\sqrt{2}$ with respect to metric d_Σ. The mapping g satisfies (DOPP) but is not an isometry.

Remark. In the general case for E^n, $n > 2$, a rotation as in E^2 does not do the job. This happens because the balls in metrics d_m and d_Σ are of the same shape only for $n = 1, 2$. In E^2 one has squares in both cases, but in E^3 one has cubes for d_m and octahedrons for d_Σ.

Example 2.6. For (E^n, d_m), $n > 2$, a mapping satisfying (DOPP) need not be an isometry. For this it is enough to consider the mapping $f : E^n \to E^n$ defined by $f(x_1, \ldots, x_n) = ([x_1], \ldots, [x_n])$.

For d_Σ the following problem is still open:

Problem. Must the mapping $f : (E^n, d_\Sigma) \to (E^n, d_\Sigma)$ satisfying (DOPP) be an isometry for $n \ge 3$?

Th. M. Rassias and P. Šemrl [**18**] introduced the following condition: Let X and Y be two real normed vector spaces. A mapping $f : X \to Y$ satisfies the **strong distance one preserving property** (SDOPP) if and only if for all $x, y \in X$ with $\|x - y\| = 1$ it follows that $\|f(x) - f(y)\| = 1$ and conversely.

The following two theorems were proved in [**18**]:

Theorem 2.7. *Let X and Y be real normed vector spaces such that one of them has dimension greater than one. Suppose that $f : X \to Y$ is a surjective mapping satisfying* (SDOPP). *Then f is an injective mapping satisfying*

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| < 1$$

for all $x, y \in X$. Moreover, f preserves distance n in both directions for any positive integer n.

The assumption that one of the spaces has dimension greater than one cannot be omitted in the theorem.

In the theorem (SDOPP) cannot be replaced by (DOPP).

The inequality

$$|\,\|f(x) - f(y)\| - \|x - y\|\,| < 1 \quad \text{for all} \quad x, y \in X$$

in the theorem is sharp.

Theorem 2.8. ([**18**]) *Let X and Y be real normed vector spaces such that one of them has dimension greater than one. Suppose that $f : X \to Y$ is a Lipschitz mapping with $k = 1$:*

$$\|f(x) - f(y)\| \leq \|x - y\| \quad \textit{for all} \quad x, y \in X.$$

Assume also that f is a surjective mapping satisfying (SDOPP). *Then f is an isometry. Thus f is a linear isometry up to translation.*

Corollary 2.9. *Let X and Y be real normed vector spaces such that one of them has dimension greater than one. Assume also that one of the spaces is strictly convex. Suppose that $f : X \to Y$ is a surjective mapping satisfying* (SDOPP). *Then f is a linear isometry up to translation.*

Corollary 2.10. *Let X and Y be real normed vector spaces with* $\dim X > 1$*, such that one of them is strictly convex. Suppose that $f : X \to Y$ is a homeomorphism satisfying* (DOPP). *Then f is a linear isometry up to translation.*

Open problems

1. Let X and Y be Banach spaces such that Y is strictly convex, $\dim Y > 2$, and $f : X \to Y$ be a mapping. Suppose that f preserves the two distances a and λa for some non-integer $\lambda > 2$. It is an open problem whether f must be an isometric mapping.

2. Examine whether a mapping $f : S^n \to S^n$ for $1 < n \leq \infty$, which preserves two distances, both different from $\pi/2$ and π, can be an isometry (S^n denotes the n-sphere in $\mathbb{R}^{n+1}$).

If $f : S^n \to S^n$ maps every point of S^n onto itself, except the north and south poles, and maps these two points onto each other, then f is not an isometry. This mapping f does preserve the two distances $\pi/2$ and π. The mapping is not continuous.

Let f be a mapping of a metric space X into itself. A nonnegative number r is called a **nonexpanding** (or **contractive**) distance of f if and only if for any $x, y \in X$, $d(x,y) = r$ implies $d\big(f(x), f(y)\big) \le r$. A nonnegative number r is called a **nonshrinking** (or **extensive**) distance of f if and only if for all $x, y \in X$, $d(x,y) = r$ implies $d\big(f(x), f(y)\big) \ge r$. The distance r is called **preserved** (or **conservative**) by f if and only if for all $x, y \in X$ with $\|x - y\| = r$, it follows that $\|f(x) - f(y)\| = r$.

Th. M. Rassias and S. Xiang [**19**] proved the following two theorems:

Theorem 2.11. *Let X and Y be real Hilbert spaces with the dimension of X greater than one. Suppose that $f : X \to Y$ satisfies* (DOPP) *and the distances a, b are contractive by f, where a and b are positive numbers with $|a - b| < 1$. Then the distance $\sqrt{2a^2 + 2b^2 - 1}$ is contractive by f. Especially, if the distance $\sqrt{2a^2 + 2b^2 - 1}$ is extensive by f, then the distances a, b and $\sqrt{2a^2 + 2b^2 - 1}$ are preserved by f.*

Theorem 2.12. *Let X and Y be real Hilbert spaces with the dimension of X greater than one. Suppose that $f : X \to Y$ satisfies* (DOPP). *Assume that the distance $n\sqrt{4^m k^2 - \frac{4^m - 1}{3}}$ is extensive by f for some positive integers n, k and m. Then f must be a linear isometry up to translation.*

Recently, S.-M. Jung and K.-S. Lee [**10**] proved a general inequality for distances between points: Let X be a real (or complex) inner product space, let n be an integer not less than 2, and let p_{ik}, $i \in \{1, \dots, n\}$ and $k \in \{1, 2\}$, be any distinct $2n$ points of X.

(a) It holds that

$$\sum_{\substack{1 \le i < j \le n \\ k, \ell \in \{1,2\}}} \|p_{ik} - p_{j\ell}\|^2 \ge (n-1) \sum_{i \in \{1,\dots,n\}} \|p_{i1} - p_{i2}\|^2.$$

(b) The equality sign holds true in the above inequality if and only if for all $i, j \in \{1, \dots, n\}$ with $i < j$, the pair of four points $\{p_{i1}, p_{i2}, p_{j1}, p_{j2}\}$ comprises the vertices of an appropriate (possibly degenerate) parallelogram such that p_{i1} and p_{j1} are the opposite vertices to p_{i2} and p_{j2}, respectively.

(Inequality (a) for $n = 2$ was proved in Lemma 1 of [**9**] and the case for $n = 3$ was treated in Theorem 2 of [**9**].)

We will label the vertices of any (possibly degenerate) parallelogram by p_{11}, p_{12}, p_{21}, and p_{22} as we see in the left-hand side of Fig. 1. We label the vertices of any (possibly degenerate) octahedron by p_{11}, p_{12}, p_{21}, p_{22}, p_{31}, and p_{32} as we see in the right-hand side of Fig. 1.

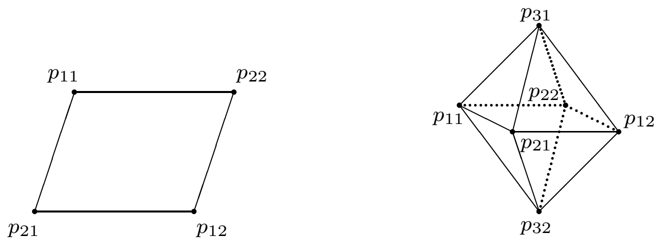

Fig. 1

We can continue this construction for the general case. Assume that we have constructed an n-dimensional polyhedron with $2n$ vertices, p_{11}, $p_{12}, \ldots$, p_{n1}, p_{n2}. Now, we add two more points, denoted by $p_{(n+1)1}$ and $p_{(n+1)2}$, to construct an $(n+1)$-dimensional polyhedron in the following manner: Each of the new points, $p_{(n+1)1}$ and $p_{(n+1)2}$, is connected to the existing $2n$ vertices, p_{11}, p_{12}, $\ldots$, p_{n1}, p_{n2}.

For a given n-dimensional polyhedron constructed as above, we will denote its $2n$ vertices by p_{11}, p_{12}, $\ldots$, p_{n1}, p_{n2} as the above construction. We define

$$\alpha_{ij} = \|p_{i1} - p_{j1}\|, \quad \beta_{ij} = \|p_{i2} - p_{j2}\|, \quad \gamma_{ij} = \|p_{i1} - p_{j2}\|$$

for all $i, j \in \{1, \ldots, n\}$. In the following theorem, we will assume that for any $i, j \in \{1, \ldots, n\}$ with $i < j$, each pair of four points, p_{i1}, p_{i2}, p_{j1}, p_{j2}, comprises the vertices of a corresponding parallelogram.

With these notations Jung and Lee [**10**] obtained the following

Theorem 2.13. *Let X and Y be either real inner product spaces or complex inner product spaces with $\dim X \geq n$ and $\dim Y \geq n$, where $n \geq 2$. Assume that the distances α_{ij}, β_{ij}, γ_{ij} are contractive by a mapping $f : X \to Y$ for all $i, j \in \{1, \ldots, n\}$ with $i < j$ and that the distances γ_{ii} are extensive by f for each $i \in \{1, \ldots, n\}$. Then f preserves the distances α_{ij}, β_{ij}, γ_{ij} for all $i, j \in \{1, \ldots, n\}$ with $i \leq j$.*

Sketch of the proof. First, we denote by p'_{ik} the image of p_{ik} under f. Since $\gamma_{ii} = \|p_{i1} - p_{i2}\|$ are extensive by f and α_{ij}, β_{ij}, γ_{ij} are contractive by f for all $1 \leq i < j \leq n$, we have

$$\begin{aligned}
(n-1) \sum_{i \in \{1,\ldots,n\}} \|p'_{i1} - p'_{i2}\|^2 &\geq (n-1) \sum_{i \in \{1,\ldots,n\}} \|p_{i1} - p_{i2}\|^2 \\
&= \sum_{\substack{1 \leq i < j \leq n \\ k, \ell \in \{1,2\}}} \|p_{ik} - p_{j\ell}\|^2 \\
&\geq \sum_{\substack{1 \leq i < j \leq n \\ k, \ell \in \{1,2\}}} \|p'_{ik} - p'_{j\ell}\|^2 \\
&\geq (n-1) \sum_{i \in \{1,\ldots,n\}} \|p'_{i1} - p'_{i2}\|^2,
\end{aligned}$$

where the last inequality follows from inequality (a). Hence, we get

$$\textstyle\sum_{i\in\{1,\ldots,n\}}\|p'_{i1}-p'_{i2}\|^2=\sum_{i\in\{1,\ldots,n\}}\|p_{i1}-p_{i2}\|^2,$$

$$\sum_{\substack{1\le i<j\le n\\ k,\ell\in\{1,2\}}}\|p_{ik}-p_{j\ell}\|^2=\sum_{\substack{1\le i<j\le n\\ k,\ell\in\{1,2\}}}\|p'_{ik}-p'_{j\ell}\|^2.$$

Since $\|p'_{i1}-p'_{i2}\|\ge\|p_{i1}-p_{i2}\|$ and $\|p_{ik}-p_{j\ell}\|\ge\|p'_{ik}-p'_{j\ell}\|$ for all $1\le i<j\le n$ and $k,\ell\in\{1,2\}$, we may conclude that

$$\|p'_{i1}-p'_{i2}\|=\|p_{i1}-p_{i2}\|=\gamma_{ii}$$

and

$$\|p'_{ik}-p'_{j\ell}\|=\|p_{ik}-p_{j\ell}\|=\begin{cases}\alpha_{ij} & (\text{for } k=\ell=1)\\ \beta_{ij} & (\text{for } k=\ell=2)\\ \gamma_{ij} & (\text{for } k=1 \text{ and } \ell=2)\\ \gamma_{ij} & (\text{for } k=2 \text{ and } \ell=1)\end{cases}$$

for any $1\le i<j\le n$.

As we see in Theorem 4 and Corollary 5 of [9], if we set $n=3$, $\alpha_{ij}=\beta_{ij}=\gamma_{ij}=\rho$ for $1\le i<j\le 3$, and $\gamma_{ii}=\sqrt{2}\rho$ for $i\in\{1,2,3\}$, then we obtain the following

Corollary 2.14. *Let X and Y be real Hilbert spaces with $\dim X\ge 3$ and $\dim Y\ge 3$. For a given $\rho>0$, assume that the distance ρ is contractive and the distance $\sqrt{2}\rho$ is extensive by a mapping $f:X\to Y$. Then, f is a linear isometry up to translation.*

We now consider an octahedron determined by the six vertices

$$p_{11}=\left(\frac{\sqrt{3}}{2}\rho,0,0,0,\ldots,0\right),\qquad p_{12}=\left(-\frac{\sqrt{3}}{2}\rho,0,0,0,\ldots,0\right),$$
$$p_{21}=\left(0,\frac{1}{2}\rho,0,0,\ldots,0\right),\qquad p_{22}=\left(0,-\frac{1}{2}\rho,0,0,\ldots,0\right),$$
$$p_{31}=\left(0,0,\frac{1}{2}\rho,0,\ldots,0\right),\qquad p_{32}=\left(0,0,-\frac{1}{2}\rho,0,\ldots,0\right),$$

where ρ is a given positive number. Applying Theorem 2.13 for $n=3$ to the above octahedron and using Theorem 2.1 of S. Xiang [**22**], we can prove the following

Corollary 2.15. *Let X and Y be real Hilbert spaces with $\dim X\ge 3$ and $\dim Y\ge 3$. For a given $\rho>0$, assume that the distance ρ is preserved, $\frac{1}{\sqrt{2}}\rho$ is contractive, and that the distance $\sqrt{3}\rho$ is extensive by a mapping $f:X\to Y$. Then, f is a linear isometry up to translation.*

Now, let X and Y denote n-dimensional Euclidean spaces, where $n\ge 3$, for which there exists a unit vector $w\in X$ and a subspace X_s of X such that

$X = X_s \oplus Sp(w)$ and X_s is orthogonal to $Sp(w)$, where $Sp(w)$ is the subspace of X which is spanned by w.

We define

$$r_0 = \theta, \quad r_1 = \theta + \rho, \quad r_2 = \theta + \rho + \rho_1, \quad r_3 = \theta + \left(1 + \frac{1}{n}\right)\rho + \rho_1,$$

where θ is a real number, ρ is a positive real number and

$$\rho_1 = \sqrt{\frac{2(n+1)}{n}}\,\rho.$$

By using these r_k's we define

$$E_k = \{x + \lambda w : x \in X_s\,;\, \lambda > r_k\}$$

for $k \in \{0, 1, 2, 3\}$.

Using these notations, S.-M. Jung and Th. M. Rassias [**11**] have proved the classical theorem of Beckman and Quarles for a restricted domain (see also [**12**]):

Theorem 2.16. *If a mapping $f : E_0 \to Y$ preserves the distance ρ, then the restriction $f|_{E_3}$ is an isometry. In particular, for any $x, y \in E_2$ with $x_s \neq y_s$, it holds that $\|f(x) - f(y)\| = \|x - y\|$, where x_s and y_s denote the X_s-components of x and y, respectively.*

Sketch of the proof. Lemma 13 of [**11**] implies that the distance $\frac{2(n+1)}{n}\rho$ preserved (extensive) by $f|_{E_2}$, while Lemma 14 of [**11**] shows the contractive property of the distance $\frac{2}{n}\rho$ under $f|_{E_2}$. Thus, in view of Theorem 9 of [**11**], we can conclude that the restriction $f|_{E_3}$ is an isometry. The second part of this theorem also follows from the second part of Theorem 9 of [**11**]. (We may remark that the proofs of Theorem 9 and Lemmas 13 and 14 are strongly based on the papers [**3, 4**] of W. Benz.)

B. Mielnik and Th. M. Rassias [**15**] have proved the following

Theorem 2.17. *Let f be a homeomorphism of the unit sphere X in a real Hilbert space H ($3 \leq \dim H \leq \infty$) which preserves the angular distance $\pi/2$. Then f is an isometry.*

The proof of the above theorem is based on a very fundamental theorem that was proposed by Eugene Wigner [**21**].

This theorem asserts that mappings from a Hilbert space to itself which preserve the absolute values of inner products are in a certain sense equivalent to isometries (for a precise statement and proof of Wigner's theorem see [**20**]).

Absolute values of inner products are related to probabilities of transitions between states of a quantum system and the time evolution of such a system is supposed to preserve these probabilities.

Wigner used his theorem to define two linear mappings from a Hilbert space to itself which have played very fundamental roles in the development of quantum theory. These mappings are known to physicists as **time reversal** and **charge conjugation operators**.

It is an *open problem* to examine if the above theorem holds when f satisfies a condition weaker than that of a homeomorphism.

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WHAT IS HADAMARD'S INEQUALITY?

A. M. Fink, Zsolt Páles

The Hadamard inequality usually stated as a result valid for convex functions only, actually holds for many other functions. We argue that an attempt ought to be made to close this gap by either changing the inequality or considering the measures in the integrals as a "second variable."

Hadamard's Inequality [**1**] is usually stated as

$$f\left(\frac{a+b}{2}\right) \le \frac{1}{b-a}\int_a^b f\,\mathrm{d}x \le \frac{f(a)+f(b)}{2} \tag{1}$$

or in Fejer's version (left hand side only)(and change the interval)

$$\int_{-1}^{1} p(x)\,\mathrm{d}x\ f(0) \le \int_{-1}^{1} p(x)f(x)\,\mathrm{d}x \tag{2}$$

$$\text{provided } p \ge 0 \text{ and } p(x) = p(-x). \tag{3}$$

In both cases, f is presumed to be convex and continuous and p integrable. It is the convexity of f that we wish to challenge. Let us look at the proof on $[-1,1]$. Since *the graph of* f *lies above any supporting line at* $(0, f(0))$ *and below the chord joining* $(-1, f(-1))$ *and* $(1, f(1))$, we have for some c

$$f(0) + cx \le f(x) \le \frac{f(1)+f(-1)}{2} + \frac{f(1)-f(-1)}{2}\,x. \tag{4}$$

Integrating with respect to Lebesque measure yields (1) and (2) by integrating with respect to $p(x)\,\mathrm{d}x$.

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Although any convex function satisfies (4), there are a host of other non-convex functions that do also. Merely the emphasized assumption is required. But even that is not necessary. Take any function that is odd about the center of the interval, then both (1) and (2) are satisfied trivially. But more. Since the inequalities are linear, the class of all functions satisfying (1) or (2) is a cone, i.e. closed under sums and positive multiples. For example, $1+\sin 1000\pi x$ satisfies these inequalities. From our point of view, citing (1) or (2) as a basic inequality for convex function is unsatisfactory.

There are several remedies available. The first is suggested by passing from (1) to (2). That is, we should consider bounding

$$\frac{1}{P_0}\int_{-1}^{1} f(x)\,\mathrm{d}\mu(x)$$

where μ is any (say Borel) measure and $P_0=\int_{-1}^{1}\mathrm{d}\mu>0$, and μ is any signed measure. That is, we now have 2 "variables," the class of functions, and the class of measures. Typically, we would want to prove a bound for the "largest" class of functions and the "largest" class of measures.

In [**3**] one of us solved this dilemma by changing the inequality, fixing the class of functions as the convex ones, and finding an appropriate class of measures. We gave a definition of the "largest" class referred to above. Specifically, let all functions f be continuous and

$$\widehat{M_0}=\{\mu|\mu(-1,1)\ge 0,\ \int_{t}^{1}(x-1)\,\mathrm{d}\mu(x)\ge 0,\ \int_{-1}^{t}(t-x)\,\mathrm{d}\mu(x)\ge 0\ \text{ for }\ t\in[-1,1]\}$$

and $M_0=\{f|\quad f \text{ is a continuous convex function}\}$. Then we proved in [**3**] that

$$P_0 f\left(\frac{P_1}{P_0}\right)\le\int_{-1}^{1} f\,\mathrm{d}\mu,\qquad P_1=\int_{-1}^{1}x\,\mathrm{d}\mu,\quad P_0=\int_{-1}^{1}\mathrm{d}\mu \tag{6}$$

holds for all $f\in M_0$ if and only if $\mu\in\widehat{M_0}$ and (6) holds for all $\mu\in\widehat{M_0}$ if and only if $f\in M_0$.

We call instances where we can prove both these statements a "best possible inequality." That is, the inequality for all convex f characterizes $\widehat{M_0}$ and the inequality for all measures in $\widehat{M_0}$ characterizes M_0. The reason this works, is that (6) uses the fact that the graph lies above all supporting lines by varying the measure. As a comment, we proved in [**3**] a version of the right hand inequality of (4) which at the time we did not know that it is also a best possible inequality. But we can give a proof that it is.

For the right hand inequality

$$\int_{-1}^{1} f(x)\,\mathrm{d}\mu(x)\le P_0\,\frac{f(-1)+f(1)}{2}+P_1\,\frac{f(1)-f(-1)}{2}\,, \tag{6'}$$

the result was that this hold for all convex f and measures μ such that the boundary value problem

$$y''\mathrm{d}t = \mathrm{d}\mu(t); \quad y(-1) = y(1) = 0 \quad \text{has a solution}$$

$y(x) \le 0$ on $[-1, 1]$. This means that if $G(x,t)$ is the GREEN's function for the problem $Ly = y''$, $y(\pm 1) = 0$ then $y(x) = \int_{-1}^{1} G(x,t)\,\mathrm{d}\mu(t)$. In [**3**] it is stated that it is unlikely that $(6')$ is a best possible inequality. In fact it is. We state this as our first new theorem.

Theorem 1. *Let $M_0^1 = \{\mu|$ if $y''(t)\,\mathrm{d}t = d\mu(t) \quad y(\pm 1) = 0$ then $y(x) \le 0$ on $[-1,1]\}$. Then*

$$(6') \quad \textit{holds for all convex } f \ \textit{ if and only if } \ \mu \in M_0^1;$$

and

$$(6') \quad \textit{holds for all} \quad \mu \in M_0^1 \ \textit{ if and only if } \quad f \ \textit{ is convex.}$$

Proof. The sufficiency in both cases is Theorem 5 of [**3**].

The GREEN's function is given by

$$G(x,t) = \begin{cases} (t-1)x; & -1 \le x \le t \le 1 \\ (x-1)t; & -1 \le t \le x \le 1 \end{cases}.$$

If we take $\mu = \delta_{\frac{a+b}{2}} - \frac{1}{2}\,\delta_a - \frac{1}{2}\,\delta_b$ when δ_t is the unit mass at t and $-1 \le a \le b \le 1$ then $\int_{-1}^{1} G(x,t)\,\mathrm{d}\mu(t) = G\left(x, \frac{a+b}{2}\right) - \frac{1}{2}\,G(x,a) - G(x,b) \le 0$ by the convexity of G. So $\mu \in M_0^1$ and $(6')$ becomes

$$f\left(\frac{a+b}{2}\right) - \frac{1}{2}\,f(a) - \frac{1}{2}\,f(b) \le 0 \quad \text{since} \quad P_0 = P_1 = 0.$$

Since f is always assumed to be continuous, this make f convex, proving the second statement of the theorem. For the other part observe that $f(t) = G(x,t)$ for fixed x is convex and $f(\pm 1) = 0$ so $(6')$ becomes $\int_{-1}^{1} G(x,t)\,\mathrm{d}\mu(t) \le 0$ and μ is necessarily in M_0^1. □

But suppose we do not want to change the inequality (1) and retain some best possible inequality. Two preliminary comments are in order for the inequality

$$f(0)P_0 \le \int_{-1}^{1} f\,\mathrm{d}\mu. \tag{7}$$

First, this inequality does not respect $\mu\{0\}$. For if we replace μ for $\mu + c\delta_0$, δ_0 the unit mass at 0, this just adds $cf(0)$ to both sides of the inequality. Therefore we can never deduce anything about $\mu\{0\}$. Secondly, if we add a constant to f, that constant times P_0 appears on both sides. In particular, we may require $f(0) = 0$. We may consider the inequality

$$(8) \qquad 0 \le \int_{-1}^{1} f \,\mathrm{d}\mu \quad \text{with the proviso that} \quad f(0) = 0.$$

As a test case, let us look at FEJER's version. Let $\widehat{M_1} = \{\mu| \quad \text{for} \quad A \subset (0,1], \mu(A) = \mu(-A) \ge 0\}$.

Theorem 2. *Let* $M_1 = \{f|\ f$ *is continuous,* $f(0) = 0$ *and* $f(t) + f(-t) \ge 0$ *for* $t \in (0,1]\}$

then (8) *holds for all* $\mu \in \widehat{M_1}$ *if and only if* $f \in M_1$, *and*
(8) *holds for all* $f \in M_1$ *if and only if* $\mu \in \widehat{M_1}$.

Proof. For $\mu \in \widehat{M_1}$, $\int_{-1}^{1} f \,\mathrm{d}\mu = \int_{0}^{1} \big(f(t) + f(-t)\big) \,\mathrm{d}\mu$ so if $f \in M_1$ (8) holds. For the converse take $\mu = \delta_t + \delta_{-t}, 0 < t \le 1$, then $0 \le \int_{-1}^{1} f \,\mathrm{d}\mu = f(t) + f(-t)$ so $f \in M_1$. This proves the first if only if and the "if" part of the second statement. To prove that $\mu \in \widehat{M_1}$ if (8) holds for all $f \in M_1$, we take $f = \chi_A - \chi_{-A}$ for a measurable $A \subset (0,1]$. Then f is odd so $f(0) = 0$ and $f \in M_1$. Now (8) leads to $\mu(A) - \mu(-A) \ge 0$. Taking $-f$ we get the reverse inequality. Finally we take $f = \chi_A + \chi_{-A}$ to get $\mu(A) \ge 0$. These f are not continuous but can be approximated in L_1.

For the original inequality (7) the class M_1 now becomes those functions f for which

$$(9) \qquad \frac{f(t) + f(-t)}{2} \ge f(0)$$

i.e. the even part has its minimum at 0. □

It is a triviality to prove

Theorem 3. *The inequality* (8) *holds for all* $\mu \ge 0$ *if and only if* $f \ge 0$ *and the inequality* (8) *holds for all* $f \ge 0$ *if and only if* $\mu \ge 0$. *Consequently,* (7) *is a best possible inequality for the classes*
$M^* = \{f|\ f$ *has a minimum at* $0\}$ *and*
$M^* = \{\mu | \mu \ge 0\}$.

The reduction of HADAMARD's inequality (7) to (8) allows us to prove a variety of best possible inequalities. We always work inside this class of functions which are continuous with $f(0) = 0$, but state the results in terms of the inequality (7) with the measures always in $\widehat{U}$, the class of regular BOREL measures.

Thus HADAMARD's (and FEJER's) inequality (1) and (2) are in fact about functions which have their minimum at the center of the interval. As a second simple example, let $M_2 = \{f | f$ is non-decreasing on $[-1,1]\}$ and $\widehat{M_2} = \{\mu | \int_{-1}^{t} \mathrm{d}\mu \leq 0$ for $-1 \leq t < 0$ and $\int_{t}^{1} \mathrm{d}\mu \geq 0$ for $0 < t \leq 1\}$.

Theorem 4. *The Hadamard inequality* (7) *is best possible with the pairs* M_2 *and* $\widehat{M_2}$.

Proof. If f is increasing then there are measures λ, σ such that $\lambda \leq 0$ $f(x) = \int_{x}^{0} d\lambda$ for $-1 \leq x < 0$ and $f(x) = \int_{0}^{x} \mathrm{d}\sigma$ for $0 < x \leq 1$ and $\sigma \geq 0$. (Recall $f(0) = 0$). Then

$$\int_{-1}^{1} f(x)\,\mathrm{d}\mu(x) = \int_{-1}^{0} \int_{x}^{0} \mathrm{d}\lambda(t)\mathrm{d}\mu(x) + \int_{0}^{1} \int_{0}^{x} \mathrm{d}\sigma(t)\mathrm{d}\mu(x) = \int_{-1}^{0} \int_{-1}^{t} \mathrm{d}\mu(x)\mathrm{d}\lambda(t) + \int_{0}^{1} \int_{t}^{1} \mathrm{d}\mu(x)\mathrm{d}\sigma(t).$$

Consequently, if $\mu \in \widehat{M_2}$ and $f \in M_2$,(8) is satisfied. On the other hand if we take $f = \chi_{[t.1]}$ for $0 < t \leq 1$ then (8) becomes $\int_{t}^{1} \mathrm{d}\mu \geq 0$ and if $f = -\chi_{[-1,t]}$ for $-1 \leq t < 0$ then $\int_{-1}^{t} \mathrm{d}\mu \leq 0$. These f are not in M_2 but can be approximated in L_1 by elements of M_2. So $\mu \in \widehat{M_2}$. If we take $\mu = \delta_t - \delta_s$ for $0 < s < t < 1$ then $\mu \in \widehat{M_2}$ and (8) becomes $f(s) \leq f(t)$ and if $-1 < s < t < 0$ we take $\mu = \delta_g - \delta_s$ then $\mu \in \widehat{M_2}$ and $f(s) \leq f(t)$. So $f \in M_2$. □

To add one multiplicity suppose

$$C = \{f | f(0) = 0,\ f \text{ is convex on } [-1,1] \subset U\}$$

and we seek the measures for which (8) holds. For example $f(x) \equiv \pm|x|$ is in C. So that $\int_{-1}^{1} x\,\mathrm{d}\mu = 0$ is necessary. We may subtract a supporting line at 0 since $\int_{-1}^{1} x\,\mathrm{d}\mu = 0$ and we do not change the inequality. That is f may be assumed to be increasing and convex on [0.1]. In particular for any such f we have $f = f_1 + f_2$ where

$$f_1 = \begin{cases} f & 0 \leq x \leq 1 \\ 0 & -1 \leq x \leq 0 \end{cases} \quad \text{and} \quad f_2 = \begin{cases} 0 & 0 \leq x \leq 1 \\ f & -1 \leq 1 \leq 0 \end{cases}.$$

Inequality (7) holds for all f if and only if it holds for all such f_1 and f_2 since all these are convex. We have (10)

$$f_1(x) = \int_{0}^{x} (x-t)\,\mathrm{d}\sigma(t) \text{ for some } \sigma \geq 0. \tag{10}$$

Then

$$\int_0^1 f_1(x)\,\mathrm{d}\mu(x) = \int_0^1\int_0^x (x-t)\,\mathrm{d}\sigma(t)\mathrm{d}\mu(x) = \int_0^1\int_t^1 (x-t)\,\mathrm{d}\mu(x)\mathrm{d}\sigma(t) \geq 0 \tag{11}$$

if $\int_t^1 (x-t)\,\mathrm{d}\mu(x) = \int_t^1\int_t^x \mathrm{d}s\mathrm{d}\mu(x) = \int_t^1\int_s^1 \mathrm{d}\mu ds \geq 0$. This condition is

$$\int_t^1 \mu[s,1]\,\mathrm{d}s \geq 0, \quad 0 < t \leq 1, \tag{12}$$

and is necessary and sufficient for (8) to hold. The necessity is obtained by taking σ to be a point mass in (10) and (11), i.e. f is an angle. In a similar way the condition on $[-1,0]$ is $\int_{-1}^t \mu(-1,s)\,\mathrm{d}s \geq 0$, $-1 \leq t < 0$. $\big(f_2(x) = \int_x^0 (t-x)\,\mathrm{d}\sigma(t).\big)$

To complete the best possible statements, one has to prove the convexity of f. Take $\mu = \frac{1}{2}\,\delta_x + \frac{1}{2}\,\delta_y - \delta_{\frac{x+y}{2}}$ for $-1 \leq x < y < 1$. Then

$$\mu(s,1] = \begin{cases} 0 & s \leq x,\ s \geq y \\ \dfrac{1}{2} & \dfrac{x+y}{2} \leq s < y \\ -\dfrac{1}{2} & x < s < \dfrac{x+y}{2} \end{cases};$$

so that $\int_t^1 \mu(s,1]\,\mathrm{d}s = \begin{cases} 0 & t \geq y,\ t \leq x \\ \dfrac{1}{2}\,(y-t) & \dfrac{x+y}{2} < t < y \\ \dfrac{1}{2}\,(t-x) & x < t < \dfrac{x+y}{2} \end{cases}$. This is ≥ 0 for $t \in [0,1]$ regardless of where x, $\frac{x+y}{2}$, and y are.

Consequently μ satisfies (10) and (8) become $\frac{1}{2}\,f(x) + \frac{1}{2}\,f(y) \geq f\left(\frac{x+y}{2}\right)$. Note that the function $\int_t^1 \mu[s,1]\,\mathrm{d}s$ is a spline with support $[x,y]$. So taking

$$\widehat{C} = \left\{\mu\Big|\int_{-1}^1 x\,\mathrm{d}\mu = 0;\ \int_{-1}^t \mu[-1,s]\,\mathrm{d}s \geq 0 \text{ for } -1 \leq t \leq 0;\ \int_t^1 \mu[s,1]\,\mathrm{d}s \geq 0 \text{ for } 0 \leq t \leq 1\right\}$$

we have the best possible inequality (9).

We give one other simple example. Suppose we consider the class $M = \{f|f$ for even and $f \geq 0$ on $[-1,1]$, $f(0) = 0\}$. What is the appropriate class of measures for (8) to hold?

Theorem 5. *Let M be as above and $M^* = \{\mu|\mu(A)+\mu(-A) \geq 0$ for all $A \subset (0,1]\}$. Then* (8) *is a best possible inequality.*

Proof. Since $\int_{-1}^{1} f(x)\,\mathrm{d}\mu(x) = \int_{0}^{1} f(x)\big(\mathrm{d}\mu(x) + \mathrm{d}\mu(-x)\big)$ for $f \in M$, (8) hold for $f \in M$ and $\mu \in M^*$. To get the necessity first pick $f(x) = \chi_A(x)$ for any $A \subset (0,1]$. Evidently f is even and $f(0) = 0$. The inequality (8) yield $\mu(A) + \mu(-A) \geq 0$ so $\mu \in M^*$. On the other hand if $\mu = \delta_x - \delta_{-x}$ then clearly $\mu \in M^*$ and (8) yields $f(x) + f(-x) \geq 0$. If we take $\mu = \delta_x - \delta_{-x}$ then $\mu(A) + \mu(-A) = 0$ so $\mu \in M^*$ and (8) yields $f(x) - f(-x) \geq 0$. Combining the two inequalities give $f(x) \geq 0$. Finally taking $\mu = \delta_{-x} - \delta_x$ we also get $f(-x) - f(x) \geq 0$, so that $f(x) = f(-x)$. $\square$

For a modification of Hadamard's inequality that adds a term, see Fink [**5**]. For other views of the inequality, see [**6**] or [**7**].

We have argued that at minimum, Hadamard's inequality should be stated for a class of measures. Then it is about functions with minimums at the center of the interval (Theorem 3) or about functions whose even part about the center has a minimum there (Theorem 4).

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COLOR-BOUNDED HYPERGRAPHS, III: MODEL COMPARISON

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Generalizing previous models of hypergraph coloring — due to Voloshin, Drgas-Burchardt and Łazuka, and the present authors – in this paper we introduce and study the structure class that we call *stably bounded hypergraphs*. In this model, a hypergraph is viewed as a six-tuple $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$, where $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b} : \mathcal{E} \to \mathbb{N}$ are given integer-valued functions on the edge set. A mapping $\varphi : X \to \mathbb{N}$ is a *proper vertex coloring* if it satisfies the following conditions for each edge $E \in \mathcal{E}$: the number of colors in E is at least $\boldsymbol{s}(E)$ and at most $\boldsymbol{t}(E)$, while the largest number of vertices having the same color inside E is at least $\boldsymbol{a}(E)$ and at most $\boldsymbol{b}(E)$.

Taking different subsets of $\{\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}\}$ (as combinations of nontrivial conditions on colorability) result in a hierarchy of structure classes with respect to vertex coloring. The main issue of this paper is to carry out a detailed analysis of how those classes are related. This includes the study of possible chromatic polynomials and 'feasible sets' — that is, the set $\Phi(\mathcal{H})$ of integers k such that $\mathcal{H}$ has a proper vertex coloring with exactly k colors — with or without assuming that the number of vertices is the same under the different combinations of color-bound conditions, or restricting the edge sizes. Furthermore, substantial change is observed concerning the algorithmic complexity of recognizing hypergraphs that are uniquely colorable and $\Phi(\mathcal{H}) = \{\,|X| - 1\}$.

1. INTRODUCTION

The main goal of this paper is to describe a unified framework for various concepts in the coloring theory of hypergraphs, and to study how some of its naturally arising subclasses are interrelated. The model presented here includes, as

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particular cases, the proper (vertex) colorings in the classical sense, moreover the class of mixed hypergraphs, and also the color-bounded hypergraphs that have been introduced recently.

We assume throughout that $X = \{x_1, \ldots, x_n\}$ is a finite vertex set and $\mathcal{E} = \{E_1, \ldots, E_m\}$ is the edge set, where each E_i is a nonempty subset of X. Under a given mapping $\varphi : X \to \mathbb{N}$ – for which we shall use the term *vertex coloring*, or simply *coloring* – a set $Y \subseteq X$ is *monochromatic* if $\varphi(y) = \varphi(y')$ for all $y, y' \in Y$; and Y is said to be *polychromatic* if $\varphi(y) \neq \varphi(y')$ for any two distinct $y, y' \in Y$. Introducing the notation $\varphi(Y)$ for the image of Y under φ (i.e., the set of colors appearing in Y), 'monochromatic' and 'polychromatic' mean $|\varphi(Y)| = 1$ and $|\varphi(Y)| = |Y|$, respectively.

The classical theory of graph and hypergraph coloring proceeds by *excluding monochromatic edges*. Around the mid-1990's, Voloshin [**18, 19**] introduced the more general structures of *mixed hypergraphs*, in which some edges – that are called $\mathcal{D}$*-edges* – are not allowed to be monochromatic, while some edges – the $\mathcal{C}$*-edges* – are not allowed to be polychromatic. A *bi-edge* is a vertex subset (edge) which is a $\mathcal{C}$-edge and a $\mathcal{D}$-edge at the same time. In the past decade, the theory of mixed hypergraphs developed rapidly. For a concise survey on the subject, we refer to [**16**]; see also the research monograph [**20**] and the regularly updated web site [**21**].

In this paper we view hypergraphs as six-tuples $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$, where

$$\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b} : \mathcal{E} \to \mathbb{N}$$

are given integer-valued functions on the edge set; they will play important role concerning colorings. To simplify notation, we define

$$s_i := \boldsymbol{s}(E_i), \qquad t_i := \boldsymbol{t}(E_i), \qquad a_i := \boldsymbol{a}(E_i), \qquad b_i := \boldsymbol{b}(E_i)$$

and assume throughout that the inequalities

$$1 \le s_i \le t_i \le |E_i|, \qquad 1 \le a_i \le b_i \le |E_i|$$

are valid for all edges E_i.

Given a coloring $\varphi : X \to \mathbb{N}$ and any $Y \subseteq X$, we denote by $\mu(Y)$ (by $\pi(Y)$, respectively) the largest cardinality of a monochromatic (resp. polychromatic) subset of Y. A *stable coloring* of $\mathcal{H}$ is a mapping such that

$$s_i \le \pi(E_i) \le t_i \qquad \text{and} \qquad a_i \le \mu(E_i) \le b_i \qquad \text{for all} \qquad E_i \in \mathcal{E}$$

If these conditions are met, we may also call φ a *proper (vertex) coloring* of $\mathcal{H}$. Hence, the bounds s_i and t_i force that the largest *polychromatic* subset of each E_i has at least s_i and at most t_i vertices, whereas the largest *monochromatic* subset of E_i must have at least a_i and at most b_i colors.

We shall refer to $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$ as *color-bound functions*, and to s_i, t_i, a_i, b_i as *color-bounds* on edge E_i. In this setting the pair $(\boldsymbol{s}, \boldsymbol{t})$ restricts the *maximum number* of colors inside an edge, while $(\boldsymbol{a}, \boldsymbol{b})$ is responsible for the *maximum multiplicity* of colors on the edge. We introduce the terminology *stably bounded hypergraph* for $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$; this phrase may be viewed as an alternative rewritten form of '$(\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$-ly bounded'.

Our paper is a continuation of the study of *color-bounded hypergraphs*, introduced and investigated in [**2**–**4**], in which only the functions $\boldsymbol{s}, \boldsymbol{t}$ were considered as restrictions. In those works, strongly motivated by the paper of Drgas-Burchardt and Lazuka [**6**] (concerning the function $\boldsymbol{s}$ itself) and the concept of mixed hypergraphs, substantial differences have been pointed out between the classes of mixed and color-bounded hypergraphs. Though more general than all those, our present model with its four color-bound functions is still a subclass of the 'pattern hypergraphs' introduced by Dvořák et al. in [**7**], since in the latter the collection of feasible coloring patterns may be specified for each edge separately. Compared to that, however, our more restrictive conditions allow us to prove stronger results.

In the present paper we give a detailed analysis of the relations among the four color-bound functions. The subsets of $\{\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}\}$, as combinations of nontrivial conditions on colorability, form a hierarchy with respect to the strength of models concerning vertex coloring. In a way, the pair $(\boldsymbol{s}, \boldsymbol{a})$ is universal; but, interestingly enough, the partial order among the classes is not always the same, as it may depend on the aspect under which the allowed colorings are compared. Our results indicate that concerning the possible numbers of colors on a given number of vertices, the more restrictive function is the *monochromatic* upper bound $\boldsymbol{b}$ (cf. Theorems 1 and 2), while with respect to the number of color partitions in general the stronger restriction is the *polychromatic* upper bound $\boldsymbol{t}$ (see Section 2.3).

Although the decision problem whether a hypergraph admits any proper coloring is NP-complete for all nontrivial combinations of the conditions, nevertheless some algorithmic questions exhibit further substantial differences among the color-bound types. This fact is demonstrated concerning unique colorability in Section 3. On the other hand, there are subclasses of stably bounded hypergraphs that admit efficient coloring algorithms. Some of them will be discussed in the forthcoming paper [**5**].

While revising the manuscript, we have learned that the subclass of hypergraphs with bound $\boldsymbol{b}$ was studied previously in [**14**], [**11**] and [**1**], especially concerning approximation algorithms for the minimum number of colors in a proper coloring.

Further notation and terminology. Conditions of the types $s_i = 1$, $t_i = |E_i|$, $a_i = 1$, and $b_i = |E_i|$ have no effect on the colorability properties of $\mathcal{H}$, because they are trivially satisfied in every coloring. For this reason, we may restrict our attention to the subset of $\{\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}\}$ that really means some conditions on at least one edge. We shall use Capital letters to indicate them. For instance, by an (S, T)-hypergraph we mean one where $a_i = 1$ and $b_i = |E_i|$ hold for all edges. In such hypergraphs it is usually the case – though not required by definition – that there is at least one edge $E_{i'}$ with $s_{i'} > 1$ and at least one edge $E_{i''}$ with $t_{i''} < |E_{i''}|$.

Otherwise, e.g. if $s_i = 1$ also holds for all i, we may simply call it a T-hypergraph.

Colorability, feasible sets, chromatic spectrum. A hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ may admit no colorings at all; this is the case already with the mixed hypergraphs. We call $\mathcal{H}$ *colorable* if it has at least one proper coloring; and otherwise we say that it is *uncolorable*.

Assume that $\mathcal{H}$ is colorable. By a *k-coloring* we mean a proper vertex coloring with *exactly* k colors; that is, a coloring $\varphi : X \to \mathbb{N}$ with $|\varphi(X)| = k$. (Observe that this is a slight deviation from standard terminology.) The set

$$\Phi(\mathcal{H}) := \{k \mid \mathcal{H} \text{ has a } k\text{-coloring}\}$$

is termed the *feasible set* of $\mathcal{H}$. Assuming that $\mathcal{H}$ is colorable, the largest and smallest possible numbers of colors in a feasible coloring are termed the *upper chromatic number* and *lower chromatic number* of $\mathcal{H}$, respectively. In notation,

$$\chi(\mathcal{H}) = \min \Phi(\mathcal{H}), \qquad \overline{\chi}(\mathcal{H}) = \max \Phi(\mathcal{H}).$$

Each k-coloring φ of $\mathcal{H}$ induces a *color partition*, $X = X_1 \cup \cdots \cup X_k$, where the X_i are the inclusionwise maximal monochromatic subsets of X in the coloring φ. We shall denote by r_k the number of proper color partitions with precisely k nonempty classes. By the *chromatic spectrum* of $\mathcal{H}$ we mean the $\overline{\chi}$-tuple $(r_1, r_2, \ldots, r_{\overline{\chi}})$. We should mention, however, that in other parts of the literature (see [**20**]) the chromatic spectrum is defined as the n-tuple $(r_1, r_2, \ldots, r_n)$. Nevertheless, it is clear that the two representations are equivalent whenever the number of vertices is irrelevant; therefore we prefer to keep the shorter notation in the present context.

2. SMALL VALUES AND REDUCTIONS

Here we point out some simple relations among the color-bound functions $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$. It will turn out that on 3-uniform hypergraphs without further restrictions, four different models are equivalent. On the other hand, for hypergraphs with arbitrary edge sizes, one of them is universal.

Proposition 1. *Let E_i be an edge in a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$. If $|E_i| \le 3$, then $\pi(E_i) + \mu(E_i) = |E_i| + 1$.*

Proof. It suffices to observe that any E_i has a unique partition into 1 or $|E_i|$ classes, verifying $\pi(E_i) + \mu(E_i) = |E_i| + 1$ for such trivial partitions; moreover, if $|E_i| = 3$, then the size distribution in precisely two nonempty partition classes is uniquely determined as $(2, 1)$, so that $\pi(E_i) = \mu(E_i) = 2$ in this case. $\square$

Corollary 1. *Let $E_i \in \mathcal{E}$ be an edge with at most three vertices.*

1. *If $|E_i| = 1$, then $s_i = t_i = a_i = b_i = 1$ necessarily holds, and the edge may be deleted without changing the coloring properties of $\mathcal{H}$.*

2. *If* $|E_i| = 2$, *then between the local conditions the following equivalences are valid for* $k = 1, 2$.

   (i) $s_i = k \Leftrightarrow b_i = 3 - k$,

   (ii) $a_i = k \Leftrightarrow t_i = 3 - k$.

3. *If* $|E_i| = 3$, *then between the local conditions the following equivalences are valid for* $k = 1, 2, 3$.

   (i) $s_i = k \Leftrightarrow b_i = 4 - k$,

   (ii) $a_i = k \Leftrightarrow t_i = 4 - k$.

An important consequence is that, in the restricted class of 3-uniform hypergraphs, each pair in $(\boldsymbol{s}, \boldsymbol{b}) \times (\boldsymbol{t}, \boldsymbol{a})$ represents any nontrivial combination of $\boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b}$ in full generality:

Corollary 2. *If each edge of* $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ *has at most three vertices, then* $\mathcal{H}$ *has an equivalent description as an*

- (S, T)*-hypergraph,*
- (S, A)*-hypergraph,*
- (T, B)*-hypergraph,*
- (A, B)*-hypergraph.*

Proof. Based on Corollary 1, every s-condition and a-condition can be transcribed to an equivalent b-condition and t-condition, respectively; and vice versa. □

The coincidences of conditions above do not carry over for edges with $|E_i| > 3$. Indeed, a 4-element set admits 2-partitions of both types $2 + 2$ and $3 + 1$ (and the situation is even worse for larger edges), hence there is no strict relation between $\pi(E_i)$ and $\mu(E_i)$ in either direction. Nevertheless, the following implications remain valid for edges of any size, by the pigeon-hole principle.

Proposition 2. *Let* E_i *be any edge in a hypergraph* $\mathcal{H} = (X, \mathcal{E},\ \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$. *Then, between the conditions the following equivalences are valid*:

(i) $s_i = 2 \Leftrightarrow b_i = |E_i| - 1$,

(ii) $a_i = 2 \Leftrightarrow t_i = |E_i| - 1$. □

In particular, for mixed hypergraphs we obtain

Corollary 3. *Every mixed hypergraph is a member of all the four classes of* (S, T)-, (S, A)-, (T, B)-, *and* (A, B)*-hypergraphs at the same time.*

Proof. Every $\mathcal{C}$-edge E_i can be interpreted as $(s_i, a_i) = (1, 2)$, a $\mathcal{D}$-edge E_i corresponds to the bounds $(s_i, a_i) = (2, 1)$, whereas a bi-edge E_i is equivalent to the bounds $(s_i, a_i) = (2,2)$. This yields membership in (S, A). Transcription to the other three models can be done via Proposition 2. $\square$

REMARK. Contrary to mixed hypergraphs, in the general model the edges E_i of cardinality 2 with $t_i = 1$ or $a_i = 2$ usually cannot be contracted, despite their two vertices must get the same color in every proper coloring. The reason is that in $(\boldsymbol{a}, \boldsymbol{b})$ the multiplicities of colors are of essence. To keep track of them, one would need to introduce weighted vertices and interpret $(\boldsymbol{a}, \boldsymbol{b})$ as weighted conditions. We do not study weighted hypergraphs in the present paper.

2.1. CLASS REDUCTIONS AND COLORABILITY

Some combinations between color-bound conditions can be done; moreover, some of their combinations always admit a proper coloring. We summarize these facts as follows.

Table 1

1. *Colorable pairs*:
   - (S, B)-hypergraphs allow every edge to be polychromatic, therefore the upper chromatic number equals the number of vertices.
   - (T, A)-hypergraphs allow every edge to be monochromatic, therefore the lower chromatic number equals 1.
2. *Combinations admitting uncolorability*:

   These are the sets of color-bound functions intersecting both (S, B) and (T, A); i.e., the minimal such sets are the pairs (S, T), (S, A), (A, B), and (T, B). The following operations show that all of them can be reduced to (S, A).
   - $b_i < |E_i|$: insert all $(b_i + 1)$-subsets of E_i with lower color-bound $s = 2$, and omit the condition b_i from E_i. This eliminates the function $\boldsymbol{b}$.
   - $t_i < |E_i|$: insert all $(t_i + 1)$-subsets of E_i with lower color-bound $a = 2$, and omit the condition t_i from E_i. This eliminates the function $\boldsymbol{t}$.
3. *Universal classes for colorability problems*:
   - S-hypergraphs [**6**] are universal models for n-colorable stably bounded structures where the question is to determine χ.
   - A-hypergraphs are universal models for 1-colorable stably bounded structures where the question is to determine $\overline{\chi}$.
   - (S, A)-hypergraphs are universal models for stably bounded structures where both χ and $\overline{\chi}$ are of interest.

Concerning feasible sets, the following assertions are valid.

Proposition 3. *If a hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{s}, \boldsymbol{t}, \boldsymbol{a}, \boldsymbol{b})$ has $\chi(\mathcal{H}) = 1$ or $\overline{\chi}(\mathcal{H}) = |X|$, then its chromatic spectrum is continuous. Moreover, every interval of positive integers can be realized as the feasible set of hypergraphs with just one edge in each of the four types (S, T), (S, A), (T, B), and (A, B); and such a realization is possible even with an S-hypergraph and with a B-hypergraph.*

Proof. If $\chi(\mathcal{H}) = 1$, then we have non-restrictive bounds $s_i = 1$ and $b_i = |E_i|$ for all edges E_i. Let φ be a k-coloring of $\mathcal{H}$ with $k > 1$. Taking the union of two arbitrarily chosen color classes of φ as just one new color class, no edge E_i will have smaller $\mu(E_i)$ or larger $\pi(E_i)$, hence a proper $(k-1)$-coloring is constructed. Starting from $k = \overline{\chi}(\mathcal{H})$, we obtain that all numbers of colors between 1 and $\overline{\chi}$ admit a proper coloring.

Similarly, for $\overline{\chi}(\mathcal{H}) = |X|$ we have $t_i = |E_i|$ and $a_i = 1$ for all E_i. If φ is a k-coloring of $\mathcal{H}$ with $k < |X|$, then some color class has more than one vertex, and splitting it into two nonempty classes in an arbitrary way we cannot violate the conditions s_i and b_i, so that a proper $(k+1)$-coloring is obtained. Starting from $k = \chi(\mathcal{H})$, all numbers of colors between χ and $|X|$ admit a proper coloring.

In order to construct a hypergraph $\mathcal{H} = (X, \mathcal{E})$ with feasible set $\Phi(\mathcal{H}) = \{\ell \mid p \le \ell \le q\}$ for any given $q \ge p \ge 1$, we let $|X| = q$ that will ensure $\overline{\chi} = q$ for both S- and B-hypergraphs. To satisfy the equation $\chi = p$, we may simply assign $s = p$ to an edge whose cardinality is between p and q. In type B, this edge should have cardinality exactly p, assigned with the color-bound $b = 1$ that makes it polychromatic. □

Remark 2. Conditions involving S are more flexible than those with B. Namely, for the types (S, T) and (S, A) we may take $\boldsymbol{s}(X) = p$ with any number n of vertices, because either of the conditions $\boldsymbol{t}(X) = q$ and $\boldsymbol{a}(X) = n - q + 1$ yields then $\overline{\chi} = q$, as the total number of colors cannot be larger than $n - \boldsymbol{a}(X) + 1$. On the other hand, concerning the hypergraph $\mathcal{H} = (X, \{X\})$ the only possible choice for $\boldsymbol{b}(X)$ is $\lceil n/p \rceil$, which guarantees $\chi = p$ if and only if $(p-1)\lceil n/p \rceil \le n - 1$. Thus, for (T, B) and (A, B) the set of feasible orders n is precisely $\bigcup_{b \ge 1} \{n \in \mathbb{N} \mid pb - b + 1 \le n \le pb\}$.

Proposition 4. *For every finite sequence $(r_2, \ldots, r_k)$ of nonnegative integers with $r_k > 0$, there exist (S, T)-, (S, A)-, $(A, B)-$, and (T, B)-hypergraphs whose upper chromatic number is k and chromatic spectrum is $(r_1 = 0, r_2, \ldots, r_k)$.*

Proof. As proved by Král' in [**9**], every spectrum $(r_2, \ldots, r_k)$ occurs in non-1-colorable (i.e., with $r_1 = 0$) mixed hypergraphs. Since every mixed hypergraph belongs to all of the four types by Corollary 3, the assertion follows. □

Hence, in hypergraphs $\mathcal{H}$ belonging to class types other than the trivially colorable ones which are subsets of $\{S, B\}$ and $\{T, A\}$, it remains a substantial question to determine the feasible set $\Phi(\mathcal{H})$. On the other hand, chromatic spectra and chromatic polynomials are of interest for trivially colorable classes, too.

2.2. LARGE GAPS IN THE CHROMATIC SPECTRUM

JIANG et al. constructed in [**8**] a *mixed* hypergraph on $2k+4$ vertices and with a gap of size k in the chromatic spectrum, for all $k \geq 1$. This $2k+4$ is the smallest possible order, what follows from another result of the same paper (though this consequence is not formulated there explicitly).

In this subsection we extend this result by pointing out that the minimality of $2k+4$ for a gap of size k remains valid in the more general class of (T, A, B)-hypergraphs, too. In contrast to this, in [**3**] the exact minimum for (S, T)-hypergraphs has been proved to be $k+5$, and at the end of this subsection we show that the same is valid for (S, A)-hypergraphs as well.

First, we prove an assertion that we shall use as a lemma but it can be of interest in itself, too. It contains, as subcases, all the types of (T, B)-, (A, B)-, and mixed hypergraphs.

Proposition 5. *If a (T, A, B)-hypergraph on n vertices has a gap at g, then its lower chromatic number χ is at least $2g-n+2$.*

Proof. By definition, there exists an integer $j \geq g+1$ such that the hypergraph has a coloring φ with exactly j colors but there is no proper $(j-1)$-coloring.

Suppose first that $2j-2 \geq n$. Then there occur at least $2j-n \geq 2$ singleton color classes in φ. Considering two of them, say $\{x\}$ and $\{y\}$, their union yields a non-feasible $(j-1)$-coloring. After the identification of $\varphi(x)$ and $\varphi(y)$, however, all the bounds t_i and a_i remain fulfilled. Consequently, the obtained $(j-1)$-coloring can be non-feasible only because of an edge E_i containing both vertices x and y and having bound $b_i = 1$. Thus, x and y must have different colors in every feasible coloring. For the same reason, any two vertices from the at least $2j-n$ singletons are differently colored in any χ-coloring, too. This implies $\chi \geq 2j-n$. Due to the condition $j \geq g+1$, the inequality $\chi \geq 2g-n+2$ follows.

On the other hand, if $2j-2 < n$, considering the gap at $g \leq j-1$ we obtain the upper bound $2g-n+2 \leq 2j-n \leq 1$, thus the inequality $\chi \geq 2g-n+2$ automatically holds. ☐

We mention the following consequence that was proved for *mixed* hypergraphs in [**8**]. Tightness follows from a construction of the same paper.

Corollary 4. *If a (T, A, B)-hypergraph is ℓ-colorable and has a gap at g, then it has at least $2g+2-\ell$ vertices.*

Theorem 1. *If a (T, A, B)-hypergraph has a gap of size $k \geq 1$ in its chromatic spectrum, then it has at least $2k+4$ vertices. Moreover, this bound is sharp; that is, for every positive integer k there exist mixed, (T, B)- and (A, B)-hypergraphs on $|X| = 2k+4$ vertices, whose chromatic spectrum has a gap of size k.*

Proof. Suppose that a (T, A, B)-hypergraph has an ℓ-coloring and an $(\ell+k+1)$-coloring, but all integers in between are gaps. Then we can apply Proposition 5 with $g = \ell + k$, so that $2\ell + 2k - n + 2 \leq \chi \leq \ell$ is obtained. Moreover, every

1-colorable hypergraph has continuous chromatic spectrum, hence $\ell \geq 2$ holds and the above facts imply that the lower bound $n \geq \ell + 2k + 2 \geq 2k + 4$ is valid.

To show that the bound is sharp, we consider the construction from [**8**]. The hypergraph $\mathcal{H}_{2,k+3}$ is defined on the $(2k+4)$-element vertex set $\{x_1, x_2, a_1, a_2, \dots, a_{k+1}, b_1, b_2, \dots, b_{k+1}\}$, with the following edges:

- Triples of the form $\{x_i, a_j, b_j\}$ for $i = 1, 2$ and for all $1 \leq j \leq k+1$. They are bi-edges in the mixed hypergraph and have bounds $(t, b) = (2, 2)$ or $(a, b) = (2, 2)$ in the other models.
- Quadruples of the form $\{a_i, a_j, b_i, b_j\}$ for all $1 \leq i < j \leq k+1$, as $\mathcal{D}$-edges, with bounds $(t, b) = (4, 3)$ or $(a, b) = (1, 3)$.
- Triples of the form $\{a_i, a_j, b_i\}$ and $\{a_i, b_i, b_j\}$ for any two distinct indices $i, j \in \{1, 2, \dots, k+1\}$. They are $\mathcal{C}$-edges or equivalently have bounds $(t, b) = (2, 3)$ and $(a, b) = (2, 3)$, respectively.
- The pair $\{x_1, x_2\}$ as a $\mathcal{D}$-edge, with bounds $(t, b) = (2, 1)$ or $(a, b) = (1, 1)$.

The feasible set of this hypergraph is $\{2, k+3\}$, as it was proved in [**8**]. This fact remains valid in all of the three models considered, thus the assertion follows. $\square$

In [**3**] we proved that (S, T)-hypergraphs can have a gap of size k only if the number of vertices is at least $k + 5$, and this bound is tight. Now, we extend the lower bound of this result to all stably bounded hypergraphs, and show that it is tight already for (S, A)-hypergraphs.

Theorem 2. *If a stably bounded hypergraph has a gap of size $k \geq 1$ in its chromatic spectrum, then it has at least $k + 5$ vertices. Moreover, this estimate is sharp, already for the type (S, A); that is, for every positive integer k there exists an (S, A)-hypergraph on $|X| = k + 5$ vertices, whose chromatic spectrum has a gap of size k.*

Proof. We have proved in Proposition 3 that if a stably bounded hypergraph has a 1-coloring or a totally polychromatic n-coloring, then its chromatic spectrum is continuous. Hence, the only possibility for having a gap of size $k \geq 1$ on fewer than $k + 5$ vertices would be with $n = k + 4$ and with the feasible set $\{2, k+3\}$.

Assume for a contradiction that this is the case. Because of 2-colorability, the inequality $s_i \leq 2$ is valid for every edge E_i. Let now φ be a coloring with precisely $k + 3 = n - 1$ colors. Then $k + 2$ of the color classes (i.e., all but one) in φ are singletons. Taking the union of two arbitrarily chosen 1-element classes, say $\{x\}$ and $\{y\}$, we get a non-feasible color partition. This change can never decrease the size of monochromatic subsets or increase the number of distinct colors occurring inside any edge, therefore all of the bounds a_i and t_i are kept satisfied.

Hence, there exists an edge E_i for which either the bound $s_i \leq 2$ gets violated, or its monochromatic subset becomes larger than b_i. The former means, however, that E_i becomes monochromatic. That is, we have $s_i = 2$ and $E_i = \{x, y\}$, hence

x and y are colored differently in every feasible coloring. On the other hand, since $\varphi(x) \neq \varphi(y)$, in the modified coloring the color of $\{x, y\}$ does not occur on any other vertex. Hence, if b_i gets violated, then $b_i = 1$ must hold, and again we can conclude that x and y are colored differently in every feasible coloring. This property is valid for any two of the $k + 2 \geq 3$ singletons, what contradicts to the assumption that the hypergraph is 2-colorable. Hence, there cannot exist any color-bounded hypergraphs with a gap of size k on fewer than $k + 5$ vertices.

To show that a gap of size k is realizable on $k + 5$ vertices, we refer to the corresponding construction from [**3**], that has feasible set $\{3, k + 4\}$. The (S, T)-hypertree described there has edges only of size 4, with bounds $(s, t) = (3, 3)$, what can be interpreted with bounds $(s, a) = (3, 2)$ in an equivalent way. Hence, an (S, A)-hypergraph with the required properties is obtained. □

Theorems 1 and 2 together characterize the minimum order of a hypergraph of any nontrivial type for a gap of size k: the minimum is $k + 5$ if and only if the type contains (S, T) or (S, A), and it is $2k + 4$ if and only if it does not contain S but contains B and at least one of T and A. In any other case, the spectrum is gap-free.

2.3. COMPARISON OF THE SETS OF CHROMATIC POLYNOMIALS

We have already seen that any type of nontrivial combinations of $\boldsymbol{s}$, $\boldsymbol{t}$, $\boldsymbol{a}$, $\boldsymbol{b}$ can be expressed with $(\boldsymbol{s}, \boldsymbol{a})$ on applying part 2 of Table 1, if no structural conditions are imposed; and, furthermore, for 3-uniform hypergraphs each pair in $(\boldsymbol{s}, \boldsymbol{b}) \times (\boldsymbol{t}, \boldsymbol{a})$ would work equally nicely. Here we prove that this latter equivalence is not valid in general.

To formulate observations providing a more detailed information, let us denote by $\mathcal{P}_{X,Y}$ and $\mathcal{P}_X$ the sets of chromatic polynomials belonging to the classes of hypergraphs of type (X, Y) and of type X, respectively, for any $X, Y \in \{S, T, A, B\}$. Similarly, the set of chromatic polynomials appearing in the case of mixed hypergraphs will be denoted by $\mathcal{P}_m$.

Theorem 3. *For the sets of chromatic polynomials belonging to* (S, A)-, (A, B)-, (S, T)-, (T, B)-, *and mixed hypergraphs, the relations* $\mathcal{P}_{S,A} = \mathcal{P}_{A,B} \supsetneqq \mathcal{P}_{S,T} = \mathcal{P}_{T,B} = \mathcal{P}_m$ *hold.*

Proof.

1. According to Corollary 3, each mixed hypergraph has a chromatic equivalent in each of those four stably bounded subclasses. Thus, the set $\mathcal{P}_m$ is contained in each of $\mathcal{P}_{S,A}$, $\mathcal{P}_{A,B}$, $\mathcal{P}_{S,T}$, and $\mathcal{P}_{T,B}$.

2. On the other hand, as it has been shown, the bound $b_i < |E_i|$ can be replaced by some (b_i+1)-element $\mathcal{D}$-edges, whilst the elimination of the bound $t_i < |E_i|$ can be done by inserting some $(t_i + 1)$-element $\mathcal{C}$-edges. Therefore, every (T, B)-hypergraph has a chromatically equivalent mixed hypergraph (on the

same vertex set). Taking into consideration the observation 1, the equality of $\mathcal{P}_{T,B}$ and $\mathcal{P}_m$ is obtained.

3. It was proved in [**3**] that $\mathcal{P}_{S,T} = \mathcal{P}_m$. It worth noting that there exists an (S,T)-hypergraph with no chromatic equivalent mixed hypergraph on the same number of vertices. For instance, due to [**3**], there exists an (S,T)-hypergraph with a gap of size 2 on seven vertices, whilst in the case of mixed hypergraphs it needs at least eight vertices, by a result of [**8**].

4. By the elimination of $\boldsymbol{t}$, the (S,T)-hypergraphs can be modeled in (S,A), hence $\mathcal{P}_{S,A} \supseteq \mathcal{P}_{S,T}$. We are going to show that the sets of chromatic spectra, and consequently also the chromatic polynomials, of (S,A)- and (S,T)-hypergraphs are not equal.

   Let $\mathcal{H}_{s,a}$ have four vertices and just one 4-element edge with bounds $a = 3$ and $s = 1$. Obviously, $r_1 = 1$ and $r_2 = 4$. On the other hand, it was proved in [**3**] that in 1-colorable (S,T)-hypergraphs the value of r_2 always is of the form $2^{n-1} - 1$. Since this property is not valid for $\mathcal{H}_{s,a}$, it cannot have a chromatically equivalent (S,T)-hypergraph.

5. Since any chromatic spectrum with $r_1 = 0$ belongs to some mixed hypergraphs, the same holds for (S,A)- and (A,B)-hypergraphs, too. Thus, a difference between $\mathcal{P}_{S,A}$ and $\mathcal{P}_{A,B}$ might occur only on hypergraphs with $r_1 = 1$. The assumption of 1-colorability in an (S,A)-hypergraph implies that every edge E_i has bounds $(s_i, a_i) = (1, a_i)$, whereas in an (A,B)-hypergraph it implies $(a_i, b_i) = (a_i, |E_i|)$ for every edge. These two color-bound conditions clearly are equivalent on each edge. Hence, the possible chromatic spectra and consequently the chromatic polynomials are the same: $\mathcal{P}_{S,A} = \mathcal{P}_{A,B}$.

   Nevertheless, there exist some (S,A)-hypergraphs not having chromatic equivalent (A,B)-hypergraphs on the same number of vertices. Similarly to the example in step 4 of the proof, one can see that there exists an (S,A)-hypergraph on seven vertices with feasible set $\{3,6\}$, but to generate this feasible set in (A,B)-hypergraphs needs at least eight (in fact, at least nine) vertices. □

As regards modeling with the same number of vertices, the previous proof yields the following observation.

Remark 3. Every mixed hypergraph has a chromatically equivalent (T,B)-hypergraph such that their vertex sets are of the same cardinality, and vice versa. This stronger condition does not hold for any other pairs of the models listed above.

We close this subsection with supplements of Theorem 3 regarding other types of stably bounded hypergraphs.

Proposition 6. *Concerning the possible chromatic polynomials of* S-, T-, A-, B-, $\mathcal{C}$- (*'mixed', without* $\mathcal{D}$-*edges*) *and* $\mathcal{D}$- (*classical*) *hypergraphs the following relations hold*:

1. $\mathcal{P}_m \supsetneqq \mathcal{P}_S = \mathcal{P}_{S,B} \supsetneqq \mathcal{P}_{\mathcal{D}} = \mathcal{P}_B$,

2. $\mathcal{P}_{S,A} \supsetneqq \mathcal{P}_A = \mathcal{P}_{A,T} \supsetneqq \mathcal{P}_{\mathcal{C}} = \mathcal{P}_T$,

3. *$\mathcal{P}_m$ and $\mathcal{P}_A$ are incomparable.*

Proof.

1. Every $\mathcal{D}$-edge E_i can be interpreted equivalently with bound $b_i = |E_i| - 1$, whilst a bound $b_i < |E_i|$ can be replaced by some $(b_i + 1)$-element $\mathcal{D}$-edges, therefore $\mathcal{P}_{\mathcal{D}} = \mathcal{P}_B$ holds.

   Every $\mathcal{D}$-edge evidently means an edge with bound $s = 2$, hence $\mathcal{P}_{\mathcal{D}} \subseteq \mathcal{P}_S$ is clear. On the other hand, let us consider the S-hypergraph $\mathcal{H} = (X, \{X\}, \boldsymbol{s})$ with $|X| = 5$ vertices and with color-bound $\boldsymbol{s}(X) = 3$. Its chromatic spectrum is $(0, 0, 25, 10, 1)$. Assuming a $\mathcal{D}$-hypergraph with this spectrum, it should have five vertices and each of its 3-partitions should yield a proper coloring. In particular, for any three vertices there should exist a coloring where they get the same color, implying that there can occur $\mathcal{D}$-edges only of sizes 4 and 5. Consequently, the 2-partitions with color classes of size 2 and 3 are not forbidden, what contradicts $r_2 = 0$. Therefore, this S-hypergraph has no equivalent $\mathcal{D}$-hypergraph, implying $\mathcal{P}_S \supsetneqq \mathcal{P}_{\mathcal{D}}$.

   By the elimination of $\boldsymbol{b}$, we can transform the structures of type (S, B) to type S, hence $\mathcal{P}_S = \mathcal{P}_{S,B}$. It is also clear that $\mathcal{P}_S \subseteq \mathcal{P}_{S,T} = \mathcal{P}_m$, and that mixed hypergraphs having gaps in their chromatic spectra cannot be modeled in S-hypergraphs. That is, $\mathcal{P}_S \subsetneqq \mathcal{P}_m$ is obtained.

2. Any $\mathcal{C}$-edge E_i can be considered as an edge with bound $t_i = |E_i| - 1$, whilst any bound $t_i < |E_i|$ can be expressed by $\mathcal{C}$-edges, hence $\mathcal{P}_{\mathcal{C}} = \mathcal{P}_T$.

   By eliminating $\boldsymbol{t}$, every (A, T)-hypergraph can be rewritten only with the bound $\boldsymbol{a}$, thus $\mathcal{P}_A = \mathcal{P}_{A,T}$. Moreover $\mathcal{P}_{S,A} \supseteq \mathcal{P}_A$ trivially holds.

   To show that there exist A-hypergraphs having no chromatically equivalent $\mathcal{C}$-hypergraphs, we recall the example from step 4 in the proof of Theorem 3. This (S, A)-hypergraph can be considered as just an A-hypergraph, and since it has no equivalent of type (S, T), the same is true for mixed- and $\mathcal{C}$-hypergraphs, too. Consequently, $\mathcal{P}_A \supsetneqq \mathcal{P}_{\mathcal{C}}$, and because of the 1-colorability of every A-hypergraph, $\mathcal{P}_{S,A} \neq \mathcal{P}_A$ is valid as well.

3. By the previous example there exist A-hypergraphs that have no equivalent mixed hypergraphs whereas mixed hypergraphs admitting no 1-coloring cannot be equivalent to any A-hypergraphs. □

Proposition 7. *Concerning the possible chromatic polynomials of stably bounded hypergraphs involving at least three types of conditions, the following equations hold:*

1. $\mathcal{P}_{S,T,A,B} = \mathcal{P}_{S,A,B} = \mathcal{P}_{S,A,T} = \mathcal{P}_{S,A}$,

2. $\mathcal{P}_{A,B,T} = \mathcal{P}_{A,B} = \mathcal{P}_{S,A}$,

3. $\mathcal{P}_{S,T,B} = \mathcal{P}_{S,T}$.

Proof. The reductions described in part 2 of Table 1 yield:

- The color-bound function $\boldsymbol{t}$ can be expressed by the function $\boldsymbol{a}$. Consequently, if a type contains T and A together, then omitting T the set of possible chromatic polynomials does not change.
- Similarly, the function $\boldsymbol{b}$ can be reduced to $\boldsymbol{s}$, therefore in the presence of S the cancelation of B cannot make a change in the set of possible chromatic polynomials.

These observations immediately imply the statements listed above, except for the last equation in part 2, what has been proved in Theorem 3. □

3. COMPLEXITY OF TESTING COLORABILITY

In this section we investigate the time complexity of the following two algorithmic problems.

COLORABILITY

Instance: A hypergraph $\mathcal{H}$ of a given type.

Question: Is $\mathcal{H}$ colorable?

UNIQUE k-COLORABILITY

Instance: A hypergraph $\mathcal{H}$ of a given type, together with a proper k-coloring φ.

Question: Does $\mathcal{H}$ admit any proper coloring other than φ?

For the former, we simply extend the NP-hardness result of [**4**] from (S,T)-hypergraphs to all nontrivial combinations of the color-bound functions. On the other hand, the situation with the latter problem is more interesting. We choose the value $k = n-1$ and prove that two of the non-trivial pairs, namely those containing S, lead to intractability; but the other two, containing B, admit a good characterization and polynomial-time algorithms.

In general, it should be noted that COLORABILITY clearly belongs to NP, whereas UNIQUE k-COLORABILITY is in co-NP. Moreover, since a hypergraph on n vertices cannot have more than $\binom{n}{2}$ proper $(n-1)$-colorings, we can see that for $k = n-1$ (and also if k is as large as n minus a constant) it does not change the complexity status of the problem if a k-coloring is not given in the input.

3.1. COLORABILITY OF 3-UNIFORM HYPERGRAPHS

It was first observed in [**17**] that the recognition problem of colorable mixed hypergraphs is NP-complete in general, and also when restricted to 3-uniform mixed hypergraphs. There are some important classes with a nice structure, however, that admit efficient algorithms.

A hypergraph $\mathcal{H} = (X, \mathcal{E})$ is called a *hypertree* if there exists a tree graph T on the same vertex set X as $\mathcal{H}$, such that each edge $E_i \in \mathcal{E}$ induces a subtree in T. In [**15**] a simple necessary and sufficient condition was given for the colorability of *mixed hypertrees*, from which an efficient algorithm is obtained, too.

On the other hand, we have shown in [**4**] that the colorability of 3-uniform (S, T)-hypertrees is NP-complete. We have also seen in Corollary 2 that every 3-uniform stably bounded hypergraph has equivalent representations with all the types of (S, T)-, (S, A)-, (T, B)-, and (A, B)-hypergraphs, and those can be constructed in linear time. In this way, an input of any of these types can efficiently be transformed to an (S, T)-hypergraph. Consequently, the result of [**4**] can be extended as follows.

Theorem 4. *The* COLORABILITY *problem is* NP*-complete on each of the following classes of hypergraphs*:

- 3*-uniform* (S, T)*-hypertrees*,
- 3*-uniform* (S, A)*-hypertrees*,
- 3*-uniform* (T, B)*-hypertrees*,
- 3*-uniform* (A, B)*-hypertrees.*

It is worth comparing Theorem 4 with the following results: there are linear-time algorithms for deciding whether a *mixed* hypertree is colorable, and also for finding a proper coloring if there exists one [**15**], whereas determining the upper chromatic number of a mixed hypertree without edges larger than three is NP-complete [**10**].

Let us note further that NP-completeness remains valid if we assume that the host tree is a star. On the other hand, it will be proved in the forthcoming paper [**5**] that 3-uniform stably bounded *interval* hypergraphs admit a linear-time colorability test and a linear-time coloring algorithm, too.

3.2. UNIQUELY $(n-1)$-COLORABLE (S, T)- AND (S, A)-HYPERGRAPHS

Although it is hard to test whether an unrestricted mixed hypergraph is uniquely colorable [**17**], this is not the case if $\overline{\chi}$ is very large. For the latter case, NICULITSA and VOSS [**12**] described a characterization of uniquely $(n-1)$-colorable, and also of uniquely $(n-2)$-colorable mixed hypergraphs.

In sharp contrast to this, we have proved in [**3**] that the recognition of uniquely $(n-1)$-colorable (S,T)-hypergraphs is hard. Here we show how the construction can be extended to (S,A)-hypergraphs.

Theorem 5. *The* Unique $(n-1)$-Colorability *problem is co-*NP*-complete on (S,A)-hypergraphs.*

Proof. As we have already mentioned, membership in co-NP is clear. To prove hardness, let us recall from [**3**] the reduction for (S,T)-hypergraphs, from the problem of determining the *chromatic number of Steiner triple systems*.

Phelps and Rödl proved in [**13**] that it is NP-complete to decide whether a Steiner triple system – viewed as a 3-uniform $\mathcal{D}$- (classical) hypergraph – is colorable with 14 colors. Given an input Steiner triple system $\mathcal{S} = STS(n-2) = (X,\mathcal{B})$ of order $n-2$ with vertex set $X = \{x_1,\ldots,x_{n-2}\}$ and edge set $\mathcal{B}$, an (S,T)-hypergraph $\mathcal{H} = (X',\mathcal{E},\boldsymbol{s},\boldsymbol{t})$ is constructed as follows. We set $X' = X \cup \{z_1,z_2\}$, where z_1, z_2 are two new vertices, and consider the following edges with respective color-bounds:

- $B' = B \cup \{z_1,z_2\}$ with $\boldsymbol{s}(B') = 4$ and $\boldsymbol{t}(B') = 5$, for all blocks $B \in \mathcal{B}$;
- $W' = W \cup \{z_1,z_2\}$ with $\boldsymbol{s}(W') = 1$ and $\boldsymbol{t}(W') = 16$, for all 15-element subsets of X;
- $e_{i,j} = \{x_i, z_j\}$ with $\boldsymbol{s}(e_{i,j}) = \boldsymbol{t}(e_{i,j}) = 2$, for all $1 \le i \le n-2$ and $j = 1,2$.

In this (S,T)-hypergraph, every t_i is either $|E_i|$ or $|E_i| - 1$. Hence, it is easy to eliminate $\boldsymbol{t}$ along the lines of Proposition 2 and obtain an equivalent (S,A)-hypergraph: we simply define

$$\boldsymbol{a}(B') = 1, \qquad \boldsymbol{a}(W') = 2, \qquad \boldsymbol{a}(e_{i,j}) = 1$$

for all edges $B', W', e_{i,j} \in \mathcal{E}$. From the argument in [**3**] it follows that $\mathcal{H}$ is *not* uniquely $(n-1)$-colorable if and only if $\mathcal{S}$ has a proper coloring with at most 14 colors; and certainly the same holds for the derived (S,A)-hypergraph, too. Thus, co-NP-hardness follows. □

3.3. UNIQUELY $(n-1)$-COLORABLE (T,B)- AND (A,B)-HYPERGRAPHS

In this subsection we characterize the uniquely $(n-1)$-colorable (T,B)- and (A,B)-hypergraphs. In the models (S,A) and (S,T) studied in the previous subsection, the decision problem of unique $(n-1)$-colorability was proved to be co-NP-complete. In contrast to this, the characterization presented below yields polynomial-time algorithms for (T,B)- and (A,B)-hypergraphs.

Before the characterization, let some terminology be introduced:

- $\{x,y\}$ is called $\mathcal{B}_1$-edge if x and y are contained in a common hyperedge E_i having bound $b_i = 1$. (This corresponds to a graph-edge in the usual sense.)

- E_i is called $\mathcal{B}_2$-edge if $b_i = 2$.
- E_i is called $\mathcal{C}$-edge if $t_i = |E_i| - 1$.

Concerning a given (T, B)-hypergraph, the set of $\mathcal{C}$-, $\mathcal{B}_1$- and $\mathcal{B}_2$-edges will be denoted by $\mathcal{C}$, $\mathcal{B}_1$ and $\mathcal{B}_2$, respectively. As a side-product of the characterization theorem, it will turn out that if an edge has bound $b_i \geq 3$, then the exact value of b_i has no influence on unique $(n-1)$-colorability.

Theorem 6. *A (T, B)-hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{t}, \boldsymbol{b})$ on $|X| = n$ vertices is uniquely $(n-1)$-colorable if and only if the following conditions hold*:

(α) $\max\limits_{E_i \in \mathcal{E}}(|E_i| - t_i) = 1$.

(β) *The set $C^* := \bigcap\limits_{C \in \mathcal{C}} C$ contains at least two vertices and induces a complete $\mathcal{B}_1$-graph minus one $\mathcal{B}_1$-edge.*

Moreover, denoting by y_1 and y_2 the vertices from the missing $\mathcal{B}_1$-edge,

(γ) *$X \setminus \{y_1, y_2\}$ is a complete $\mathcal{B}_1$-graph.*

(δ) *For each vertex $x \in X \setminus C^*$, at least one of the relations $\{x, y_1\} \in \mathcal{B}_1$, $\{x, y_2\} \in \mathcal{B}_1$ and $\{x, y_1, y_2\} \subseteq E_i \in \mathcal{B}_2$ holds.*

(ϵ) *For each pair of vertices $x_j, x_k \in X \setminus C^*$, if $\{x_j, x_k\}$ intersects every $\mathcal{C}$-edge, then either there exist $\mathcal{B}_1$-edges $\{z, x_j\}$ and $\{z, x_k\}$ for a $z \in \{y_1, y_2\}$, or there exist $\mathcal{B}_1$-edges $\{z, y_1\}$ and $\{z, y_2\}$ for a $z \in \{x_j, x_k\}$.*

Proof. Consider a uniquely $(n-1)$-colorable (T, B)-hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{t}, \boldsymbol{b})$. Since it admits an $(n-1)$-coloring, where each edge has at least $|E_i| - 1$ colors, $t_i \geq |E_i| - 1$ holds. On the other hand, since the n-coloring is not feasible, there is some hyperedge with bound $t_i = |E_i| - 1$. Consequently, we have $\max\limits_{E_i \in \mathcal{E}}(|E_i| - t_i) = 1$, according to ($\alpha$).

Let φ be a proper $(n-1)$-coloring, and assume without loss of generality that its color classes are $\{x_1\}, \{x_2\}, \ldots, \{x_{n-2}\}$, and $\{y_1, y_2\}$. Every $\mathcal{C}$-edge has to involve vertices with a common color by φ, moreover the color class $\{y_1, y_2\}$ cannot be a $\mathcal{B}_1$-edge, therefore:

$$(\beta_1) \qquad \{y_1, y_2\} \subseteq C^* \qquad \text{and} \qquad \{y_1, y_2\} \notin \mathcal{B}_1.$$

Taking the union of any two color classes from φ, the obtained $(n-2)$-coloring is not feasible, what can be caused only by breaking some bound b_i. We are going to analyze the various vertex partitions with $n-2$ classes.

- The contraction of any two singletons $\{x_j\}$ and $\{x_k\}$ is forbidden, hence there exists an edge $E_i \supseteq \{x_j, x_k\}$ with bound $b_i = 1$. That is, $\{x_j, x_k\} \in \mathcal{B}_1$ for every $1 \leq j < k \leq n-2$, so that (γ) holds.

- The contraction of any singleton $\{x_j\}$ and $\{y_1, y_2\}$ is also forbidden by some bound b_i, consequently at least one of the alternatives from (δ) holds.
- Any $(n-2)$-partition containing the two non-singleton color classes $\{x_j, y_1\}$ and $\{x_k, y_2\}$ is non-feasible, hence either a $\mathcal{C}$-edge omits both x_j and x_k (since it includes both y_1 and y_2), or there occur $\mathcal{B}_1$ edges in both sets $\{\{x_j, y_1\}, \{x_k, y_2\}\}$ and $\{\{x_j, y_2\}, \{x_k, y_1\}\}$. This means, the implication of (ϵ) is valid.

By assumption, φ is the unique coloring of $\mathcal{H}$; thus, the coloring with singletons and the only two-element color-class $\{x_j, y_k\}$ is non-feasible for all $1 \le j \le n-2$ and $1 \le k \le 2$. If x_j belongs to each $\mathcal{C}$-edge, the bounds t_i are fulfilled, hence in this case there surely occurs $\{x_j, y_k\}$ as a $\mathcal{B}_1$-edge:

(β_2) If $x_j \in C^*$, then $\{x_j, y_1\} \in \mathcal{B}_1$ and $\{x_j, y_2\} \in \mathcal{B}_1$ hold.

The properties (β_1), (β_2) and (γ) together ensure the existence of a complete $\mathcal{B}_1$-graph minus one $\mathcal{B}_1$-edge on the intersection of $\mathcal{C}$-edges, implying that (β) is fulfilled, too.

Now, assume a hypergraph $\mathcal{H}$ satisfying the conditions $(\alpha) - (\epsilon)$ of the theorem. Unique $(n-1)$-colorability is verified as follows:

- By the requirement (α), the hypergraph admits no n-coloring.
- Consider the $(n-1)$-coloring φ, where the only monochromatic vertex pair is $\{y_1, y_2\}$. According to (β), both y_1 and y_2 are contained in each $\mathcal{C}$-edge, and hence, due to (α) all the bounds from $\boldsymbol{t}$ are satisfied. Since $\{y_1, y_2\} \notin \mathcal{B}_1$, every hyperedge E_i containing both y_1 and y_2, has bound $b_i \ge 2$, whilst each of the remaining hyperedges involves no monochromatic vertex pair. Therefore, all bounds from $\boldsymbol{b}$ are fulfilled, the color partition $\{x_1\}, \{x_2\}, \ldots, \{x_{n-2}\}, \{y_1, y_2\}$ is feasible.
- According to (γ):

  $(\star)$ There is no feasible partition with a color class containing both x_j and x_k (for all $1 \le j < k \le n-2$).

  Thus, the only possibility for a second $(n-1)$-coloring would be a partition with 2-element color class $\{x_j, y_k\}$ (for some $1 \le j \le n-2$ and $1 \le k \le 2$). But if $x_j \in C^*$, there is contained a forbidden $\mathcal{B}_1$-edge due to (β), whilst if $x_j \notin C^*$, then some forbidden polychromatic $\mathcal{C}$-edge would arise. Consequently, no $(n-1)$-coloring different from φ can be feasible.
- To prove that no $(n-2)$-colorings exist:

  $(\star\star)$ There is no feasible partition containing $\{x_j, y_1, y_2\}$ as a color class. If $x_j \in C^*$, the class contains two forbidden $\mathcal{B}_1$-edges, due to (β). And if $x_j \notin C^*$, the property (δ) ensures that there occurs either a forbidden $\mathcal{B}_1$-edge in the 3-element color class, or this class is involved in a $\mathcal{B}_2$-edge. All these cases are impossible.

($\star\star\star$) The pairs $\{x_j, y_1\}$ and $\{x_k, y_2\}$ cannot be color classes simultaneously. Such a coloring is trivially non-feasible if there exists a $\mathcal{C}$-edge containing neither x_j nor x_k. Also, if at least one of the vertices x_j and x_k is contained in C^*, the partition is forbidden by a $\mathcal{B}_1$-edge according to (β). In the third case, when all $\mathcal{C}$-edges meet $\{x_j, x_k\}$ but their intersection doesn't, the conditions of (ϵ) are satisfied, hence its conclusion excludes the feasibility of this partition.

The claims ($\star$), ($\star\star$) and ($\star\star\star$) together imply that the hypergraph $\mathcal{H}$ admits no $(n-2)$-coloring.

- Because of ($\star$), the vertices $x_1, x_2, \ldots, x_{n-2}$ have mutually distinct colors in every feasible coloring, therefore $\mathcal{H}$ admits no coloring with fewer than $n-2$ colors.

Thereupon, the hypergraph is uniquely $(n-1)$-colorable, and this completes the proof. □

There is no restriction for the exact value of bounds $b_i \geq 3$ in the characterization, therefore we immediately get the following corollary:

Corollary 5. *Let $\mathcal{H}$ and $\mathcal{H}'$ be (T, B)-hypergraphs on n vertices, and suppose that $\mathcal{H}'$ can be obtained from $\mathcal{H}$ by replacing each bound $b_i \geq 3$ with some bound $3 \leq b_i' \leq |E_i|$. Then $\mathcal{H}$ is uniquely $(n-1)$-colorable if and only if so is $\mathcal{H}'$. In particular, concerning unique $(n-1)$-colorability, $\mathcal{H}$ can be reduced to a T-hypergraph supplemented with some $\mathcal{B}_1$- edges and 3-element $\mathcal{B}_2$-edges, that is, with a classical ($\mathcal{D}$-) hypergraph of rank at most three.*

Except for the first property (α), the above characterization gives conditions only for the color-bound function $\boldsymbol{b}$ and for the edges having bound $t_i = |E_i| - 1$. Since the restriction $\max_{E_i \in \mathcal{E}} (|E_i| - t_i) = 1$ can be equivalently expressed with the bound $\boldsymbol{a}$, we get an analogous characterization for uniquely $(n-1)$-colorable (A, B)-hypergraphs, too. The terms $\mathcal{B}_1$ and $\mathcal{B}_2$ are used as above; $\mathcal{C}$-edge means a hyperedge E_i with bound $a_i = 2$.

Theorem 7. *An (A, B)-hypergraph $\mathcal{H} = (X, \mathcal{E}, \boldsymbol{a}, \boldsymbol{b})$ with $|X| = n$ vertices is unique $(n-1)$-colorable if and only if the following conditions hold:*

(α') $\max\limits_{E_i \in \mathcal{E}} a_i = 2.$

(β) *The set $C^* := \bigcap\limits_{C \in \mathcal{C}} C$ contains at least two vertices and induces a complete $\mathcal{B}_1$-graph minus one $\mathcal{B}_1$-edge.*

Moreover, denoting by y_1 and y_2 the vertices of the omitted $\mathcal{B}_1$-edge,

(γ) *$X \setminus \{y_1, y_2\}$ is a complete $\mathcal{B}_1$-graph.*

(δ) *For each vertex $x \in X \setminus C^*$, at least one of the relations $\{x, y_1\} \in \mathcal{B}_1$, $\{x, y_2\} \in \mathcal{B}_1$ and $\{x, y_1, y_2\} \subseteq E_i \in \mathcal{B}_2$ holds.*

(ϵ) *For each pair of vertices $x_j, x_k \in X \setminus C^*$, if $\{x_j, x_k\}$ intersects every $\mathcal{C}$-edge, then either there exist $\mathcal{B}_1$-edges $\{z, x_j\}$ and $\{z, x_k\}$ for some $z \in \{y_1, y_2\}$, or there exist $\mathcal{B}_1$-edges $\{z, y_1\}$ and $\{z, y_2\}$ for some $z \in \{x_j, x_k\}$.*

Proof. If the condition (α') is valid for a given (A, B)-hypergraph, we can replace each bound $a_i = 2$ by $t_i = |E_i| - 1$, whilst the non-restricting $a_i = 1$ can be rewritten as $t_i = |E_i|$, and we get a chromatically equivalent (T, B)-hypergraph on the same vertex set. The obtained (T, B)-hypergraph is uniquely $(n-1)$-colorable if and only if so is the original (A, B)-hypergraph. Also, the $\mathcal{B}_1$-, $\mathcal{B}_2$- and $\mathcal{C}$-edges are the same, hence in this case the conditions (α')–(ϵ) give an exact characterization for the (A, B)-hypergraph.

On the other hand, if the condition (α') does not hold, then either $\overline{\chi} < n - 1$ or $\overline{\chi} = n$ or the hypergraph is uncolorable, so it is not uniquely $(n - 1)$-colorable in either case. Hence, (α') is indeed necessary for unique $(n - 1)$-colorability. □

As it was our purpose, the characterization theorems make it possible to design polynomial-time algorithms for testing unique $(n-1)$-colorability in the two hypergraph classes in question. As a matter of fact, on the one hand it is obvious that the condition $a_i \leq 2$ is necessary for unique $(n-1)$-colorability in *every* stably bounded hypergraph; while, on the other hand, the proof of Theorem 7 shows that if this condition holds, then the color-bound function $\boldsymbol{a}$ can completely be replaced with a suitably chosen $\boldsymbol{t}$. Thus, the following more general result is obtained.

Theorem 8. *The decision problem* UNIQUE $(n-1)$-COLORABILITY *can be solved in polynomial time for (T, A, B)-hypergraphs.*

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SOME FOX-WRIGHT GENERALIZED HYPERGEOMETRIC FUNCTIONS AND ASSOCIATED FAMILIES OF CONVOLUTION OPERATORS

H. M. Srivastava

Here, in this lecture, we aim at presenting a systematic account of the basic properties and characteristics of several subclasses of analytic functions (with *Montel's normalization*), which are based upon some convolution operators on Hilbert space involving the Fox-Wright generalization of the *classical* hypergeometric ${}_qF_s$ function (with q numerator and s denominator parameters). The various results presented in this lecture include (for example) normed coefficient inequalities and estimates, distortion theorems, and the radii of convexity and starlikeness for each of the analytic function classes which are investigated here. We also briefly indicate the relevant connections of the some of the results considered here with those involving the Dziok-Srivastava operator.

1. INTRODUCTION, DEFINITIONS AND PRELIMINARIES

Following the usual notations, we let $\mathcal{A}$ denote the class of functions f of the form:

$$f(z) = \sum_{n=1}^{\infty} a_n z^n \qquad (a_1 > 0), \tag{1.1}$$

which are *analytic* in $\mathbb{U} := \mathbb{U}(1)$, where

$$\mathbb{U}(r) := \{z : z \in \mathbb{C} \qquad \text{and} \qquad |z| < r\}.$$

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For the class $\mathcal{A}$, the normalization:

$$f(0) = f'(0) - 1 = 0, \tag{1.2}$$

is *classical*. As already observed by Dziok and Srivastava [**6**], one can obtain interesting results by applying *Montel's normalization* of the form (*cf.* Montel [**13**]):

$$f(0) = f'(\rho) - 1 = 0 \tag{1.3}$$

or

$$f(0) = f(\rho) - \rho = 0, \tag{1.4}$$

where ρ is a fixed point of the *punctured* unit disk

$$\mathbb{U}^* := \mathbb{U} \setminus \{0\} = \{z : z \in \mathbb{C} \qquad \text{and} \qquad 0 < |z| < 1\}.$$

The classes of functions with the normalizations (1.3) and (1.4) will henceforth be called the *classes of functions with two fixed points* (see Dziok and Srivastava [**6**, p. 8]).

A function f belonging to the class $\mathcal{A}$ is said to be *convex* in $\mathbb{U}(r)$ if and only if (*cf.* [**17**] and [**18**])

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0 \qquad \big(z \in \mathbb{U}(r);\ 0 < r \leqq 1\big).$$

On the other hand, a function f belonging to the class $\mathcal{A}$ is said to be *starlike* in $\mathbb{U}(r)$ if and only if (*cf.* [**17**] and [**18**])

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0 \qquad \big(z \in \mathbb{U}(r);\ 0 < r \leqq 1\big).$$

Suppose now that $\mathcal{B}$ is a subclass of the class $\mathcal{A}$. We define the radius of starlikeness $R^*(\mathcal{B})$ and the radius of convexity $R^c(\mathcal{B})$ for the class $\mathcal{B}$ by

$$R^*(\mathcal{B}) := \inf_{f \in \mathcal{B}} \big(\sup\{r \in (0,1] : f \quad \text{is starlike in} \quad \mathbb{U}(r)\}\big)$$

and

$$R^c(\mathcal{B}) := \inf_{f \in \mathcal{B}} \big(\sup\{r \in (0,1] : f \quad \text{is convex in} \quad \mathbb{U}(r)\}\big),$$

respectively.

For two given analytic functions

$$f(z) = \sum_{n=0}^{\infty} a_n\, z^n \qquad \text{and} \qquad g(z) = \sum_{n=0}^{\infty} b_n\, z^n,$$

we denote by $f * g$ the *Hadamard product* (or *convolution*) of f and g defined by

$$(f * g)(z) := \sum_{n=0}^{\infty} a_n\, b_n\, z^n =: (g * f)(z)\,. \tag{1.5}$$

For complex parameters

$$\alpha_1, \ldots, \alpha_q \qquad \left(\frac{\alpha_j}{A_j} \neq 0, -1, -2, \ldots;\ j = 1, \ldots, q\right)$$

and

$$\beta_1, \ldots, \beta_s \qquad \left(\frac{\beta_j}{B_j} \neq 0, -1, -2, \ldots;\ j = 1, \ldots, s\right),$$

we define the Fox-Wright generalization ${}_q\Psi_s$ of the hypergeometric ${}_qF_s$ function by (*cf.* Fox [**8**] and Wright ([**20**] and [**21**]; see also [**15**, p. 21] and [**14**, p. 19])

$$\begin{aligned} {}_q\Psi_s \left[\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_q, A_q); \\ \\ (\beta_1, B_1), \ldots, (\beta_s, B_s); \end{array} z\right] &= {}_q\Psi_s \left[(\alpha_j, A_j)_{1,q}; (\beta_j, B_j)_{1,s}; z\right] \\ &:= \sum_{n=0}^{\infty} \frac{\Gamma(\alpha_1 + A_1 n) \cdots \Gamma(\alpha_q + A_q n)}{\Gamma(\beta_1 + B_1 n) \cdots \Gamma(\beta_s + B_s n)}\, \frac{z^n}{n!} \end{aligned} \tag{1.6}$$

$$\left(A_j > 0\ \ (j = 1, \ldots, q);\ B_j > 0\ \ (j = 1, \ldots, s);\ 1 + \sum_{j=1}^{s} B_j - \sum_{j=1}^{q} A_j \geqq 0\right)$$

for suitably bounded values of $|z|$. In particular, when

$$A_j = 1\ \ (j = 1, \ldots, q) \qquad \text{and} \qquad B_j = 1\ \ (j = 1, \ldots, s)\,,$$

we have the following obvious relationship:

$${}_qF_s(\alpha_1, \ldots, \alpha_q;\ \beta_1, \ldots, \beta_s; z) = \omega\, {}_q\Psi_s\left[(\alpha_j, 1)_{1,q}; (\beta_j, 1)_{1,s}; z\right] \tag{1.7}$$

$$(q \leqq s + 1;\ q, s \in \mathbb{N}_0 := \mathbb{N} \cup \{0\};\ z \in \mathbb{U})\,,$$

where, *and in what follows*, $\mathbb{N}$ denotes the set of *positive* integers and

$$\omega := \frac{\Gamma(\beta_1) \cdots \Gamma(\beta_s)}{\Gamma(\alpha_1) \cdots \Gamma(\alpha_q)}. \tag{1.8}$$

Moreover, in terms of Fox's H-function [**9**], we have (*cf.*, *e.g.*, [**14**, p. 19])

$$\begin{aligned} {}_q\Psi_s &\left[\begin{array}{c} (\alpha_1, A_1), \ldots, (\alpha_q, A_q); \\ \\ (\beta_1, B_1), \ldots, (\beta_s, B_s); \end{array} z\right] \\ &= H^{1,q}_{q,s+1}\left[-z \,\middle|\, \begin{array}{c} (1 - \alpha_1, A_1), \ldots, (1 - \alpha_q, A_q) \\ \\ (0, 1), (1 - \beta_1, B_1), \ldots, (1 - \beta_s, B_s) \end{array}\right]. \end{aligned}$$

It should be remarked in passing that a *further* generalization of FOX's H-function is provided by the $\bar{H}$-function which was encountered in the physics literature while investigating and illustrating the use of certain FEYNMAN integrals that arise naturally in perturbation calculations of the equilibrium properties of a magnetic model of phase transitions (see, for example, [**16**]).

Other interesting and useful special cases of the FOX-WRIGHT generalized hypergeometric ${}_q\Psi_s$ function defined by (1.6) include (for example) the generalized BESSEL function $J_\nu^\mu(z)$ defined by (*cf.* WRIGHT [**19**])

$$J_\nu^\mu(z) := \sum_{n=0}^{\infty} \frac{(-z)^n}{n!\,\Gamma(\mu n + \nu + 1)} = {}_0\Psi_1\left[\text{—};(\nu+1,\mu);-z\right],$$

which, for $\mu = 1$, corresponds essentially to the classical BESSEL function $J_\nu(z)$, and the generalized MITTAG-LEFFLER function $E_{\lambda,\mu}(z)$ defined by

$$E_{\lambda,\mu}(z) := \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\lambda n + \mu)} = {}_1\Psi_1\left[(1,1);(\mu,\lambda);z\right],$$

whose *further* special cases appeared recently as solutions of several families of fractional differential equations with physical applications (see, for details, GORENFLO *et al.* [**10**]; see also the recent monograph on the subject of *Fractional Differential Equations* [**11**]).

Now let $q, s \in \mathbb{N}$ and suppose that the parameters $\alpha_1, \ldots, \alpha_q$ and $\beta_1, \ldots, \beta_s$ are *also* positive real numbers. Then, corresponding to a function

$$\vartheta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s};z\right]$$

defined by

$$\vartheta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s};z\right] := \omega z\,{}_q\Psi_s\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s};z\right],$$

we consider a linear operator

$$\Theta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s}\right] : \mathcal{A} \longrightarrow \mathcal{A}$$

defined by the following HADAMARD product (or convolution) (*cf.* DZIOK *et al.* [**3**, p. 45 *et seq.*]):

$$\Theta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s}\right] f(z) := \vartheta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s};z\right] * f(z). \tag{1.9}$$

REMARK 1. The linear operator $\Theta\left[(\alpha_j, A_j)_{1,q};(\beta_j, B_j)_{1,s}\right]$ includes (as its special cases) various other linear operators which were investigated, in a *unified* manner, by DZIOK and SRIVASTAVA ([**4**], [**5**] and [**6**]), who made appropriate use of the hypergeometric ${}_qF_s$ function (in place of the FOX-WRIGHT ${}_q\Psi_s$ function) in the definition (1.9) (see also [**2**] and [**12**]). Indeed, by setting

$$A_j = 1 \quad (j = 1, \ldots, q) \qquad \text{and} \qquad B_j = 1 \quad (j = 1, \ldots, s)$$

in the definition (1.9), we are led immediately to the aforementioned DZIOK-SRIVASTAVA operator

$$\Theta\left[(\alpha_j, 1)_{1,q}\,;\ (\beta_j, 1)_{1,s}\right],$$

which contains, as its *further* special cases, such other linear operators of Geometric Function Theory as the *Hohlov operator*, the *Carlson-Shaffer operator*, the *Ruscheweyh derivative operator*, the *generalized Bernardi-Libera-Livingston operator*, the *fractional derivative operator*, and so on (see, for the *precise* relationships, DZIOK and SRIVASTAVA [**4**, pp. 3-4]).

For convenience, we write

$$\Theta\left[\alpha_1\right] f(z) := \Theta\left[(\alpha_1, A_1), \ldots, (\alpha_q, A_q); (\beta_1, B_1), \ldots, (\beta_s, B_s)\right] f(z). \tag{1.10}$$

Let $\mathcal{H}$ be a *complex Hilbert space* and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all *bounded linear operators* on $\mathcal{H}$. For a complex-valued function f analytic in a domain $\mathbb{E}$ of the complex z-plane containing the spectrum $\sigma(\mathbb{P})$ of the bounded linear operator $\mathbb{P}$, let $f(\mathbb{P})$ denote the operator on $\mathcal{H}$ defined by [**1**, p. 568]

$$f(\mathbb{P}) = \frac{1}{2\pi i} \int_{\mathcal{C}} (z\mathbb{I} - \mathbb{P})^{-1}\ f(z)\, \mathrm{d}z,$$

where $\mathbb{I}$ is the identity operator on $\mathcal{H}$ and $\mathcal{C}$ is a positively-oriented simple rectifiable closed contour containing the spectrum $\sigma(\mathbb{P})$ in the interior domain. The operator $f(\mathbb{P})$ can also be defined by the following series:

$$f(\mathbb{P}) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}\ \mathbb{P}^n,$$

which converges in the normed topology (*cf.* [**7**]).

Let $\mathcal{E}(q, s; A, B; \mathbb{P})$ denote the class of functions f of the form:

$$f(z) = a_1 z - \sum_{n=2}^{\infty} a_n\ z^n \qquad \big(a_1 > 0;\ a_n \geqq 0;\ n \in \mathbb{N} \setminus \{1\}\big), \tag{1.11}$$

which also satisfy the following subordination condition:

$$\alpha_1 \frac{\Theta\left[\alpha_1 + 1\right] f(\mathbb{P})}{\Theta\left[\alpha_1\right] f(\mathbb{P})} + A_1 - \alpha_1 \prec A_1\ \frac{1 + A\mathbb{P}}{1 + B\mathbb{P}} \quad (0 \leqq B \leqq 1;\ \ -B \leqq A < B) \tag{1.12}$$

for all operators $\mathbb{P}$ such that $\mathbb{P} \neq \mathbb{O}$ and $\|\mathbb{P}\| < 1$, $\mathbb{O}$ being the null operator on $\mathcal{H}$.

Finally, for a real parameter ρ $(0 < |\rho| < 1)$, we define the following subclasses of the class $\mathcal{E}\,(q,s;A,B;\mathbb{P})$:

$$\mathcal{E}_{\rho}\,(q,s;A,B;\mathbb{P}) := \{f : f \in \mathcal{E}\,(q,s;A,B;\mathbb{P}) \quad \text{and satisfies (1.4)}\} \tag{1.13}$$

and

$$\mathcal{E}_{\rho}^{*}\,(q,s;A,B;\mathbb{P}) := \{f : f \in \mathcal{E}\,(q,s;A,B;\mathbb{P}) \quad \text{and satisfies (1.3)}\}\,. \tag{1.14}$$

In particular, for $q = s+1$ and $\alpha_{s+1} = A_{s+1} = 1$, we write

$$\begin{aligned}\mathcal{E}\,(s;A,B;\mathbb{P}) &= \mathcal{E}\,(s+1,s;A,B;\mathbb{P})\,,\\ \mathcal{E}_{\rho}\,(s;A,B;\mathbb{P}) &= \mathcal{E}_{\rho}\,(s+1,s;A,B;\mathbb{P})\,,\end{aligned}$$

and

$$\mathcal{E}_{\rho}^{*}\,(s;A,B;\mathbb{P}) = \mathcal{E}_{\rho}^{*}\,(s+1,s;A,B;\mathbb{P})\,. \tag{1.15}$$

In this lecture, we propose to present a systematic investigation of such basic properties and charateristics of each of the analytic function classes which we have introduced here as (for example) the normed coefficient estimates, distortion theorems, and the radii of convexity and starlikeness. We also briefly indicate the relevant connections of some of the results considered here with those involving the aforementioned Dziok-Srivastava operator.

2. A SET OF COEFFICIENT INEQUALITIES AND COEFFICIENT ESTIMATES

We begin by stating and proving the following result involving coefficient inequalities and estimates (*cf.* Dziok *et al.* [**3**]).

Theorem 1. *A function f of the form* (1.11) *belongs to the class $\mathcal{E}\,(q,s;A,B;\mathbb{P})$ if and only if*

$$\sum_{n=2}^{\infty} \delta_n\ a_n \leqq a_1\ \delta_1 \qquad \big(\delta_n := [(B+1)\,n - (A+1)]\,\sigma_n\big), \tag{2.1}$$

where σ_n is given by

$$\sigma_n := \frac{\Gamma\,[\alpha_1 + A_1\,(n-1)]\cdots\Gamma\,[\alpha_q + A_q\,(n-1)]}{(n-1)!\cdot\Gamma\,[\beta_1 + B_1\,(n-1)]\cdots\Gamma\,[\beta_s + B_s\,(n-1)]} \qquad (n \in \mathbb{N})\,. \tag{2.2}$$

Proof. Let a function f of the form (1.11) belong to the class $\mathcal{E}\,(q,s;A,B;\mathbb{P})$. Then, in view of (1.12), we have

$$\alpha_1 \frac{\Theta\,[\alpha_1+1]\,f\,(\mathbb{P})}{\Theta\,[\alpha_1]\,f\,(\mathbb{P})} + A_1 - \alpha_1 = A_1 \frac{1+Aw\,(\mathbb{P})}{1+Bw\,(\mathbb{P})} \qquad (0 \leqq B \leqq 1;\ \ -B \leqq A < B)\,,$$

where $w(\mathbb{O}) = \mathbb{O}$ ($\mathbb{O}$ being the null operator on $\mathcal{H}$) and $\|w(\mathbb{P})\| < 1$ for all operators $\mathbb{P} \neq \mathbb{O}$. It follows that

$$\left\| \frac{\alpha_1 \{\Theta[\alpha_1+1] f(\mathbb{P}) - \Theta[\alpha_1] f(\mathbb{P})\}}{\alpha_1 \, B\Theta[\alpha_1+1] f(\mathbb{P}) - \{AA_1 + (\alpha_1 - A_1) B\} \Theta[\alpha_1] f(\mathbb{P})} \right\| < 1. \tag{2.3}$$

Making use of (1.6), (1.9), and (1.10), the normed inequality (2.3) simplifies to the form:

$$\left\| \frac{\sum_{n=2}^{\infty} (n-1) \sigma_n \, a_n \, \mathbb{P}^{n-1}}{a_1 \, \delta_1 - \sum_{n=2}^{\infty} (Bn - A) \sigma_n \, a_n \, \mathbb{P}^{n-1}} \right\| < 1, \tag{2.4}$$

where δ_1 and σ_n are defined by (2.1) and (2.2), respectively.

Putting $\mathbb{P} = r\mathbb{I}$ $(0 < r < 1)$, we find from (2.4) that

$$\sum_{n=2}^{\infty} (n-1) \sigma_n \, a_n \, r^{n-1} \leqq a_1 \, \delta_1 - \sum_{n=2}^{\infty} (Bn - A) \sigma_n \, a_n \, r^{n-1} \qquad (0 < r < 1),$$

which, upon letting $r \to 1-$, yields the assertion (2.1) of Theorem 1.

Conversely, let a function f of the form (1.11) satisfy the condition (2.1). Then it is sufficient to prove that

$$\begin{aligned} &\|\alpha_1 \Theta[\alpha_1+1] f(\mathbb{P}) - \Theta[\alpha_1] f(\mathbb{P})\| \\ &\qquad - \|\alpha_1 B\Theta[\alpha_1+1] f(\mathbb{P}) - \{AA_1 + (\alpha_1 - A_1) B\} \Theta[\alpha_1] f(\mathbb{P})\| < 0. \end{aligned}$$

Choosing $\mathbb{P} = r\mathbb{I}$ $(0 < r < 1)$, we have

$$\begin{aligned} &\|\alpha_1 \Theta[\alpha_1+1] f(\mathbb{P}) - \Theta[\alpha_1] f(\mathbb{P})\| \\ &\qquad\qquad - \|\alpha_1 B\Theta[\alpha_1+1] f(\mathbb{P}) - \{AA_1 + (\alpha_1 - A_1) B\} \Theta[\alpha_1] f(\mathbb{P})\| \\ &\quad = \left\| \sum_{n=2}^{\infty} (n-1) \sigma_n \, a_n \, \mathbb{P}^n \right\| - \left\| a_1 \, \delta_1 - \sum_{n=2}^{\infty} (Bn - A) \sigma_n \, a_n \, \mathbb{P}^n \right\| \\ &\quad \leqq \sum_{n=2}^{\infty} (n-1) \sigma_n \, a_n \, r^n - \left(a_1 \, \delta_1 - \sum_{n=2}^{\infty} (Bn - A) \sigma_n \, a_n \, r^n \right) \\ &\quad = \sum_{n=2}^{\infty} \delta_n \, a_n \, r^n - a_1 \, \delta_1 \\ &\quad < \sum_{n=2}^{\infty} \delta_n \, a_n - a_1 \, \delta_1 \leqq 0, \end{aligned}$$

which shows that f belongs to the class $\mathcal{E}(q, s; A, B; \mathbb{P})$. This evidently completes the proof of Theorem 1.

Corollary 1. *A function f of the form* (1.11) *belongs to the class $\mathcal{E}_\rho(q, s; A, B; \mathbb{P})$ if and only if it satisfies* (1.4) *and*

$$\sum_{n=2}^{\infty} \left(\delta_n - \delta_1 \, \rho^{n-1} \right) a_n \leqq \delta_1, \tag{2.5}$$

where δ_n *is defined by* (2.1).

Corollary 2. *A function* f *of the form* (1.11) *belongs to the class* $\mathcal{E}^*_\rho\,(q,s;A,B;\mathbb{P})$ *if and only if it satisfies* (1.3) *and*

$$\sum_{n=2}^{\infty}\left(\delta_n - n\ \delta_1\ \rho^{n-1}\right)a_n \leqq \delta_1, \tag{2.6}$$

where δ_n *is defined by* (2.1).

Corollary 1 and Corollary 2 can be obtained by observing that, for a function f of the form (1.11) with the normalization (1.4), we have

$$a_1 = 1 + \sum_{n=2}^{\infty} a_n\ \rho^{n-1}, \tag{2.7}$$

and that, for a function f of the form (1.11) with the normalization (1.3), we have

$$a_1 = 1 + \sum_{n=2}^{\infty} na_n\ \rho^{n-1}. \tag{2.8}$$

By applying (2.7) and (2.8), the inequality (2.1) yields the assertions (2.5) and (2.6), respectively.

The following lemmas are easy consequences of Corollary 1 and Corollary 2.

Lemma 1. *If there exists a positive integer* n_0 $(n_0 \in \mathbb{N}\setminus\{1\})$ *such that*

$$\delta_{n_0} - \delta_1\ \rho^{n_0-1} \leqq 0, \tag{2.9}$$

then the function

$$f_{n_0}(z) = \left(1 + a\rho^{n_0-1}\right)z - az^{n_0}$$

belongs to the class $\mathcal{E}_\rho\,(q,s;A,B;\mathbb{P})$ *for any positive real number* a. *Moreover, for all* n $(n \in \mathbb{N}\setminus\{1\})$ *such that*

$$\delta_n - \delta_1\ \rho^{n-1} > 0,$$

the functions

$$f_n(z) = \left(1 + a\rho^{n_0-1} + b\rho^{n-1}\right)z - az^{n_0} - bz^n \tag{2.10}$$

$$\left(n \in \mathbb{N}\setminus\{1\};\ b := \frac{\delta_1 + a\left(\delta_1\ \rho^{n_0-1} - \delta_{n_0}\right)}{\delta_n - \delta_1\ \rho^{n-1}}\right)$$

belong to the class $\mathcal{E}_\rho\,(q,s;A,B;\mathbb{P})$.

Lemma 2. *If there exists a positive integer* n_0 $(n_0 \in \mathbb{N}\setminus\{1\})$ *such that*

$$\delta_{n_0} - n_0\ \delta_1\ \rho^{n_0-1} \leqq 0,$$

then the function

$$f_{n_0}(z) = \left(1 + an_0\ \rho^{n_0-1}\right) z - az^{n_0}$$

belongs to the class $\mathcal{E}_\rho(q,s;A,B;\mathbb{P})$ *for any positive real number* a. *Moreover, for all* n $(n \in \mathbb{N}\setminus\{1\})$ *such that*

$$\delta_n - n\delta_1\ \rho^{n-1} > 0,$$

the functions

$$f_n(z) = \left(1 + an_0\ \rho^{n_0-1} + bn\rho^{n-1}\right) z - az^{n_0} - bz^n \tag{2.11}$$

$$\left(n \in \mathbb{N}\setminus\{1\}\,;\ b := \frac{\delta_1 + a\left(n_0\ \delta_1\ \rho^{n_0-1} - \delta_{n_0}\right)}{\delta_n - n\delta_1\ \rho^{n-1}}\right)$$

belong to the class $\mathcal{E}^*_\rho(q,s;A,B;\mathbb{P})$.

Applying Lemma 1 and Corollary 1, we obtain

Corollary 3. *If there exists a positive integer* n_0 $(n_0 \in \mathbb{N}\setminus\{1\})$ *such that*

$$\delta_{n_0} - \delta_1\ \rho^{n_0-1} < 0,$$

then the coefficients a_n *of a function* f *of the form* (1.11) *and belonging to the class* $\mathcal{E}_\rho(q,s;A,B;\mathbb{P})$ *are unbounded. Moreover, all of these coefficients* a_n *are unbounded also when*

$$\delta_n - \delta_1\ \rho^{n-1} = 0 \qquad (n \in \mathbb{N}\setminus\{1\}).$$

In all other cases, if a function f *of the form* (1.11) *belongs to the class* $\mathcal{E}_\rho(q,s;A,B;\mathbb{P})$, *then*

$$a_n \leqq \frac{\delta_1}{\delta_n - \delta_1\ \rho^{n-1}} \qquad (n \in \mathbb{N}\setminus\{1\}). \tag{2.12}$$

The result is sharp for the functions given by

$$f_n(z) = \frac{\delta_n\ z - \delta_1\ z^n}{\delta_n - \delta_1\ \rho^{n-1}} \qquad (n \in \mathbb{N}\setminus\{1\}). \tag{2.13}$$

Applying Lemma 2 and Corollary 2, we have

Corollary 4. *If there exists a positive integer* n_0 $(n_0 \in \mathbb{N}\setminus\{1\})$ *such that*

$$\delta_{n_0} - n_0\ \delta_1\ \rho^{n_0-1} < 0,$$

then the coefficients a_n *of a function* f *of the form* (1.11) *and belonging to the class* $\mathcal{E}^*_\rho(q,s;A,B;\mathbb{P})$ *are unbounded. Moreover, all of these coefficients* a_n *are unbounded also when*

$$\delta_n - n\delta_1\ \rho^{n-1} = 0 \qquad (n \in \mathbb{N}\setminus\{1\}).$$

In all other cases, if a function f of the form (1.11) *belongs to the class* $\mathcal{E}_{\rho}^{*}(q, s; A, B; \mathbb{P})$, *then*

$$a_n \leqq \frac{\delta_1}{\delta_n - n\delta_1\ \rho^{n-1}} \qquad (n \in \mathbb{N}\setminus\{1\}). \tag{2.14}$$

The result is sharp for the functions given by

$$f_n(z) = \frac{\delta_n\ z - \delta_1\ z^n}{\delta_n - n\delta_1\ \rho^{n-1}} \qquad (n \in \mathbb{N}\setminus\{1\}). \tag{2.15}$$

Each of the following results (Corollary 5 and Corollary 6) follows from Corollary 3 and Corollary 4 above.

Corollary 5. *For δ_n given by* (2.1), *let the sequence* $\left\{\delta_n - \delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty}$ *be positive. If a function f of the form* (1.11) *belongs to the class* $\mathcal{E}_{\rho}(q, s; A, B; \mathbb{P})$, *then the assertion* (2.12) *holds true for all n* ($n \in \mathbb{N}\setminus\{1\}$). *The result is sharp for the functions given by* (2.13).

Corollary 6. *For δ_n given by* (2.1), *let the sequence* $\left\{\delta_n - n\delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty}$ *be positive. If a function f of the form* (1.11) *belongs to the class* $\mathcal{E}_{\rho}^{*}(q, s; A, B; \mathbb{P})$, *then the assertion* (2.14) *holds true for all n* ($n \in \mathbb{N}\setminus\{1\}$). *The result is sharp for the functions given by* (2.15).

Remark 2. For

$$q = s + 1, \quad \alpha_{s+1} = A_{s+1} = 1, \quad \beta_1 \leqq \alpha_1 + 1, \quad A_1 \leqq \alpha_1,$$
$$\beta_j \leqq \alpha_j \quad (j = 2, \ldots, s), \quad \text{and} \quad B_j = A_j \quad (j = 1, \ldots, s),$$

the sequences

$$\left\{\delta_n - \delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty} \qquad \text{and} \qquad \left\{\delta_n - n\delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty}$$

are positive and nondecreasing. Moreover, if $\beta_1 \leqq \alpha_1$, then the sequences

$$\left\{\frac{\delta_n - n\delta_1\ \rho^{n-1}}{n}\right\}_{n=2}^{\infty} \qquad \text{and} \qquad \left\{\frac{\delta_n - \delta_1\ \rho^{n-1}}{n}\right\}_{n=2}^{\infty}$$

are positive and nondecreasing.

3. DISTORTION THEOREMS AND THEIR APPLICATIONS

In this section, we first state and prove the following distortion theorem (*cf.* Dziok *et al.* [**3**]).

Theorem 2. *Let a function f of the form* (1.11) *belong to the class* $\mathcal{E}_{\rho}(q, s; A, B; \mathbb{P})$. *Also let δ_n be defined by* (2.1). *If the sequence* $\left\{\delta_n - \delta_1\ \rho^{n-1}\right\}_{n=2}^{\infty}$ *is positive and nondecreasing, then*

$$\mathcal{J}(r) \leqq \|f(\mathbb{P})\| \leqq \frac{\delta_2\ r + \delta_1\ r^2}{\delta_2 - \delta_1\ \rho} \qquad \big(\|\mathbb{P}\| = r\ \ (0 < r < 1)\big), \tag{3.1}$$

where

$$\mathcal{J}(r) = \begin{cases} r & (r \leqq \rho) \\ \dfrac{\delta_2 \ r - \delta_1 \ r^2}{\delta_2 - \delta_1 \ \rho} & (r > \rho). \end{cases} \tag{3.2}$$

If the sequence

$$\left\{ \frac{\delta_n - \delta_1 \ \rho^{n-1}}{n} \right\}_{n=2}^{\infty}$$

is positive and nondecreasing, then

$$a_1 - \frac{2\delta_1 \ r}{\delta_2 - \delta_1 \ \rho} \leqq \| f'(\mathbb{P}) \| \leqq \frac{\delta_2 \ r + 2\delta_1 \ r}{\delta_2 - \delta_1 \ \rho} \qquad \big(\|\mathbb{P}\| = r \ \ (0 < r < 1) \big). \tag{3.3}$$

The result is sharp, with the extremal function f_2 *given by* (2.13) (*with* $n = 2$) *and* $f(z) = z$.

Proof. Let a function f of the form (1.11) belong to the class $\mathcal{E}_\rho(q, s; A, B; \mathbb{P})$. If the sequence $\left\{ \delta_n - \delta_1 \ \rho^{n-1} \right\}_{n=2}^{\infty}$ is positive and nondecreasing, by Corollary 1, we have

$$\sum_{n=2}^{\infty} a_n \leqq \frac{\delta_1}{\delta_2 - \delta_1 \ \rho}. \tag{3.4}$$

Moreover, if the sequence

$$\left\{ \frac{\delta_n - \delta_1 \ \rho^{n-1}}{n} \right\}_{n=2}^{\infty}$$

is positive and nondecreasing, by Corollary 2, we have

$$\sum_{n=2}^{\infty} n a_n \leqq \frac{2\delta_1}{\delta_2 - 2\delta_1 \ \rho}. \tag{3.5}$$

Using (2.7) and (3.4), we find for

$$\mathbb{P} = r\mathbb{I} \qquad (0 < r < 1)$$

that

$$\begin{aligned} \| f(\mathbb{P}) \| &= \left\| a_1 \ \mathbb{P} - \sum_{n=2}^{\infty} a_n \ \mathbb{P}^n \right\| \leqq r \left(a_1 + \sum_{n=2}^{\infty} a_n \ r^{n-1} \right) \\ &\leqq r \left(1 + \sum_{n=2}^{\infty} a_n \ \rho^{n-1} + \sum_{n=2}^{\infty} a_n \ r^{n-1} \right) \\ &\leqq r \left(1 + (\rho + r) \sum_{n=2}^{\infty} a_n \right) \leqq \frac{\delta_2 \ r + \delta_1 \ r^2}{\delta_2 - \delta_1 \ \rho} \end{aligned} \tag{3.6}$$

and

$$\|f(\mathbb{P})\| = \left\| a_1\ \mathbb{P} - \sum_{n=2}^{\infty} a_n\ \mathbb{P}^n \right\| \geqq r \left(a_1 - \sum_{n=2}^{\infty} a_n\ r^{n-1} \right) \tag{3.7}$$

$$= r \left(1 + \sum_{n=2}^{\infty} a_n \left(\rho^{n-1} - r^{n-1} \right) \right).$$

If $r \leqq \rho$, then we have $\|f(\mathbb{P})\| \geqq r$. If $r > \rho$, then the sequence $\left\{ \rho^{n-1} - r^{n-1} \right\}_{n=2}^{\infty}$ is negative and decreasing. Hence, by (3.7), we obtain

$$\|f(\mathbb{P})\| \geqq r \left(1 + (\rho - r) \sum_{n=2}^{\infty} a_n \right) \geqq \frac{\delta_2\ r - \delta_1\ r^2}{\delta_2 - \delta_1\ \rho},$$

which, in conjunction with (3.6), yields the assertion (3.1) of Theorem 2.

Similarly, by using (3.5) in conjunction with (2.7), we arrive at the assertion (3.3) of Theorem 2.

The proof of the following result is analogous to that of Theorem 2.

Theorem 3. *Let a function f of the form* (1.11) *belong to the class $\mathcal{E}_\rho^*(q, s; A, B; \mathbb{P})$. Also let δ_n be defined by* (2.1). *If the sequence $\left\{ \delta_n - n\delta_1\ \rho^{n-1} \right\}_{n=2}^{\infty}$ is positive and nondecreasing, then*

$$a_1\ r - \frac{\delta_1\ r^2}{\delta_2 - \delta_1\ \rho} \leqq \|f(\mathbb{P})\| \leqq \frac{\delta_2\ r + \delta_1\ r^2}{\delta_2 - n\delta_1\ \rho} \qquad \big(\|\mathbb{P}\| = r\ \ (0 < r < 1) \big). \tag{3.8}$$

If the sequence

$$\left\{ \frac{\delta_n - n\delta_1\ \rho^{n-1}}{n} \right\}_{n=2}^{\infty}$$

is positive and nondecreasing, then

$$\mathcal{J}'(r) \leqq \|f'(\mathbb{P})\| \leqq \frac{\delta_2 + 2\delta_1\ r}{\delta_2 - n\delta_1\ \rho} \qquad \big(\|\mathbb{P}\| = r\ \ (0 < r < 1) \big), \tag{3.9}$$

where $\mathcal{J}(r)$ is defined by (3.2). *The result is sharp, with the extremal function f_2 given by* (2.15) *with $n = 2$ and $f(z) = z$.*

Applying Lemma 1, we deduce the following result.

Corollary 7. *If there exists an integer n_0 $(n_0 \in \mathbb{N} \setminus \{1\})$ such that* (2.9) *holds true, then $\|f(\mathbb{P})\|$ and $\|f'(\mathbb{P})\|$ $\big(\|\mathbb{P}\| = r\ \ (0 < r < 1) \big)$ for functions of the class $\mathcal{E}_\rho(q, s; A, B; \mathbb{P})$ are unbounded.*

Next, by applying Lemma 2, we have

Corollary 8. *If there exists an integer n_0 $(n_0 \in \mathbb{N} \setminus \{1\})$ such that* (2.10) *holds true, then $\|f(\mathbb{P})\|$ and $\|f'(\mathbb{P})\|$ $\big(\|\mathbb{P}\| = r\ \ (0 < r < 1) \big)$ for functions of the class $\mathcal{E}_\rho^*(q, s; A, B; \mathbb{P})$ are unbounded.*

By virtue of Remark 2, Theorem 2 and Theorem 3 give the following results.

Corollary 9. *Let a function f of the form* (1.11) *belong to the class* $\mathcal{E}_\rho(s;A,B;\mathbb{P})$. *If*

$$\beta_1 \leqq \alpha_1+1, \quad A_1 \leqq \alpha_1, \quad \beta_j \leqq \alpha_j \quad (j=2,\ldots,s), \quad \text{and} \quad B_j = A_j \quad (j=1,\ldots,s),$$

then the assertion (3.1) *holds true. Further, if* $\beta_1 \leqq \alpha_1$, *then the assertion* (3.3) *holds true.*

Corollary 10. *Let a function f of the form* (1.11) *belong to the class* $\mathcal{E}^*_\rho(s;A,B;\mathbb{P})$. *If*

$$\beta_1 \leqq \alpha_1+1, \quad A_1 \leqq \alpha_1, \quad \beta_j \leqq \alpha_j \quad (j=2,\ldots,s), \quad \text{and} \quad B_j = A_j \quad (j=1,\ldots,s),$$

then the assertion (3.8) *holds true. Further, if* $\beta_1 \leqq \alpha_1$, *then the assertion* (3.9) *holds true.*

4. COMPUTATION OF THE ASSOCIATED RADII OF CONVEXITY AND STARLIKENESS

Our first set of results involving the radius of starlikeness can be stated as Theorem 4 below (*cf.* Dziok *et al.* [**3**]).

Theorem 4. *If a function f of the form* (1.11) *belongs to the class* $\mathcal{E}(q,s;A,B;\mathbb{P})$, *then f is starlike in the disk*

$$\left\|R^*\left(\mathcal{E}(q,s;A,B;\mathbb{P})\right)\right\| < r_1 := \inf_{n\in\mathbb{N}\setminus\{1\}} \left(\frac{\delta_n}{n\delta_1}\right)^{1/(n-1)}, \tag{4.1}$$

where δ_n is defined by (2.1). *The result is sharp for the function f_a^* given by*

$$f_a^*(z) = a\left(z - \frac{\delta_1}{\delta_n}\, z^n\right) \qquad (a>0). \tag{4.2}$$

Proof. It suffices to show that

$$\left\|\frac{\mathbb{P}\, f'(\mathbb{P})}{f(\mathbb{P})} - 1\right\| < 1 \qquad \big(\mathbb{P} = r_1\mathbb{I} \ \ (0<r_1<1)\big). \tag{4.3}$$

Since

$$\left\|\frac{\mathbb{P}\, f'(\mathbb{P})}{f(\mathbb{P})} - 1\right\| = \left\|\frac{\sum\limits_{n=2}^{\infty}(n-1)\,a_n\,\mathbb{P}^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} a_n\,\mathbb{P}^{n-1}}\right\|,$$

the condition (4.3) holds true if

$$\left\|\frac{\mathbb{P}\, f'(\mathbb{P})}{f(\mathbb{P})} - 1\right\| \leqq \frac{\sum\limits_{n=2}^{\infty}(n-1)\,a_n\,r_1^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} a_n\,r_1^{n-1}} \leqq 1,$$

that is, if

$$\sum_{n=2}^{\infty} na_n\ r_1^{n-1} \leqq a_1. \tag{4.4}$$

By Theorem 1, we also have

$$\sum_{n=2}^{\infty} \frac{\delta_n\ a_n}{\delta_1} \leqq a_1, \tag{4.5}$$

where δ_n is defined by (2.1). Comparing (4.4) and (4.5), we obtain the desired result (4.1). The sharpness of the result (4.1) can easily be verified for the function f_a^* given by (4.2).

Theorem 5. *If a function f of the form* (1.11) *belongs to the class $\mathcal{E}\,(q, s; A, B; \mathbb{P})$, then f is convex in the disk*

$$R^c\big(\mathcal{E}\,(q, s; A, B; \mathbb{P})\big) < r_2 := \inf_{n\in\mathbb{N}\setminus\{1\}} \left(\frac{\delta_n}{n^2\ \delta_1}\right)^{1/(n-1)}, \tag{4.6}$$

where δ_n is defined by (2.1). *The result is sharp for the function f_a^c given by*

$$f_a^c\,(z) = a\left(z - \frac{n\delta_1}{\delta_n}\ z^n\right) \qquad (a > 0)\,. \tag{4.7}$$

Proof. It suffices to show that

$$\left\|\frac{\mathbb{P}\ f''\,(\mathbb{P})}{f'\,(\mathbb{P})}\right\| < 1 \qquad \big(\mathbb{P} = r_2\ \mathbb{I}\ \ (0 < r_2 < 1)\big). \tag{4.8}$$

Since

$$\left\|\frac{\mathbb{P}\ f''\,(\mathbb{P})}{f'\,(\mathbb{P})}\right\| = \left\| -\frac{\sum\limits_{n=2}^{\infty} n\,(n-1)\,a_n\ \mathbb{P}^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} na_n\ \mathbb{P}^{n-1}} \right\|,$$

the condition (4.8) holds true if

$$\left\|\frac{\mathbb{P}\ f''\,(\mathbb{P})}{f'\,(\mathbb{P})}\right\| \leqq \frac{\sum\limits_{n=2}^{\infty} n\,(n-1)\,a_n\ r_2^{n-1}}{a_1 - \sum\limits_{n=2}^{\infty} na_n\ r_2^{n-1}} \leqq 1,$$

that is, if

$$\sum_{n=2}^{\infty} n^2\ a_n\ r_2^{n-1} \leqq a_1. \tag{4.9}$$

By comparing (4.9) with (4.5) again, we arrive at the desired result (4.6), with the extremal function f_a^c given by (4.7).

Remark 3. Just as we pointed out in Remark 1, the various results presented in this lecture would provide interesting extensions and generalizations of those considered earlier for *simpler* analytic function classes. The details involved in the derivations of such specializations of the results presented here are fairly straightforward.

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SPECIAL FUNCTIONS: APPROXIMATIONS AND BOUNDS

P. Cerone

The Steffensen inequality and bounds for the Čebyšev functional are utilised to obtain bounds for some classical special functions. The technique relies on determining bounds on integrals of products of functions. The above techniques are used to obtain novel and useful bounds for the Bessel function of the first kind, the Beta function, and the Zeta function.

1. INTRODUCTION AND REVIEW OF SOME RECENT RESULTS

There are a number of results that provide bounds for integrals of products of functions. The main techniques that shall be employed in the current article involve the Steffensen inequality and a variety of bounds related to the Čebyšev functional. There have been some developments in both of these in the recent past with which the current author has been involved. These have been put to fruitful use in a variety of areas of applied mathematics including quadrature rules, in the approximation of integral transforms, as well as in applied probability problems (see [**31**], [**22**] and [**11**]. This article is a review of these developments and some new results are also presented.

It is intended that in the current article the techniques will be utilised to obtain useful bounds for special functions. The methodologies will be demonstrated through obtaining bounds for the Bessel function of the first kind, the Beta function and the Zeta function.

It is instructive to introduce some techniques for approximating and bounding integrals of the product of functions. We first present inequalities due to Steffensen and then review bounds for the Čebyšev functional.

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The following theorem is due to sc Steffensen [**45**] (see also [**11**] and [**16**]).

Theorem 1. *Let* $h:[a,b]\to\mathbb{R}$ *be a nonincreasing mapping on* $[a,b]$ *and* $g:[a,b]\to\mathbb{R}$ *be an integrable mapping on* $[a,b]$ *with*

$$-\infty<\phi\le g(t)\le\Phi<\infty\quad\textit{for all } x\in[a,b],$$

then

$$\phi\int_a^{b-\lambda}h(x)\,\mathrm{d}x+\Phi\int_{b-\lambda}^b h(x)\,\mathrm{d}x\le\int_a^b h(x)\,g(x)\,\mathrm{d}x \tag{1.1}$$
$$\le\Phi\int_a^{a+\lambda}h(x)\,\mathrm{d}x+\phi\int_{a+\lambda}^b h(x)\,\mathrm{d}x,$$

where

$$\lambda=\int_a^b G(x)\,\mathrm{d}x,\qquad G(x)=\frac{g(x)-\phi}{\Phi-\phi},\quad\Phi\ne\phi. \tag{1.2}$$

Remark 1. We note that the result (1.1) may be rearranged to give Steffensen's better known result that

$$\int_{b-\lambda}^b h(x)\,\mathrm{d}x\le\int_a^b h(x)\,G(x)\,\mathrm{d}x\le\int_a^{a+\lambda}h(x)\,\mathrm{d}x,$$

where λ is as given by (1.2) and $0\le G(x)\le 1$.

Equation (1.3) has a very pleasant interpretation, as observed by Steffensen, that if we divide by λ then

$$\frac{1}{\lambda}\int_{b-\lambda}^b h(x)\,\mathrm{d}x\le\frac{\int_a^b G(x)\,h(x)\,\mathrm{d}x}{\int_a^b G(x)\,\mathrm{d}x}\le\frac{1}{\lambda}\int_a^{a+\lambda}h(x)\,\mathrm{d}x. \tag{1.4}$$

Thus, the weighted integral mean of $h(x)$ is bounded by the integral means over the end intervals of length λ, the total weight.

Now, for two measurable functions $f,g:[a,b]\to\mathbb{R}$, define the functional, which is known in the literature as Čebyšev's functional, by

$$T(f,g):=\mathcal{M}(fg)-\mathcal{M}(f)\,\mathcal{M}(g), \tag{1.5}$$

where the integral mean is given by

$$\mathcal{M}(f):=\frac{1}{b-a}\int_a^b f(x)\,\mathrm{d}x. \tag{1.6}$$

The integrals in (1.5) are assumed to exist.

The weighted ČEBYŠEV functional is defined by

$$T(f,g;p) := \mathcal{M}(fg;p) - \mathcal{M}(f;p)\,\mathcal{M}(g;p), \tag{1.7}$$

where the weighted integral mean $\mathcal{M}(f;p)$ is given by

$$P \cdot \mathcal{M}(f;p) = \int_a^b p(x)\,f(x)\,\mathrm{d}x, \quad P = \int_a^b p(x)\,\mathrm{d}x \tag{1.8}$$

with the weight P satisfying $0 < P < \infty$.

We note that

$$T(f,g;1) \equiv T(f,g) \quad \text{and} \quad \mathcal{M}(f;1) \equiv \mathcal{M}(f).$$

We further note that bounds for (1.5) and (1.7) may be looked upon as approximating the integral mean of the product of functions in terms of the product of integral means which are more easily calculated explicitly. Bounds are perhaps best procured from identities. It is worthwhile noting that a number of identities relating to the ČEBYŠEV functional already exist. (The reader is referred to [**40**] Chapters IX and X.) KORKINE's identity is well known, see [**40**, p. 296] and is given by

$$T(f,g) = \frac{1}{2(b-a)^2}\int_a^b\int_a^b \big(f(x)-f(y)\big)\big(g(x)-g(y)\big)\,\mathrm{d}x\mathrm{d}y. \tag{1.9}$$

It is identity (1.9) that is often used to prove an inequality due to GRÜSS for functions bounded above and below, [**40**].

The GRÜSS inequality [**35**] is given by

$$|T(f,g)| \le \frac{1}{4}(\Phi_f - \phi_f)(\Phi_g - \phi_g), \tag{1.10}$$

where $\phi_f \le f(x) \le \Phi_f$ for $x \in [a,b]$, with ϕ_f, Φ_f constants and similarly for $g(x)$.

The interested reader is also referred to DRAGOMIR [**30**] and FINK [**34**] for extensive treatments of the GRÜSS and related inequalities.

Identity (1.9) may also be used to prove the ČEBYŠEV inequality which states that for $f(\cdot)$ and $g(\cdot)$ synchronous, namely $(f(x)-f(y))(g(x)-g(y)) \ge 0$, a.e. $x, y \in [a,b]$, then

$$T(f,g) \ge 0. \tag{1.11}$$

As mentioned earlier, there are many identities involving the ČEBYŠEV functional (1.5) or more generally (1.7). Recently, CERONE [**11**] obtained, for $f, g : [a,b] \to \mathbb{R}$ where f is of bounded variation and g continuous on $[a,b]$, the identity

$$T(f,g) = \frac{1}{(b-a)^2}\int_a^b \psi(t)\,\mathrm{d}f(t), \tag{1.12}$$

where

$$\psi(t) = (t-a)\,G(t,b) - (b-t)\,G(a,t) \tag{1.13}$$

with

$$G(c,d) = \int_c^d g(x)\;\mathrm{d}x. \tag{1.14}$$

The following theorem was proved in [**11**].

Theorem 2. *Let $f, g : [a,b] \to \mathbb{R}$, where f is of bounded variation and g is continuous on $[a,b]$. Then*

$$(b-a)^2\,|T(f,g)| \le \begin{cases} \sup\limits_{t\in[a,b]} |\psi(t)| \bigvee\limits_a^b (f), \\ L\int_a^b |\psi(t)|\,dt, & \text{for } f\ L-\text{Lipschitzian}, \\ \int_a^b |\psi(t)|\,df(t), & \text{for } f \text{ monotonic nondecreasing}, \end{cases} \tag{1.15}$$

where $\bigvee\limits_a^b (f)$ is the total variation of f on $[a,b]$.

The bounds for the Čebyšev functional were utilised to procure approximations to moments and moment generating functions in [**11**] and [**24**].

The reader is referred to [**31**] and the references therein for applications to numerical quadrature of trapezoidal and Ostrowski functionals, which were shown to be related to the Čebyšev functional in [**15**].

For other Grüss type inequalities, see the books [**9**] and [**40**], and the papers [**19**], [**23**], [**26**], [**29**], [**30**], where further references are given.

Recently, Cerone and Dragomir [**19**]–[**23**] have pointed out generalisations of the above results for integrals defined on two different intervals and more generally in a measurable space setting (see also, [**8**] and [**14**]).

The functional $T(f,g;p)$ defined in (1.7) satisfies a number of identities including that due to Sonin [**42**]

$$P\cdot|T(f,g;p)| = \left|\int_a^b p(x)\big(f(x)-\gamma\big)\big(g(x)-\mathcal{M}(g;p)\big)\,\mathrm{d}x\right| \tag{1.16}$$

from which the following bounds may be procured. Namely,

$$P\cdot|T(f,g;p)| \le \begin{cases} \inf\limits_{\gamma\in\mathbb{R}} \|f(\cdot)-\gamma\| \int_a^b p(x)\,|g(x)-\mathcal{M}(g;p)|\;\mathrm{d}x, \\ \left(\int_a^b p(x)\,(f(x)-\mathcal{M}(f;p))^2\,\mathrm{d}x\right)^{1/2} \\ \qquad\qquad \times\left(\int_a^b p(x)\,(g(x)-\mathcal{M}(g;p))^2\,\mathrm{d}x\right)^{1/2}, \end{cases} \tag{1.17}$$

where

$$\int_a^b p(x)\,(h(x)-\mathcal{M}(h;p))^2\,\mathrm{d}x = \int_a^b p(x)\,h^2(x)\,\mathrm{d}x - P\cdot\mathcal{M}^2(h;p) \tag{1.18}$$

and P is as defined in (1.8). Further, it may be easily shown by direct calculation that,

$$\inf_{\gamma\in\mathbb{R}}\left[\int_a^b p(x)\,(f(x)-\gamma)^2\,\mathrm{d}x\right] = \int_a^b p(x)\,\bigl(f(x)-\mathcal{M}(f;p)\bigr)^2\mathrm{d}x. \tag{1.19}$$

Some of the above results are used to find bounds for the Bessel function (Section 2), the Beta function (Section 3), the Zeta function (Section 4) (see also [**9**] for further details).

2. BOUNDING THE BESSEL FUNCTION

In this section we investigate techniques for determining bounds on the Bessel function of the first kind (see also [**12**], [**13**]).

In Abramowitz and Stegun [**1**] equation (9.1.21) defines the Bessel of the first kind

$$J_\nu(z) = \gamma_\nu(z)\int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}}\cos(zt)\,\mathrm{d}t, \quad \mathrm{Re}\,(\nu) > -\frac{1}{2}, \tag{2.1}$$

where

$$\gamma_\nu(z) = \frac{2\left(\frac{z}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}. \tag{2.2}$$

For the current work the interest is in both z and ν real.

Theorem 3. *For z real then*

$$\begin{aligned}&\frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right) - B\left(\frac{1}{2},\nu+\frac{1}{2};(1-\lambda)^2\right)\\ &\le \frac{J_\nu(z)}{\gamma_\nu(z)}\\ &\le B\left(\frac{1}{2},\nu+\frac{1}{2};\lambda^2\right) - \frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right), \quad \nu > \frac{1}{2}\end{aligned} \tag{2.3}$$

and

$$B\left(\frac{1}{2},\nu+\frac{1}{2};\lambda^2\right) - \frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right) \tag{2.4}$$

$$\leq \frac{J_\nu(z)}{\gamma_\nu(z)}$$
$$\leq \frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right) - B\left(\frac{1}{2},\nu+\frac{1}{2};(1-\lambda)^2\right), \quad -\frac{1}{2}<\nu<\frac{1}{2},$$

where

$$B(\alpha,\beta;x) = \int_0^x u^{\alpha-1}(1-u)^{\beta-1}\,\mathrm{d}u, \qquad \textit{the incomplete Beta function,} \tag{2.5}$$

$$B(\alpha,\beta) = B(\alpha,\beta;1) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \textit{the Beta function,} \tag{2.6}$$

and

$$2\lambda - 1 = \frac{\sin z}{z}. \tag{2.7}$$

Taking $\nu = \frac{1}{2}$ *produces equality in* (2.3) *and* (2.4), *namely,* $J_{\frac{1}{2}}(z) = \gamma_{\frac{1}{2}}(z)\frac{\sin z}{z}$.

Proof. Consider the case $\nu > \frac{1}{2}$ then $h(t) = (1-t^2)^{\nu-\frac{1}{2}}$ is nonincreasing for $t \in [0,1]$. Further, taking $g(t) = \cos zt$ we have that $-1 \leq g(t) \leq 1$ for $t \in [0,1]$ and, from (1.2)

$$\lambda = \frac{1}{2}\int_0^1 (\cos zt + 1)\,\mathrm{d}t = \frac{1}{2}\left(1+\frac{\sin z}{z}\right).$$

Utilising Theorem 1 and after some algebra, the above results are procured. □

Remark. We note from (2.1) that we may obtain a classical bound (see [**1**, p. 362]) for $J_\nu(z)$, namely

$$|J_\nu(z)| \leq \frac{2\left(\frac{|z|}{2}\right)^\nu}{\sqrt{\pi}\,\Gamma\left(\nu+\frac{1}{2}\right)}\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\,\mathrm{d}t,$$

where from (2.5) and (2.6)

$$\int_0^1 (1-t^2)^{\nu-\frac{1}{2}}\,\mathrm{d}t = \frac{1}{2}B\left(\frac{1}{2},\nu+\frac{1}{2}\right) = \frac{1}{2}\cdot\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\nu+\frac{1}{2}\right)}{\Gamma(\nu+1)} \tag{2.8}$$

to give

$$|J_\nu(z)| \leq \left|\frac{z}{2}\right|^\nu \frac{1}{\Gamma(\nu+1)}. \tag{2.9}$$

The following theorem gives a bound on the deviation of the Bessel function from an approximant (see also [**17**]). This is accomplished via bounds on the Čebyšev functional for which there are numerous results.

Theorem 4. *The following result holds for the Bessel function of the first kind* $J_\nu(z)$. *Namely,*

$$\left| J_\nu(z) - \frac{\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1)} \cdot \frac{\sin z}{z} \right| \le \left(\frac{|z|}{2}\right)^\nu \left[\frac{2}{\sqrt{\pi}} \cdot \frac{\Gamma(2\nu)}{\Gamma^2\left(\nu+\frac{1}{2}\right)\Gamma\left(2\nu+\frac{1}{2}\right)} - \frac{1}{\Gamma^2(\nu+1)} \right]^{1/2} \times \left[\left(\frac{\cos z}{4}\right)^2 + \frac{1}{2} - \left(\frac{\sin z}{z} - \frac{\cos z}{4}\right)^2 \right]^{1/2}. \tag{2.10}$$

Proof. (Sketch) We use the 2−norm result for the Čebyšev functional. From (2.1) and (2.2) consider,

$$Q_\nu(z) = \frac{J_\nu(z)}{\gamma_\nu(z)} = \int_0^1 \left(1-t^2\right)^{\nu-\frac{1}{2}} \cos(zt)\, \mathrm{d}t. \tag{2.11}$$

Let $f(t) = \left(1-t^2\right)^{\nu-\frac{1}{2}}$ and $g(t) = \cos zt$. □

3. BOUNDING THE BETA FUNCTION

The incomplete beta function is defined by

$$B(x,y;z) = \int_0^z t^{x-1}(1-t)^{y-1}\, \mathrm{d}t, \quad 0 < z \le 1. \tag{3.1}$$

We shall restrict our attention to $x > 1$ and $y > 1$.

In this region we observe that

$$0 \le t^{x-1} \le z^{x-1} \quad \text{and} \quad (1-z)^{y-1} \le (1-t)^{y-1} \le 1 \tag{3.2}$$

with t^{x-1}, an increasing function and $(1-t)^{y-1}$, a decreasing function, for $t \in [0,z]$.

The following theorem follows from utilizing Steffensen's result as depicted in Theorem 1 [**12**], see also [**17**] for details.

Theorem 5. *For* $x > 1$ *and* $y > 1$ *with* $0 \le z \le 1$ *we have the incomplete Beta function defined by* (3.1) *satisfying the following bounds*

$$\max\{L_1(z), L_2(z)\} \le B(x,y;z) \le \min\{U_1(z), U_2(z)\}, \tag{3.3}$$

where

$$L_1(z) = \frac{z^{x-1}}{y}\left[\left(1 - z + \frac{z}{x}\right)^y - (1-z)^y\right], \quad U_1(z) = \frac{z^{x-1}}{y}\left[1 - \left(1 - \frac{z}{x}\right)^y\right] \tag{3.4}$$

and

$$L_2(z) = \frac{\lambda_2^x(z)}{x} + (1-z)^{y-1}\frac{z^x - \lambda_2^x(z)}{x}, \tag{3.5}$$

$$U_2(z) = (1-z)^{y-1}\frac{\left(x - \lambda_2(z)\right)^x}{x} + \frac{z^x - \left(z - \lambda_2(z)\right)^x}{x}$$

with

$$\lambda_2(z) = \frac{1 - (1-z)\left[1 - z(1-y)\right]}{y\left[1 - (1-z)^{y-1}\right]}. \tag{3.6}$$

Proof. (Using Steffensen's inequality) If we take $h(t) = (1-t)^{y-1}$ and $g(t) = t^{x-1}$, then for $y > 1$ and $x > 1$, $h(t)$ is a decreasing function of t and $0 \le g(t) \le z^{x-1}$. Thus, from (1.1)

$$z^{x-1}\int_{z-\lambda_1}^{z}(1-t)^{y-1}\,\mathrm{d}t \le \int_0^z t^{x-1}(1-t)^{y-1}\,\mathrm{d}t \le z^{x-1}\int_0^{\lambda_1}(1-t)^{y-1}\,\mathrm{d}t, \tag{3.7}$$

where

$$\lambda_1 = \lambda_1(z) = \int_0^z \frac{t^{x-1}}{z^{x-1}}\,\mathrm{d}t = \frac{z}{x}.$$

□

Corollary 1. *For $x > 1$ and $y > 1$ we have the Beta function*

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,\mathrm{d}t,$$

which is symmetric in x and y, satisfies the following bounds,

$$\max\left\{\frac{1}{xy^x}, \frac{1}{yx^y}\right\} \le B(x,y) \le \min\left\{\frac{1}{y}\left[1 - \left(1 - \frac{1}{x}\right)^y\right], \frac{1}{x}\left[1 - \left(1 - \frac{1}{y}\right)^x\right]\right\}. \tag{3.8}$$

Proof. Put $z = 1$ in (3.6) to give $\lambda_2(1) = \dfrac{1}{y}$ followed by the obvious correspondences from (3.3)–(3.5). □

The following theorem relates to the Beta function [**17**] and is a correction of the result in [**12**].

Theorem 6. *For $x > 1$ and $y > 1$ the following bounds hold for the Beta function, namely,*

$$0 \le \frac{1}{xy} - B(x,y) \le 2\min\{A(x), A(y)\}, \tag{3.9}$$

where

$$A(x) = \frac{x-1}{x^{\left(1+\frac{x}{x-1}\right)}}. \tag{3.10}$$

Proof. (Sketch. Using the Čebyšev functional and Sonin identity). We have from (1.16)–(1.17) with $p(\cdot) \equiv 1$,

$$0 \le |T(f,g)| = |\mathcal{M}(fg) - \mathcal{M}(f)\mathcal{M}(g)| \le \mathcal{M}(|f(\cdot) - \gamma|\,|g(\cdot) - \mathcal{M}(g)|).$$

That is,

$$|T(f,g)| \le \inf_{\gamma} \|f(\cdot) - \gamma\|_{\infty} \mathcal{M}\,|g(\cdot) - \mathcal{M}(g)|. \tag{3.11}$$

If we take $f(t) = t^{x-1}$, $g(t) = (1-t)^{y-1}$ then $\mathcal{M}(f) = \frac{1}{x}$ and $\mathcal{M}(g) = \frac{1}{y}$. □

The following pleasing result is valid ([**12**], [**17**]).

Theorem 7. *For $x > 1$ and $y > 1$ we have*

$$0 \le \frac{1}{xy} - B(x,y) \le \frac{x-1}{x\sqrt{2x-1}} \cdot \frac{y-1}{y\sqrt{2y-1}} \le 0.090169437\ldots, \tag{3.12}$$

where the upper bound is obtained at $x = y = \frac{3+\sqrt{5}}{2} = 2.618033988\ldots$.

Proof. (Using the 2-norm bound for the Čebyšev functional) We have from (1.17)–(1.19)

$$(b-a)|T(f,g)| \le \left(\int_a^b f^2(t)\,\mathrm{d}t - \mathcal{M}^2(f)\right)^{1/2} \times \left(\int_a^b g^2(t)\,\mathrm{d}t - \mathcal{M}^2(g)\right)^{1/2}.$$

That is, taking $f(t) = t^{x-1}$, $g(t) = (1-t)^{y-1}$.

Now, consider

$$C(x) = \frac{x-1}{x\sqrt{2x-1}}. \tag{3.13}$$

The maximum occurs when $x = x^* = \frac{3+\sqrt{5}}{2}$ to give $C(x^*) = 0.3002831\ldots$. Hence, because of the symmetry we have the upper bound as stated in (3.12). □

Remark 3. In a recent paper Alzer [**4**] shows that

$$0 \le \frac{1}{xy} - B(x,y) \le b_A = \max_{x \ge 1}\left(\frac{1}{x^2} - \frac{\Gamma^2(x)}{\Gamma(2x)}\right) = 0.08731\ldots,$$

where 0 and b_A are shown to be the best constants. This uniform bound of Alzer is only smaller for a small area around $\left(\frac{3+\sqrt{5}}{2}, \frac{3+\sqrt{5}}{2}\right)$ while the first upper bound in (3.12) provides a better bound over a much larger region of the $x-y$ plane.

We may state the following corollary given the results above.

Corollary 2. *For $x > 1$ and $y > 1$ we have*

$$0 \le \frac{1}{xy} - B(x,y) \le \min\{C(x)C(y), b_A\},$$

where $C(x)$ is defined by (3.13) *and b_A by* (3.14).

Remark 4. The upper bound in Theorem 6 by numerical investigation, seems not to be as good as that given in Theorem 7. Analytically, the transformation $\chi = \frac{x-1}{x}$ and $\eta = \frac{y-1}{y}$ in (3.9)–(3.12) results in requiring to show that

$$H(\chi,\eta) = 2(1-\chi)^{\frac{1}{\chi}} - \eta\sqrt{\frac{1-\chi}{1+\chi}\cdot\frac{1-\eta}{1+\eta}} \ge 0$$

for $0 \le \chi, \eta \le 1$.

4. BOUNDS FOR THE EULER ZETA AND RELATED FUNCTIONS

4.1. BACKGROUND TO ZETA AND RELATED FUNCTIONS

The Zeta function ([**10**])

$$\zeta(x) := \sum_{n=1}^{\infty} \frac{1}{n^x}, \qquad x > 1 \tag{4.1}$$

was originally introduced in 1737 by the Swiss mathematician Leonhard Euler (1707-1783) for real x who proved the identity

$$\zeta(x) := \prod_p \left(1 - \frac{1}{p^x}\right)^{-1}, \qquad x > 1, \tag{4.2}$$

where p runs through all primes. It was Riemann who allowed x to be a complex variable z and showed that even though both sides of (4.1) and (4.2) diverge for $\operatorname{Re}(z) \le 1$, the function has a continuation to the whole complex plane with a simple pole at $z = 1$ with residue 1. The function plays a very significant role in the theory of the distribution of primes (see [**5**], [**7**], [**27**], [**32**], [**37**] and [**46**]). One of the most striking properties of the zeta function, discovered by RIEMANN himself, is the functional equation

$$\zeta(z) = 2^z \pi^{z-1} \sin\left(\frac{\pi z}{2}\right) \Gamma(1-z)\zeta(1-z) \tag{4.3}$$

that can be written in symmetric form to give

$$\pi^{-\frac{z}{2}} \Gamma\left(\frac{z}{2}\right) \zeta(z) = \pi^{-\left(\frac{1-z}{2}\right)} \Gamma\left(\frac{1-z}{2}\right) \zeta(1-z). \tag{4.4}$$

$\zeta(s)$ is commonly referred to as the RIEMANN Zeta function and if s is restricted to a real variable x, it is referred to as the EULER Zeta function.

In addition to the relation (4.3) between the zeta and the gamma function, these functions are also connected via the integrals [**32**]

$$\zeta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t - 1}\,\mathrm{d}t, \qquad x > 1, \tag{4.5}$$

and

$$\zeta(x) = \frac{1}{C(x)} \int_0^\infty \frac{t^{x-1}}{e^t + 1}\,\mathrm{d}t, \qquad x > 0, \tag{4.6}$$

where

$$C(x) := \Gamma(x)\left(1 - 2^{1-x}\right) \quad \text{and} \quad \Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\mathrm{d}t. \tag{4.7}$$

In the series expansion

$$\frac{te^{xt}}{e^t - 1} = \sum_{m=0}^{\infty} B_m(x) \frac{t^m}{m!}, \tag{4.8}$$

where $B_m(x)$ are the BERNOULLI polynomials (after JACOB BERNOULLI), $B_m(0) = B_m$ are the BERNOULLI numbers. They occurred for the first time in the formula [**1**, p. 804]

$$\sum_{k=1}^{m} k^n = \frac{B_{n+1}(m+1) - B_{n+1}}{n+1}, \qquad n, m = 1, 2, 3, \dots . \tag{4.9}$$

One of EULER's most celebrated theorems discovered in 1736 (Institutiones Calculi Differentialis, Opera (1), Vol. 10) is

$$\zeta(2n) = (-1)^{n-1} \frac{2^{2n-1}\pi^{2n}}{(2n)!} B_{2n}; \qquad n = 1, 2, 3, \dots . \tag{4.10}$$

The Zeta function is also explicitly known at the non-positive integers by

$$\zeta(-n) = (-1)^n \frac{B_{n+1}}{n+1}, \qquad \text{for } n = 1, 2, \ldots$$

The result may also be obtained in a straight forward fashion from (4.6) and a change of variable on using the fact that

$$B_{2n} = (-1)^{n-1} \cdot 4n \int_0^\infty \frac{t^{2n-1}}{e^{2\pi t} - 1}\, \mathrm{d}t \tag{4.11}$$

from Whittaker and Watson [**48**, p. 126].

We note here that

$$\zeta(2n) = A_n \pi^{2n},$$

where

$$A_n = (-1)^{n-1} \cdot \frac{n}{(2n+1)!} + \sum_{j=1}^{n-1} \frac{(-1)^{j-1}}{(2j+1)!} A_{n-j}$$

and $A_1 = \frac{1}{3!}$.

Further, the Zeta function for even integers satisfy the relation (Borwein et al. [**7**], Srivastava [**43**])

$$\zeta(2n) = \left(n + \frac{1}{2}\right)^{-1} \sum_{j=1}^{n-1} \zeta(2j)\, \zeta(2n-2j), \quad n \in \mathbb{N} \setminus \{1\}.$$

Despite several efforts to find a formula for $\zeta(2n+1)$, there seems to be no elegant closed form representation for the zeta function at the odd integer values. Several series representations for the value $\zeta(2n+1)$ have been proved by Srivastava and co-workers in particular, see [**43**], [**44**].

There are also integral representations for $\zeta\,(n+1)$, see [**1**, p. 807] and [**28**].

Both series representations and the integral representations are however somewhat difficult in terms of computational aspects and time considerations.

We note that there are functions that are closely related to $\zeta\,(x)$. Namely, the Dirichlet $\eta\,(\cdot)$ and $\lambda\,(\cdot)$ functions given by

$$\eta\,(x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^x} = \frac{1}{\Gamma\,(x)} \int_0^\infty \frac{t^{x-1}}{e^t + 1}\, \mathrm{d}t, \quad x > 0 \tag{4.12}$$

and

$$\lambda\,(x) = \sum_{n=0}^{\infty} \frac{1}{(2n+1)^x} = \frac{1}{\Gamma\,(x)} \int_0^\infty \frac{t^{x-1}}{e^t - e^{-t}}\, \mathrm{d}t, \quad x > 0. \tag{4.13}$$

These are related to $\zeta(x)$ by

$$\eta(x) = \left(1 - 2^{1-x}\right)\zeta(x) \quad \text{and} \quad \lambda(x) = \left(1 - 2^{-x}\right)\zeta(x) \tag{4.14}$$

satisfying the identity

$$\zeta(x) + \eta(x) = 2\lambda(x). \tag{4.15}$$

It should be further noted that explicit expressions for both of $\eta(2n)$ and $\lambda(2n)$ exist as a consequence of the relation to $\zeta(2n)$ via (4.14).

4.2. RESULTS FOR THE ZETA FUNCTION

Lemma 1. *The following identity involving the Zeta function holds. Namely,*

$$\int_0^\infty \frac{t^x}{\left(e^t + 1\right)^2}\,\mathrm{d}t = C(x+1)\,\zeta(x+1) - xC(x)\,\zeta(x), \qquad x > 0, \tag{4.16}$$

where $C(x)$ is as given by (4.7).

Based on the identity in Lemma 1, the following theorem was developed (see Alzer [**2**], Cerone et al. [**18**], and also [**10**] where the constants in the bounds of (4.17) were developed.

Theorem 8. *For real numbers $x > 0$ we have*

$$\left(\ln 2 - \frac{1}{2}\right) b(x) < \zeta(x+1) - \big(1 - b(x)\big)\zeta(x) < \frac{b(x)}{2}, \tag{4.17}$$

where

$$b(x) = \frac{1}{2^x - 1}, \tag{4.18}$$

and the constants $\ln 2 - \frac{1}{2}$ and $\frac{1}{2}$ are sharp.

The following is a correction of a result obtained by the author [**13**] by utilising the Čebyšev functional bounds given by (1.17) and (4.5).

Theorem 9. *For $\alpha > 0$ the Zeta function satisfies the inequality*

$$\left|\zeta(\alpha+1) - \frac{2^{\alpha-1}}{\alpha}\cdot\frac{\pi^2}{6}\right| \le \frac{\kappa\cdot 2^{\alpha-\frac{1}{2}}}{\Gamma(\alpha+1)}\big(\Gamma(2\alpha-1) - \Gamma^2(\alpha)\big)^{\frac{1}{2}}, \tag{4.19}$$

where

$$\kappa = \left[\pi^2\left(1 - \frac{\pi^2}{72}\right) - 7\zeta(3)\right]^{\frac{1}{2}} = 0.319846901\ldots \tag{4.20}$$

with equality obtained at $\alpha = 1$.

The following theorem was obtained in [**17**] utilising bounds for the ČEBYŠEV functional.

Theorem 10. *For* $\alpha > 1$ *and* $m = \lfloor \alpha \rfloor$ *the zeta function satisfies the inequality*

$$\left| \zeta(\alpha+1) - 2^{\alpha-m} \frac{\Gamma(m+1)}{\Gamma(\alpha+1)} \zeta(m+1) \Gamma(\alpha-m+1) \right| \tag{4.21}$$

$$\leq \frac{2^{\left(\alpha-m+\frac{1}{2}\right)}}{\Gamma(\alpha+1)} \cdot E_m \cdot \left(\Gamma(2\alpha-2m+1) - \Gamma^2(\alpha-m+1)\right)^{\frac{1}{2}},$$

where

$$E_m^2 = 2^{2m}\Gamma(2m+1)\left(\lambda(2m) - \lambda(2m+1)\right) - \frac{1}{2}\Gamma^2(m+1)\zeta^2(m+1), \tag{4.22}$$

with $\lambda(\cdot)$ *given by* (4.13). *Equality in* (4.21) *results when* $\alpha = m$.

Proof. (Sketch using the ČEBYŠEV Functional Approach). Let

$$\tau(\alpha) = \Gamma(\alpha+1)\zeta(\alpha+1) = \int_0^\infty \frac{x^\alpha}{e^x - 1}\,\mathrm{d}x \tag{4.23}$$

$$= \int_0^\infty e^{-x/2} \frac{x^m}{e^{x/2} - e^{-x/2}} \cdot x^{\alpha-m}\,\mathrm{d}x, \quad \alpha > 1$$

where $m = \lfloor \alpha \rfloor$.

Make the associations

$$p(x) = e^{-x/2}, \quad f(x) = \frac{x^m}{e^{x/2} - e^{-x/2}}, \quad g(x) = x^{\alpha-m} \tag{4.24}$$

then we have from (1.17)

$$\begin{cases} P = \displaystyle\int_0^\infty e^{-x/2}\,\mathrm{d}x = 2, \\ \mathcal{M}(f;p) = \displaystyle\frac{1}{2}\int_0^\infty \frac{e^{-x/2}x^m}{e^{x/2} - e^{-x/2}}\,\mathrm{d}x = \frac{1}{2}\Gamma(m+1)\zeta(m+1), \\ \mathcal{M}(g;p) = \displaystyle\frac{1}{2}\int_0^\infty e^{-x/2}x^{\alpha-m}\,\mathrm{d}x = 2^{\alpha-m}\Gamma(\alpha-m+1). \end{cases} \tag{4.25}$$

$\square$

The following corollary provides upper bounds for the zeta function at odd integers.

Corollary 3. *The inequality*

$$\Gamma(2m+1)\left(2\cdot\left(2^{2m}-1\right)\zeta(2m)-\left(2^{2m+1}-1\right)\zeta(2m+1)\right) - \Gamma^2(m+1)\zeta^2(m+1)>0 \tag{4.26}$$

holds for $m=1,2,\dots$.

Proof. From equation (4.22) of Theorem 10, we have $E_m^2>0$. Utilising the relationship between $\lambda(\cdot)$ and $\zeta(\cdot)$ given by (4.14) readily gives the inequality (4.26). □

Remark 5. In (4.26), if m is odd, then $2m$ and $m+1$ are even so that an expression in the form

$$\alpha(m)\zeta(2m)-\beta(m)\zeta(2m+1)-\gamma(m)\zeta^2(m+1)>0, \tag{4.27}$$

results, where

$$\begin{aligned}\alpha(m)&=2\left(2^{2m}-1\right)\Gamma(2m+1),\\ \beta(m)&=\left(2^{2m+1}-1\right)\Gamma(2m+1)\quad\text{and}\\ \gamma(m)&=\Gamma^2(m+1).\end{aligned} \tag{4.28}$$

Thus for m **odd** we have

$$\zeta(2m+1)<\frac{\alpha(m)\zeta(2m)-\gamma(m)\zeta^2(m+1)}{\beta(m)}. \tag{4.29}$$

That is, for $m=2k-1$, we have from (4.29)

$$\zeta(4k-1)<\frac{\alpha(2k-1)\zeta(4k-2)-\gamma(2k-1)\zeta^2(2k)}{\beta(2k-1)} \tag{4.30}$$

giving for $k=1,2,3$, for example,

$$\begin{aligned}\zeta(3)&<\frac{\pi^2}{7}\left(1-\frac{\pi^2}{72}\right)=1.21667148,\\ \zeta(7)&<\frac{2\pi^6}{1905}\left(1-\frac{\pi^2}{2160}\right)=1.00887130,\\ \zeta(11)&<\frac{62\pi^{10}}{5803245}\left(1-\frac{\pi^2}{492150}\right)=1.00050356.\end{aligned}$$

The above bound for $\zeta(3)$ was obtained previously by the author in [**13**] from (4.20).

If m is **even** then for $m=2k$ we have from (4.29)

$$\zeta(4k+1)<\frac{\alpha(2k)\zeta(4k)-\gamma(2k)\zeta^2(2k+1)}{\beta(2k)},\quad k=1,2,\dots. \tag{4.31}$$

We notice that in (4.31), or equivalently (4.27) with $m = 2k$ there are two zeta functions with odd arguments. There are a number of possibilities for resolving this, but firstly it should be noticed that $\zeta(x)$ is monotonically decreasing for $x > 1$ so that $\zeta(x_1) > \zeta(x_2)$ for $1 < x_1 < x_2$.

Firstly, we may use lower bounds obtained in [**10**], namely

$$L(x) = (1 - b(x))\,\zeta(x) + \left(\ln 2 - \frac{1}{2}\right) b(x) \quad \text{or} \quad L_2(x) = \frac{\zeta(x+2) - \dfrac{b(x+1)}{2}}{1 - b(x+1)},$$

where $b(x)$ is given by (4.18).

However, from numerical investigation in [**10**], it seems that $L_2(x) > L(x)$ for positive integer x and so we have from (4.31)

$$\zeta_L(4k+1) < \frac{\alpha(2k)\,\zeta(2k) - \gamma(2k)\,L_2^2(2k)}{\beta(2k)}, \tag{4.32}$$

where we have used the fact that $L_2(x) < \zeta(x+1)$.

Secondly, since the even argument $\zeta(2k+2) < \zeta(2k+1)$, then from (4.31) we have

$$\zeta_E(4k+1) < \frac{\alpha(2k)\,\zeta(4k) - \gamma(2k)\,\zeta^2(2k+2)}{\beta(2k)}. \tag{4.33}$$

Finally, we have that $\zeta(m+1) > \zeta(2m+1)$ so that from (4.27) we have, with $m = 2k$ on solving the resulting quadratic equation that

$$\zeta_Q(4k+1) < \frac{-\beta(2k) + \sqrt{\beta^2(2k) + 4\gamma(2k)\,\alpha(2k)\,\zeta(4k)}}{2\gamma(2k)}. \tag{4.34}$$

For $k = 1$ we have from (4.32)–(4.34) that

$$\begin{aligned}
\zeta_L(5) &< \frac{\pi^4}{93} - \frac{1}{186}\left(\frac{7\pi^4}{540} - \frac{1}{12}\right)^2 = 1.039931461,\\
\zeta_E(5) &< \frac{\pi^4}{93}\left(1 - \frac{\pi^4}{16200}\right) = 1.041111605,\\
\zeta_Q(5) &< -93 + \sqrt{8649 + 2\pi^4} = 1.04157688;
\end{aligned}$$

and for $k = 2$

$$\begin{aligned}
\zeta_L(9) &< \frac{17}{160965}\pi^8 - \frac{1}{35770}\left(\frac{31}{28350}\pi^6 - \frac{1}{60}\right)^2 = 1.002082506,\\
\zeta_E(9) &< \frac{17}{160965}\pi^8\left(1 - \frac{\pi^4}{337650}\right) = 1.0020834954,\\
\zeta_Q(9) &< -17885 + \frac{1}{3}\sqrt{2878859025 + 34\pi^8} = 1.00208436.
\end{aligned}$$

It should be noted that the above results give tighter upper bounds for the odd zeta function evaluations than were possible using the methodology utilising techniques based around Theorem 8 as demonstrated by the numerics which are presented in Table 1 of [**10**].

Numerical experimentation using Maple seems to indicate that the upper bounds for

$$\zeta_L(4k+1), \zeta_E(4k+1) \quad \text{and} \quad \zeta_Q(4k+1)$$

are in increasing order. Analytic demonstration that $\zeta_L(4k+1)$ is better remains an open problem.

5. CONCLUDING REMARKS

In the paper the usefulness of some recent results in the analysis of inequalities, has been demonstrated through application to some special functions. Although these techniques have been applied in a variety of areas of applied mathematics, their application to special functions does not seem to have received much attention to date. There are many special functions which may be represented as the integral of products of functions. The investigation in the current article has restricted itself to the investigation of the Bessel function of the first kind, the Beta function and the Zeta function.

It may be surmised from the above investigations that the accuracy of the bounds over particular regions of parameters cannot be ascertained *a priori.* It has been demonstrated, however, that some useful bounds may be obtained which have hitherto do not seem to have been discovered. The approach of utilising developments in the field of inequalities to special functions has been shown to have the potential for further development.

A general investigation of Dirichlet series has also been undertaken in [**20**], [**21**] utilising convexity arguments and it is shown that in particular

$$\zeta(s+1) \le \mathcal{A}\left(\frac{1}{\zeta(s)}, \frac{1}{\zeta(s+2)}\right) \le \mathcal{G}\big(\zeta(s), \zeta(s+2)\big) \tag{5.1}$$

where $\mathcal{A}(\cdot,\cdot)$ is the arithmetic mean and $\mathcal{G}(\cdot,\cdot)$ the geometric mean.

Specifically, for $s = 2n$, then

$$\zeta(2n+1) \le \mathcal{H}\big(\zeta(2n), \zeta(2n+2)\big) \le \mathcal{G}\big(\zeta(2n), \zeta(2n+2)\big), \tag{5.2}$$

where the harmonic mean

$$\mathcal{H}(\alpha,\beta) = \frac{\mathcal{G}^2(\alpha,\beta)}{\mathcal{A}(\alpha,\beta)} = \mathcal{A}\left(\frac{1}{\alpha}, \frac{1}{\beta}\right).$$

The reader may also wish to refer to the papers [**3**] and [**6**] which provide some results using monotonicity and convexity arguments.

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INEQUALITIES FOR NORMAL OPERATORS IN HILBERT SPACES

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Some inequalities for normal operators in Hilbert spaces are established. For this purpose, classical and new vector inequalities due to Buzano, Dunkl-Williams, Hile, Goldstein-Ryff-Clarke, Dragomir-Sándor and the author are employed.

1. INTRODUCTION

Let $(H;\langle\cdot,\cdot\rangle)$ be a complex Hilbert space and $T:H\to H$ a bounded linear operator on H. Recall that T is a *normal operator* if $T^*T=TT^*$. Normal operators may be regarded as a generalisation of self-adjoint operator T in which T^* need not be exactly T but commutes with T [**11**, p. 15].

The *numerical range* of an operator T is the subset of the complex numbers $\mathbb{C}$ given by [**11**, p. 1]:

$$W(T)=\{\langle Tx,x\rangle\,,\ x\in H,\ \|x\|=1\}\,.$$

For various properties of the numerical range see [**11**].

We recall here some of the ones related to normal operators.

Theorem 1. *If $W(T)$ is a line segment, then T is normal.*

We denote by $r(T)$ the operator *spectral radius* [**11**, p. 10] and by $w(T)$ its *numerical radius* [**11**, p. 8]. The following result may be stated as well [**11**, p. 15].

Theorem 2. *If T is normal, then $\|T^n\|=\|T\|^n$, $n=1,\dots$. Moreover, we have*:

$$r(T)=w(T)=\|T\|\,. \tag{1.1}$$

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An important fact about the normal operators that will be used frequently in the sequel is the following one [**12**, p. 42]:

Theorem 3. *A necessary and sufficient condition that an operator T be normal is that $\|Tx\| = \|T^*x\|$ for every vector $x \in H$.*

We observe that, if one uses the SCHWARZ inequality

$$|\langle u, v\rangle| \leq \|u\| \|v\|, u, v \in H,$$

for the choices $u = Tx, v = T^*x$ with $x \in H$, then that one gets the following simple inequality for the normal operator T :

$$\|Tx\|^2 \geq \left|\left\langle T^2x, x\right\rangle\right|, \ x \in H. \tag{1.2}$$

It is then natural to look for upper bounds for the quantity $\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right|$, $x \in H$ under various assumptions for the normal operator T, which would give a measure of the closeness of the terms involved in the inequality (1.2).

Motivated by this problem, the aim of the paper is to establish some reverse inequalities for (1.2). Norm inequalities for various expressions with normal operators and their adjoints are also provided. For both purposes, some inequalities for vectors in inner product spaces due to BUZANO, DUNKL-WILLIAMS, HILE, GOLDSTEIN-RYFF-CLARKE, DRAGOMIR-SÁNDOR and the author, are employed.

2. INEQUALITIES FOR VECTORS

The following result may be stated.

Theorem 4. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a normal linear operator on H. Then*

$$\left(\|Tx\|^2\right) \geq \frac{1}{2}\left(\|Tx\|^2 + \left|\left\langle T^2x, x\right\rangle\right|\right) \geq |\langle Tx, x\rangle|^2, \tag{2.1}$$

for any $x \in H$, $\|x\| = 1$. The constant $\frac{1}{2}$ is best possible in (2.1).

Proof. The first inequality is obvious.

For the second inequality, we need the following refinement of SCHWARZ's inequality obtained by the author in 1985 [**2**, Theorem 2] (see also [**8**] and [**4**]):

$$\|a\| \|b\| \geq |\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| + |\langle a, e\rangle \langle e, b\rangle| \geq |\langle a, b\rangle|, \tag{2.2}$$

provided a, b, e are vectors in H and $\|e\| = 1$.

Observing that

$$|\langle a, b\rangle - \langle a, e\rangle \langle e, b\rangle| \geq |\langle a, e\rangle \langle e, b\rangle| - |\langle a, b\rangle|,$$

then by the first inequality in (2.2) we deduce

$$\frac{1}{2}\left(\|a\|\|b\|+|\langle a,b\rangle|\right)\geq|\langle a,e\rangle\langle e,b\rangle|. \tag{2.3}$$

This inequality was obtained in a different way earlier by M. L. Buzano in [**1**].

Now, choose in (2.3), $e=x$, $\|x\|=1$, $a=Tx$ and $b=T^{*}x$ to get

$$\frac{1}{2}\left(\|Tx\|\|T^{*}x\|+\left|\left\langle T^{2}x,x\right\rangle\right|\right)\geq|\langle Tx,x\rangle|^{2} \tag{2.4}$$

for any $x\in H$, $\|x\|=1$. Since T is normal, then $\|Tx\|=\|T^{*}x\|$, and by (2.4) we deduce the desire result (2.1).

The fact that, the constant $\frac{1}{2}$ is best possible in (2.1) is obvious since for $T=I$, the identity operator, we get equality in (2.1). □

From a different perspective, we can state the following result:

Theorem 5. *Let $T:H\to H$ be a normal operator on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. If $\lambda\in\mathbb{C}$, then*

$$(0\leq)\,\|Tx\|^{2}-\left|\left\langle T^{2}x,x\right\rangle\right|\leq\frac{2}{(1+|\lambda|)^{2}}\|Tx-\lambda T^{*}x\|^{2} \tag{2.5}$$

for any $x\in H$, $\|x\|=1$.

Proof. We use the following inequality [**9**]:

$$\|a-b\|\geq\frac{1}{2}\left(\|a\|+\|b\|\right)\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|,\quad a,b\in H\setminus\{0\},$$

which is well known in the literature as the *Dunkl-Williams inequality.*

This inequality, by taking the square, is clearly equivalent to

$$\frac{4\|a-b\|^{2}}{(\|a\|+\|b\|)^{2}}\geq\left\|\frac{a}{\|a\|}-\frac{b}{\|b\|}\right\|^{2}=2-2\cdot\frac{\operatorname{Re}\langle a,b\rangle}{\|a\|\|b\|},$$

which shows that (see [**3**, Eq. (2.5)])

$$\frac{\|a\|\|b\|-|\langle a,b\rangle|}{\|a\|\|b\|}\leq\frac{2\|a-b\|^{2}}{(\|a\|+\|b\|)^{2}}.$$

Now, for $x\in H\setminus\ker(T)$, $\|x\|=1$, choose $a=Tx$ and $b=\lambda T^{*}x$ $(\lambda\neq0)$ to obtain

$$\|Tx\|\|T^{*}x\|-\left|\left\langle T^{2}x,x\right\rangle\right|\leq\frac{2\|Tx\|\|T^{*}x\|}{(\|Tx\|+|\lambda|\|T^{*}x\|)^{2}}\|Tx-\lambda T^{*}x\|^{2}. \tag{2.6}$$

Since $\|Tx\|=\|T^{*}x\|$, T being a normal operator, we get from (2.6) that (2.5) holds true for any $x\in H\setminus\ker(T)$, $\|x\|=1$.

For $\lambda = 0$ the inequality (2.5) is obvious.

Since for normal operators $\ker(T) = \ker(T^*)$ then for $x \in \ker(T)$, $\|x\| = 1$ the inequality (2.5) also holds true. □

The following result which provides a different upper bound for the nonnegative quantity

$$\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right|, x \in H, \|x\| = 1$$

may be stated as well:

Theorem 6. *Let $T : H \to H$ be a normal operator on the Hilbert space H and $\alpha, \lambda \in \mathbb{C} \setminus \{0\}$. Then*

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leq \frac{1}{2} \cdot \frac{\left[|\operatorname{Re}\alpha| \left\|Tx - \frac{\alpha}{\bar{\alpha}}\lambda T^*x\right\| + |\operatorname{Im}\alpha| \left\|Tx + \frac{\alpha}{\bar{\alpha}}\lambda T^*x\right\|\right]^2}{|\lambda| |\alpha|^2} \tag{2.7}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following inequality (see [**3**, Theorem 2.11]):

$$\|a\| \|b\| - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2}\langle a, b\rangle\right] \leq \frac{1}{2} \cdot \frac{[|\operatorname{Re}\alpha| \|a - b\| + |\operatorname{Im}\alpha| \|a + b\|]^2}{|\alpha|^2} \tag{2.8}$$

for the choices:

$$a = \frac{Tx}{\alpha}, \quad b = \frac{\lambda}{\bar{\alpha}}T^*x, \quad x \in H$$

to obtain:

$$\frac{|\lambda| \|Tx\| \|T^*x\|}{|\alpha|^2} - \operatorname{Re}\left[\frac{\alpha^2}{|\alpha|^2} \cdot \frac{\bar{\lambda}}{\alpha^2}\langle Tx, T^*x\rangle\right] \leq \frac{1}{2} \cdot \frac{\left[|\operatorname{Re}\alpha| \left\|\frac{Tx}{\alpha} - \frac{\lambda}{\alpha}T^*x\right\| + |\operatorname{Im}\alpha| \left\|\frac{Tx}{\alpha} + \frac{\lambda}{\alpha}T^*x\right\|\right]^2}{|\alpha|^2}. \tag{2.9}$$

Since T is normal, we get from (2.9) the desired result (2.7). The details are omitted. □

Another result of this type is incorporated in:

Theorem 7. *Let $T : H \to H$ be a normal operator on the Hilbert space H, $s \in [0, 1]$ and $t \in \mathbb{R}$. Then*

$$(0 \leq) \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \leq \|Tx\|^2 \left[s \|tT^*x - Tx\|^2 + (1 - s) \|T^*x - tTx\|^2\right]. \tag{2.10}$$

In particular

$$(0 \leq) \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \leq \frac{1}{2} \|Tx\|^2 \inf_{t \in \mathbb{R}} \left[\|tT^*x - Tx\|^2 + \|T^*x - tTx\|^2\right].$$

Proof. We use the inequality obtained in [**4**, Theorem 2], to state that

$$\left[(1-s)\|a\|^2+s\|b\|^2\right]\left[(1-s)\|b\|^2+s\|a\|^2\right]-|\langle a,b\rangle|^2 \le \left[(1-s)\|a\|^2+s\|b\|^2\right]\left[(1-s)\|b-ta\|^2+s\|tb-a\|^2\right] \tag{2.11}$$

for any $s\in[0,1]$, $t\in\mathbb{R}$ and $a,b\in H$.

If in (2.11) we choose $a=Tx$, $b=T^*x$, $x\in H$ and $\|x\|=1$, then we get

$$\|Tx\|^4-\left|\left\langle T^2x,x\right\rangle\right|^2 \le \|Tx\|^2\left[s\|tT^*x-Tx\|^2+(1-s)\|T^*x-tTx\|^2\right]$$

for any $s\in[0,1]$, $t\in\mathbb{R}$, from where we deduce the desired inequality (2.10). □

From a different perspective, we can state the following result as well.

Theorem 8. *Let $T:H\to H$ be a normal operator on the Hilbert space $(H;\langle\cdot,\cdot\rangle)$. If $\lambda\in\mathbb{C}\setminus\{0\}$ and $r>0$ are such that*

$$\|T-\lambda T^*\|\le r, \tag{2.12}$$

then:

$$(0\le)\,\|Tx\|^4-\left|\left\langle T^2x,x\right\rangle\right|^2 \le \frac{r^2}{|\lambda|^2}\|Tx\|^2 \tag{2.13}$$

for any $x\in H$, $\|x\|=1$.

Proof. We use the following reverse of the quadratic Schwarz inequality obtained by the author in [**4**]

$$(0\le)\,\|a\|^2\|b\|^2-|\langle a,b\rangle|^2 \le \frac{1}{|\alpha|^2}\|a\|^2\|a-\alpha b\|^2 \tag{2.14}$$

provided $a,b\in H$ and $\alpha\in\mathbb{C}\setminus\{0\}$.

Choosing in (2.14) $a=Tx$, $\alpha=\lambda$, $b=T^*x$, we get

$$\begin{aligned}\|Tx\|^4 &\le \left|\left\langle T^2x,x\right\rangle\right|^2+\frac{1}{|\lambda|^2}\|Tx\|^2\|Tx-\lambda T^*x\|^2 \\ &\le \left|\left\langle T^2x,x\right\rangle\right|^2+\frac{1}{|\lambda|^2}r^2\|Tx\|^2\end{aligned} \tag{2.15}$$

which is the desired result (2.13). □

Finally, on utilising the following result obtained in [**4**]:

Lemma 1. *Let $a,b\in H\setminus\{0\}$ and $\varepsilon\in(0,1/2]$. If*

$$(0\le)\,1-\varepsilon-\sqrt{1-2\varepsilon}\le\frac{\|a\|}{\|b\|}\le 1-\varepsilon+\sqrt{1-2\varepsilon}, \tag{2.16}$$

then

$$(0 \le) \|a\| \|b\| - Re \langle a, b\rangle \le \varepsilon \|a - b\|^2 . \tag{2.17}$$

We can state:

Theorem 9. *Let $T : H \to H$ be a normal operator on H. If $\lambda \in \mathbb{C}$ is such that*

$$(0 \le) 1 - \varepsilon - \sqrt{1 - 2\varepsilon} \le |\lambda| \le 1 - \varepsilon + \sqrt{1 - 2\varepsilon}, \quad \varepsilon \in (0, 1/2] \tag{2.18}$$

then

$$(0 \le) \|Tx\|^2 - \left|\left\langle T^2 x, x\right\rangle\right| \le \frac{\varepsilon}{|\lambda|} \|Tx - \lambda T^* x\|^2 \tag{2.19}$$

for any $x \in H$, $\|x\| = 1$.

Proof. Utilising Lemma 1 for $a = \lambda T^* x$, $b = Tx$, $x \in H \setminus \ker(T)$, $\|x\| = 1$, we have

$$|\lambda| \|Tx\|^2 - |\lambda| \left|\left\langle T^2 x, x\right\rangle\right| \le \varepsilon \|Tx - \lambda T^* x\|^2 . \tag{2.20}$$

For $x \in \ker(T)$, $\|x\| = 1$ the inequality (2.19) also holds, and the proof is completed. □

3. INEQUALITIES FOR OPERATOR NORM

The purpose of this section is to point out some norm inequalities for normal operators that can be naturally obtained from various vector inequalities in inner product spaces, such as the ones due to Hile, Goldstein-Ryff-Clarke, Dragomir-Sándor and the author.

Theorem 10. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $\lambda \in \mathbb{C}$, $|\lambda| \ne 1$, then*:

$$\left\|T - |\lambda|^{v+1} T^*\right\| \le \frac{1 - |\lambda|^{v+1}}{1 - |\lambda|} \|T - \lambda T^*\| , \tag{3.1}$$

for any $v > 0$.

Proof. We use the following inequality:

$$\|\|a\|^v a - \|b\|^v b\| \le \frac{\|a\|^{v+1} - \|b\|^{v+1}}{\|a\| - \|b\|} \|a - b\| \tag{3.2}$$

provided $v > 0$ and $\|a\| \ne \|b\|$, which is known in the literature as the *Hile inequality* [**13**].

Now, if we choose in (3.2) $a = Tx$, $b = \lambda T^* x$, since T is normal, we have $\|a\| = \|Tx\|$, $\|b\| = |\lambda| \|Tx\|$ and by (3.2) we get

$$\|Tx\|^v \left\| Tx - |\lambda|^{v+1} T^* x \right\| \leq \|Tx\|^v \frac{\left(1 - |\lambda|^{v+1}\right)}{1 - |\lambda|} \|Tx - \lambda T^* x\| \tag{3.3}$$

for any $x \in H \setminus \ker(T)$.

If $x \notin \ker(T)$, then from (3.3) we get

$$\left\| Tx - |\lambda|^{v+1} T^* x \right\| \leq \frac{1 - |\lambda|^{v+1}}{1 - |\lambda|} \|Tx - \lambda T^* x\|. \tag{3.4}$$

If $x \in \ker(T)$ and since $\ker(T) = \ker(T^*)$, T being normal, then the inequality (3.4) is also valid. Therefore, (3.4) holds for any $x \in H$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we get the desired inequality (3.1). □

Remark 1. For $v = 1$, we get the inequality:

$$\left\| T - |\lambda|^2 T^* \right\| \leq (1 + |\lambda|) \|T - \lambda T^*\|. \tag{3.5}$$

Utilising the second inequality due to Hile (see [**13**, Eq. (5.2)]):

$$\left\| \frac{a}{\|a\|^{v+2}} - \frac{b}{\|b\|^{v+2}} \right\| \leq \frac{\|a\|^{v+2} - \|b\|^{v+2}}{\|a\| - \|b\|} \cdot \frac{\|a - b\|}{\|a\|^{v+1} \cdot \|b\|^{v+1}}$$

for $a, b \in H$, $a, b \neq 0$ and $\|a\| \neq \|b\|$, and making use of an argument similar to the one in the proof of the above theorem, we can state the following result:

Theorem 11. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $\lambda \in \mathbb{C}$, $|\lambda| \neq 0, 1$, then:*

$$\left\| T - \frac{\lambda}{|\lambda|^{v+2}} T^* \right\| \leq \frac{1 - |\lambda|^{v+1}}{(1 - |\lambda|) |\lambda|^{v+1}} \|T - \lambda T^*\|, \tag{3.6}$$

where $v > 0$.

The following result may be stated as well.

Theorem 12. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $|\lambda| \leq 1$, then*

$$(1 - |\lambda|^\rho)^2 \|T\|^2 \leq \begin{cases} \rho^2 \|T - \lambda T^*\|^2 & \text{if } \rho \geq 1, \\ |\lambda|^{2\rho - 2} \|T - \lambda T^*\|^2 & \text{if } \rho < 1. \end{cases} \tag{3.7}$$

Proof. We use the following inequality due to Goldstein, Ryff and Clarke [**10**]

$$\|a\|^{2\rho} + \|b\|^{2\rho} - 2\|a\|^{\rho-1}\|b\|^{\rho-1}\operatorname{Re}\langle a,b\rangle \leq \begin{cases} \rho^2 \|a\|^{2\rho-2}\|a-b\|^2 & \text{if } \rho \geq 1, \\ \|b\|^{2\rho-2}\|a-b\|^2 & \text{if } \rho < 1, \end{cases} \tag{3.8}$$

provided $\rho \in \mathbb{R}$ and $a, b \in H$ with $\|a\| \geq \|b\|$.

Since $\operatorname{Re}\langle a,b\rangle \leq |\langle a,b\rangle|$, then, from (3.8), we have the inequality

$$\|a\|^{2\rho} + \|b\|^{2\rho} \leq 2\|a\|^{\rho-1}\|b\|^{\rho-1}|\langle a,b\rangle| + \begin{cases} \rho^2 \|a\|^{2\rho-2}\|a-b\|^2 & \text{if } \rho \geq 1, \\ \|b\|^{2\rho-2}\|a-b\|^2 & \text{if } \rho < 1. \end{cases} \tag{3.9}$$

We choose $a = Tx$, $b = \lambda T^*x$ and since $|\lambda| \leq 1$, we have $\|a\| \geq \|b\|$. From (3.9), on taking into account that $\|Tx\| = \|T^*x\|$, we deduce

$$\|Tx\|^{2\rho} + |\lambda|^{2\rho}\|Tx\|^{2\rho} \leq 2\|Tx\|^{2\rho-2}|\lambda|^{\rho}\left|\left\langle T^2x, x\right\rangle\right| + \begin{cases} \rho^2 \|Tx\|^{2\rho-2}\|Tx-\lambda T^*x\|^2 & \text{if } \rho \geq 1, \\ |\lambda|^{2\rho-2}\|Tx\|^{2\rho-2}\|Tx-\lambda T^*x\|^2 & \text{if } \rho < 1, \end{cases}$$

which implies that:

$$\left(1+|\lambda|^{2\rho}\right)\|Tx\|^2 \leq 2|\lambda|^{\rho}\left|\left\langle T^2x, x\right\rangle\right| + \begin{cases} \rho^2 \|Tx-\lambda T^*x\|^2 & \text{if } \rho \geq 1, \\ |\lambda|^{2\rho-2}\|Tx-\lambda T^*x\|^2 & \text{if } \rho < 1, \end{cases} \tag{3.10}$$

for any $x \in H$, $\|x\| = 1$.

This inequality is of interest in itself.

Taking the supremum over $x \in H$, $\|x\| = 1$, and using the fact that

$$\sup_{\|x\|=1}\left|\left\langle T^2x, x\right\rangle\right| = w\left(T^2\right) = \|T\|^2,$$

we get the desired inequality (3.7). □

Remark 2. If $|\lambda| > 1$, on choosing in (3.9) $a = \lambda T^*x$, $b = Tx$ we get:

$$\left(|\lambda|^{2\rho}+1\right)\|Tx\|^2 \leq 2|\lambda|^{\rho}\left|\left\langle T^2x, x\right\rangle\right| + \begin{cases} \rho^2 |\lambda|^{2\rho-2}\|Tx-\lambda T^*x\|^2 & \text{if } \rho \geq 1, \\ \|Tx-\lambda T^*x\|^2 & \text{if } \rho < 1, \end{cases}$$

which implies the "dual" inequality:

$$(1-|\lambda|^{\rho})^2\|T\|^2 \leq \begin{cases} \rho^2 |\lambda|^{2\rho-2}\|T-\lambda T^*\|^2 & \text{if } \rho \geq 1, \\ \|T-\lambda T^*\|^2 & \text{if } \rho < 1, \end{cases} \tag{3.11}$$

for any $\lambda \in \mathbb{C}$, $|\lambda| > 1$.

The following result concerning operator norm inequalities may be stated as well:

Theorem 13. *Let $T : H \to H$ be a normal operator on the Hilbert space $(H; \langle\cdot,\cdot\rangle)$ and $\alpha, \beta \in \mathbb{C}$. Then*:

$$\|T\|^p \left[(|\alpha| + |\beta|)^p + ||\alpha| - |\beta||^p\right] \le \|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \tag{3.12}$$

if $p \in (1,2)$ and

$$\|\alpha T + \beta T^*\|^p + \|\alpha T - \beta T^*\|^p \ge 2\left(|\alpha|^p + |\beta|^p\right)\|T\|^p \tag{3.13}$$

if $p \ge 2$.

Proof. We use the following result obtained by Dragomir and Sándor in [**8**]:

$$\|a + b\|^p + \|a - b\|^p \ge (\|a\| + \|b\|)^p + |\|a\| - \|b\||^p \tag{3.14}$$

if $p \in (1,2)$ and

$$\|a + b\|^p + \|a - b\|^p \ge 2\left(\|a\|^p + \|b\|^p\right) \tag{3.15}$$

if $p \ge 2$, where a, b are arbitrary vectors in the inner product space $(H; \langle\cdot,\cdot\rangle)$.

We choose $a = \alpha T x$, $b = \beta T^* x$ to get:

$$\begin{aligned} \|(\alpha T + \beta T^*)(x)\|^p &+ \|(\alpha T - \beta T^*)(x)\|^p \\ &\ge (|\alpha| + |\beta|)^p \|Tx\|^p + ||\alpha| - |\beta||^p \|Tx\|^p \\ &= \left[(|\alpha| + |\beta|)^p + ||\alpha| - |\beta||^p\right] \|Tx\|^p \end{aligned} \tag{3.16}$$

if $p \in (1,2)$ and

$$\|(\alpha T + \beta T^*)(x)\|^p + \|(\alpha T - \beta T^*)(x)\|^p \ge 2\left(|\alpha|^p + |\beta|^p\right)\|Tx\|^p \tag{3.17}$$

if $p \ge 2$.

Taking the supremum over $x \in H$, $\|x\| = 1$, we deduce (3.12) and (3.13). □

Remark 3. The case $p = 2$ produces the following inequality:

$$\|\alpha T + \beta T^*\|^2 + \|\alpha T - \beta T^*\|^2 \ge 2\left(|\alpha|^2 + |\beta|^2\right)\|T\|^2,$$

that can also be obtained by utilising the parallelogram identity.

The following general result may be stated as well:

Theorem 14. *Let $T : H \to H$ be a normal operator on the Hilbert space H. If $\alpha, \beta \in \mathbb{C}$ and $r, \rho > 0$ are such that*

$$\|T - \bar{\alpha} I\| \le r \quad \text{and} \quad \|T^* - \beta I\| \le \rho, \tag{3.18}$$

then

$$\|T\|^2 + \frac{1}{2}\left(|\alpha|^2 + |\beta|^2\right) \leq \frac{1}{2}\left(r^2 + \rho^2\right) + \|\alpha T + \beta T^*\|. \tag{3.19}$$

Proof. The condition (3.18) obviously implies that

$$\|Tx\|^2 + |\alpha|^2 \leq 2\operatorname{Re}\langle(\alpha T)x, x\rangle + r^2 \tag{3.20}$$

and

$$\|T^*x\|^2 + |\beta|^2 \leq 2\operatorname{Re}\left\langle(\beta T)^* x, x\right\rangle + \rho^2 \tag{3.21}$$

for any $x \in H$, $\|x\| = 1$.

Adding (3.20) and (3.21) and taking into account that $\|Tx\| = \|T^*x\|$, we obtain

$$\begin{aligned} 2\|Tx\|^2 + |\alpha|^2 + |\beta|^2 &\leq 2\operatorname{Re}\langle(\alpha T + \beta T^*)x, x\rangle + r^2 + \rho^2 \\ &\leq 2|\langle(\alpha T + \beta T^*)x, x\rangle| + r^2 + \rho^2. \end{aligned} \tag{3.22}$$

Taking the supremum on (3.22) over $x \in H$, $\|x\| = 1$, and utilising the fact that for the normal operator T we have

$$w(\alpha T + \beta T^*) = \|\alpha T + \beta T^*\|$$

then we get the desired inequality (3.19). □

Remark. If $\alpha, \beta \in \mathbb{C}$ and $r, \rho > 0$ are such that $|\alpha|^2 + |\beta|^2 = \rho^2 + r^2$, then from (3.19) we have:

$$\|T\|^2 \leq \|\alpha T + \beta T^*\|. \tag{3.23}$$

4. SOME REVERSE INEQUALITIES

The following result may be stated.

Theorem 15. *Let $(H; \langle\cdot,\cdot\rangle)$ be a Hilbert space and $T : H \to H$ a normal operator on H. If $\lambda \in \mathbb{C}\setminus\{0\}$ and $r > 0$ are such that*

$$\|T - \lambda T^*\| \leq r, \tag{4.1}$$

then

$$\frac{1 + |\lambda|^2}{2|\lambda|}\|Tx\|^2 \leq \left|\left\langle T^2x, x\right\rangle\right| + \frac{r^2}{2|\lambda|} \tag{4.2}$$

for any $x \in H, \|x\| = 1$.

Proof. The inequality (4.1) is obviously equivalent to

$$\|Tx\|^2 + |\lambda|^2 \|T^*x\|^2 \le 2\operatorname{Re}\left[\overline{\lambda}\langle Tx, T^*x\rangle\right] + r^2 \tag{4.3}$$

for any $x \in H$, $\|x\| = 1$.

Since T is a normal operator, then $\|Tx\| = \|T^*x\|$ for any $x \in H$ and by (4.3) we get

$$\left(1 + |\lambda|^2\right)\|Tx\|^2 \le 2\operatorname{Re}\left[\overline{\lambda}\left\langle T^2x, x\right\rangle\right] + r^2 \tag{4.4}$$

for any $x \in H$, $\|x\| = 1$.

Now, on observing that $\operatorname{Re}\left[\overline{\lambda}\left\langle T^2x, x\right\rangle\right] \le |\lambda|\left|\left\langle T^2x, x\right\rangle\right|$, then by (4.4) we deduce (4.2). □

Remark. Observe that, since $|\lambda|^2 + 1 \ge 2|\lambda|$ for any $\lambda \in \mathbb{C}\setminus\{0\}$, hence by (4.2) we get the simpler (yet coarser) inequality:

$$(0 \le)\,\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \le \frac{r^2}{2|\lambda|}, \quad x \in H, \quad \|x\| = 1, \tag{4.5}$$

provided $\lambda \in \mathbb{C}\setminus\{0\}$, $r > 0$ and T satisfy (4.1).

If $r > 0$ and $\|T - \lambda T^*\| \le r$, with $|\lambda| = 1$, then by (4.2) we have

$$(0 \le)\,\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \le \frac{1}{2}r^2, \quad x \in H, \quad \|x\| = 1. \tag{4.6}$$

The following improvement of (2.5) should be noted:

Corollary 1. *With the assumptions of Theorem* 15, *we have the inequality*

$$(0 \le)\,\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \le \frac{r^2}{1 + |\lambda|^2}\left(\le \frac{2r^2}{(1 + |\lambda|)^2}\right) \tag{4.7}$$

for any $x \in H, \|x\| = 1$.

Proof. The inequality (4.2) is obviously equivalent to:

$$\|Tx\|^2 \le \frac{2|\lambda|}{1 + |\lambda|^2}\left|\left\langle T^2x, x\right\rangle\right| + \frac{r^2}{1 + |\lambda|^2} \le \left|\left\langle T^2x, x\right\rangle\right| + \frac{r^2}{1 + |\lambda|^2}$$

and the first part of the inequality (4.7) is obtained. The second part is obvious. □

For a normal operator T we observe that

$$\left|\left\langle T^2x, x\right\rangle\right| = |\langle Tx, T^*x\rangle| \le \|Tx\|\,\|T^*x\| = \|Tx\|^2$$

for any $x \in H$, hence

$$\|Tx\| - |\langle Tx, T^*x\rangle|^{\frac{1}{2}} \ge 0$$

for any $x \in H$.

Define $\delta(T) := \inf_{\|x\|=1}\left[\|Tx\| - \left|\left\langle T^2x, x\right\rangle\right|^{1/2}\right] \geq 0$. The following inequality may be stated:

Theorem 16. *With the assumptions of Theorem* 15, *we have the inequality*:

$$(0 \leq)\, \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leq r^2 - 2|\lambda|\,\delta(T)\,\mu(T), \tag{4.8}$$

for any $x \in H, \|x\| = 1,$ *where* $\mu(T) = \inf_{\|x\|=1}\left|\left\langle T^2x, x\right\rangle\right|^{1/2}$.

Proof. From the inequality (4.3) we obviously have

$$\|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leq 2\operatorname{Re}\left[\overline{\lambda}\left\langle T^2x, x\right\rangle\right] - \left|\left\langle T^2x, x\right\rangle\right| - |\lambda|^2\|Tx\|^2 + r^2 \tag{4.9}$$

for any $x \in H$, $\|x\| = 1$.

Now, observe that the right hand side of (4.9) can be written as:

$$I := r^2 + 2\operatorname{Re}\left[\overline{\lambda}\left\langle T^2x, x\right\rangle\right] - 2|\lambda|\left|\left\langle T^2x, x\right\rangle\right|^{1/2}\|Tx\| - \left(\left|\left\langle T^2x, x\right\rangle\right|^{1/2} - |\lambda|\,\|Tx\|\right)^2.$$

Since, obviously,

$$\operatorname{Re}\left[\overline{\lambda}\left\langle T^2x, x\right\rangle\right] \leq |\lambda|\left|\left\langle T^2x, x\right\rangle\right|$$

and

$$\left(\left|\left\langle T^2x, x\right\rangle\right|^{1/2} - |\lambda|\,\|Tx\|\right)^2 \geq 0,$$

then

$$\begin{aligned} I &\leq r^2 - 2|\lambda|\left|\left\langle T^2x, x\right\rangle\right|^{1/2}\left(\|Tx\| - \left|\left\langle T^2x, x\right\rangle\right|^{1/2}\right) \\ &\leq r^2 - 2|\lambda|\,\delta(T)\left|\left\langle T^2x, x\right\rangle\right|^{1/2}. \end{aligned}$$

Utilising (4.9) we get

$$\|Tx\|^2 \leq \left|\left\langle T^2x, x\right\rangle\right| - 2|\lambda|\,\delta(T)\left|\left\langle T^2x, x\right\rangle\right|^{1/2} + r^2$$

for any $x \in H$, $\|x\| = 1$, which implies the desired result. □

5. INEQUALITIES UNDER MORE RESTRICTIONS

Now, observe that, for a normal operator $T : H \to H$ and for $\lambda \in \mathbb{C}\setminus\{0\}$, $r > 0$, the following two conditions are equivalent

$$\|Tx - \lambda T^*x\| \leq r \leq |\lambda|\,\|T^*x\| \quad \text{for any } x \in H,\ \|x\| = 1 \tag{c}$$

and

$$\|T - \lambda T^*\| \leq r \quad \text{and} \quad \xi(T) := \inf_{\|x\|=1}\|Tx\| \geq \frac{r}{|\lambda|}. \tag{cc}$$

We can state the following result.

Theorem 17. *Assume that the normal operator $T : H \to H$ satisfies either* (c) *or, equivalently,* (cc) *for a given $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$. Then:*

$$(0 \le) \|Tx\|^4 - \left|\left\langle T^2 x, x\right\rangle\right|^2 \le r^2 \|Tx\|^2 \tag{5.1}$$

and

$$\|Tx\| \left(\xi^2(T) - \frac{r^2}{|\lambda|^2}\right)^{1/2} \le \left|\left\langle T^2 x, x\right\rangle\right|, \tag{5.2}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following elementary reverse of Schwarz's inequality for vectors in inner product spaces (see [**6**] or [**5**]):

$$\|y\|^2 \|a\|^2 - [\operatorname{Re}\langle y, a\rangle]^2 \le r^2 \|y\|^2 \tag{5.3}$$

provided $\|y - a\| \le r \le \|a\|$.

If in (5.3) we choose $x \in H$, $\|x\| = 1$ and $y = Tx$, $a = \lambda T^* x$, then we have:

$$\|Tx\|^2 \|\lambda T^* x\|^2 - |\langle Tx, \lambda T^* x\rangle|^2 \le r^2 \|\lambda T^* x\|^2$$

giving

$$\|Tx\|^4 \le \left|\left\langle T^2 x, x\right\rangle\right|^2 + r^2 \|T^* x\|^2, \tag{5.4}$$

from where we deduce (5.1).

We also know that, if $\|y - a\| \le r \le \|a\|$, then (see [**6**] or [**5**])

$$\|y\| \left(\|a\|^2 - r^2\right)^{1/2} \le \operatorname{Re}\langle y, a\rangle,$$

which gives:

$$\|Tx\| \left(|\lambda|^2 \|Tx\|^2 - r^2\right)^{1/2} \le \operatorname{Re}\langle Tx, \lambda T^* x\rangle \le |\lambda| \left|\left\langle T^2 x, x\right\rangle\right|$$

i.e.,

$$\|Tx\| \left(\|Tx\|^2 - \frac{r^2}{|\lambda|^2}\right)^{1/2} \le \left|\left\langle T^2 x, x\right\rangle\right| \tag{5.5}$$

for any $x \in H$, $\|x\| = 1$. Since, obviously

$$\left(\|Tx\|^2 - \frac{r^2}{|\lambda|^2}\right)^{1/2} \ge \left(\xi^2(T) - \frac{r^2}{|\lambda|^2}\right)^{1/2},$$

hence, by (5.5) we get (5.2). □

Theorem 18. *Assume that the normal operator $T : H \to H$ satisfies either* (c) *or, equivalently,* (cc) *for a given $\lambda \in \mathbb{C} \setminus \{0\}$ and $r > 0$. Then*:

$$\begin{aligned}
(0 \le)\, \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \\
&\le 2\left|\left\langle T^2x, x\right\rangle\right| \|Tx\| \left[|\lambda| \|T\| - \left(|\lambda|^2 \xi^2(T) - r^2\right)^{1/2}\right] \\
&\left(\le 2|\lambda| \left|\left\langle T^2x, x\right\rangle\right| \|T\|^2\right),
\end{aligned} \tag{5.6}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We use the following reverse of the Schwarz inequality obtained in [**5**]:

$$0 \le \|y\|^2 \|a\|^2 - |\langle y, a\rangle|^2 \le 2|\langle y, a\rangle| \|a\| \left(\|a\| - \sqrt{\|a\|^2 - r^2}\right)$$

provided $\|y - a\| \le r \le \|a\|$.

Now, let $x \in H$, $\|x\| = 1$ and choose $y = Tx$, $a = \lambda T^* x$ to get from (5.6) that:

$$\begin{aligned}
&\|Tx\|^2 |\lambda|^2 \|T^*x\|^2 - |\lambda|^2 \left|\left\langle T^2x, x\right\rangle\right|^2 \\
&\le 2|\lambda|^2 \left|\left\langle T^2x, x\right\rangle\right| \|T^*x\| \left[|\lambda| \|T^*x\| - \left(|\lambda|^2 \|T^*x\|^2 - r^2\right)^{1/2}\right]
\end{aligned}$$

giving

$$\|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \le 2\left|\left\langle T^2x, x\right\rangle\right| \|Tx\| \left[|\lambda| \|Tx\| - \left(|\lambda|^2 \|Tx\|^2 - r^2\right)^{1/2}\right],$$

which, by employing a similar argument to that used in the previous theorem, gives the desired inequality (5.6). □

6. OTHER RESULTS FOR ACCRETIVE OPERATORS

For a bounded linear operator $T : H \to H$ the following two statements are equivalent

$$\operatorname{Re}\langle \Gamma T^*x - Tx, Tx - \gamma T^*x\rangle \ge 0 \quad \text{for any } \ x \in H,\ \|x\| = 1; \tag{d}$$

and

$$\left\|Tx - \frac{\gamma + \Gamma}{2} T^*x\right\| \le \frac{1}{2} |\Gamma - \gamma| \|T^*x\| \quad \text{for any } \ x \in H,\ \|x\| = 1. \tag{dd}$$

This follows by the elementary fact that in any inner product space $(H;\langle\cdot,\cdot\rangle)$ we have, for $x,z,Z\in H$, that

$$\operatorname{Re}\langle Z-x,x-z\rangle\geq 0 \tag{6.1}$$

if and only if

$$\left\|x-\frac{z+Z}{2}\right\|\leq\frac{1}{2}\left\|Z-z\right\|. \tag{6.2}$$

An operator $B:H\to H$ is called *accretive* [**11**, p. 26] if $\operatorname{Re}\langle Bx,x\rangle\geq 0$ for any $x\in H$. We observe that, the condition (d) is in fact equivalent with the condition that

(ddd) *the operator* $(T^{*}-\bar{\gamma}T)(\Gamma T^{*}-T)$ *is accretive.*

Now, if $T:H\to H$ is a normal operator, then the following statements are equivalent

$$(T^{*}-\bar{\gamma}T)(\Gamma T^{*}-T)\geq 0 \tag{e}$$

and

$$\Gamma\left[T^{*}\right]^{2}-(\bar{\gamma}\Gamma+1)\,T^{*}T+\bar{\gamma}T^{2}\geq 0. \tag{ee}$$

This is obvious since for T a normal operator we have $T^{*}T=TT^{*}$.

We also must remark that (e) implies that

$$0\leq\langle\Gamma T^{*}x-Tx,Tx-\bar{\gamma}T^{*}x\rangle\quad\text{for any}\quad x\in H,\|x\|=1.$$

Therefore, (e) (or equivalently (ee)) is a sufficient condition for (d) (or equivalently (dd) [or (ddd)]) to hold true.

The following result may be stated.

Theorem 19. *Let $\gamma,\Gamma\in\mathbb{C}$ with $\Gamma\neq-\gamma$. For a normal operator $T:H\to H$ assume that* (ddd) *holds true. Then*:

$$(0\leq)\left\|Tx\right\|^{2}-\left|\left\langle T^{2}x,x\right\rangle\right|\leq\frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\left\|Tx\right\|^{2} \tag{6.3}$$

for any $x\in H$, $\|x\|=1$.

Proof. We use the following reverse of the Schwarz inequality established in [**7**] (see also [**5**]):

$$\|z\|\,\|y\|-\frac{\operatorname{Re}(\Gamma+\gamma)\operatorname{Re}\langle z,y\rangle+\operatorname{Im}(\Gamma+\gamma)\operatorname{Im}\langle z,y\rangle}{|\Gamma+\gamma|}\leq\frac{1}{4}\cdot\frac{\left|\Gamma-\gamma\right|^{2}}{\left|\Gamma+\gamma\right|}\left\|y\right\|^{2}, \tag{6.4}$$

provided $\gamma, \Gamma \in \mathbb{C}$, $\Gamma \neq -\gamma$ and $z, y \in H$ satisfy either the condition

$$(\ell) \qquad \operatorname{Re}\langle \Gamma y - z, z - \gamma y\rangle \geq 0,$$

or, equivalently the condition

$$(\ell\ell) \qquad \left\| z - \frac{\gamma+\Gamma}{2} y \right\| \leq \frac{1}{2} |\Gamma - \gamma| \, \|y\| \, .$$

Now, if in (6.4) we choose $z = Tx$, $y = T^*x$ for $x \in H$, $\|x\| = 1$, then we obtain

$$\|Tx\| \, \|T^*x\| - |\langle Tx, T^*x\rangle| \leq \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \|T^*x\|^2 ,$$

which is equivalent with (6.3). □

Remark 6. The second inequality in (6.3) is equivalent with

$$\|Tx\|^2 \left(1 - \frac{1}{4} \cdot \frac{|\Gamma - \gamma|^2}{|\Gamma + \gamma|} \right) \leq \left| \left\langle T^2 x, x \right\rangle \right|$$

for any $x \in H$, $\|x\| = 1$. This inequality is of interest if $4|\Gamma + \gamma| \geq |\Gamma - \gamma|^2$.

The following result may be stated as well.

Theorem 20. *Let $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$. If $T : H \to H$ is a normal operator such that* (ddd) *holds true, then*:

$$(6.5) \qquad \|Tx\|^2 \leq \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} \left| \left\langle T^2 x, x \right\rangle \right|$$

for any $x \in H$, $\|x\| = 1$.

Proof. We can use the following reverse of the Schwarz inequality:

$$(6.6) \qquad \|z\| \, \|y\| \leq \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle z, y\rangle| ,$$

provided $\gamma, \Gamma \in \mathbb{C}$ with $\operatorname{Re}(\Gamma\bar{\gamma}) > 0$ and $z, y \in H$ are satisfying either the condition (ℓ) or, equivalently the condition $(\ell\ell)$.

Now, if in (6.6) we choose $z = Tx$, $y = T^*x$ for $x \in H$, $\|x\| = 1$, then we get

$$\|Tx\| \, \|T^*x\| \leq \frac{|\Gamma + \gamma|}{2\sqrt{\operatorname{Re}(\Gamma\bar{\gamma})}} |\langle Tx, T^*x\rangle|$$

which is equivalent with (6.5). □

Also, we have:

Theorem 21. *If γ, Γ, T satisfy the hypothesis of Theorem* 20, *then we have the inequality*:

$$(0 \leq) \|Tx\|^4 - \left|\left\langle T^2x, x\right\rangle\right|^2 \leq \left[|\Gamma + \gamma| - 2\sqrt{Re\left(\Gamma\bar{\gamma}\right)}\right] \left|\left\langle T^2x, x\right\rangle\right| \|Tx\|^2 , \tag{6.7}$$

for any $x \in H$, $\|x\| = 1$.

Proof. We make use of the following inequality [**5**]:

$$(0 \leq) \|z\|^2 \|y\|^2 - |\langle z, y\rangle|^2 \leq \left[|\Gamma + \gamma| - 2\sqrt{\mathrm{Re}\left(\Gamma\bar{\gamma}\right)}\right] |\langle z, y\rangle| \|y\|^2 \tag{6.8}$$

that holds for $\gamma, \Gamma \in \mathbb{C}$ with $\mathrm{Re}\left(\Gamma\bar{\gamma}\right) > 0$ and provided the vectors $z, y \in H$ satisfy either the condition (ℓ) or, equivalently the condition $(\ell\ell)$.

Now, if in (6.8) we choose $z = Tx$, $y = T^*x$ with $x \in H$, $\|x\| = 1$, then we get the desired result (6.7). □

Remark. If we choose $\Gamma = M \geq m = \gamma > 0$, then, obviously

$$\mathrm{Re}\left\langle MT^*x - Tx, Tx - mT^*x\right\rangle \geq 0 \quad \text{for any} \quad x \in H, \|x\| = 1 \tag{6.9}$$

is equivalent with

$$\left\|Tx - \frac{m + M}{2}T^*x\right\| \leq \frac{1}{2}\left(M - m\right) \quad \text{for any} \quad x \in H, \ \|x\| = 1, \tag{6.10}$$

or with the fact that

$$\text{the operator } \left(T^* - mT\right)\left(MT^* - T\right) \text{ is accretive.} \tag{6.11}$$

If T is normal, then the above are implied by the following two conditions that are equivalent between them:

$$\left(T^* - mT\right)\left(MT^* - T\right) \geq 0 \tag{6.12}$$

and

$$M\left[T^*\right]^2 - \left(mM + 1\right)T^*T + mT^2 \geq 0. \tag{6.13}$$

Now, if (6.11) holds, then

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leq \frac{1}{4} \cdot \frac{\left(M - m\right)^2}{M + m} \|Tx\|^2 , \tag{6.14}$$

$$\|Tx\|^2 \leq \frac{M + m}{2\sqrt{mM}} \left|\left\langle T^2x, x\right\rangle\right| \tag{6.15}$$

or, equivalently

$$(0 \leq) \|Tx\|^2 - \left|\left\langle T^2x, x\right\rangle\right| \leq \frac{\left(\sqrt{M} - \sqrt{m}\right)^2}{2\sqrt{mM}} \left|\left\langle T^2x, x\right\rangle\right| \tag{6.16}$$

and

$$(0 \leq) \|Tx\|^4 - \left|\langle T^2x, x\rangle\right|^2 \leq \left(\sqrt{M} - \sqrt{m}\right)^2 \left|\langle T^2x, x\rangle\right| \|Tx\|^2, \tag{6.17}$$

for any $x \in H$, $\|x\| = 1$.

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ON SOME MEAN SQUARE ESTIMATES IN THE RANKIN-SELBERG PROBLEM

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An overview of the classical Rankin-Selberg problem involving the asymptotic formula for sums of coefficients of holomorphic cusp forms is given. We also study the function $\Delta(x;\xi)$ $(0 \le \xi \le 1)$, the error term in the Rankin-Selberg problem weighted by ξ-th power of the logarithm. Mean square estimates for $\Delta(x;\xi)$ are proved.

1. THE RANKIN-SELBERG PROBLEM

The classical Rankin-Selberg problem consists of the estimation of the error term function

$$\Delta(x) := \sum_{n \le x} c_n - Cx, \tag{1.1}$$

where the notation is as follows. Let $\varphi(z)$ be a holomorphic cusp form of weight κ with respect to the full modular group $SL(2,\mathbb{Z})$, and denote by $a(n)$ the n-th Fourier coefficient of $\varphi(z)$ (see e.g., R. A. Rankin [**15**] for a comprehensive account). We suppose that $\varphi(z)$ is a normalized eigenfunction for the Hecke operators $T(n)$, that is, $a(1) = 1$ and $T(n)\varphi = a(n)\varphi$ for every $n \in \mathbb{N}$. In (1.1) $C > 0$ is a suitable constant (see e.g., [**9**] for its explicit expression), and c_n is the convolution function defined by

$$c_n = n^{1-\kappa} \sum_{m^2 | n} m^{2(\kappa-1)} \left| a\left(\frac{n}{m^2}\right)\right|^2 .$$

The classical Rankin-Selberg bound of 1939 is

$$\Delta(x) = O(x^{3/5}), \tag{1.2}$$

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hitherto unimproved. In their works, done independently, R. A. Rankin [**14**] derives (1.2) from a general result of E. Landau [**11**], while A. Selberg [**17**] states the result with no proof. Although the exponent 3/5 in (1.2) represents one of the longest standing records in analytic number theory, recently there have been some developments in some other aspects of the Rankin-Selberg problem. In this paper we shall present an overview of some of these new results. In addition, we shall consider the weighted sum (the so-called Riesz logarithmic means of order ξ), namely

$$\frac{1}{\Gamma(\xi+1)}\sum_{n\le x} c_n \log^\xi\left(\frac{x}{n}\right) := Cx + \Delta(x;\xi) \qquad (\xi \ge 0), \tag{1.3}$$

where C is as in (1.1), so that $\Delta(x) \equiv \Delta(x;0)$. The effect of introducing weights such as the logarithmic weight in (1.3) is that the ensuing error term (in our case this is $\Delta(x;\xi)$) can be estimated better than the original error term (i.e., in our case $\Delta(x;0)$). This was shown by Matsumoto, Tanigawa and the author in [**9**], where it was proved that

$$\Delta(x;\xi) \ll_\varepsilon x^{(3-2\xi)/5+\varepsilon} \qquad (0 \le \xi \le 3/2). \tag{1.4}$$

Here and later ε denotes arbitrarily small constants, not necessarily the same ones at each occurrence, while $a \ll_\varepsilon b$ means that the constant implied by the $\ll$-symbol depends on ε. When $\xi = 0$ we recover (1.2) from (1.4), only with the extra 'ε' factor present. In this work we shall pursue the investigations concerning $\Delta(x;\xi)$, and deal with mean square bounds for this function.

2. THE FUNCTIONAL EQUATIONS

In view of (1.1) and (1.2) it follows that the generating Dirichlet series

$$Z(s) := \sum_{n=1}^\infty c_n n^{-s} \qquad (s = \sigma + it) \tag{2.1}$$

converges absolutely for $\sigma > 1$. The arithmetic function c_n is multiplicative and satisfies $c_n \ll_\varepsilon n^\varepsilon$. Moreover, it is well known (see e.g., R. A. Rankin [**14**], [**15**]) that $Z(s)$ satisfies for all s the functional equation

$$\Gamma(s+\kappa-1)\Gamma(s)Z(s) = (2\pi)^{4s-2}\Gamma(\kappa-s)\Gamma(1-s)Z(1-s), \tag{2.2}$$

which provides then the analytic continuation of $Z(s)$. In modern terminology $Z(s)$ belongs to the Selberg class $\mathcal{S}$ of L-functions of degree four (see A. Selberg [**18**] and the survey paper of Kaczorowski–Perelli [**10**]). An important feature, proved by G. Shimura [**19**] (see also A. Sankaranarayanan [**16**]) is

$$Z(s) = \zeta(s)\sum_{n=1}^\infty b_n n^{-s} = \zeta(s)B(s), \tag{2.3}$$

where $B(s)$ is holomorphic for $\sigma > 0$, $b_n \ll_\varepsilon n^\varepsilon$ (in fact $\sum_{n\le x} b_n^2 \le x\log^A x$ holds, too). It also satisfies the functional equation

$$\begin{aligned} B(s)\Delta_1(s) &= B(1-s)\Delta_1(1-s),\\ \Delta_1(s) &= \pi^{-3s/2}\Gamma\big(\tfrac12(s+\kappa-1)\big)\Gamma\big(\tfrac12(s+\kappa)\big)\Gamma\big(\tfrac12(s+\kappa+1)\big), \end{aligned}$$

and actually $B(s) \in \mathcal{S}$ with degree three. The decomposition (2.3) (the so-called 'Shimura lift') allows one to use, at least to some extent, results from the theory of $\zeta(s)$ in connection with $Z(s)$, and hence to derive results on $\Delta(x)$.

3. THE COMPLEX INTEGRATION APPROACH

A natural approach to the estimation of $\Delta(x)$, used by the author in [**8**], is to apply the classical complex integration technique. We shall briefly present this approach now. On using PERRON's inversion formula (see e.g., the Appendix of [**3**]), the residue theorem and the convexity bound $Z(s) \ll_\varepsilon |t|^{2-2\sigma+\varepsilon}$ $(0 \le \sigma \le 1,\ |t| \ge 1)$, it follows that

$$\Delta(x) = \frac{1}{2\pi i}\int_{\frac12 - iT}^{\frac12+iT} \frac{Z(s)}{s} x^s\,\mathrm{d}s + O_\varepsilon\left(x^\varepsilon\left(x^{1/2} + \frac{x}{T}\right)\right) \quad (1 \ll T \ll x). \tag{3.1}$$

If we suppose that

$$\int_X^{2X} \big|B\big(\tfrac12 + it\big)\big|^2\,\mathrm{d}t \ll_\varepsilon X^{\theta+\varepsilon} \qquad (\theta \ge 1), \tag{3.2}$$

and use the elementary fact $\big($see [**3**] for the results on the moments of $\big|\zeta\big(\tfrac12+it\big)\big|\big)$ that

$$\int_X^{2X} \big|\zeta\big(\tfrac12 + it\big)\big|^2\,\mathrm{d}t \ll X\log X, \tag{3.3}$$

then from (2.3), (3.2), (3.3) and the CAUCHY-SCHWARZ inequality for integrals we obtain

$$\int_X^{2X} \big|Z\big(\tfrac12 + it\big)\big|\,\mathrm{d}t \ll_\varepsilon X^{(1+\theta)/2+\varepsilon}.$$

Therefore (3.1) gives

$$\Delta(x) \ll_\varepsilon x^\varepsilon(x^{1/2}T^{\theta/2-1/2} + xT^{-1}) \ll_\varepsilon x^{\frac{\theta}{\theta+1}+\varepsilon} \tag{3.4}$$

with $T = x^{1/(\theta+1)}$. This was formulated in [**8**] as

Theorem A. *If θ is given by* (3.2), *then*

$$\Delta(x) \ll_\varepsilon x^{\frac{\theta}{\theta+1}+\varepsilon}. \tag{3.5}$$

To obtain a value for θ, note that $B(s)$ belongs to the Selberg class of degree three, hence $B\left(\frac{1}{2}+it\right)$ in (3.2) can be written as a sum of two Dirichlet polynomials (e.g., by the reflection principle discussed in [**3**, Chapter 4]), each of length $\ll X^{3/2}$. Thus by the mean value theorem for Dirichlet polynomials (op. cit.) we have $\theta \le 3/2$ in (3.2). Hence (3.5) gives (with unimportant ε) the Rankin-Selberg bound $\Delta(x) \ll_\varepsilon x^{3/5+\varepsilon}$. Clearly improvement will come from better values of θ. Note that the best possible value of θ in (3.2) is $\theta = 1$, which follows from general results on Dirichlet series (see e.g., [**3**, Chapter 9]). It gives $1/2+\varepsilon$ as the exponent in the Rankin-Selberg problem, which is the limit of the method (the conjectural exponent $3/8+\varepsilon$, which is best possible, is out of reach; see the author's work [**4**]). To attain this improvement one faces ssentially the same problem as in proving the sixth moment for $\left|\zeta\left(\frac{1}{2}+it\right)\right|$, namely

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^6 \mathrm{d}t \ll_\varepsilon T^{1+\varepsilon},$$

only this problem is even more difficult, because the arithmetic properties of the coefficients b_n are even less known than the properties of the divisor coefficients

$$d_3(n) = \sum_{abc=n;a,b,c\in\mathbb{N}} 1,$$

generated by $\zeta^3(s)$. If we knew the analogue of the strongest sixth moment bound

$$\int_0^T \left|\zeta\left(\tfrac{1}{2}+it\right)\right|^6 \mathrm{d}t \ll T^{5/4}\log^C T \qquad (C>0),$$

namely the bound (3.2) with $\theta = 5/4$, then (3.1) would yield $\Delta(x) \ll_\varepsilon x^{5/9+\varepsilon}$, improving substantially (1.2).

The essential difficulty in this problem may be seen indirectly by comparing it with the estimation of $\Delta_4(x)$, the error term in the asymptotic formula for the summatory function of $d_4(n) = \sum_{abcd=n;a,b,c,d\in\mathbb{N}} 1$. The generating function in this case is $\zeta^4(s)$. The problem analogous to the estimation of $\Delta(x)$ is to estimate $\Delta_4(x)$, given the product representation

$$\sum_{n=1}^\infty d_4(n)n^{-s} = \zeta(s)G(s) = \zeta(s)\sum_{n=1}^\infty g(n)n^{-s} \quad (\sigma>1) \tag{3.6}$$

with $g(n) \ll_\varepsilon n^\varepsilon$ and $G(s)$ of degree three in the Selberg class (with a pole of order three at $s=1$). By the complex integration method one gets $\Delta_4(x) \ll_\varepsilon x^{1/2+\varepsilon}$ (here 'ε' may be replaced by a log-factor) using the classical elementary bound $\int_0^T \left|\zeta\left(\frac{1}{2}+it\right)\right|^4 \mathrm{d}t \ll T\log^4 T$. Curiously, this bound for $\Delta_4(x)$ has never been improved; exponential sum techniques seem to give a poor result here. However, if one knows only (3.6), then the situation is quite analogous to the Rankin–Selberg problem, and nothing better than the exponent $3/5$ seems obtainable. The bound $\Delta(x) \ll_\varepsilon x^{1/2+\varepsilon}$ follows also directly from (3.1) if the Lindelöf hypothesis for $Z(s)$ (that $Z\left(\frac{1}{2}+it\right) \ll_\varepsilon |t|^\varepsilon$) is assumed.

4. MEAN SQUARE OF THE RANKIN–SELBERG ZETA–FUNCTION

Let, for a given $\sigma \in \mathbb{R}$,

$$\mu(\sigma) = \limsup_{t\to\infty} \frac{\log|\zeta(\sigma+it)|}{\log t} \tag{4.1}$$

denote the LINDELÖF function (the famous, hitherto unproved, LINDELÖF conjecture for $\zeta(s)$ is that $\mu(\sigma) = 0$ for $\sigma \geq \frac{1}{2}$, or equivalently that $\zeta\bigl(\frac{1}{2}+it\bigr) \ll_\varepsilon |t|^\varepsilon$). In [**8**] the author proved the following

Theorem B. *If $\beta = 2/\bigl(5-\mu\bigl(\frac{1}{2}\bigr)\bigr)$, then for fixed σ satisfying $\frac{1}{2} < \sigma \leq 1$ we have*

$$\int_1^T |Z(\sigma+it)|^2\,\mathrm{d}t = T\sum_{n=1}^\infty c_n^2 n^{-2\sigma} + O_\varepsilon(T^{(2-2\sigma)/(1-\beta)+\varepsilon}). \tag{4.2}$$

This result is the sharpest one yet when σ is close to 1. For σ close to $\frac{1}{2}$ one cannot obtain an asymptotic formula, but only the upper bound (this is [**7**, eq. (9.27)])

$$\int_T^{2T} |Z(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon T^{2\mu(1/2)(1-\sigma)+\varepsilon}(T+T^{3(1-\sigma)}) \quad \bigl(\tfrac{1}{2} \leq \sigma \leq 1\bigr). \tag{4.3}$$

The upper bound in (4.3) follows easily from (2.3) and the fact that, as already mentioned, $B(s) \in \mathcal{S}$ with degree three, so that $B\bigl(\frac{1}{2}+it\bigr)$ can be approximated by DIRICHLET polynomials of length $\ll t^{3/2}$, and the mean value theorem for DIRICHLET polynomials yields

$$\int_T^{2T} |B(\sigma+it)|^2\,\mathrm{d}t \ll_\varepsilon T^\varepsilon(T+T^{3(1-\sigma)}) \qquad \bigl(\tfrac{1}{2} \leq \sigma \leq 1\bigr).$$

Note that with the sharpest known result (see M. N. HUXLEY [**2**]) $\mu(1/2) \leq 32/205$ we obtain $\beta = 410/961 = 0.426638917\ldots$. The limit is the value $\beta = 2/5$ if the LINDELÖF hypothesis $\bigl(\text{that } \mu\bigl(\frac{1}{2}\bigr) = 0\bigr)$ is true. Thus (4.2) provides a true asymptotic formula for

$$\sigma > \frac{1+\beta}{2} = \frac{1371}{1922} = 0.7133194\ldots.$$

The proof of (4.2), given in [**8**], is based on the general method of the author's paper [**6**], which contains a historic discussion on the formulas for the left-hand side of (4.2) (see also K. MATSUMOTO [**12**]).

We are able to improve (4.2) in the case when $\sigma = 1$. The result is contained in

Theorem 1. *We have*

$$\int_1^T |Z(1+it)|^2\,\mathrm{d}t = T\sum_{n=1}^\infty c_n^2 n^{-2} + O_\varepsilon((\log T)^{2+\varepsilon}). \tag{4.4}$$

Proof. For $\sigma = \Re\mathrm{e}\, s > 1$ and $X \ge 2$ we have

$$Z(s) = \sum_{n\le X} c_n n^{-s} + \int_X^\infty x^{-s}\,\mathrm{d}\Bigl(\sum_{n\le x} c_n\Bigr) \tag{4.5}$$

$$= \sum_{n\le X} c_n n^{-s} + \frac{CX^{1-s}}{s-1} - \Delta(x)X^{-s} - s\int_X^\infty \Delta(x)x^{-s-1}\,\mathrm{d}x.$$

By using (1.2) it is seen that the last integral converges absolutely for $\sigma = \Re\mathrm{e}\, s > 3/5$, so that (4.5) provides the analytic continuation of $Z(s)$ to this region. Taking $s = 1+it, 1 \le t \le T, X = T^{10}$, it follows that

$$\int_1^T |Z(1+it)|^2\,\mathrm{d}t = \int_1^T \Bigl\{\Bigl|\sum_{n\le X} c_n n^{-1-it}\Bigr|^2 - 2C\Im\mathrm{m}\Bigl(\sum_{n\le X}\frac{c_n}{nt}\Bigl(\frac{X}{n}\Bigr)^{it}\Bigr)\Bigr\}\,\mathrm{d}t + O(1). \tag{4.6}$$

By the mean value theorem for Dirichlet polynomials we have

$$\int_1^T \Bigl|\sum_{n\le X} c_n n^{-1-it}\Bigr|^2\,\mathrm{d}t = T\sum_{n\le X} c_n^2 + O\Bigl(\sum_{n\le X} c_n^2 n^{-1}\Bigr) = T\sum_{n=1}^\infty c_n^2 + O_\varepsilon\Bigl((\log T)^{2+\varepsilon}\Bigr),$$

where we used the bound (see K. Matsumoto [**12**])

$$\sum_{n\le x} c_n^2 \ll_\varepsilon x(\log x)^{1+\varepsilon} \tag{4.7}$$

and partial summation. Finally we have

$$\sum_{n\le X}\frac{c_n}{n}\int_1^T \frac{1}{t}\left(\frac{X}{n}\right)^{it}\mathrm{d}t \ll \log\log T. \tag{4.8}$$

To see that (4.8) holds, note first that for $X - X/\log T \le n \le X$ the integral over t is trivially estimated as $\ll \log T$, and the total contribution of such n is

$$\ll \log T \sum_{X-X/\log T\le n\le X} \frac{c_n}{n}\,\mathrm{d}x \ll 1$$

on using (1.1)–(1.2). For the remaining n we note that the integral over t equals

$$\frac{\bigl(\frac{X}{n}\bigr)^{it}}{it\log(X/n)}\Bigg|_1^T + \frac{1}{i\log(X/n)}\int_1^T \left(\frac{X}{n}\right)^{it}\frac{\mathrm{d}t}{t^2}.$$

The contribution of those n is, using (1.1)–(1.2) again and making the change of variable $X/u = v$,

$$
\begin{aligned}
&\ll \sum_{1\le n\le X-X/\log T}\frac{c_n}{n\log(X/n)} = \int_{1-0}^{X-X/\log T}\frac{1}{u\log(X/u)}\,\mathrm{d}(Cu+\Delta(u))\\
&= \int_1^{X-X/\log T}\frac{1}{u\log(X/u)}\left(C+\frac{\Delta(u)}{u}+\frac{\Delta(u)}{u\log(X/u)}\right)\mathrm{d}u + O(1)\\
&\ll \int_1^{X-X/\log T}\frac{\mathrm{d}u}{u\log(X/u)} + 1 = \int_{(1-1/\log T)^{-1}}^{X}\frac{\mathrm{d}v}{v\log v}+1\\
&= \log\log X - \log\log(1-1/\log T)^{-1} + 1 \ll \log\log T,
\end{aligned}
$$

and (4.8) follows.

One can improve the error term in (4.4) to $O(\log^2 T)$, which is the limit of the method. I am very grateful to Prof. Alberto Perelli, who has kindly indicated this to me. The argument is very briefly as follows. Note that the coefficients c_n^2 are essentially the tensor product of the c_n's, and the c_n are essentially the tensor product of the $a(n)$'s; "essentially" means in this case that the corresponding L-functions differ at most by a "fudge factor", i.e., a Dirichlet series converging absolutely for $\sigma > 1/2$ and non-vanishing at $s = 1$. In terms of L-functions, the tensor product of the $a(n)$ (the coefficients of the tensor square L-function) corresponds to the product of $\zeta(s)$ and the L-function of Sym^2 (Shimura's lift). Moreover, Gelbart–Jacquet [**1**] have shown that Sym^2 is a cuspidal automorphic representation, so one can apply to the above product the general Rankin-Selberg theory to obtain "good properties" of the corresponding L-function. Since Sym^2 is irreducible, the L-function corresponding to c_n^2 has a double pole at $s = 1$ and a functional equation of Riemann type. It follows that the sum in (4.7) is asymptotic to $Dx\log x$ for some $D > 0$, and the assertion follows by following the preceding argument.

In concluding this section, let it be mentioned that, using (4.5), it easily follows that $Z(1 + it) \ll \log|t|$ $(t \ge 2)$.

5. MEAN SQUARE OF $\Delta(x;\xi)$

In this section we shall consider mean square estimates for $\Delta(x;\xi)$, defined by (1.3). Although we could consider the range $\xi > 1$ as well, for technical reasons we shall restrict ourselves to the range $0 \le \xi \le 1$, which is the condition that will be assumed henceforth to hold. Let

$$
\beta_\xi \;:=\; \inf\left\{\ \beta \ge 0\ :\ \int_1^X \Delta^2(x;\xi)\,\mathrm{d}x \ll X^{1+2\beta}\ \right\}. \tag{5.1}
$$

The definition of β_ξ is the natural analogue of the classical constants in mean square estimates for the generalized Dirichlet divisor problem (see [**3**, Chapter 13]). Our first result in this direction is

Theorem 2. *We have*

$$\frac{3-2\xi}{8} \;\le\; \beta_\xi \;\le\; \max\left(\frac{1-\xi}{2},\, \frac{3-2\xi}{8}\right) \qquad (0 \le \xi \le 1). \tag{5.2}$$

Proof. First of all, note that (5.2) implies that $\beta_\xi = (3-2\xi)/8$ for $\frac12 \le \xi \le 1$, so that in this interval the precise value of β_ξ is determined. The main tool in our investigations is the explicit VORONOÏ type formula for $\Delta(x;\xi)$. This is

$$\Delta(x;\xi) \;=\; V_\xi(x,N) + R_\xi(x,N), \tag{5.3}$$

where, for $N \gg 1$,

(5.4)

$$V_\xi(x,N) = (2\pi)^{-1-\xi}x^{(3-2\xi)/8}\sum_{n\le N} c_n n^{-(5+2\xi)/8}\cos\Bigl(8\pi(xn)^{1/4} + \tfrac12\bigl(\tfrac12-\xi\bigr)\pi\Bigr),$$

$$R_\xi(x,N) \ll_\varepsilon (xN)^\varepsilon\left(1 + x^{(3-\xi)/4}N^{-(1+\xi)/4} + (xN)^{(1-\xi)/4} + x^{(1-2\xi)/8}\right).$$

This follows from the work of U. VORHAUER [**20**] (for $\xi = 0$ this is also proved in [**9**]), specialized to the case when

$$A = \frac{1}{(2\pi)^2}, B = (2\pi)^4, M = L = 2, b_1 = b_2 = d_1 = d_2 = 1, \beta_1 = \kappa - \tfrac12, b_2 = \tfrac12,$$
$$\delta_1 = \kappa - \tfrac32, \delta_2 = -\tfrac12, \gamma = 1, p = B, q = 4, \lambda = 2, \Lambda = -1, C = (2\pi)^{-5/2}.$$

In (5.3)–(5.4) we take $N = x$, so that $R_\xi(x,N) \ll_\varepsilon x^{(1-\xi)/2+\varepsilon}$. Since $\frac{1-\xi}{2} \le \frac{3-2\xi}{8}$ for $\xi \ge \frac12$, the lower bound in (5.2) follows by the method of [4]. For the upper bound we use $c_n \ll_\varepsilon n^\varepsilon$ and note that ($\mathrm{e}(z) = \exp(2\pi i z)$)

$$\begin{aligned}
&\int_X^{2X}\Bigl|\sum_{K<k\le 2K} c_k k^{-(5+2\xi)/8}\mathrm{e}(4(xk)^{1/4})\Bigr|^2\,\mathrm{d}x\\
&\ll X + \sum_{k_1\ne k_2} c_{k_1}c_{k_2}(k_1k_2)^{-(5+2\xi)/8}\int_X^{2X}\mathrm{e}(4x^{1/4}(k_1^{1/4}-k_2^{1/4}))\,\mathrm{d}x\\
&\ll_\varepsilon X + X^{3/4+\varepsilon}K^{-(5+2\xi)/4}\sum_{k_1\ne k_2}\Bigl|k_1^{1/4}-k_2^{1/4}\Bigr|^{-1}\\
&\ll_\varepsilon X + X^{3/4+\varepsilon}K^{(1-\xi)/2},
\end{aligned}$$

where we used the first derivative test (cf. [**3**, Lemma 2.1]). Since $K \ll X$ and

$$\int_X^{2X}\Delta^2(x;\xi)\,\mathrm{d}x \ll \int_X^{2X}|V_\xi(x,N)|^2\,\mathrm{d}x + \int_X^{2X}|R(x,N)|^2\,\mathrm{d}x,$$

it follows that

$$\int_X^{2X}\Delta^2(x;\xi)\,\mathrm{d}x \ll_\varepsilon X^{(7-2\xi)/4+\varepsilon} + X^{2-\xi+\varepsilon},$$

which clearly proves the assertion.

Our last result is a bound for β_ξ, which improves on (5.2) when ξ is small. This is

Theorem 3. *We have*

$$\beta_\xi \le \frac{2-2\xi}{5-2\mu(\frac12)} \qquad \Big(0 \le \xi \le \tfrac16\big(1+2\mu(\tfrac12)\big)\Big). \tag{5.5}$$

Proof. We start from

$$\Delta(x;\xi) = \frac{1}{2\pi i}\lim_{T\to\infty}\int_{c-iT}^{c+iT} Z(s)\frac{x^s}{s^{\xi+1}}\,\mathrm{d}s, \tag{5.6}$$

where $0 < c = c(\xi) < 1$ is a suitable constant (see K. Matsumoto [**13**] for a detailed derivation of formulas analogous to (5.6)). By the Mellin inversion formula we have (see e.g., the Appendix of [**3**])

$$Z(s)s^{-\xi-1} = \int_0^\infty \Delta(1/x;\xi)x^{s-1}\,\mathrm{d}x \qquad (\Re\mathrm{e}\, s = c).$$

Hence by Parseval's formula for Mellin tranforms (op. cit.) we obtain, for $\beta_\xi < \sigma < 1$,

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{|Z(\sigma+it)|^2}{|\sigma+it|^{2\xi+2}}\,\mathrm{d}t &= \int_0^\infty \Delta^2(1/x;\xi)x^{2\sigma-1}\,\mathrm{d}x\\ &= \int_0^\infty \Delta^2(x;\xi)x^{-2\sigma-1}\,\mathrm{d}x \gg X^{-2\sigma-1}\int_X^{2X}\Delta^2(x;\xi)\,\mathrm{d}x.\end{aligned} \tag{5.7}$$

Therefore if the first integral converges for $\sigma = \sigma_0+\varepsilon$, then (5.7) gives

$$\int_X^{2X}\Delta^2(x;\xi)\ \mathrm{d}x \ \ll X^{2\sigma+1},$$

namely $\beta_\xi \le \sigma_0$. The functional equation (2.2) and Stirling's formula in the form

$$|\Gamma(s)| = \sqrt{2\pi}|t|^{\sigma-1/2}\mathrm{e}^{-\pi|t|/2}\big(1+O(|t|^{-1})\big) \qquad (|t| \ge t_0 > 0)$$

imply that

$$Z(s) \ = \ \mathcal{X}(s)Z(1-s), \quad \mathcal{X}(\sigma+it) \asymp |t|^{2-4\sigma} \quad (s=\sigma+it,\, 0\le\sigma\le1, |t|\ge 2). \tag{5.8}$$

Thus it follows on using (4.3) that

$$\begin{aligned}\int_T^{2T}|Z(\sigma+it)|^2\,\mathrm{d}t \ &\ll\ T^{4-8\sigma}\int_T^{2T}|Z(1-\sigma+it)|^2\,\mathrm{d}t\\ &\ll_\varepsilon\ T^{4-8\sigma+2\mu(\frac12)\sigma+\max(1,3\sigma)+\varepsilon}.\end{aligned}$$

But we have $4-8\sigma+2\mu(\frac12)\sigma+\max(1,3\sigma)=4-5\sigma+2\mu\big(\frac12\big)\sigma<2\xi+2$ for

$$\sigma>\sigma_0=\frac{2-2\xi}{5-2\mu\big(\frac12\big)},\tag{5.9}$$

provided that $\sigma_0\ge 1/3$, which occurs if $0\le\xi\le\frac16\big(1+2\mu\big(\frac12\big)\big)$. Thus the first integral in (5.7) converges if (5.9) holds, and Theorem 3 is proved. Note that this result is a generalization of Theorem 7 in [**8**], which says that $\beta_0\le(2-2\xi)/\big(5-2\mu\big(\frac12\big)\big)$.

In the case when $\beta_\xi=(3-2\xi)/8$ we could actually derive an asymptotic formula for the integral of the mean square of $\Delta(x;\xi)$, much in the same way that this was done in [**9**] for the square of $\Delta_1(x) := \int_0^x\Delta(u)\,\mathrm{d}u$, where it was shown that

$$\int_1^X\Delta_1^2(x)\,\mathrm{d}x=DX^{13/4}+O_\varepsilon(X^{3+\varepsilon})\tag{5.10}$$

with explicit $D>0$ (in [**12**] the error term was improved to $O_\varepsilon(X^3(\log X)^{3+\varepsilon})$). In the case of $\Delta(x;1)$ the formula (5.10) may be used directly, since

$$\frac1x\Delta_1(x)=\frac1x\int_0^x\Delta(u)\,\mathrm{d}u=\Delta(x;1)+O_\varepsilon(x^\varepsilon).\tag{5.11}$$

To see that (5.11) holds, note that with $c=1-\varepsilon$ we have

$$\begin{aligned}\Delta(x;1)&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}Z(s)\,\frac{x^s}{s^2}\,\mathrm{d}s\\&=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}Z(s)\,\frac{x^s}{s(s+1)}\,\mathrm{d}s+\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}Z(s)\,\frac{x^s}{s^2(s+1)}\,\mathrm{d}s\\&=\frac1x\int_0^x\Delta(u)\,\mathrm{d}u+\frac{1}{2\pi i}\int_{\varepsilon-i\infty}^{\varepsilon+i\infty}Z(s)\,\frac{x^s}{s^2(s+1)}\,\mathrm{d}s\\&=\frac1x\Delta_1(x)+O_\varepsilon(x^\varepsilon),\end{aligned}$$

on applying (5.8) to the last integral above.

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METHOD OF FACTORIZATION OF ORDINARY DIFFERENTIAL OPERATORS AND SOME OF ITS APPLICATIONS

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The paper is dedicated to analytical and algebraic approaches to the problem of the integration of ordinary differential equations. The first part is devoted to linear ordinary differential equations of the second and nth orders, while the second deals with nonlinear ordinary differential equations. Factorization of nonlinear equations of the second and the third orders both through commutative and noncommutative nonlinear differential operators are considered. The method of the exact linearization for nonlinear equations is explained. Some applications are also considered.

1. INTRODUCTION

The contents of this paper are closely connected to the problem of the integration of ordinary differential equations. Factorization of differential operators is a very effective method for analyzing both linear and nonlinear ordinary differential equations. It uses analogies between differential operators and algebraic polynomials.

The prehistory of this method goes back to investigations of G. Frobenius [**29**], E. Landau, [**43**] and G. Mammana [**47**].

The most efficacious is simultaneously using factorization method and variables transformation.

A great contribution to the problem of integrating ordinary differential equations was made by mathematicians of Serbia and the former Yugoslavia: M. Petrović, T. Pejović, D. S. Mitrinović, B. Popov, I. Šapkarev, I. Bandić, P.

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Vasić, J. Kečkić, V. Kocić and others. The journal “Publications of the Faculty of Electrical Engineering - Series Mathematics” (1956–2006), which was founded by Professor D. S. Mitrinović, have played significant role in the regeneration of interest in the problem of the solution of ordinary differential equations iin closed form.

At the present time the importance of this problem has increased considerably. Closed-form solutions are necessary both for new mathematical models in the natural sciences and for the testiing of numerical and analytic algorithms.

In Section 2 we consider differential algebras of differential operators. We place the main emphasis on their factorization.

In Section 3 it is shown how to use the method of LODE-2 and LODE-n transformation. The Kummer-Liouville transformation, that is applied in this work, is the most general transformation of variables that preserves the order and the linearity of the given equation.

The solutions of the classical Kummer's and Halphen's problems of LODE-2 and LODE-n equivalence are given.

The criteria of LODE-n reducibility to equations with constant coefficients are pointed out.

In Section 4 we consider the method of autonomization for nonlinear differential equations. It is applicable for equations that can be representad as a sum of linear and nonlinear parts. The test for autonomization is also adduced. The generalized Emden-Fowler's equation and generalized Ermakov's equation, which frequently appear in different applications, are considered. The very important idea of a nonlinear superposition principle for nonlinear differential equations is given.

In Section 5 the method of linearization of nonlinear differential equations (see Berkovich [**18, 22**]) is applied to the equations of the second and third orders. A nonlinear oscillator and the Euler-Poinsot case in the problem of the gyroscope are good examples of the effectiveness of this method.

In Section 6 we simultaneously apply the method of transformation of variables and factorization of nonlinear differential operators to the generalized Emden-Fowler's equation of the third order, to Lienard's equation and to the equation of the anharmonic oscillator.

2. DIFFERENTIAL ALGEBRA OF DIFFERENTIAL OPERATORS

Definitions of the main concepts can be found in the following books: Kaplansky [**32**], Magid [**46**], Singer [**56**] and Berkovich [**13, 22**].

2.1. Differential field

Definition 1. *A differential field is a pair* (F, δ), *where* F *is a functional field and* δ *is a derivation. Let* K *be a number field of characteristic 0* (*i.e. constant field* F). *It may be algebraically closed, or it may be not.*

$$a' := \delta(a), \quad a \in F,$$

$$a \in F_0 \Leftrightarrow a' \in F_0, \quad c \in K \Leftrightarrow c' = 0.$$

Example 1. Field (F, δ), where $\delta = \dfrac{\mathrm{d}}{\mathrm{d}x} = D$, $\delta = x\,\dfrac{\mathrm{d}}{\mathrm{d}x}$. Further let δ be D.

Example 2. Field $(\mathbb{C}(x), D)$, where $\mathbb{C}(x)$ is the field of rational functions over the field of complex numbers $\mathbb{C}$.

2.2. Ring of differential operators

Consider the set of differential operators of arbitrary order

$$L = a_n D^n + \cdots + a_1 D + a_0,$$

where $n \in \mathbb{N}$, $a_i \in F_0$, $\forall i$. Multiplication in F_0 is determined by the rule:

$$Da = aD + D(a) = aD + a'. \tag{2.1}$$

From (2.1) Leibnitz' formula follows:

$$D^i b = \sum_{k=0}^{i} \binom{i}{k} b^{(i-k)} D^k.$$

$F_0[D]$ is an associative but not a commutative ring.

2.3. Factorization of differential operators

Definition 2. *An operator, L, is factorizable in F_0 if it can be represented as the product of differential operators of lower order. The latter operators have coefficients in F_0. Under factorization the source number field may be extended to the algebraically closed field $\overline{K}$.*

Equivalent definition:

Definition 3. *The equation, $Ly = 0$, of order n is factorizable in F_0 if both this equation and the equation, $My = 0$, of order less than n have a common nontrivial integral.*

Otherwise L is said to be not factorizable in F_0.

2.4. Right differential analogue of Bezout's theorem

Theorem 1. *Dividing L by $D - \alpha$ from the right we get*

$$f(x) = \exp\Big(-\int \alpha \,\mathrm{d}x\Big) L \exp\Big(\int \alpha \,\mathrm{d}x\Big).$$

In the ring $F_0[D]$ Horner-type schemes take place by analogy with algebraic polynomials.

Using the right differential analogue of Horner's scheme one can make an expansion

$$L = \sum_{s=0}^{n-1} \beta_s D^s (D - \alpha), \quad \beta_{n-1} = 1.$$

Using the left differential analogue of HORNER's scheme one can make an expansion:

$$L = (D - \alpha) \sum_{s=0}^{n-1} \beta_s D^s, \quad \beta_{n-1} = 1.$$

2.5. Conjugation operator and its properties

Definition 4. *Transformation of conjugation, τ, is linear operator that acts on the Linear Ordinary Differential Operator (LODO) as it pointed out below*:

$$\tau\big(p(x)D^n\big) = (-1)^n D^n p(x) = (-1)^n \sum_{k=0}^{n} \binom{n}{k} p^{(k)} D^{n-k},$$

$$\tau\Big(\sum_{s=0}^{n} C_s p_s(x) D^s\Big) = \sum_{s=0}^{r} C_s \tau\big(p_s D^s\big), \quad C_s = \text{const}.$$

Let L^* be the operator, τL, that is formally conjugated to L

$$L^* \equiv \tau\Big(\sum_{k=0}^{n} a_k D^k\Big) = \sum_{k=0}^{n} (-1)^k D^k a_k = \sum_{k=0}^{n} \sum_{s=0}^{k} (-1)^k \binom{k}{s} a_k^{(s)} D^{k-s}.$$

Let L and M be LODOes. Then

$$\tau(LM) = \tau(M)\tau(L) = M^* L^*.$$

2.6. Left differential analogue of Bezout's theorem

Theorem 2. *Dividing L by $D - \alpha$ from the left we get*

$$g(x) = \exp\Big(\int \alpha \, \mathrm{d}x\Big) L^* \exp\Big(-\int \alpha \, \mathrm{d}x\Big).$$

2.7. Selfconjugated and antiselfconjugated operators

Theorem 3. *A selfconjugated operator, L_{2n}, can be represented as*

$$L_{2n} \equiv \prod_{k=2n}^{1} (\beta_k D - \alpha_k) = \prod_{k=1}^{n} (\beta_k D + \beta_k' + \alpha_k) \prod_{k=n}^{1} (\beta_k D - \alpha_k).$$

Theorem 4. *An antiselfconjugated operator, L_{2n+1}, can be represented as*

$$\begin{aligned} L_{2n+1} &\equiv \prod_{s=2n+1}^{1} (\beta_s D - \alpha_s) \\ &= \prod_{k=1}^{n} (\beta_k D + \beta_k' + \alpha_k)\Big(-2\int \alpha_{n+1} \, \mathrm{d}x D - \alpha_{n+1}\Big) \prod_{k=n}^{1} (\beta_k D - \alpha_k). \end{aligned}$$

2.8. Reducible selfconjugated and antiselfconjugated operators

Theorem 5 (see Berkovich, Rozov and Eishinsky [**4**]). *A selfconjugated operator that admits the factorization,*

$$L_{2n} = \prod_{k=1}^{n} \Big(D + \frac{2n+1-2k}{2n-1}\,\alpha\Big) \prod_{k=n}^{1} \Big(D - \frac{2n+1-2k}{2n-1}\,\alpha\Big), \tag{2.2}$$

can be represented as

$$\exp\Big(\frac{4n}{2n-1}\int \alpha\,\mathrm{d}x\Big) L_{2n} = \bigg(\exp\Big(\frac{2}{2n-1}\int \alpha\,\mathrm{d}x\Big)(D-\alpha)\bigg)^{2n}.$$

Theorem 6 [**4**]. *An antiselfconjugated operator that admits the factorization,*

$$L_{2n+1} = \prod_{k=1}^{n} \Big(D + \frac{n+1-k}{n}\,\alpha\Big) D \prod_{k=n}^{1} \Big(D - \frac{n+1-k}{n}\alpha\Big), \tag{2.3}$$

can be represented as

$$\exp\Big(\frac{2n+1}{n}\int \alpha\,\mathrm{d}x\Big) L_{2n+1} = \bigg(\exp\Big(\frac{1}{n}\int \alpha\,\mathrm{d}x\Big)(D-\alpha)\bigg)^{2n+1}.$$

The operator, (2.2), is called a *reducible selfconjugated operator.*

The operator, (2.3), is called a *reducible antiselfconjugated operator.*

2.9. Liouvillian and Euler expansions

A set Λ is a generalized Liouvillian (Euler) expansion of the field F_0 if there is a tower of fields,

$$F_0 \subset F_1 \subset \ldots \subset F_n = \Lambda,$$

such that one of the following conditions is fulfilled

- a. $F_i = F_{i-1}(\alpha)$, where $F_{i-1}(\alpha)$ is the field of rational functions of α with coefficients from F_{i-1} and $\alpha' \in F_{i-1}$.
- b. $F_i = F_{i-1}(\alpha)$, $\alpha \neq 0$, $\alpha'/\alpha \in F_{i-1}$.
- c. $F_i = F_{i-1}(\alpha)$, where α satisfies an algebraic equation of order $n \geq 2$.
- d. $F_i = F_{i-1}(y_1, y_2)$, where y_1 and y_2 constitute a basis of the equation

$$y'' + a_1 y' + a_0 y = 0, \quad a_1, a_0 \in F_{i-1}. \tag{2.4}$$

If (a), (b) or (c) is satisfied, then we get a Liouvillian expansion Λ_0. If in addition condition (d) is satisfied, then we have a generalized Liouvillian (Euler) expansion Λ of the field F_0.

2.10. Picard-Vessiot expansion

Definition 5. *The Picard-Vessiot expansion for the equation*

$$Ly \equiv \sum_{s=0}^{n} a_s y^{(s)} = 0, \quad a_s \in F_0 \tag{2.5}$$

is the differential field $F_0(y_1, \ldots, y_n)$, *where* $y_1, y_2, \ldots, y_n$ *is a basis of equation* (2.5).

Definition 6. *Equation* (2.5) *can be integrated in quadratures if* $PV \subset \Lambda_0$.

Definition 7 *Equation* (2.5) *has an Euler solution if* $PV \subset \Lambda$.

2.11. Mammana's theorems

Theorem 7. *It is always possible to factorize the equation* $Ly = 0$ *by an infinite number of ways through operators of the first order*

$$Ly \equiv \prod_{k=n}^{1} (D - \alpha_k) y = 0, \tag{2.6}$$

where α_k *are complex-valued functions of* x.

Example 3.

$$D^2 + 1 \equiv \left(D + \frac{i(c_1 e^{ix} - c_2 e^{-ix})}{c_1 e^{ix} + c_2 e^{-ix}} \right) \left(D - \frac{i(c_1 e^{ix} - c_2 e^{-ix})}{c_1 e^{ix} + c_2 e^{-ix}} \right).$$

Theorem 8. *Suppose that we have an equation* $Ly = 0$, $a_s \in \mathbb{C}^s(I)$, $I = \{x | a < x < b\}$. *Let* α_k *be real-valued functions in* I.

Factorization of (2.6) *in* I *exists if and only if any solution* $y(x)$ *of the equation* $Ly = 0$ *is nonoscillating, i.e. it has no more than* $n - 1$ *zeroes (counted according to their multiplicity) in* I.

Example 4.

$$D^2 + 1 \equiv \left(D + \frac{-c_1 \sin x + c_2 \cos x}{c_1 \cos x + c_2 \sin x} \right) \left(D - \frac{-c_1 \sin x + c_2 \cos x}{c_1 \cos x + c_2 \sin x} \right).$$

2.12. Factorization in ground differential field

The equation

$$y'' + a_0 y = 0 \tag{2.7}$$

admits the factorization

$$(D + \alpha)(D - \alpha) y = 0, \tag{2.8}$$

where $\alpha(x)$ satisfies the Riccati equation

$$\alpha' + \alpha^2 + a_0 = 0, \quad a_0 \in \mathbb{C}(x).$$

They also have the form

$$\alpha = \sum_{i=1}^{n} \sum_{j=1}^{m_i} \frac{c_{ij}}{(x - r_i)^j} + p(x), \quad \alpha \in \mathbb{C}(x),$$

where $p(x)$ is a polynomial.

Example 5 (Kovacic [**41**]). Equation

$$y'' = \Big(x^2 - 2x + 3 + \frac{1}{x} + \frac{7}{4x^2} - \frac{5}{x^3} + \frac{1}{x^4}\Big) y$$

admits the factorization

$$\Big(D + \frac{1}{x+1} + \frac{1}{x-1} - \frac{3}{2x} + \frac{1}{x^2} + x - 1\Big)\Big(D - \frac{1}{x+1} - \frac{1}{x-1} + \frac{3}{2x} - \frac{1}{x^2} - x + 1\Big) y = 0$$

and has the particular solution

$$y = (x^2 - 1)\, x^{-3/2} \exp\Big(-\frac{1}{x} + \frac{1}{2}x^2 - x\Big).$$

The factorization of differential operators of order n

$$L = a_n y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_1 y' + a_0,$$

namely representation as

$$L = (\beta_n D - \alpha_n)(\beta_{n-1} D - \alpha_{n-1}) \cdots (\beta_2 D - \alpha_2)(\beta_1 D - \alpha_1),$$

was considered in works by (Mitrinović [**48**], Popov [**54**], Berkovich [**13, 21, 22, 25**] and others).

2.13. Factorization in the quadratic expansion of the field F_0

Lemma 1 (see, for e.g., Kaplansky [**32**]). *The factorization,* (2.8)*, takes place in the quadratic expansion of the field* F_0*, or in other words the condition*

$$\alpha^2 - p(x)\alpha + q(x) = 0, \quad p, q \in F_0, \quad p \neq 0$$

is fullfiled if and only if the following relations are satisfied:

$$p'' + 3pp' + p^3 + 2a' + 4ap = 0, \qquad 2q = p' + p^2 + 2a.$$

EXAMPLE 6. The equation (see [**22, 25**])

$$Ly \equiv y'' + \left(\frac{3}{16}x^{-2} - bx^{-1}\right) y = 0$$

admits the factorization

$$Ly \equiv \left(D + \frac{1}{4}\,x^{-1} \pm \sqrt{b}\,x^{-1/2}\right)\left(D - \frac{1}{4}\,x^{-1} \mp \sqrt{b}\,x^{-1/2}\right) y = 0,\ b > 0,$$

and has solutions

$$y = x^{1/4}\big(c_1 \exp(2\sqrt{bx}\,) + c_2 \exp(-2\sqrt{bx}\,)\big).$$

2.14. Analogues of Vieta's formulæ and LODE-2 solutions

Suppose we have linear ordinary differential equation of the second order (LODE-2)

$$Ly \equiv y'' + a_1 y' + a_0 y = 0, \tag{2.9}$$

where the operator L admits the factorization

$$L \equiv (D - \alpha_2)(D - \alpha_1). \tag{2.10}$$

From formulæ (2.9) and (2.10) the analogues of VIÈTE's formulæ follow

$$a_1 = -(\alpha_1 + \alpha_2), \qquad a_0 = \alpha_2\alpha_1 - \alpha_1',$$

where α_1 and α_2 satisfy the RICCATI equations [1]

$$\alpha_1' + {\alpha_1}^2 + a_1\alpha_1 + a_0 = 0, \quad \alpha_2' - {\alpha_2}^2 - a_1\alpha_2 - a_0 = 0.$$

Linearly independent solutions of equations (2.9) and (2.10) have the form

$$y_1 = e^{\int \alpha_1\,\mathrm{d}x}, \quad y_2 = e^{\int \alpha_1\,\mathrm{d}x}\!\int\! e^{\int (\alpha_2 - \alpha_1)\,\mathrm{d}x}\mathrm{d}x.$$

The linear nonhomogeneous equation, $Ly = f(x)$, where L admits the factorization (2.10), has the particular solution

$$\overline{y} = e^{\int \alpha_1\,\mathrm{d}x}\!\int \left(e^{\int (\alpha_2 - \alpha_1)\,\mathrm{d}x}\!\int\! e^{-\int \alpha_2\,\mathrm{d}x} f(x)\,\mathrm{d}x\right)\mathrm{d}x.$$

2.15. Factorization of Lamé's operator

Suppose we have LAMÉ's equation,

$$Ly \equiv y'' - (2\wp(x) + \lambda)y = 0, \quad \lambda = \wp(\varepsilon), \tag{2.11}$$

[1] We remark that criteria of an integrability of RICCATI's equation were considered, in particular, in the papers (MITRINOVIĆ and VASIĆ [**49, 50**]).

where $\wp(x)$ is the Weierstrass elliptic function. Lamé's operator admits the factorization

$$L=\big(D+\zeta(x\pm\varepsilon)-\zeta(X)\mp\zeta(\varepsilon)\big)\big(D-\zeta(x\pm\varepsilon)+\zeta(x)\pm\zeta(\varepsilon)\big)$$

and equation (2.11) the has general solution

$$y(x)=c_1\,\frac{\sigma(x+\varepsilon)}{\sigma(x)}\,e^{-\zeta(\varepsilon)x}+c_2\,\frac{\sigma(x-\varepsilon)}{\sigma(x)}\,e^{\zeta(\varepsilon)x},$$

where the Weierstrass functions $\wp(x)$, $\sigma(x)$ and $\zeta(x)$ are connected by the relations

$$\wp(x)=-\zeta'(x),\ \zeta(x)=\frac{\sigma'(x)}{\sigma(x)}\,,\quad \zeta(x+\varepsilon)-\zeta(x)-\zeta(\varepsilon)=\frac{\wp'(x)-\wp'(\varepsilon)}{\wp(x)-\wp(\varepsilon)}\,.$$

Example 7. The degenerate case: $\wp(x)=\dfrac{1}{x^2}\,.$

Equation (2.11) takes the form

$$Ly\equiv y''-\left(\frac{2}{x^2}+\frac{1}{\alpha^2}\right)y=0,$$

where L admits the factorization

$$L=\left(D+\frac{1}{x+\alpha}-\frac{1}{x}\mp\frac{1}{\alpha}\right)\left(D-\frac{1}{x+\alpha}+\frac{1}{x}\pm\frac{1}{\alpha}\right),$$

has the general solution

$$y=c_1\,\frac{x+\alpha}{x}\,e^{-x/\alpha}+c_2\,\frac{x-\alpha}{x}\,e^{x/\alpha}.$$

Example 8. Degenerate case. Suppose that

$$\wp(x)=\frac{1}{\sin^2 x}-\frac{1}{3}\,.$$

Lamé's equation has the form

$$Ly\equiv y''-\left(\frac{2}{\sin^2 x}+\operatorname{ctg}^2\varepsilon\right)y=0, \tag{2.12}$$

where the operator L admits the factorization

$$L=\big(D+\operatorname{ctg}(x\pm\varepsilon)-\operatorname{ctg}x\pm\operatorname{ctg}\varepsilon\big)\big(D-\operatorname{ctg}(x\pm\varepsilon)+\operatorname{ctg}x\mp\operatorname{ctg}\varepsilon\big).$$

Equation (2.12) has general solution

$$y=c_1\,\frac{\sin(x+\varepsilon)}{\sin x}\,e^{-x\operatorname{ctg}\varepsilon}+c_2\,\frac{\sin(x-\varepsilon)}{\sin x}\,e^{x\operatorname{ctg}\varepsilon}.$$

2.16. Factorization of the third-order Halphen's 0perator

HALPHEN's equation of the third order is

$$Ly \equiv y''' - 3\wp(x)y' - \left(\frac{3}{2}\wp'(x) + \frac{1}{2}\wp'(\alpha)\right) y = 0, \tag{2.13}$$

where the operator L admits the factorization

$$\begin{aligned} L = &\big(D + \zeta(x+\alpha+\beta) - \zeta(x) - \zeta(\alpha) - \zeta(\beta)\big) \times \\ &\times \big(D - \zeta(x+\alpha+\beta) + \zeta(x+\alpha) + \zeta(\beta)\big)\big(D - \zeta(x+\alpha) + \zeta(x) + \zeta(\alpha)\big). \end{aligned}$$

The general solution of HALPHEN's equation, (2.13), has the form

$$y(x) = c_1 \frac{\sigma(x+\alpha)}{\sigma(x)} e^{-x\zeta(\alpha)} + c_2 \frac{\sigma(x+\beta)}{\sigma(x)} e^{-x\zeta(\beta)} + c_3 \frac{\sigma(x+\gamma)}{\sigma(x)} e^{-x\zeta(\gamma)},$$

where

$$\wp'^2(x) - \wp'^2(\alpha) = 0, \quad \wp(\alpha) + \wp(\beta) + \wp(\gamma) = 0.$$

EXAMPLE 9. The degenerate case: $\wp = \dfrac{1}{x^2}$.

The equation

$$Ly \equiv y''' - \frac{3}{x^2}y' + \left(\frac{3}{x^3} + \frac{1}{\alpha^3}\right) y = 0,$$

where the operator L admits the factorization

$$\begin{aligned} L = &\left(D + \frac{1}{x+\alpha+\beta} + \frac{1}{x} - \frac{1}{\alpha} - \frac{1}{\beta}\right) \times \\ &\times \left(D - \frac{1}{x+\alpha+\beta} + \frac{1}{x+\alpha} + \frac{1}{\beta}\right)\left(D - \frac{1}{x+\alpha} + \frac{1}{x} + \frac{1}{\alpha}\right), \end{aligned}$$

has the general solution

$$y = c_1 \frac{x+\alpha}{x} e^{-x/\alpha} + c_2 \frac{x+\beta}{x} e^{-x/\beta} + c_3 \frac{x+\gamma}{x} e^{-x/\gamma}.$$

Note. LAMÉ's operator and HALPHEN's operator are commutative. The KORTEWEG-DE VRIES's equation, well-known in the theory of solitons, $u_t = 6uu_x + u_{xxx}$ is generated by commutative condition of the corresponding pair of operators of the second and third orders.

2.17. Operational identities

Differential operators of higher orders may admits a factorization not only through operators of the first order but also operators of other orders. Operational identities in this case are useful. In the paper, (Berkovich, Kval'wasser [**3**]), such identities are constructed, for example

$$(xD^2 + aD)^m = \sum_{k=0}^{m} \binom{m}{k} \frac{\Gamma(a+m)}{\Gamma(a+m-k)}\, x^{m-k} D^{2m-k}, \tag{2.14}$$

where $\Gamma(a+m) = (a+m-1)\cdots(a+1)a\Gamma(a)$;

$$\left(xD^2 + \left(m - \frac{2n+1}{2}\right) D\right)^{\frac{2n+1}{2}}$$

$$= \pm \sum_{k=0}^{m-1} \binom{m-1}{k} \frac{\Gamma\left(\frac{2n+1}{2}+1\right)}{\Gamma\left(\frac{2n+1}{2}-k+1\right)}\, x^{\frac{2n+1}{2}-k} D^{2n+1-k}.$$

The identity (2.14) was generalized in the paper of (Klamkin and Newman [**37**]).

3. TRANSFORMATION OF LODE

3.1. Statement of Kummer's problem

Suppose that we have the equations

$$y'' + a_1(x)y' + a_0(x)y = 0, \quad a_k \in \mathbb{C}^k(I), \quad I = \{x | a < x < b\}, \quad k = 0, 1, \tag{3.1}$$

$$\ddot{z} + b_1(t)\,\dot{z} + b_0\,(t)z = 0, \quad b_k \in \mathbb{C}^k(J), \quad J = \{t | \alpha < t < \beta\}, \tag{3.2}$$

and the Kummer-Liouville transformation

$$y = v(x)z, \quad \mathrm{d}t = u(x)\mathrm{d}x, \quad v, u \in \mathbb{C}^2(I), uv \neq 0. \tag{3.3}$$

It is an invertible transformation, that is, the Jacobian

$$J = \left(\frac{\partial(y,\ \ x)}{\partial(z,\ \ t)}\right) = \begin{vmatrix} \dfrac{\partial y}{\partial t} & \dfrac{\partial y}{\partial z} \\ \dfrac{\partial x}{\partial t} & \dfrac{\partial x}{\partial z} \end{vmatrix} \neq 0.$$

Is it possible to transform (3.1) to (3.2) with the help of KL-transformation (3.3)?

3.2. Solution of Kummer's problem

Theorem 9 (see Berkovich [**10**], Berkovich and Rozov [**15**]). *Equation* (3.1) *can be transformed to* (3.2) *with transformation* (3.3) *if and only if the following conditions for the KL-transformation are satisfied*:

$$v(x) = |u(x)|^{-1/2} \exp\left(-\frac{1}{2}\int a_1\,\mathrm{d}x + \frac{1}{2}\int b_1(t)\,\mathrm{d}t\right),$$

$$\frac{1}{2}\frac{t'''}{t'} - \frac{3}{4}\left(\frac{t''}{t'}\right)^2 + B_0(t)\,t'^2 = A_0(x), \tag{3.4}$$

where (3.4) *is the Kummer-Schwartz equation of the third order* (*KS*-3), *and*

$$A_0(x) = a_0 - \frac{1}{4}\,{a_1}^2 - \frac{1}{2}\,a_1', \qquad B_0(t) = b_0 - \frac{1}{4}\,{b_1}^2 - \frac{1}{2}\,\dot{b}_1$$

are semiinvariants of equations (3.1) *and* (3.2) *respectively* (see Pejović [**51**]), *and* v *and* u *also satisfy the equation*

$$v'' + a_1 v' + a_0 v - b_0 u^2 v = 0. \tag{3.5}$$

Example 10 (see Šapkarev [**55**], Vasić [**57**]):

$$y'' + \left(\frac{ff'}{f^2+b^2} - \frac{f''}{f'}\right) y' - \frac{a^2 f'^2}{f^2+b^2}\, y = 0,\ f = f(x).$$

By the transformation

$$\mathrm{d}t = \frac{f'}{\sqrt{f^2+b^2}}\,\mathrm{d}x$$

this equation is reduced to the equation $\ddot{y} - \dfrac{a^2}{b^2}\, y = 0$.

3.3. Kummer-Schwartz and Ermakov equations

Suppose $a_1 = b_1 = 0$. The equation (Ermakov [**26**], see Berkovich and Rozov [**8**])

$$v'' + a_0 v - b_0 v^{-3} = 0 \tag{3.5$'$}$$

has the general solution (see also Pinney [**53**])

$$v(x) = \sqrt{A{Y_2}^2 + BY_1Y_2 + C{Y_1}^2}, \quad B^2 - 4AC = -4b_0, \tag{3.6}$$

where $Y_1, Y_2 = Y_1 \int Y_1^{-2}\,\mathrm{d}x$ forms a basis of the second-order equation

$$Y'' + a_0 Y = 0. \tag{3.7}$$

The Kummer-Schwarz equation of the second order (KS-2),

$$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + b_0 u^2 = a_0, \tag{3.8}$$

has general solution of the form

$$u(x) = (A{Y_2}^2 + BY_1Y_2 + C{Y_1}^2)^{-1}, \quad B^2 - 4AC = -4b_0. \tag{3.9}$$

3.4. LODE-2 Related by KL transformation

Equation (2.5) with a "carrier", a_0, generates the next sequence of related equations [**12, 25**]

$$y_k'' + a_k y_k = 0,$$

where

$$a_k = a_0 - \sum_{s=1}^{k} b_{0s} u_s^{\,2}, \quad a_k = a_{k-1} - b_{0k} u_k^{\,2},$$
$$\frac{1}{2}\,\frac{u_s''}{u_s} - \frac{3}{4}\left(\frac{u_s'}{u_s}\right)^2 - \frac{1}{4}\,\delta_s u_s^{\,2} = a_{s-1},\ \delta_s = b_{1s}^2 - 4b_{0s},$$
$$y_k^{(1,2)} = |u_k|^{-1/2} \exp\left(\pm\frac{1}{2}\, b_{1k} \int u_k \,\mathrm{d}x\right),\ b_{1k} \neq 0.$$

3.5. Examples of related equations

The following equations are related to the equation $y'' = 0$ [**12, 25**].

Example 11. $y'' - \bigl(m(m+1)x^{-2} + T^4\bigr)y = 0, \quad T = \alpha x^{-m} + \beta x^{m+1},\ m \neq \dfrac{1}{2}$.

General solution: $y(x) = T\left(M \operatorname{ch}\left(\dfrac{x^{-m}}{\gamma T}\right) + N \operatorname{sh}\left(\dfrac{x^{-m}}{\gamma T}\right)\right), \quad \gamma = (2m+1)\beta.$

Example 12. $y'' + \left(\dfrac{1}{4x^2} + \dfrac{1}{x^2 S^4}\right) y = 0, \quad S = \alpha \log x + \beta.$

General solution: $y = \sqrt{x}\, S\Bigl(M \cos\dfrac{1}{\alpha S} + N \sin\dfrac{1}{\alpha S}\Bigr).$

3.6. Halphen's problem for LODE-n

Suppose the equations (Halphen [**30**], Berkovich [**11, 22**])

$$L_n y \equiv y^{(n)} + \sum_{k=0}^{n} \binom{n}{k} a_k y^{(n-k)} = 0, \quad a_k \in \mathbb{C}^{n-k}(I), \tag{3.10}$$

$$M_n Z \equiv z^{(n)}(t) + \sum_{k=1} \binom{n}{k} b_k z^{(n-k)}(t) \quad b_k \in \mathbb{C}^{n-k}(J), \tag{3.11}$$

and the KL-transformation

$$y = v(x) z,\ \ \mathrm{d}t = u(x)\,\mathrm{d}x,\ vu \neq 0,\ v, u \in \mathbb{C}^n(I). \tag{3.12}$$

Problem 1: Find necessary and sufficient conditions of equivalence of (3.10) and (3.11) under the KL transformation (3.12).

Problem 2: Classify equations (3.10) with the help of canonical forms.

3.7. Lemmas of LODE-n equivalence

Lemma 2. *Equations* (3.10) *and* (3.11) *are equivalent if and only if the following system is compatible*

$$\{t,x\} + \frac{3}{n+1}\,B_2 t'^2 = \frac{3}{n+1}\,A_2(x),$$

$$\frac{t^{IV}}{t'} - 6\,\frac{t'''\,t''}{t'^2} + 6\left(\frac{t''}{t'}\right)^3 + \frac{12}{n+1}\,A_2\,\frac{t''}{t'} + \frac{4}{n+1}\,B_3 t'^3 = \frac{4}{n+1}A_3, \dots,$$

$$\left((t')^{\frac{1-n}{2}}\right)^{(n)} + \sum_{k=2}^{n}\binom{n}{k}A_k\left((t')^{\frac{1-n}{2}}\right)^{(n-k)} - B_n(t')^{\frac{n+1}{2}} = 0,$$

where A_k, B_k *are semiinvariants of equations* (3.8) *and* (3.9) *respectively*:

$$A_2 = a_2 - a_1^2 - a_1', \quad A_3 = a_3 + 2a_1^3 - 3a_1a_2 - a_1'', \dots$$

Lemma 3. *Equations* (3.10) *and* (3.11) *are equivalent if the following conditions are satisfied*

$$v'' - \frac{n-2}{n-1}\,\frac{v'^2}{v} + 3\,\frac{n-1}{n+1}\,A_2 v - 3\,\frac{n-1}{n+1}\,B_2 v^{\frac{n-5}{n-1}} = 0,$$

$$v''' - 3\,\frac{n-3}{n-1}\,\frac{v'v''}{v} + 2\,\frac{(n-2)(n-3)}{(n-1)^2}\,\frac{v'^3}{v^2}\,\frac{12}{n+1}\,A_2 v' + 2\,\frac{n-1}{n+1}\,A_3 v - 2\,\frac{n-1}{n+1}\,B_3 v^{\frac{n-7}{n-1}} = 0,$$

$$\vdots$$

$$v^{(n)} + \sum_{k=2}^{n}\binom{n}{k}A_k v^{(n-k)} - B_n v^{-\frac{n+1}{n-1}} = 0.$$

3.8. Theorem of LODE-n equivalence

Theorem 10. *Equations* (3.10) *and* (3.11) *are equivallent if and only if the following relations between their invariants are satisfied*: $I_0(A) = u^3 I_0(B)$, $J_{n,1}(A) = u^4 J_{n,1}(B)$, $J_{n,2}(A) = u^5 J_{n,2}(B), \dots, J_{n,n-3}(A) = u^n J_{n,n-3}(B)$, *where* $\int u(x)\,\mathrm{d}x = t(x)$ *satisfies the equation (KS-3)*

$$\{t,x\} + \frac{3}{n+1}B_2 t'^2 = \frac{3}{n+1}A_2, \{t,x\} = \frac{1}{2}\frac{t'''}{t'} - \frac{3}{4}\left(\frac{t''}{t'}\right)^2,$$

and $I_0(A)$ *is Laguerre's invariant* (Laguerre [**42**])

$$I_0(A) = A_3 - \frac{3}{2}\,A_2' = a_3 - 3a_1a_2 + 2a_1^3 + 3a_1a_1' + \frac{1}{2}\,a_1'' - \frac{3}{2}\,a_2',$$

and

$$J_{n,1}(A) = A_1 - 2A_3 + \frac{6}{5}\,A_2'' - \frac{3(5n+7)}{5(n+1)}\,A_2^2,$$

$$J_{n,2}(A), \dots, J_{n,n-3}(A)$$

are Halphen's invariants.

3.9. Halphen's canonical forms

Class	Invariants	Transformation $y = u_k^{-\frac{n-1}{2}} z,\ \mathrm{d}t = u_k\,\mathrm{d}x$	Halphen's Canonical forms
Y_0	$I_0 \neq 0$	$u_0 = \sqrt[3]{I_0}$	Principal (H_{n0}), depends on $n-2$ parameters
Y_k, $k = \overline{1, n-3}$	$I_0 = I_{n,1} = \cdots = I_{n,k-1} = 0,\ I_{n,k} = J_{n,k} \neq 0$	$u_k = \sqrt[k+3]{I_{n,k}}$	Degenerate (H_{nk}), depends of $n-k-2$ parameters
Y_{n-2}	$I_0 = I_{n,1} = \cdots$ $= I_{n,n-3} = 0$	$\frac{1}{2}\frac{u''_{n-2}}{u_{n-2}} - \frac{3}{4}\left(\frac{u'_{n-2}}{u_{n-2}}\right)^2 = \frac{3}{n+1}A_2$	Elementary denegerate $(H_{nn-2}) : z^{(n)}(t) = 0$

3.10. Forsythe's canonical forms

Class	Invariants	Transformation $y = u^{-\frac{n-1}{2}} z,\ \mathrm{d}t = u\,\mathrm{d}x$	Forsythe's canonical forms
Y_0	$I_0 \neq 0$	$\frac{1}{2}\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 = \frac{3}{n+1}A_2$	Principal (F_{n0}), depends on $n-2$ parameters
Y_k, $k = \overline{1, n-3}$	$I_0 = I_{n,1} = \cdots = I_{n,k-1}$ $=0,\ I_{n,k} = J_{n,k} \neq 0$		Degenerate (F_{nk}), depends on $n-k-2$ parameters
Y_{n-2}	$I_0 = I_{n,1} = \cdots$ $= I_{n,n-3} = 0$		Elementary degenerate $(F_{nn-2}) : z^{(n)}(t) = 0$

Note. Halphen [**30**] found canonical forms for the equations of orders $n = 3$ and $n = 4$.

Forsyth [**28**] found the canonical form F_{n0}.

3.11. Criteria of LODE-n reducibility

Equation (3.10) is locally reducible (by HALPHEN) if it can be transformed to the following form

$$M_n z \equiv z^{(z)} + \sum_{k=1}^{n} \binom{n}{k} b_k z^{(n-k)}(t) = 0, \quad b_k = \text{const} \tag{3.13}$$

by the KL-transformation, (3.12).

Theorem 11 [**13, 22**]. *The followiing conditions are equivalent*:

1. *Equation* (3.10) *is reducible*;
2. *The operator* L_n *admits noncommutative factorization*

$$L_n = \prod_{k=n}^{1} \left(D - \frac{v'}{v} - (k-1)\,\frac{u'}{u} - r_k u \right),$$

where

$$v = |u|^{-\frac{n-1}{2}} \exp\left(-\textstyle\int \alpha_1 \,\mathrm{d}x + b_1 \int u \,\mathrm{d}x\right),$$
$$\frac{1}{2}\,\frac{u''}{u} - \frac{3}{4}\left(\frac{u'}{u}\right)^2 + \frac{3}{n+1}\,B_2 u^2 = \frac{3}{n+1}\,A_2,$$

r_k *are roots of the characteristic equation*

$$M_n(r) \equiv r^n + \sum_{k=1}^{n} \binom{n}{k} b_k\, r^{n-k} = 0; \tag{3.14}$$

3. *The operator* $u^{-n} L_n$ *admits the commutative factorization*

$$u^{-n} L_n = \prod_{k=1}^{n} \left(\frac{1}{u}\,D - \frac{v'}{vu} - r_k \right);$$

4. *there exist four functions*, ω, w, λ *and* μ, *namely*

$$\omega = v^{-1}u^{1-n},\ w = v^{-1}u^{-n},\ \lambda = u^{-1},\ \mu = -v'v^{-1}u^{-1}, \text{(see FAYET [\textbf{27}])}$$

such that

$$\omega L_n(\lambda D + \mu) y = D\left(\omega L_n y\right);$$

5. $Y(x)$ *is solution of* (3.10) *if* $y(x)$ *is solution of* (3.10) : (see KAKEYA [**31**])

$$Y(x) = \frac{1}{u}\,y' - \frac{v'}{vu}\,y;$$

6. *I_0 and $J_{n,k}$ are connected in a special way, namely in Theorem* 10 $I_0(B)$, $J_{n,1}(B), \dots, J_{n,n-3}(B)$ *are constants;*

7. *Absolute Halphen's invariants* $h_k = \text{const}$ (Halphen [**30**]);

8. *Equation* (3.10) *admits a point symmetry with a generator*

$$X = \frac{1}{u}\frac{\partial}{\partial x} + \frac{v'}{uv}\, y\, \frac{\partial}{\partial y}.$$

4. AUTONOMIZATION OF NODE

We consider nonautonomous nonlinear ordinary differential equations (NODE) [**5**,**6**].

4.1. Nonlinear equations with reducible linear part

$$y^{(n)} + \sum_{k=1}^{n} \binom{n}{k} a_k\, y^{(n-k)} + F(x, y, y', \dots, y^{(m)}) = 0. \tag{4.1}$$

Theorem 12. *Equation* (4.1) *can be reduced to an autonomous form*

$$z^{(n)}(t) + \sum_{k=1}^{n} \binom{n}{k} b_k\, z^{(n-k)}(t) + a\,\Phi\bigl(z, z'(t), \dots, z^{m}(t)\bigr) = 0$$

by the KL-transformation (3.12) *if and only if the nonlinear part F can be represented as*

$$F = au^n v\Phi\left(\frac{y}{v}, \frac{1}{v}\left(\frac{1}{u}D - \frac{v'}{vu}\right)y, \dots, \frac{1}{v}\left(\frac{1}{u}D - \frac{v'}{vu}\right)^m y\right).$$

Bandić (see for example [**2**]) transformed nonlinear equations by applying the so-called relative derivatives $\Delta_k = y^{(k)}/y$ (Petrovich [**52**]).

4.2. Test for autonomization

1. Using the criteria for reducibility, verify whether $L_n y = 0$ is reducible.

2. If $L_n y = 0$ is reducible (it always is for $n = 2$), represent the general solution in the form:

$$y = v \sum_{k=1}^{n} c_k \exp(r_k U), \quad U = \textstyle\int u\, \mathrm{d}x,$$

where r_k are distinct roots of the characteristic equation (3.14), or in the form

$$y = v \sum_{k=1}^{m} \sum_{s=1}^{\ell_k} \frac{1}{(s-1)!}\, U^{s-1} \exp(r_k U), \quad \sum_{k=1}^{m} \ell_k = n,$$

where r_k are multiple roots of characteristic equation (3.14),

$$u(x) = (A{Y_2}^2 + BY_2Y_1 + C{Y_1}^2)^{-1}, \quad B^2 - 4AC = -\frac{12}{n+1}\, B_2,$$

and Y_1, $Y_2 = Y_1 \int \mathrm{d}x/{Y_1}^2$ are linearly independent solutions of

$$Y'' + \frac{3}{n+1}\, A_2 Y = 0. \tag{4.2}$$

4.3. Principles of nonlinear superposition

Let there be given the equation

$$f(x, y, y', \ldots, y^{(n)}) = 0. \tag{4.3}$$

A system of functions

$$\{Y_1(x), \ldots, Y_m(x)\} \tag{4.4}$$

(see Lie [**45**]) forms a fundamental system of solutions (FSS) of equation (4.3) if its general solution can be represented in the form

$$y = F(Y_1, Y_2, \ldots, Y_m; c_1, \ldots, c_n), \tag{4.5}$$

where (4.4) are particular solutions of (4.3), particular solutions of the adjoint nonlinear equation

$$\varphi(X, Y, Y', \ldots, Y^{(m)}) = 0$$

or they (4.4) are FSS of the adjoint linear equation

$$Y^{(m)} + \sum_{k=1}^{m} \binom{m}{k} a_k(x)\, Y^{(m-k)} = 0.$$

Function (4.5) is called a *nonlinear superposition principle* for equation (4.3) (see Winternitz [**58**], Berkovich [**13**]).

Note. Formulas (3.6) and (3.9) are nonlinear superposition principles for the Ermakov equation, (3.5'), and for the Kummer-Schwartz equation (KS-2), (3.8), respectively.

4.4. Generalized Emden-Fowler equation of the second order

Theorem 13. *In order that the equation*

$$y'' + f(x)\, y^n = 0, \quad n \neq 0, \quad n \neq 1,$$

lead to

$$\ddot{z} \pm b_1 \dot{z} + b_0 z + c z^n = 0,$$

it is necessary and sufficient that

$$
\begin{aligned}
&f_1(x) = (\alpha_1 x+\beta_1)^{-\frac{3+n}{2}\pm\frac{b_1(1-n)}{2\sqrt{\delta_1}}}(\alpha_2 x+\beta_2)^{-\frac{3+n}{2}\mp\frac{b_1(1-n)}{2\sqrt{\delta_1}}}\ \delta_1>0,\\
&f_2(x) = (Ax^2+Bx+C)^{-\frac{3+n}{2}}\exp\left(\pm\frac{(1-n)\,b_1}{\sqrt{\delta_2}}\arctan\frac{2AX+B}{\sqrt{-\delta_2}}\right)\ \delta_2<0,\\
&f_3(x) = (\alpha x+\beta)^{-(n+3)}\exp\left(\pm\frac{(1-n)b_1}{2\alpha(\alpha x+\beta)}\right),\quad \delta_3=0,\ \alpha\neq 0,\\
&f_4(x) = (\alpha x+\beta)^{-\frac{n+3}{2}+b_1\frac{1-n}{2\alpha}},\quad \delta_4=\alpha^2>0,\\
&f_5(x) = C\exp\left(\pm\frac{1-n}{2}\,b_1 x\right),\quad \delta_5=0.
\end{aligned}
$$

(see also Kečkić [**35**], Kocić [**38**], Berkovich [**7, 16**], Leach [**44**]).

4.5. Ermakov systems

The system (Ermakov [**26**])

$$
\begin{cases}
\ddot{x}+a_0(t)x=0\\
\ddot{y}+a_0 y=b_0 y^{-3}
\end{cases}
$$

havs the integral (invariant):

$$
\frac{1}{2}\,(\dot{x}y-\dot{y}x)^2+\frac{1}{2}\,b_0\left(\frac{x}{y}\right)^2=C.
$$

The generalized Ermakov system (Berkovich [**22**]) is

$$
\begin{cases}
\ddot{x}+a_1(t)\dot{x}+a_0(t)x=af(t)x^m y^n F(x,y)\\
\ddot{y}+a_1(t)\dot{y}+a_0(t)y=bf(t)x^n y^m G(x,y),
\end{cases}\qquad a,b=\text{const.}\tag{4.5}
$$

If the left part of system (4.5) is reduced to constant coefficients by the KL-transformation

$$
x=v(t)X,\ y=v(t)Y,\ \mathrm{d}T=u(t)\,\mathrm{d}t
$$

and thus

$$
F=F(y/x),\ G=G(x/y),\ f(t)=v^{1-m-n}u^2,\ m=-(n+3)
$$

, system (4.5) possesses the first integral (invariant)

$$
\begin{aligned}
&I=\frac{1}{2}\,\varphi^2(x\dot{y}-y\dot{x})^2+a\int^{x/y}u^{n+1}F(u)\,\mathrm{d}u+b\int^{y/x}u^{n+1}G(u)\,\mathrm{d}u,\\
&\varphi=\ \exp\left(\int^{x}a_1(t)\,\mathrm{d}t\right).
\end{aligned}
$$

4.6. Generalized Ermakov's equation of the n-order

Theorem 14. *Equation*

$$y^{(n)} + \sum_{k=2}^{n} \binom{n}{k} a_n y^{(n-k)} + b_n y^{\frac{1+n}{2-n}} = 0,$$

where $L_n y = 0$ *is reducible, has the two-parameter solution*

$$y = p\left(AY_1{}^2 + BY_1Y_2 + CY_2{}^2\right)^{\frac{n-1}{2}}, \quad B^2 - 4AC = q,$$

where Y_1 *and* Y_2 *are linearly independent solutions of equation*

$$Y'' + \frac{3}{n+1}\, a_2 Y = 0$$

and admits three-dimensional LIE *algebra with generators*

$$\begin{aligned}
X_1 &= Y_1{}^2 \frac{\partial}{\partial x} + (n-1) Y_1 Y_1' \, y \frac{\partial}{\partial y},\\
X_2 &= Y_1 Y_2 \frac{\partial}{\partial x} + \frac{n-1}{2} \left(Y_1 Y_2' + Y_2 Y_1'\right) y \frac{\partial}{\partial y},\\
X_3 &= Y_2{}^2 \frac{\partial}{\partial x} + (n-1) Y_2 Y_2' \, y \frac{\partial}{\partial y}
\end{aligned}$$

and commutators

$$[X_1, X_2] = X_1, \ [X_2, X_3] = X_3, \ [X_3, X_1] = -2X_2.$$

5. LINEARIZATION OF NODE

In the papers [**17-21, 23**] and in the book [**22**] we have already investigated autonomous nonlinear ordinary differential equations (NODE)

$$y^{(n)} = F(y, y', \ldots, y^{(n-1)}). \tag{5.1}$$

Lemma 4. *In order that equation* (5.1) *can be linearized by the nonlinear transformation*

$$y = v(y) z, \quad \mathrm{d}t = u(y)\,\mathrm{d}x \tag{5.2}$$

to equation (3.13), *it is necessary and sufficient that equation* (5.1) *admit the noncommutative factorization*

$$\prod_{k=n}^{1} \left(D - \left(\frac{v^*}{v} - (k-1)\frac{u^*}{u}\right) y' - r_k u\right) y = 0$$

or the commutative factorization

$$\prod_{k=1}^{n}\left(\frac{1}{u}D-\frac{v^*}{uv}y'-r_k\right)y=0,$$

where r_k *are roots of the characteristic equation* (3.14).

5.1. Linearization of second-order equations

The equation

$$y''+f(y)y'^2+b_1\varphi(y)y'+\psi(y)=0,\quad b_1=\mathrm{const} \tag{5.3}$$

can be linearized by the transformation (5.2) to the equation

$$\ddot z+b_1\dot z+b_0z+c=0,\ a,\ b,\ c=\mathrm{const}, \tag{5.4}$$

if and only if

$$\psi(y)=\varphi\exp\left(-\textstyle\int f\,\mathrm{d}y\right)\left(b_0\textstyle\int\varphi e^{\int f\,\mathrm{d}y}\,\mathrm{d}y+\dfrac{c}{\beta}\right). \tag{5.5}$$

Here the transformation (5.2) is

$$z=\beta\int\varphi\exp\left(\textstyle\int f\,\mathrm{d}y\right)\mathrm{d}y,\quad \mathrm{d}t=\varphi(y)\,\mathrm{d}x,$$

where $\beta=$const is a normalizing factor. One-parameter solutions of the equations (5.3) and (5.5), where $c=0$, are

$$r_kx+C_k=\int\frac{\exp\left(\int f\,\mathrm{d}y\right)\mathrm{d}y}{\int\varphi\exp\left(\int f\,\mathrm{d}y\right)\mathrm{d}y},$$

where distinct r_k, $(k=1,2)$, satisfy the equation

$$r^2+b_1r+b_0=0.$$

5.2. Nonlinear Oscillator

The equation

$$y''+f(y)y^2\pm a^2\psi(y)=0 \tag{5.6}$$

by transformation

$$z=\sqrt{2\int\psi\exp\left(2\textstyle\int f\,\mathrm{d}y\right)\mathrm{d}y},\quad \mathrm{d}t=z^{-1}\psi\exp\left(\textstyle\int f\,\mathrm{d}y\right)\mathrm{d}x$$

is reduced to the form

$$\ddot z\pm a^2z=0.$$

Equation (5.6) has the first integrals:

$$y'^2 = a^2 \left(C \mp 2\int \psi \exp\left(2\int f \, \mathrm{d}y\right) \mathrm{d}y\right) \exp\left(-2\int f \, \mathrm{d}y\right)$$

and also the one-parameter solutions:

$$\int \frac{\exp\left(2\int f \, \mathrm{d}y\right) \mathrm{d}y}{z} = \pm \sqrt{\mp a^2}\, x + C.$$

5.3. Linearization of third-order equations

We find conditions for linearization of the equation

$$y''' + f_5(y)y'y'' + f_4(y)y'' + f_3(y)y'^3 + f_2(y)y'^2 + f_1(y)y' + f_0(y) = 0 \tag{5.7}$$

to the equation

$$\dddot{z} + b_2\ddot{z} + b_1\dot{z} + b_0 z + c = 0 \tag{5.8}$$

by a transformation of the form (5.2).

Theorem 15. *Equation* (5.7) *can be linearized if and only if it can be represented in the form*

$$\begin{aligned} y''' + f(y)y'y'' + \frac{1}{9}\left(3\,\frac{\varphi^{**}}{\varphi} - 5\,\frac{\varphi^{*2}}{\varphi^2} - f\,\frac{\varphi^*}{\varphi} + f^2 + 3f^*\right) y'^3 \\ + b_2\varphi y'' + \frac{1}{3}\, b_2\varphi\left(f + \frac{\varphi^*}{\varphi}\right) y'^2 + b_1\varphi^2 y' \\ + \varphi^{5/3}\left(b_0\int \varphi^{4/3} \exp\left(\frac{1}{3}\int f \, \mathrm{d}y\right) \mathrm{d}y + \frac{c}{\beta}\right) \exp\left(-\frac{1}{3}\int f \, \mathrm{d}y\right) = 0. \end{aligned} \tag{5.9}$$

Equation (5.9) by the transformation

$$z = \beta\varphi^{4/3} \exp\left(\frac{1}{3}\int f \, \mathrm{d}y\right) \mathrm{d}y, \quad \mathrm{d}t = \varphi(y)\, \mathrm{d}x$$

is reduced to the linear form (5.8) and, if $c = 0$, has the distinct one-parameter solutions

$$r_k x + c_k = \int \frac{\varphi^{4/3} \exp\left(\frac{1}{3}\int f \, \mathrm{d}y\right) \mathrm{d}y}{\int \varphi^{4/3} \exp\left(\frac{1}{3}\int f \, \mathrm{d}y\right) \mathrm{d}y}\,,$$

where r_k satisfy the equation

$$r^3 + b_2 r^2 + b_1 r + b_0 = 0.$$

We remark that Kečkić [**34–36**] and Kocić [**38, 39**] investigated nonlinear equations of the second and third orders in another way.

5.4. Euler-Poinsot case in the problem of the gyroscope

Suppose we have the coupled system

$$\begin{cases} A\dot{p} - (B - C)qr = 0, \\ B\dot{q} - (C - A)rp = 0, \\ C\dot{r} - (A - B)pq = 0, \end{cases} \tag{5.10}$$

where p, q and r are the components of the angular velocity in the directions of its principal axes of inertia, A, B and C are its principal moments of inertia. Eliminating the variables we get a noncoupled system of nonlinear third-order equations:

$$y_i''' - \frac{1}{y_i}\, y_i' y_i'' + b_i y_i' {y_i}^2 = 0, \quad (') = \mathrm{d}/\mathrm{d}x_i, \tag{5.11}$$

where b_i is expressed through A, B and C. By the transformations

$$z_i = {y_i}^2,\ \mathrm{d}s_i = y_i\,\mathrm{d}x_i.$$

equations (5.11) are reduced to the linear equations

$$z_i'''(s_i) + b_i z(s_i)' = 0.$$

As a result equations (5.11) have the parametrical solutions:

$$y_i = \Big(2\Big(A_{1i}\cos\big(\sqrt{b_i}s_i{+}\theta\big){+}A_{2i}\Big)\Big)^{1/2},\ x_i = \int \frac{\mathrm{d}s_i}{\Big(2\big(A_{1i}\cos(\sqrt{b_i}s_i + \theta) + A_{2i}\big)\Big)^{1/2}}.$$

6. SIMULTANEOUS USING OF DIFFERENT METHODS

6.1. Generalized Emden-Fowler equations of the third order

The equation

$$y''' + bx^s y^n = 0,\ n \neq 0, n \neq 1$$

can be reduced by the transformation

$$y = v_1(x)v_2\big(y/v_1(x)\big)z, \quad \mathrm{d}t = u_1(x)u_2\big(y/v_1(x)\big)\,\mathrm{d}x$$

to a linear equarion if and only if $n = -5/2$, $s = 1$ or $n = -7/2$, $s = 3$ respectively. The equations

$$y''' + bxy^{-5/2} = 0,$$
$$y''' + bx^3 y^{-7/2} = 0$$

by the transformation

$$z = x^2 y^{-1}, \quad \mathrm{d}t = xy^{-3/2}\,\mathrm{d}x$$

are reduced to the linear forms

$$z_t''' - b = 0,$$
$$z_t''' - bz = 0$$

respectively.

6.2. Factorization of Lienard's equation

The equation

$$y'' + a_1(y)y' + a_0(y)y = 0 \tag{6.1}$$

admits factorization of the form

$$\big(D - \alpha_2(y)\big)\big(D - \alpha_1(y)\big)y = 0, D = \mathrm{d}/\mathrm{d}x,$$
$$a_1 = -(\alpha_1 + \alpha_2 + \alpha_1^* y),\ \ \alpha_0 = \alpha_1\alpha_2,\ \ (*) = \mathrm{d}/\mathrm{d}y,$$

where α_1 satisfies the ABEL equation of the second kind

$$y\alpha_1 \frac{d\alpha_1}{dy} + {\alpha_1}^2 + a_1\alpha_1 + a_0 = 0,$$

and α_2 satisfies the ABEL equation of the first kind

$$a_0 y \frac{\mathrm{d}\alpha_2}{\mathrm{d}y} = \alpha_2^3 + a_1\alpha_2^2 + \alpha_2(a_0 + a_0^* y).$$

Equation (6.1) was considered by BANDIĆ [**1**] in a different way.

6.3. Anharmonic oscillator

If

$$(n+3)^2 b_0 = 2(n+1)b_1^2,$$

the equation

$$y'' + b_1 y' + b_0 y + by^n = 0$$

admits the factorization

$$\left(D - r_2 - k_2 y^{\frac{n-1}{2}}\right)\left(D - r_2 - k_2 y^{\frac{n-1}{2}}\right)y = 0, \quad D = \mathrm{d}/\mathrm{d}x,$$

where

$$r_1 = -\frac{2b_1}{n+3}\,,\ r_2 = -\frac{n+1}{n+3}b_1\,,\ k_1 = \pm\sqrt{-\frac{2b}{n+1}}\,,\ k_2 = \mp\sqrt{-\frac{b(n+1)}{2}}\,,$$

and has the one-parameter system of solutions

$$y = \left(\pm\frac{n+3}{b_1}\sqrt{-\frac{b}{2(n+1)}} + C\exp\left(\frac{b_1(n+1)}{n+3}\,x\right)\right)^{\frac{2}{1-n}}.$$

7. CONCLUSION

The methods, discussed in the present work, do not minimize the importance of the other analytical methods, nor the methods of numerical analysis, nor the qualitative theory of differential equations. Only by simultaneously using all of them shall we get the best effect, but the construction of algorithms for solving ordinary differential equations in closed form is the most important goal for any effective theory of ordinary differential equations. Explicit formulas concentrate all the information about the given ordinary differential equation. In this connection we mention the following works: L. Berkovich and F. Berkovich [**14**], Berkovich and Evlakhov [**24**], in which some algorithms of LODE-2 factorization and variables transformation were implemented in REDUCE. Further implementation of such algorithms for nonlinear equations and linear high-order equations is an actual problem. It is the author's opinion that further elaboration of factorization and variable transformation can cast new light on many solved and unsolved questions of natural science.

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NONLOCAL SYMMETRIES PAST, PRESENT AND FUTURE

PGL Leach, K Andriopoulos

Nonlocal symmetries entered the literature in the Eighties of the last century largely through the work of Peter Olver. It was observed that there could be gain of symmetry in the reduction of order of an ordinary differential equation. Subsequently the reverse process was also observed. In each case the source of the 'new' symmetry was a nonlocal symmetry, *ie* a symmetry with one or more of the coefficient functions containing an integral. A considerable number of different examples and occurrences were reported by Abraham-Shrauner and Guo in the early Nineties. The role of nonlocal symmetries in the integration, indeed integrability, of differential equations was excellently illustrated by Abraham-Shrauner, Govinder and Leach with the equation $yy'' - y'^2 + f'(x)y^{p+2} + pf(x)y'y^{p+1} = 0$ which had been touted as a trivially integrable equation devoid of any point symmetry. Further theoretical contributions were made by Govinder, Feix, Bouquet, Géronimi and others in the second half of the Nineties. This included their role in reduction of order using the nonnormal subgroup. The importance of nonlocal symmetries was enhanced by the work of Krause on the Complete Symmetry Group of the Kepler Problem. Krause's work was furthered by Nucci and there has been considerable development of the use of nonlocal symmetries by Nucci, Andriopoulos, Cotsakis and Leach. The determination of the Complete Symmetry Group for integrable systems such as the simplest version of the Ermakov equation, $y'' = y^{-3}$, which possesses the algebra $sl(2, R)$ has proven to be highly nontrivial and requires some nonintuitive nonlocal symmetries. The determination of the nonlocal symmetries required to specify completely the differential equations of nonintegrable and/or chaotic systems remains largely an open question.

1. INTRODUCTION

New ideas, concepts and objects tend to originate in esoteric contexts. The commonplace does not invite deep thinking since the solution of problems there proceeds via methods upon which the experienced practitioner need not dwell for

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their execution. This does not mean that new ideas, concepts and objects cannot be found in the commonplace. One is sometimes pushed to think uncommonly in the context of the commonplace to see that the new has been under our noses since the beginnings of time if not earlier. In a pedagogical context the elimination of the esoteric origin is essential to lead the neophyte to understanding. So it is with nonlocal symmetries.

Barbara Abraham-Shrauner firstly heard of nonlocal symmetries from Peter Olver around 1990. She found them of interest and produced a series of papers with her student, Ann Guo, and some others [**1, 2, 3, 4, 5, 23**] chronicling their occurrence and characteristics. The collaboration spread and a number of papers [**6, 7, 19, 20, 21, 22, 42, 55, 56, 57, 58, 39, 41**] devoted to the subject has appeared over the years. During that time there has been some change in the relative importance of the elements of the subject of nonlocal symmetries. This has been a simple consequence of the movement from the esoteric to the commonplace.

In this paper we give an indication of the evolution of the subject of nonlocal symmetries. Firstly we look to an esoteric example and then to the removal of 'esoteric' with the presentation of some classic examples for which commonplace is too grand a word. Next we consider a standard result of the Lie theory and show that it is ill-based in its implication. Our third point is the necessity of nonlocal symmetries in a relatively new area which is the complete specification of a differential equation by means of symmetries.

That more or less covers the Past and the Present. What of the Future? Perhaps it would be better to reveal a little so that together we may seek that which is to come.

2. FROM ESOTERICA TO BANALITY

The differential equation

$$2yy'''' + 5y'y''' = 0 \tag{1}$$

arises in the study of the symmetries of the Emden-Fowler equation [**28, 30, 29, 44, 14**] and is easily shown to possess only the three obvious Lie point symmetries

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = x\partial_x \quad \text{and} \quad \Gamma_3 = y\partial_y. \tag{2}$$

We use a symmetry, Γ_1 with invariants $u = y$ and $v = y'$, to reduce the order by one and obtain

$$2u\left(v^2v''' + 4vv'v'' + v'^3\right) + 5\left(v^2v'' + vv'^2\right) = 0 \tag{3}$$

which inherits

$$\Sigma_2 = v\partial_v \quad \text{and} \quad \Sigma_3 = u\partial_u + v\partial_v \tag{4}$$

from Γ_2 and Γ_3, respectively. Note that the second term of Σ_3 is redundant due to Σ_2. However, when we calculate the Lie point symmetries of (3), we find also

$$\Sigma_4 = 2u^2\partial_u + uv\partial_v \tag{5}$$

as an unexpected but very pleasant surprise. If we continue the process of reduction using Σ_2 and Σ_3, the result is an Abel's equation of the second kind of the most hideous aspect. The invariants of Σ_4 are $r = vu^{-1/2}$ and $s = \frac{1}{2}\left(v'u^{3/2} - \frac{1}{2}vu^{1/2}\right)^2$ and the reduced equation,

$$s'' + 3s' + 2s = 0, \tag{6}$$

is pleasingly linear [**30**].

Evidently Σ_4 cannot have its origin in a point symmetry of (1) since Γ_1 was used for the reduction and Γ_2 and Γ_3 lead to Σ_2 and Σ_3. Under the reduction of (1) a symmetry

$$\Gamma_4 = \xi\partial_x + \eta\partial_y \quad \longrightarrow \quad \Sigma_4 = \eta\partial_u + (\eta' - y'\xi')\,\partial_v$$

so that

$$\eta = 2y^2 \quad \text{and} \quad \eta' - y'\xi' = yy'$$

whence $\xi = 3\int y\,\mathrm{d}x$ and we have the nonlocal symmetry

$$\Gamma_4 = 3\left(\int y\,\mathrm{d}x\right)\partial_x + 2y^2\partial_y. \tag{7}$$

The symmetry, Γ_4, is termed an 'hidden symmetry of Type II' since it appears as a point symmetry on reduction of order. Likewise an 'hidden symmetry of Type I' arises when a point symmetry becomes nonlocal on increase of order [**2, 3, 4, 23**].

The nonlinear second-order ordinary differential equation

$$y'' = \frac{y'^2}{y} + f'(x)y^{p+1} + pf(x)y'y^p \tag{8}$$

is devoid of Lie point symmetries for general f and p and yet is trivially integrable. As such it was presented as a counterexample to the need for the presence of Lie symmetries for an equation to be integrable [**18, 60**]. However, on the nonlocal changes of variable [**6**]

$$\begin{aligned} x &= x & y &= -\frac{w'}{pf(x)w} \\ X &= x & W &= \log w' \end{aligned} \tag{9}$$

the nonlinear (8) becomes

$$\frac{\mathrm{d}^2W}{\mathrm{d}X^2} = 0 \tag{10}$$

which is not only trivially integrable but also possesses eight Lie point symmetries which translate to a collection of nonlocal symmetries of (8). Unfortunately these nonlocal symmetries are complicated expressions and one could easily think it unlikely that anyone would ever essay the solution of (8) using a suitable pair of them. However, the unlikely was done recently by Nucci [**52**].

In the two examples presented the ordinary differential equations inspiring the work were somewhat special. The large number of nonlocal symmetries found for (8) suggested that 'hidden' symmetries could be of common occurrence. Several studies of both differential equations and associated first integrals/invariants [**19, 15, 16, 31, 33**] revealed that this indeed be the case. To give a flavour of the result of the studies we list the connections between the Lie point symmetries of the two equations $Y''=0$ and $y'''=0$ which are related by the nonlocal transformation $X=x$ and $Y=y'$.

The standard symmetries of

$y'''=0$	and $Y''=0$.
$\Gamma_1=\partial_y$	$\Sigma_1=\partial_Y$
$\Gamma_2=x\partial_y$	$\Sigma_2=X\partial_Y$
$\Gamma_3=x^2\partial_y$	
$\Gamma_4=\partial_x$	$\Sigma_3=\partial_X$
$\Gamma_5=x\partial_x+y\partial_y$	$\Sigma_4=X\partial_X+\frac{1}{2}Y\partial_Y$
$\Gamma_6=x^2\partial_x+2xy\partial_y$	$\Sigma_5=X^2\partial_X+XY\partial_Y$
$\Gamma_7=y\partial_y$	$\Sigma_6=Y\partial_Y$
$\Gamma_8=y'\partial_x+\frac{1}{2}y'^2\partial_y$	$\Sigma_7=Y\partial_X$
$\Gamma_9=2(xy'-y)\partial_x+xy'^2\partial_y$	$\Sigma_8=XY\partial_X+Y^2\partial_Y$
$\Gamma_{10}=(x^2y'-2xy)\partial_x$ $+(\frac{1}{2}x^2y'^2-2y^2)\partial_y$	

$y'''=0$	Fate	$Y''=0$	Source
Γ_1	annihilated	Σ_1	Γ_2
Γ_2	Σ_1	Σ_2	Γ_3
Γ_3	$(2)\Sigma_2$	Σ_3	$(\frac{1}{2})\Gamma_4$
Γ_4	Σ_3	Σ_4	$\Gamma_5-\frac{1}{2}\Gamma_7$
Γ_5	$\Sigma_4-\frac{1}{2}\Sigma_6$	Σ_5	$x^2\partial_x+3(xy-\int y\mathrm{d}x)\partial_y$
Γ_6	$X^2\partial_X+(2\int Y\mathrm{d}X)\partial_Y$	Σ_6	Γ_7
Γ_7	Σ_6	Σ_7	Γ_8
Γ_8	Σ_7	Σ_8	$xy'\partial_x+\frac{1}{2}(xy'^2+3\int y'^2\mathrm{d}x)\partial_y$
Γ_9	$-2(\int Y\mathrm{d}X-XY)\partial_X+Y^2\partial_Y$		
Γ_{10}	$-2(X\int Y\mathrm{d}X-X^2Y)\partial_X$ $-(2Y\int Y\mathrm{d}X-XY^2)\partial_Y$		

Table 1: Fates of the symmetries of $y'''=0$ and sources of the symmetries of $Y''=0$. The numerical factors in parentheses indicate the precise relationship between each pair of symmetries.

Just because the two equations are simple and are simply connected, it does not mean that their symmetries are equally so!

So far we had been looking at nonlocal symmetries in the sense of their manifestation through hidden symmetries. Theo Pillay, a thoughtful student given to Physics, made a nice job of unifying symmetry in a very direct fashion [**55**].

Everyone knows that the Lie point symmetries of

$$y'' = 0 \tag{11}$$

are eight in number and possess the Lie algebra $sl(3,R)$. How do we find them? We assume that a symmetry of (11) has the form

$$\Gamma = \xi(x,y)\partial_x + \eta(x,y)\partial_y, \tag{12}$$

apply the second extension,

$$\Gamma^{[2]} = \xi\partial_x + \eta\partial_y + \left(\eta' - y'\xi'\right)\partial_{y'} + \left(\eta'' - 2y''\xi' - y'\xi''\right)\partial_{y''}, \tag{13}$$

to (11) and determine ξ and η by separating

$$\Gamma^{[2]} y''_{|_{y''=0}} = 0 \tag{14}$$

by powers of y'. If instead of (12) one writes [**54**] [24ff]

$$\Gamma = \xi\partial_x + \eta\partial_y \tag{15}$$

without any specification of the variable dependence in ξ and η, (13) and (14) still apply, but we can no longer apply the simple rules that $\xi' = \partial\xi/\partial x + y'\partial\eta/\partial y$ *etc.* The application of (13) to (14) gives

$$\eta'' = y'\xi'' \tag{16}$$

which we may integrate by parts to obtain

$$\eta' = B + y'\xi', \tag{17}$$

$$\eta = A + Bx + \int y'\xi'\mathrm{d}x = A + Bx + y'\xi \tag{18}$$

after we take (11) into account for both integrations by parts. Alternately we could write $\xi'' = \eta''/y'$ which leads to

$$\xi = C + Dx + \int \frac{\eta'}{y'}\,\mathrm{d}x. \tag{19}$$

Relations (18) and (19) yield point symmetries only for quite specific choices of ξ and η. Obviously, if we put $\xi = 0$ in (18) and $\eta = 0$ in (19), we obtain

$$\begin{aligned} \Lambda_1 &= \partial_y \qquad & \Lambda_3 &= \partial_x \\ \Lambda_2 &= x\partial_y \qquad & \Lambda_4 &= x\partial_x, \end{aligned}$$

but that is just four of the required eight. Bear in mind that in (18) ξ must be just a function of x and y (resp (19) and η).

Two more symmetries follow easily. If in (16) we put $\xi = 0$, respectively $\eta = 0$, we obtain an equation of the same appearance as (11) for which y is a solution. Thus we have

$$\Lambda_5 = y\partial_y \quad \text{and} \quad \Lambda_6 = y\partial_x.$$

The two remaining symmetries are

$$\Lambda_7 = x^2\partial_x + xy\partial_y \quad \text{and} \quad \Lambda_8 = xy\partial_x + y^2\partial_y.$$

That these symmetries fit into the general form (18) is not obvious. We examine Λ_8; Λ_7 is treated in the same way. If ξ and η are given by

$$\xi = xy \quad \text{and} \quad \eta = y^2,$$

then (16) is automatically satisfied. In the case of (17) we obtain

$$2yy' = y'\left(xy' + y\right) + B \quad \Leftrightarrow \quad B = y'\left(y - xy'\right).$$

We recall that $I_1 = y'$ and $I_2 = y - xy'$ are first integrals of $y'' = 0$. Hence (17) is satisfied. In the case of (18) the substitution of ξ and η gives

$$y^2 = y'xy + Bx + A$$

which, when we take the integrals into account, becomes

$$y = \frac{1}{I_2}\left(Bx + A\right)$$

which is the solution of (11).

We see that, once the integration procedure is commenced, the first integrals and solutions of the equation, which are consequences of integration of the original equation, need to be taken into account. This is a case of integral consequences, as opposed to the more familiar differential consequences.

We observe that a little hoop-jumping has to be done to obtain the Lie point symmetries of (11) using (18/19). In general, given a function ξ (resp η), (18) (resp (19)) gives a nonlocal symmetry of (11). From a casual point of view the celebrated eight Lie point symmetries are lost in a sea of nonlocal symmetries.

It takes little to realise that every differential equation, indeed every differential function, possesses an infinite number of nonlocal symmetries of which a subset may be related to hidden symmetries by some nonlocal transformation. It is an interesting prospect, although scarcely conceivable of realisation, to determine the coordinate system in which the maximal number of hidden symmetries is revealed.

However, there is a subset of ordinary differential equations for which the question may be realistic. Equation (8) belongs to the Painlevé 50 for $p = 1$ and is integrable in terms of analytic functions apart from polelike singularities. We saw that it could be 'easily' transformed to a second-order differential equation of maximal point symmetry. If one examines the Painlevé 50 for symmetry, the results are somewhat mixed in that the number of Lie point symmetries ranges from eight – the 'beloved' equation[2], $y'' + 3yy' + y^3 = 0$ – to zero as for the six Painlevé transcendents. Yet they are all integrable. This suggests that somewhere there is an ordinary differential equation related one by one with a nonlocal transformation to the Painlevé 50 and that this ordinary differential equation has the requisite number of Lie point symmetries, if not more, for solution by quadrature. The resolution of this question presents something of a challenge!

3. GOING DOWN THE WRONG WAY

One of the purposes for determining the Lie point symmetries of the differential equation is to use the symmetries to reduce the order of the equation with the ultimate aim to achieve a performable quadrature. Given a set of Lie point symmetries, $\Gamma_i,\ i = 1, n$, the algebra is determined by the Lie Brackets

$$[\Gamma_i, \Gamma_j]_{LB} = C_k^{ij}\Gamma_k, \quad i, j, k = 1, n, \tag{20}$$

where the C_k^{ij} are the structure constants. To reduce the order of the equation for which the Γ_i are the set of Lie point symmetries one selects some symmetry, determines its zeroth- and first-order invariants and expresses the differential equation in terms of these invariants. The result is a differential equation of order one lower. The symmetry used for the reduction is obviously not relevant to the reduced equation. The fates of the other symmetries depend upon their Lie Brackets with the reducing symmetry. If Γ_r is the reducing symmetry and Γ_a some symmetry, one has that, if

$$[\Gamma_a, \Gamma_r]_{LB} = \lambda\Gamma_r, \tag{21}$$

λ a constant which may be zero, Γ_a becomes a point symmetry of the reduced equation. Otherwise Γ_a becomes a nonlocal symmetry of the reduced equation.

Conventional wisdom is that one wants to keep the symmetries as point symmetries and the choice for Γ_r should be such that the number of symmetries of which (21) applies is optimal. However, a closer look [**25, 17**] at the unfavoured

[2] So termed by the unfortunately late Marc Feix who grew to appreciate the fascinating properties of this equation.

option reveals a greater delicacy in the situation[3]. Suppose that

$$[\Gamma_a,\, \Gamma_r]_{LB} = \Gamma_a \tag{22}$$

(any constant multiplier is absorbed into Γ_r). Without loss of generality we may write Γ_r in the canonical form, ∂_x. If we take $\Gamma_a = \xi(x,y)\partial_x + \eta(x,y)\partial_y$, it follows from (22) that

$$\xi = \mathrm{e}^x f(y) \quad \text{and} \quad \eta = \mathrm{e}^x g(y), \tag{23}$$

where f and g are arbitrary functions. Under the reduction of order using Γ_r the invariants are y and y' so that Γ_a becomes

$$\Sigma_a = \mathrm{e}^x \left\{ g(u)\partial_u + \left[g(u) + v\left(g'(u) - f(u)\right) - v^2 f'(u) \right] \partial_v \right\} \tag{24}$$

and it becomes necessary to express x in terms of u and v as $x = \int \mathrm{d}u/v$ so that we have the nonlocal symmetry

$$\Sigma_a = \exp\left[\int \frac{\mathrm{d}u}{v}\right] \left\{ g(u)\partial_u + \left[g(u) + v(g'(u) - f(u)) - v^2 f'(u) \right] \partial_v \right\}. \tag{25}$$

The nonlocality in Σ_a occurs in the common exponential multiplier and so it is called an exponential nonlocal symmetry.

If instead of (22) one has

$$[\Gamma_a,\, \Gamma_r]_{LB} = \Gamma_b, \tag{26}$$

where Γ_b is anything but Γ_a or Γ_r, reduction by Γ_r produces a nonlocal symmetry in which the nonlocality is not conveniently separated as in the exponential nonlocal symmetry of (25)[4]. Despite the conventional wisdom of reduction by the normal subgroup, reduction using Γ_r when (22) applies is still feasible since, for the second reduction using Σ_a in the associated LAGRANGE's system for the invariants of Σ_a, the exponential terms cancel and one is left with

$$\begin{aligned} \frac{\mathrm{d}u}{g(u)} &= \frac{\mathrm{d}v}{g(u) + v\left(g'(u) - f(u)\right) - v^2 f'(u)} \\ &= \frac{\mathrm{d}v'}{g'(u) + v\left(g''(u) - f'(u)\right) - v^2 f''(u) - v'f' - 2vv'f'} \end{aligned} \tag{27}$$

[3]One notes that in the related matter of the existence of an integrating factor nonlocal symmetries often play a pivotal role [**35**].

[4]One must emphasise that the discussion here relates to scalar ordinary differential equations. In the case of systems of ordinary differential equations a nonlocal symmetry not of exponential form may not present a hindrance to reduction of order. An example of this occurs in one of the three integrable cases of the HÉNON-HEILLES Hamiltonian. The nonlocal component of the symmetry does not play a role in the reduction. This phenomenon has been observed in studies using the last multiplier of JACOBI. In general a system of ordinary differential equations invariant under time translation can have symmetries with the time-coefficient nonlocal without any adverse effect upon reducibility.

and this gives a properly defined pair of equations for the zeroth-order and first-order invariants of Σ_a.

We conclude this Section with an example which illustrates the irony of doing the wrong thing thrice.

The Chazy equation

$$y''' + yy'' - \tfrac{3}{2}y'^2 = 0 \tag{28}$$

has the three Lie point symmetries [**17**]

$$\Gamma_1 = \partial_x, \quad \Gamma_2 = x\partial_x - y\partial_y \quad \text{and} \quad \Gamma_3 = x^2\partial_x + (12 - 2xy)\partial_y \tag{29}$$

which constitute a representation of the nonsolvable algebra $sl(2, R)$. Since

$$[\Gamma_1, \Gamma_2]_{LB} = \Gamma_1, \ [\Gamma_1, \Gamma_3]_{LB} = 2\Gamma_2 \text{ and } [\Gamma_2, \Gamma_3]_{LB} = \Gamma_3, \tag{30}$$

the conventional approach would have us reduce the order of the equation by either Γ_1 or Γ_3 and certainly not Γ_2.

However, we take the unconventional approach. The invariants of Γ_2 are $u = xy$ and $v = x^2y'$ and Γ_1 and Γ_3 become, respectively,

$$\begin{aligned} \Sigma_1 &= \exp\left[-\int \frac{\mathrm{d}u}{u+v}\right]\{u\partial_u + 2v\partial_v\} \\ \Sigma_3 &= \exp\left[\int \frac{\mathrm{d}u}{u+v}\right]\big\{(12-u)\partial_u - 2(u+v)\partial_v\big\}. \end{aligned} \tag{31}$$

As both Σ_1 and Σ_3 are exponential nonlocal, both are available for reduction of order. If we take Σ_1, its invariants are

$$p = \frac{v}{u^2} \quad \text{and} \quad q = \frac{(u+v')v' - 2v}{u^3} \tag{32}$$

and Σ_3 becomes

$$\Delta_3 = -2\exp\left[\int \frac{p\,\mathrm{d}p}{q - 2p^2}\right]\big[(12p+1)\partial_p + 3(6q+p)\partial_q\big]. \tag{33}$$

Since Δ_3 is also exponential nonlocal, we may use it for a final reduction of order. The invariants are

$$r = \frac{72q + 3p + 2}{72\zeta} \text{ and } s = \big[q'(\zeta^2 - 72\zeta r + 1) - 18r\zeta^3 - 54r\zeta + 1\big]/\zeta^2, \tag{34}$$

where $\zeta = 12p + 1$.

Under these successive reductions of order the Chazy equation becomes the simple algebraic equation

$$4s + 3 = 0. \tag{35}$$

Perhaps this is a rare instance of three wrongs making a right!

4. COMPLETE SYMMETRY GROUPS

The concept of a Complete Symmetry Group was introduced by JORGE KRAUSE in 1994 [**26, 27**] in the context of the KEPLER Problem. Essentially he sought the minimum number of symmetries, $\Gamma_i,\ i = 1, N$, such that

$$\Gamma_i^{[2]} \left\{\ddot{\mathbf{x}} - \mathbf{f}(x, \dot{x}, t)\right\}_{|\ddot{\mathbf{x}}=\mathbf{f}} = 0 \tag{36}$$

required that $\mathbf{f}$ be the Newtonian force. In the case of the KEPLER Problem it was necessary to introduce nonlocal symmetries of the form

$$\boldsymbol{\Gamma} = \left(2\textstyle\int \mathbf{r}\mathrm{d}t\right)\partial_t + \mathbf{r}\mathbf{r}.\partial_{\mathbf{r}} \tag{37}$$

to complete the specification. Curiously the symmetries reflecting the conservation of angular momentum were not part of the Complete Symmetry Group of the KEPLER Problem. Subsequently NUCCI [**47**] showed that the nonlocal symmetries in (37) were a natural consequence of the LIE point symmetries of a related system. The story was completed by systematic account of the method of reduction of order [**48**] which is a group theoretic approach to the classical method [**61**] [p 78] to reduce the KEPLER Problem to an harmonic oscillator with a forcing term. A similar line of thinking showed that many integrable orbit problems related to the KEPLER Problem were essentially the same problem as far as the underlying algebraic basis is concerned [**49**]. A number of other problems [**36, 40, 50, 43, 51, 53**] also yielded to the same procedure.

In this sense the KEPLER Problem belongs to the class of problems we mentioned above. The task is to find the coordinate system in which its essential symmetries are point. One must observe that invariance under time translation makes the transition to the new coordinate system possible. Time is not a variable in the new coordinate system which means that in the reversion from the transformed system to the original KEPLER Problem the symmetry of invariance under time translation must be included as an element of the Complete Symmetry Group. Although one may hesitate to term the KEPLER Problem as an esoteric problem since it has been with us for some four centuries and its natural resolution in terms of the ERMANNO-BERNOULLI constants is about to celebrate its tercentenary [**13, 24, 11, 36, 37, 38, 49, 51**], nevertheless its defining differential equation is nonlinear and we all know the pitfalls associated with nonlinear ordinary differential equations.

In a manner of speaking Complete Symmetry Groups were the province of exotic differential equations which needed nonlocal symmetries to provide a complete specification. Fortunately the simple-minded came to provide some basic theory about complete symmetry groups which did not involve nonlocal symmetries [**9**]. The investigation of the KEPLER Problem led to the simple harmonic oscillator. The simple harmonic oscillator is related to the free particle by an easy point transformation and so the completeness of our celestial Kosmos could be explained by the analysis of a particle moving in its own universe.

These theoretical studies tended to exchange nonlocal symmetries for point symmetries by showing that the system considered could be transformed to systems possessing the appropriate number of Lie point symmetries for reduction to quadrature.

In the case of nonintegrable systems the evidence is somewhat thinner[5]. Leach *et al* [**34**] considered the third-order differential equation

$$y''' + y'' + yy' = 0 \tag{38}$$

which arises in general relativity and showed that it was completely specified by the nonlocal symmetries

$$\begin{aligned}
\Lambda_1 &= \partial_x \\
\Lambda_2 &= \left\{\int \frac{\mathrm{d}x}{y'^2\mathrm{e}^x}\right\}\partial_x \\
\Lambda_3 &= \left\{\int \frac{\mathrm{d}x}{y'^2}\left(y - \mathrm{e}^{-x}\int y\mathrm{e}^x\mathrm{d}x\right)\right\}\partial_x \\
\Lambda_4 &= \left\{\int \frac{1}{y'^2\mathrm{e}^x}\left[\int yy'\mathrm{e}^x\mathrm{d}x\right]\mathrm{d}x\right\}\partial_x + \partial_y.
\end{aligned} \tag{39}$$

For general values of the initial conditions (38) is nonintegrable. Indeed a study [**59**] of its Lyapunov exponents suggested that it exhibited chaotic behaviour away from the surface in its three-dimensional space of initial conditions on which it is demonstrably integrable in terms of analytic functions, but subsequent advice was that the solution was simply very badly behaved. The distinction between chaotic behaviour and nonintegrability can at times be visually difficult to discern. The time-dependent oscillator

$$\ddot{q} + \omega^2(t)q = 0 \tag{40}$$

is a case in point.

It has been established that the number of symmetries necessary to specify an ordinary differential equation completely is $n+1$, where n is the order of the equation [**10**].

5. THE FUTURE

We mention three recent developments.

5.1. PARTIAL DIFFERENTIAL EQUATIONS (A)

A partial differential equation with a sufficient number of Lie point symmetries can be specified completely by these symmetries [**45**]. The heat equation

[5] Although we have not detailed the results, one can expect to find the Complete Symmetry Group for an integrable equation rather more easily than for a nonintegrable equation.

and a number of equations arising in Financial Mathematics, such as the BLACK-SCHOLES equation, are so specified. In the absence of a sufficient number of suitable LIE point symmetries one must look to nonlocal symmetries to complete the specification. SENZO MYENI [**6**] has recently devised a method to deal with the problem of determining the nonlocal symmetries required. The class of partial differential equations we consider comprise the general second-order evolution partial differential equation,

$$F(x,u,u_x,u_t,u_{xx})=0. \tag{41}$$

The most important step in this type of analysis for the symmetry group is to identify at what point in the analysis a nonlocal symmetry is required. The guideline is at a point where the arbitrary function found after the application of a particular point symmetry still depends on the variable that one is trying to remove. We illustrate this by an example drawn from the Mathematics of Finance. The equation we consider is a nonlinear partial differential equation for volatility [**12**]

$$u^2u_{xx}+(r-q)xu_x+u_t-(r-q)u=0. \tag{42}$$

The economic model assumes frictionless markets, no arbitrage and that the underlying stock price process is a one-dimensional diffusion starting from a positive value. It also assumes a proportional risk-neutral drift of $r-q$, where $r\geq 0$ is the constant risk-free rate and $q\geq 0$ is the constant dividend yield. The absolute volatility rate is a positive $C^{2,1}$ function $u(x,t)$ of the stock price $x\in(0,\infty)$ and time $t\in(0,T)$, where T is some distant horizon exceeding the longest maturity of the option to be priced.

We rescale the variables to achieve an equation simpler in appearance, *videlicet*

$$u^2u_{xx}+xu_x+u_t-u=0, \tag{43}$$

and it is for this equation that we find the complete symmetry group.

The LIE point symmetries of (43) are

$$\begin{aligned}
\Sigma_1 &= \partial_t\\
\Sigma_2 &= e^t\,\partial_x\\
\Sigma_3 &= \partial_t + x\,\partial_x + u\,\partial_u\\
\Sigma_4 &= t\,\partial_t + tx\,\partial_x + \left(t-\tfrac{1}{2}\right)u\,\partial_u.
\end{aligned} \tag{44}$$

We write (41) as

$$u_t=f(x,t,u,u_x,u_{xx}). \tag{45}$$

Application of $\Sigma_1=\partial_t$ gives

$$u_t=f(x,u,u_x,u_{xx}). \tag{46}$$

The second extension of $\Sigma_2 = e^t\partial_x$ is

$$\Sigma_2^{[2]} = e^t\partial_x + (0)\partial_{u_x} - e^t u_x\,\partial_{u_t} + (0)\partial_{u_{xx}}$$

and its application to (16) yields

$$-u_x = \frac{\partial f}{\partial x} \Rightarrow f = -x\,u_x + h(u_x, u_{xx}, u).$$

This is not good since h still depends explicitly upon u_x. Before applying Σ_2 we become proactive and require that

$$u_t = f(u, xu_x, u_{xx}).$$

Obviously there is a nonlocal symmetry which allows the above constraint. We find it as follows.

The characteristics would be

$$u_t, u, u_{xx}, xu_x$$

which come from the associated Lagrange's system

$$\frac{\mathrm{d}u_x}{-u_x} = \frac{\mathrm{d}u}{0} = \frac{\mathrm{d}u_{xx}}{0} = \frac{\mathrm{d}u_t}{0} = \frac{\mathrm{d}x}{x}.$$

This suggests that the second extension of the nonlocal symmetry, say $\Sigma_5 = \xi\partial_x + \tau\partial_t + \eta\partial_u$, is

$$\Sigma_5^{[2]} = \xi\,\partial_x + \tau\,\partial_t + \eta\,\partial_u + \zeta_x\,\partial_{u_x} + \zeta_t\,\partial_{u_t} + \zeta_{xx}\,\partial_{u_{xx}},$$

where

$$\xi = x, \quad \eta = 0,$$

ζ_x, ζ_t and ζ_{xx} are the extensions of the operator Σ_5 relevant to the derivatives indicated. Specifically they are given by

$$\zeta_x = \frac{\partial\eta}{\partial x} + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\xi}{\partial x}\right]u_x - \frac{\partial\tau}{\partial x}u_t \tag{47}$$

$$\zeta_t = \frac{\partial\eta}{\partial t} + \left[\frac{\partial\eta}{\partial u} - \frac{\partial\tau}{\partial t}\right]u_t - \frac{\partial\xi}{\partial t}u_x \tag{48}$$

$$\zeta_{xx} = \frac{\partial^2\eta}{\partial x^2} + \left[2\frac{\partial^2\eta}{\partial x\partial u} - \frac{\partial^2\xi}{\partial x^2}\right]u_x - \frac{\partial^2\tau}{\partial x^2}u_t - 2\frac{\partial\xi}{\partial x}u_{xx}. \tag{49}$$

The symmetry-generating system is

$$\zeta_{xx} = 0, \quad \zeta_t = 0, \quad \zeta_x = -u_x$$

with solution

$$\tau(x,t) = 2\int\int \frac{u_{xx}}{u_t}\,\mathrm{d}x\,\mathrm{d}t, \tag{50}$$

where arbitrary functions and constants of integration have been omitted.

The nonlocal symmetry is

$$\Sigma_5 = x\,\partial_x + \tau\,\partial_t,$$

where $\tau(x,t)$ is given by (50).

Hence we have the desired result that

$$u_t = f(u, x\,u_x, u_{xx}).$$

We further proceed with the application of the remaining Lie point symmetries. The application of the second extension of $\Sigma_2 = e^t\partial_x$ gives

$$u_t + x\,u_x = h(u, u_{xx}). \tag{51}$$

The application of the second extension of Σ_4 is

$$\begin{aligned}\Sigma_4^{[2]} &= t\,\partial_t + tx\,\partial_x + \left(t - \tfrac{1}{2}\right) u\,\partial_u - \tfrac{1}{2}\,u_x\,\partial_{u_x} \\ &\quad + \left[u + \left(t - \tfrac{3}{2}\right) u_t - x\,u_x\right] \partial_{u_t} - \left(t + \tfrac{1}{2}\right) u_{xx}\,\partial u_{xx}\end{aligned} \tag{52}$$

which leads to

$$u_t + xu_x = u + \gamma u^2 u_{xx},$$

where γ is an arbitrary constant. The use of the nonlocal symmetry[6]

$$\Sigma_6 = \tau\,\partial_t$$

with τ given by

$$\tau(x,t) = 2\int\int \frac{u_{xx}}{u_t u_x}\,\mathrm{d}x\,\mathrm{d}t \tag{53}$$

requires γ to be -1.

The implicit and quasi-implicit complete symmetry group approach not only provides us with the sufficient number of symmetries to form a complete symmetry group but also provides a more direct way to find nonlocal symmetries.

[6]The calculation of which parallels the calculation given in detail above.

5.2. PARTIAL DIFFERENTIAL EQUATIONS (B)

Abraham-Shrauner and Govinder [**8**] have recently shown a new potential source of hidden symmetries for partial differential equations. The symmetries do not come from nonlocal symmetries, but are a result of the possibility that several partial differential equations could lead to the same partial differential equation on reduction of order. We illustrate their method with a simple example ([**8**], equation (2.1)),

$$u_{xxx} + u\left(u_t + cu_x\right) = 0, \tag{54}$$

which possesses the Lie point symmetries

$$\begin{aligned} \Gamma_1 &= \partial_t \\ \Gamma_2 &= \partial_x \\ \Gamma_3 &= 3t\partial_t + (x+2ct)\partial_x \\ \Gamma_4 &= t\partial_t + ct\partial_x + u\partial_u. \end{aligned} \tag{55}$$

We reduce (54) to an ordinary differential equation using the symmetry $c\Gamma_2 + \Gamma_1$ for which the invariants are $w = u$ and $y = x - ct$, *ie* we seek a travelling-wave solution. Note that this is not an invertible point transformation and so preservation of point symmetries is not guaranteed. The reduced equation is simply

$$w_{yyy} = 0 \tag{56}$$

which has the seven Lie point symmetries

$$\begin{array}{ll} \Upsilon_1 = \partial_y & \Upsilon_5 = y\partial_w \\ \Upsilon_2 = \partial_w & \Upsilon_6 = w\partial_w \\ \Upsilon_3 = y^2\partial_w & \Upsilon_7 = \frac{1}{2}y^2\partial_y + yw\partial_w. \\ \Upsilon_4 = y\partial_y & \end{array}$$

Equation (54) is not the only source of (56) under reduction. Equally it can be obtained from

$$u_{xxx} = 0, \quad u_{ttt} = 0, \quad u_{xxt} = 0 \quad \text{and} \quad u_{xtt} = 0, \tag{57}$$

where u is still a function of t and x, by means of the same invariants. For example the first of (57) has an eightfold infinity of Lie point symmetries. They are

$$\begin{array}{ll} \Delta_1 = F_1(t)\partial_x & \Delta_5 = F_5(t)x\partial_x \\ \Delta_2 = F_2(t)\partial_u & \Delta_6 = F_1(t)x\partial_u \\ \Delta_3 = F_3(t)\partial_t & \Delta_7 = F_7(t)u\partial_u \\ \Delta_4 = F_4(t)x^2\partial_u & \Delta_8 = F_8(t)\left(\frac{1}{2}x^2\partial_x + xu\partial_u\right), \end{array}$$

where the $F_i(t)$, $i = 1, 8$, are arbitrary functions. A subset of these symmetries is obtained by making specific choices for the arbitrary functions and in suitable combinations we have

$$\begin{array}{ll} \Sigma_1 = \partial_x & \Sigma_5 = (x-ct)\partial_x \\ \Sigma_2 = \partial_u & \Sigma_6 = (x-ct)\partial_u \\ \Sigma_3 = \partial_t & \Sigma_7 = u\partial_u \\ \Sigma_4 = (x-ct)^2\partial_u & \Sigma_8 = \frac{1}{2}(x-ct)^2\partial_x + (x-ct)u\partial_u \end{array}$$

which reduce to the seven Lie point symmetries of (56).

A similar result applies for the second equation in (57). However, for the third and fourth members of (57) the symmetry Υ_7 is not obtained.

In this example the invariants used for the reduction of order were the same. There is no requirement for this to be the case and Abraham-Shrauner and Govinder discuss the procedure to be used in this more general case.

5.3. WILL IT WORK FOR ORDINARY DIFFERENTIAL EQUATIONS?

We conclude with a very underdeveloped example of the application of the idea of Abraham-Shrauner and Govinder to the area of ordinary differential equations. The third-order equations

$$y''' = 0 \tag{58}$$

$$2y'y''' - 3y''^2 = 0 \tag{59}$$

are reduced to the second-order equation

$$Y'' = 0 \tag{60}$$

(now the prime denotes differentiation with respect to the transformed independent variable, X, which happens to be the same as the original independent variable in this case) by means of the transformations

$$X = x \qquad Y = y' \quad \text{and} \tag{61}$$

$$X = x \qquad Y = y'^{-1/2}, \tag{62}$$

respectively.

For (58) the symmetry generating the transformation (61) is $\Gamma_1 = \partial_y$. The remaining six Lie point symmetries are transformed as

$$\begin{array}{ll} \Gamma_2 = x\partial_y & \Lambda_2 = \partial_Y \\ \Gamma_3 = \frac{1}{2}x^2\partial_y & \Lambda_3 = X\partial_Y \\ \Gamma_4 = y\partial_y & \Lambda_4 = Y\partial_Y \\ \Gamma_5 = \partial_x & \Lambda_5 = \partial_X \\ \Gamma_6 = x\partial_x + y\partial_y & \Lambda_6 = \partial_X \\ \Gamma_7 = x^2\partial_x + 2xy\partial_y & \Lambda_7 = X^2\partial_X + \left(2\int Y\mathrm{d}X\right)\partial_Y \end{array}$$

from which it is evident that we are missing three of the LIE point symmetries of (60). The missing three are

$$\begin{aligned}\Sigma_1 &= X^2\partial_X + XY\partial_Y\\ \Sigma_2 &= Y\partial_X\\ \Sigma_3 &= XY\partial_X + Y^2\partial_Y\end{aligned}$$

and it is a simple calculation to show that they have their origins from the symmetries

$$\begin{aligned}\Delta_1 &= x^2\partial_x + 3\left[\tfrac{1}{2}x^2y' - \tfrac{1}{6}x^3y''\right]\partial_y\\ \Delta_2 &= y'\partial_x + \tfrac{1}{2}y'^2\partial_y\\ \Delta_3 &= xy'\partial_x + \left[2xy'^2 - \tfrac{3}{2}x^2y'y'' + \tfrac{1}{2}x^3y''^2\right]\partial_y\end{aligned}$$

of (58). The symmetry Δ_2 is one of the contact symmetries of (58). The other two are generalised symmetries and have been written as such instead of the nonlocal version since the integration of (58) is trivial.

In the case of (59) and the reduction (62) the LIE point symmetries of the former and their expression as symmetries of (60) are

$$\begin{array}{ll}\Gamma_1 = \partial_x & \Lambda_1 = \partial_X\\ \Gamma_2 = x\partial_x & \Lambda_2 = X\partial_X + \tfrac{1}{2}Y\partial_Y\\ \Gamma_3 = x^2\partial_x & \Lambda_3 = X^2\partial_X + XY\partial_Y\\ \Gamma_5 = y\partial_y & \Lambda_5 = Y\partial_Y\\ \Gamma_6 = y^2\partial_y & \Lambda_6 = Y\int Y^2\mathrm{d}X\partial_Y.\end{array}$$

The symmetry $\Gamma_4 = \partial_y$ is the symmetry used for the transformation (62).

The missing LIE point symmetries are

$$\begin{aligned}\Sigma_1 &= \partial_Y\\ \Sigma_2 &= X\partial_Y\\ \Sigma_3 &= Y\partial_X\\ \Sigma_4 &= XY\partial_X + Y^2\partial_Y.\end{aligned}$$

We note that in both cases the noncartan symmetries of (60) are absent in the reduction of the point symmetries of the third-order equations.

What we do wish to emphasise is that the two reductions gave us a different selection of the LIE point symmetries of (60) which is precisely the same effect reported by ABRAHAM-SHRAUNER and GOVINDER for partial differential equations.

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INTEGRABLE BOEHMIANS, FOURIER TRANSFORMS, AND POISSON'S SUMMATION FORMULA

Dennis Nemzer

The space of integrable Boehmians $\beta_\ell(\mathbb{R})$ contains a subspace which can be identified with $L^1(\mathbb{R})$. The Fourier transform can be defined for each element of $\beta_\ell(\mathbb{R})$. The Fourier transform of an integrable Boehmian is a continuous function which satisfies a growth condition. We investigate the Fourier transform on $\beta_\ell(\mathbb{R})$, and as an application, we extend Poisson's summation formula to the space $\beta_\ell(\mathbb{R})$.

1. INTRODUCTION

Boehmians are classes of generalized functions whose construction is algebraic. The first construction appeared in a paper that was published in 1981 [**6**].

In [**8**], P. Mikusiński constructs a space of Boehmians, $\beta_{L_1}(\mathbb{R})$, in which each element has a Fourier transform. Mikusiński shows that the Fourier transform of a Boehmian satisfies some basic properties, and he also proves an inversion theorem. However, the range of the Fourier transform is not investigated. Also, Mikusiński states that $\beta_{L_1}(\mathbb{R})$ contains some elements which are not Schwartz distributions, but no examples are given. We will address these problems in this paper.

In this note, we will construct a space of Boehmians $\beta_\ell(\mathbb{R})$. The space of integrable functions on the real line can be identified with a proper subspace of $\beta_\ell(\mathbb{R})$. Each element of $\beta_\ell(\mathbb{R})$ has a Fourier transform which is a continuous function and satisfies a growth condition at infinity. Conditions are given which ensure that a given function is the Fourier transform of an element of $\beta_\ell(\mathbb{R})$.

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The space $\beta_\ell(\mathbb{R})$ is slightly less general than the space MIKUSIŃSKI constructs. However, each element of $\beta_\ell(\mathbb{R})$ has local properties similar to those of a continuous function. For example, each Boehmian has a support. Also, as we will see, each element of $\beta_\ell(\mathbb{R})$ satisfies a version of POISSON's summation formula.

This article is organized as follows. Section 2 contains notation and the construction of the space of Boehmians. In Section 3, we construct and investigate the space of integrable Boehmians $\beta_\ell(\mathbb{R})$. Section 4 contains the construction and some known facts about the space of periodic Boehmians. In Section 5, as an application, we prove the POISSON summation formula for integrable Boehmians.

2. PRELIMINARIES

Let $L^1_{loc}(\mathbb{R})$ denote the space of all locally integrable functions on the real line $\mathbb{R}$, and let $\mathrm{D}(\mathbb{R})$ be the subspace of $L^1_{loc}(\mathbb{R})$ of all infinitely differentiable functions with compact support.

For $f \in L^1_{loc}(\mathbb{R})$, let

$$\gamma_n(f) = \int\limits_{|x|\le n} |f(x)|\,\mathrm{d}x, \qquad \text{for } n = 1, 2, \ldots.$$

The separating countable family of seminorms $\{\gamma_n\}$ generate a topology for $L^1_{loc}(\mathbb{R})$. A sequence of locally integrable functions $\{f_n\}$ converges in $L^1_{loc}(\mathbb{R})$ to $f \in L^1_{loc}(\mathbb{R})$ provided that for each p, $\gamma_p(f_n - f) \to 0$ as $n \to \infty$.

A sequence $\varphi_n \in \mathrm{D}(\mathbb{R})$ is called a *delta sequence* provided:

(i) $\int\limits_{-\infty}^{\infty} \varphi_n(x)\,\mathrm{d}x = 1$ for all $n \in \mathbb{N}$,

(ii) $\int\limits_{-\infty}^{\infty} |\varphi_n(x)|\,\mathrm{d}x \le M$ for some constant M and all $n \in \mathbb{N}$,

(iii) $\operatorname{supp}\varphi_n \to \{0\}$ as $n \to \infty$.

A pair of sequences (f_n, φ_n) is called a *quotient of sequences* if $f_n \in L^1_{loc}(\mathbb{R})$ for $n \in \mathbb{N}$, $\{\varphi_n\}$ is a delta sequence, and $f_k * \varphi_m = f_m * \varphi_k$ for all $k, m \in \mathbb{N}$, where $*$ denotes convolution:

$$(f * \varphi)(x) = \int\limits_{-\infty}^{\infty} f(x-u)\varphi(u)\,\mathrm{d}u.$$

Two quotients of sequences (f_n, φ_n) and (g_n, ψ_n) are said to be equivalent if $f_k * \psi_m = g_m * \varphi_k$ for all $k, m \in \mathbb{N}$. A straightforward calculation shows that this is an equivalence relation. The equivalence classes are called Boehmians. The space of all Boehmians will be denoted by $\beta(\mathbb{R})$ and a typical element of $\beta(\mathbb{R})$ will be written as $F = \left[\dfrac{f_n}{\varphi_n}\right]$.

The operations of addition, scalar multiplication, and differentiation are defined as follows:

$$\left[\frac{f_n}{\varphi_n}\right] + \left[\frac{g_n}{\psi_n}\right] = \left[\frac{f_n * \psi_n + g_n * \varphi_n}{\varphi_n * \psi_n}\right], \tag{2.1}$$

$$\alpha\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{\alpha f_n}{\varphi_n}\right], \text{where } \alpha \in \mathbb{C}, \tag{2.2}$$

$$D^k\left[\frac{f_n}{\varphi_n}\right] = \left[\frac{f_n * D^k\varphi_n}{\varphi_n * \varphi_n}\right]. \tag{2.3}$$

If f is a locally integrable function on $\mathbb{R}$, then it can be identified with the Boehmian $\left[\frac{f * \varphi_n}{\varphi_n}\right]$. Thus, we may view $L^1_{loc}(\mathbb{R})$ as a subspace of $\beta(\mathbb{R})$. Likewise, the space of SCHWARTZ distributions [**14**] can be identified with a proper subspace of $\beta(\mathbb{R})$.

For $\psi \in \mathcal{D}(\mathbb{R})$ and $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$, $F * \psi$ is defined as $F * \psi = \left[\frac{f_n * \psi}{\varphi_n}\right]$.

Definition 2.1. *A sequence of Boehmians $\{F_n\}$ is said to be δ-convergent to a Boehmian F, denoted $\delta\text{-}\lim_{n\to\infty} F_n = F$, if there exists a delta sequence $\{\varphi_n\}$ such that $F_n * \varphi_k, F * \varphi_k \in L^1_{loc}(\mathbb{R})$ for all $k, n \in \mathbb{N}$, and for each $k \in \mathbb{N}, F_n * \varphi_k \to F * \varphi_k$ in $L^1_{loc}(\mathbb{R})$ as $n \to \infty$.*

For more on δ-convergence, see [**7**].

3. INTEGRABLE BOEHMIANS

Denote by $L^1(\mathbb{R})$ the space of complex-valued LEBESGUE integrable functions on the real line $\mathbb{R}$. The space of integrable Boehmians will be denoted by $\beta_\ell(\mathbb{R})$. Thus, $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$ provided that $F \in \beta(\mathbb{R})$ and $f_n \in L^1(\mathbb{R}), n \in \mathbb{N}$.

Since each $f \in L^1(\mathbb{R})$ can be identified with $\left[\frac{f * \varphi_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$, we may consider $L^1(\mathbb{R})$ a subspace of $\beta_\ell(\mathbb{R})$. Theorems 3.4 and 3.5 show that the space $\beta_\ell(\mathbb{R})$ is considerably larger than $L^1(\mathbb{R})$. Moreover, Theorem 3.5 may be used to construct an integrable Boehmian which is not a SCHWARTZ distribution.

REMARK. The name integrable Boehmians is usually associated with the space constructed in [**8**]. Since $\beta_\ell(\mathbb{R})$ can be identified with a subspace of this space, we will call elements of $\beta_\ell(\mathbb{R})$, integrable Boehmians.

The FOURIER transform of an $L^1(\mathbb{R})$ function is given by

$$\widehat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-ixt}\,\mathrm{d}t. \tag{3.1}$$

The FOURIER transform can be extended to the space $\beta_\ell(\mathbb{R})$ as follows.

Definition 3.1. *Let* $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. *The Fourier transform of F, denoted by* $\widehat{F}$, *is the function defined for each* $x \in \mathbb{R}$ *by*

$$\widehat{F}(x) = \lim_{n\to\infty} \widehat{f}_n(x). \tag{3.2}$$

The above limit exists, and is independent of the representative. Moreover, the FOURIER transform of a Boehmian satisfies the same basic properties as the classical FOURIER transform of an L^1 function (see [**8**]).

It is not difficult to show that $\widehat{F}$ is continuous on $\mathbb{R}$. That is, $\widehat{F} \in C(\mathbb{R})$. Moreover, as the next theorem will show, $\widehat{F}$ satisfies a growth condition.

Theorem 3.2. *Let* $\theta(x)$ *be a positive increasing function such that*

$$\int_1^\infty \frac{\theta(x)}{x^2}dx = \infty.$$

If $F \in \beta_\ell(\mathbb{R})$, *then* $\liminf_{x\to\infty} e^{-\theta(x)}|\widehat{F}(x)| = 0$.

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. Thus, $F * \varphi_n = f_n \quad (n \in \mathbb{N})$, and hence,

$$\widehat{F}(x)\widehat{\varphi}_n(x) = \widehat{f}_n(x), \tag{3.3}$$

for all $x \in \mathbb{R}$ and $n \in \mathbb{N}$.

Now, suppose that there exist constants $\epsilon > 0$ and $x_0 \in \mathbb{R}$ such that

$$|\widehat{F}(x)| \geq \epsilon e^{\theta(x)}, \tag{3.4}$$

for all $x \geq x_0$.

Thus, by (3.3) and (3.4), for each $n \in \mathbb{N}$,

$$\widehat{\varphi}_n(x) = \mathrm{O}(e^{-\theta(x)}) \quad \text{ as } x \to \infty.$$

Since φ_n has compact support, Theorem XXII in [**5**] implies that $\varphi_n \equiv 0$, for all $n \in \mathbb{N}$.

This contradiction completes the proof of the theorem. □

In the previous theorem, the growth condition for $\widehat{F}$ at infinity can be replaced by an equivalent condition at negative infinity.

The proof of the following lemma is left to the reader.

Lemma 3.3. *Let* $g \in C^{(2)}(\mathbb{R})$ *such that* $g^{(j)} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R})$ *for* $j = 0, 1, 2$. *Then there exists* $f \in L^1(\mathbb{R})$ *such that* $\widehat{f}(x) = g(x)$ *for all* $x \in \mathbb{R}$.

In the above lemma, $C_0(\mathbb{R})$ denotes the space of all continuous functions which vanish at infinity.

Also, a function f is in $C^{(2)}(\mathbb{R})$ provided that f is twice differentiable and $f'' \in C(\mathbb{R})$.

In the next theorem, w is a continuous real-valued function on $\mathbb{R}$ such that

(i) $0 = w(0) \le w(x+y) \le w(x) + w(y)$ for all $x, y \in \mathbb{R}$,

(ii) $\int\limits_{-\infty}^{\infty} \frac{w(x)}{1+x^2}\,\mathrm{d}x < \infty$,

(iii) $w(x) \ge a + b\ln(1+|x|)$, for some real a and positive b and all $x \in \mathbb{R}$.

Theorem 3.4. *Let $g \in C^{(2)}(\mathbb{R})$ such that $g^{(j)}(x) = O(e^{w(x)})$ as $|x| \to \infty$ for $j = 0, 1, 2$. Then there exists $F \in \beta_\ell(\mathbb{R})$ such that $\widehat{F}(x) = g(x)$, $x \in \mathbb{R}$.*

Proof. By Theorem 1.4.1 in [**2**], there exists $\psi \in D(\mathbb{R})$ such that $\int\limits_{-\infty}^{\infty} \psi(x)\,\mathrm{d}x = 1$ and, for each $n \in \mathbb{R}$, there exists a constant $M_n > 0$ such that

$$|\widehat{\psi}(x)| \le M_n e^{-2nw(x)}, \quad x \in \mathbb{R}.$$

For $n \in \mathbb{N}$, define $\psi_n(x) = n\psi(nx)$, $\quad x \in \mathbb{R}$. Then, $\{\psi_n\}$ is a delta sequence and, for each $n \in \mathbb{N}$,

$$|\widehat{\psi_n}(x)| \le M_n e^{-2w(x)}, \quad x \in \mathbb{R}.$$

Now, let $\varphi_n = \psi_n * \psi_n * \psi_n$, $\quad n \in \mathbb{N}$. Thus, $\{\varphi_n\}$ is a delta sequence. Moreover, for $j = 0, 1, 2$ and $n \in \mathbb{N}$,

$$g\widehat{\varphi}_n \in C^{(2)}(\mathbb{R}) \text{ and } (g\widehat{\varphi}_n)^{(j)} \in L^1(\mathbb{R}) \cap C_0(\mathbb{R}).$$

Thus, by Lemma 3.3, for each $n \in \mathbb{N}$, there exists $f_n \in L^1(\mathbb{R})$ such that $\widehat{f_n} = g\widehat{\varphi}_n$. Now,

$$(f_n * \varphi_k)^\wedge = \widehat{f}_n\widehat{\varphi}_k = (g\widehat{\varphi}_n)\widehat{\varphi}_k = (g\widehat{\varphi}_k)\widehat{\varphi}_n = \widehat{f}_k\widehat{\varphi}_n = (f_k * \varphi_n)^\wedge.$$

Thus, $f_n * \varphi_k = f_k * \varphi_n$, for all $n, k \in \mathbb{N}$. Therefore, $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$.

Moreover,

$$\widehat{F}(x) = \lim_{n\to\infty} \widehat{f}_n(x) = \lim_{n\to\infty} g(x)\widehat{\varphi}_n(x) = g(x), \quad x \in \mathbb{R}. \qquad \square$$

A Boehmian F is said to vanish on an open interval (a, b) provided that there exists a delta sequence $\{\varphi_n\}$ such that $F * \varphi_n \in C(\mathbb{R}), n \in \mathbb{N}$, and $F * \varphi_n \to 0$ uniformly on compact subsets of (a, b) as $n \to \infty$. The support of F is the complement of the largest open set on which F vanishes. Every Boehmian with bounded support is an element of $\beta_\ell(\mathbb{R})$.

J. Burzyk [**3**] proved the following Paley-Wiener type theorem.

Theorem 3.5. *Suppose F is a Boehmian such that $\operatorname{supp} F \subseteq [-\sigma, \sigma]$ for some $\sigma \ge 0$. Then $\widehat{F}$ is an entire function. Moreover,*

(i) *For every $\epsilon > 0$, there exists a constant A_ϵ such that*

$$|\widehat{F}(z)| < A_\epsilon e^{(\sigma+\epsilon)|z|} \tag{3.5}$$

for $z \in \mathbb{C}$, and

(ii)

$$\int\limits_{-\infty}^{\infty} \frac{\ln_+ |\widehat{F}(x)|}{1+x^2}\,\mathrm{d}x < \infty. \tag{3.6}$$

Conversely, if an entire function g satisfies conditions (3.5) and (3.6), then it is the FOURIER transform of a Boehmian F whose support is contained in $[-\sigma, \sigma]$.

An Inversion Theorem is given in [**8**]. The next theorem gives another inversion formula, which has the form of the classical inversion formula for L^1 functions.

Theorem 3.6. *Let $F \in \beta_\ell(\mathbb{R})$. Then, $F = \int\limits_{-\infty}^{\infty} e^{ixt}\widehat{F}(t)\,\mathrm{d}t$.*

$$(\textit{That is, } F = \delta\text{-}\lim_{n\to\infty} \textstyle\int_{|t|\le n} e^{ixt}\widehat{F}(t)\,\mathrm{d}t.)$$

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. We may assume that for each $n \in \mathbb{N}$, $f_n, \widehat{f_n} \in L^1(\mathbb{R}) \cap C(\mathbb{R})$. For, if not, notice that $F = \left[\frac{f_n * \varphi_n}{\varphi_n * \varphi_n}\right]$ and $f_n * \varphi_n, (f_n * \varphi_n)^\wedge \in L^1(\mathbb{R}) \cap C(\mathbb{R})$.

Now, for each $n \in \mathbb{N}$, let

$$F_n(x) = \int\limits_{|t|\le n} e^{ixt}\widehat{F}(t)\,\mathrm{d}t, \quad x \in \mathbb{R}.$$

Thus,

$$\begin{aligned}(F_n * \varphi_k)(x) &= \int\limits_{|t|\le n} e^{ixt}\widehat{F}(t)\widehat{\varphi}_k(t)\,\mathrm{d}t \\ &= \int\limits_{|t|\le n} e^{ixt}\widehat{f}_k(t)\,\mathrm{d}t, \text{ for all } n, k \in \mathbb{N} \text{ and } x \in \mathbb{R}.\end{aligned}$$

Therefore, for each k,

$$F_n * \varphi_k \to f_k \quad \text{uniformly as } n \to \infty.$$

That is,

$$\delta\text{-}\lim_{n\to\infty} \textstyle\int_{|t|\le n} e^{ixt}\widehat{F}(t)\,\mathrm{d}t = F. \qquad \square$$

REMARKS. (i) The delta sequences used in [**8**] are more general than the delta sequences used in this paper. The space of integrable Boehmians in [**8**] is larger than $\beta_\ell(\mathbb{R})$. It can be shown that if $g \in C^{(2)}(\mathbb{R})$, then there exists an $F \in \beta_{L_1}(\mathbb{R})$ such that $\widehat{F}(x) = g(x)$, $x \in \mathbb{R}$. However, unlike the space $\beta_{L_1}(\mathbb{R})$ in [**8**], each element of $\beta_\ell(\mathbb{R})$ has local properties such as a support.

(ii) It would be of interest to find necessary and sufficient conditions for a given continuous function to be the Fourier transform of some integrable Boehmian. Since there is no nice necessary and sufficient condition which can be used to determine whether a given continuous function (which vanishes at infinity) is the Fourier transform of an $L^1(\mathbb{R})$ function, this is most likely a difficult problem.

4. PERIODIC BOEHMIANS

Let T denote the unit circle. We make no distinction between a function on T and a 2π-periodic function on $\mathbb{R}$.

In this section, we give a brief introduction to the space of periodic Boehmians $\beta(T)$. The space $\beta(T)$ is quite large. It contains a subspace which can be identified with the space of periodic Schwartz distributions, as well as some elements which can be identified with a subspace of periodic hyperfunctions.

The material in this section will be needed in Section 5. For the proofs of the theorems and for more results on $\beta(T)$, see [**9**, **10**, **11**].

For $f \in L^1_{loc}(\mathbb{R})$, let $\tau_a f(x) = f(x+a)$, $a \in \mathbb{R}$.

The translation operator τ_a can be extended to the space $\beta(\mathbb{R})$.

For $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$, define $\tau_a F = \left[\frac{\tau_a f_n}{\varphi_n}\right], a \in \mathbb{R}$. It is routine to show that $\left[\frac{\tau_a f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$.

The space of periodic Boehmians will be denoted by $\beta(T)$. That is, $F \in \beta(T)$ provided $F \in \beta(\mathbb{R})$ and $\tau_{2\pi} F = F$.

Lemma 4.1. *Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(\mathbb{R})$. Then, $F \in \beta(T)$ if and only if $f_n \in L^1(T)$, for all $n \in \mathbb{N}$.*

For $f \in L^1(T)$, the k^{th} Fourier coefficient is given by

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} \, \mathrm{d}x, \quad k \in \mathbb{Z}. \tag{4.1}$$

Definition 4.2. *Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta(T)$. The k^{th} Fourier coefficient of F is given by*

$$c_k(F) = \lim_{n \to \infty} c_k(f_n). \tag{4.2}$$

The above limit exists, and is independent of the representative.

Theorem 4.3. *Let w be a real-valued even function defined on the integers $\mathbb{Z}$ such that $0 = w(0) \leq w(n+m) \leq w(n) + w(m)$ for all $n, m \in \mathbb{Z}$ and $\sum_{n=1}^{\infty} \frac{w(n)}{n^2} < \infty$. Suppose that the set of positive integers is partitioned into two disjoint sets $\{t_n\}$ and $\{s_n\}$ such that $\sum_{n=1}^{\infty} \frac{1}{t_n} < \infty$. If $\{\xi_n\}$ is a sequence of complex numbers such that $\xi_{\pm s_n} = O(e^{w(s_n)})$ as $n \to \infty$, then there exists a periodic Boehmian F such that $c_n(F) = \xi_n$, $n \in \mathbb{Z}$.*

The next theorem is a stronger version of Theorem 3.5 in [**11**]. Since the proof is similar to that of Theorem 3.5, it is omitted.

Theorem 4.4. *Let $\theta(x)$ be an increasing function such that*

$$\int_1^{\infty} \frac{\theta(x)}{x^2} \,\mathrm{d}x = \infty.$$

Let $\{\lambda_n\}$ be an increasing sequence of positive integers such that $\lim_{n\to\infty} \frac{n}{\lambda_n} = D > 0$. Then, for each $F \in \beta(T)$, $\liminf_{n\to\infty} e^{-\theta(\lambda_n)} |c_{\lambda_n}(F)| = 0$.

By making the appropriate changes, Theorem 4.4 is also valid for a sequence of negative integers $\{\lambda_n\}$.

In the next section, Theorem 4.4 will be used to strengthen Theorem 3.2.

Theorem 4.5. *Let $F \in \beta(T)$. Then, $F = \sum_{k=-\infty}^{\infty} c_k(F) e^{ikx}$.*

$$(\textit{That is, } F = \delta\text{-}\lim_{n\to\infty} \sum_{k=-n}^{n} c_k(F) e^{ikx}.)$$

Remark. By using Theorem 4.3 it is clear that $\beta(T)$ contains a proper subspace which can be identified with the space of periodic Schwartz distributions. Theorem 4.3 also shows that there are Boehmians which are not hyperfunctions. Conversely, by using Theorem 4.4, we see that there are hyperfunctions which are not Boehmians.

5. THE POISSON SUMMATION FORMULA

The importance of the Poisson summation formula is well-known. It has been found to be useful in many areas of mathematics, such as, number theory, differential equations, and signal analysis. For a nice introduction to some applications of the Poisson summation formula, see [**12**].

One form of Poisson's summation formula, for a well-behaved function f, is given by

$$2\pi \sum_{k=-\infty}^{\infty} f(x + 2\pi k) = \sum_{k=-\infty}^{\infty} \widehat{f}(k) e^{ikx}, \tag{5.1}$$

where $\widehat{f}(\xi)=\int\limits_{-\infty}^{\infty} f(x)e^{-i\xi x}\,\mathrm{d}x$.

In this section, see Theorem 5.7, we will present a version of the POISSON summation formula for $\beta_\ell(\mathbb{R})$.

An integrable function does not necessarily satisfy POISSON's summation formula (see [**4**]). However, recall that $L^1(\mathbb{R})$ can be identified with a subspace of $\beta_\ell(\mathbb{R})$. Thus, POISSON's summation formula for integrable Boehmians, Theorem 5.7, is valid for any $L^1(\mathbb{R})$ function.

The *periodization operator* $^{\#}: L^1(\mathbb{R}) \to L^1(T)$ is given by

$$f^{\#}(x)=\sum_{k=-\infty}^{\infty} f(x+2\pi k),\ \text{ for } f\in L^1(\mathbb{R}). \tag{5.2}$$

We will see that the mapping $^{\#}$ can be extended onto the space $\beta_\ell(\mathbb{R})$ by

$$F^{\#}=\left[\frac{f_n^{\#}}{\varphi_n}\right],\ \text{where } F=\left[\frac{f_n}{\varphi_n}\right]\in\beta_\ell(\mathbb{R}). \tag{5.3}$$

Hence, $^{\#}:\beta_\ell(\mathbb{R})\to\beta(T)$.

The proof of the following lemma may be found in [**1**].

Lemma 5.1. *Let $f\in L^1(\mathbb{R})$ and $\{\varphi_n\}$ be a delta sequence. Then*

(i) $2\pi c_p(f^{\#})=\widehat{f}(p)$, *for all* $p\in\mathbb{Z}$;

(ii) $2\pi c_p(f^{\#}*\varphi_n)=\widehat{f}(p)\widehat{\varphi}_n(p)$, *for all $p\in\mathbb{Z}$ and $n\in\mathbb{N}$.*

Lemma 5.2. *Let $\left[\frac{f_n}{\varphi_n}\right]\in\beta_\ell(\mathbb{R})$. Then, $\left[\frac{f_n^{\#}}{\varphi_n}\right]\in\beta(T)$.*

Proof. $2\pi c_p(f_n^{\#}*\varphi_k)=\widehat{f}_n(p)\widehat{\varphi}_k(p)=(f_n*\varphi_k)^\wedge(p)=(f_k*\varphi_n)^\wedge(p)=\widehat{f}_k(p)\widehat{\varphi}_n(p)$ $=2\pi c_p(f_k^{\#}*\varphi_n)$. Thus, $f_n^{\#}*\varphi_k=f_k^{\#}*\varphi_n$, for all $k,n\in\mathbb{N}$.

Therefore, $\left[\frac{f_n^{\#}}{\varphi_n}\right]\in\beta(T)$. □

Since the proof of the following lemma is similar to the proof of Lemma 5.2, it is omitted.

Lemma 5.3. *Let $\left[\frac{f_n}{\varphi_n}\right],\left[\frac{g_n}{\psi_n}\right]\in\beta_\ell(\mathbb{R})$ such that $\left[\frac{f_n}{\varphi_n}\right]=\left[\frac{g_n}{\psi_n}\right]$. Then, $\left[\frac{f_n^{\#}}{\varphi_n}\right]=\left[\frac{g_n^{\#}}{\psi_n}\right]$.*

By Lemmas 5.2 and 5.3, the mapping $^{\#}$ is well-defined and maps $\beta_\ell(\mathbb{R})$ into $\beta(T)$.

Lemma 5.4. *Let $F\in\beta_\ell(\mathbb{R})$. Then, $2\pi c_k(F^{\#})=\widehat{F}(k)$, $k\in\mathbb{Z}$.*

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. Then,

$$2\pi c_k(F^{\#}) = 2\pi \lim_{n\to\infty} c_k(f_n^{\#}) = \lim_{n\to\infty} \widehat{f}_n(k) = \widehat{F}(k). \qquad \square$$

By applying Lemma 5.4 to Theorem 4.4, an improvement of Theorem 3.2 is obtained.

Theorem 5.5. *Let $\theta(x)$ be an increasing function such that*

$$\int_1^\infty \frac{\theta(x)}{x^2}\,\mathrm{d}x = \infty.$$

Let $\{\lambda_n\}$ be an increasing sequence of positive integers such that $\lim\limits_{n\to\infty} \frac{n}{\lambda_n} = D > 0$. Then, for each $F \in \beta_\ell(\mathbb{R})$, $\liminf\limits_{n\to\infty} e^{-\theta(\lambda_n)}|\widehat{F}(\lambda_n)| = 0$.

Lemma 5.6. *Let $F \in \beta_\ell(\mathbb{R})$. Then, $F^{\#} = \sum\limits_{k=-\infty}^{\infty} F(x+2\pi k)$.*

(That is, $F^{\#} = \delta\text{-}\lim\limits_{n\to\infty} \sum\limits_{|k|\le n} \tau_{2\pi k}F$.)

Proof. Let $F = \left[\frac{f_n}{\varphi_n}\right] \in \beta_\ell(\mathbb{R})$. Then, for each $p \in \mathbb{N}$, $\varphi_p * \sum\limits_{|k|\le n} \tau_{2\pi k}F = \sum\limits_{|k|\le n} \tau_{2\pi k} f_p \to f_p^{\#}$ in $L^1_{loc}(\mathbb{R})$ as $n \to \infty$ (see [**1**], Lemma 1). That is, $\delta\text{-}\lim\limits_{n\to\infty} \sum\limits_{|k|\le n} \tau_{2\pi k}F = F^{\#}$. $\square$

The following is the POISSON summation formula for integrable Boehmians.

Theorem 5.7. *Let $F \in \beta_\ell(\mathbb{R})$. Then,*

$$2\pi \sum_{k=-\infty}^{\infty} F(x+2\pi k) = \sum_{k=-\infty}^{\infty} \widehat{F}(k)e^{ikx}. \tag{5.4}$$

Proof.

$$\sum_{k=-\infty}^{\infty} \widehat{F}(k)e^{ikx} = \sum_{k=-\infty}^{\infty} 2\pi c_k(F^{\#})e^{ikx} = 2\pi F^{\#} = 2\pi \sum_{k=-\infty}^{\infty} F(x+2\pi k). \qquad \square$$

Corollary 5.8. *Let $f \in L^1(\mathbb{R})$ and $\{\varphi_n\}$ be a delta sequence. Then,*

$$2\pi \sum_{k=-\infty}^{\infty} f(x+2\pi k) = \lim_{n\to\infty} \sum_{k=-\infty}^{\infty} \widehat{\varphi}_n(k)\widehat{f}(k)e^{ikx} \tag{5.5}$$

in $L^1_{loc}(\mathbb{R})$.

It can be shown that if $\delta\text{-}\lim\limits_{n\to\infty} F_n = F$ and $F_n = 0$ on (a,b) for all $n \in \mathbb{N}$, then $F = 0$ on (a,b). Combining this with POISSON's summation formula and Theorem 3.5, we obtain the following.

Let g be an entire function satisfying the following conditions.

(i) For each $\epsilon > 0$, there exists a constant A_ϵ such that $|g(z)| < A_\epsilon e^{(\sigma+\epsilon)|z|}$, for all $z \in \mathbb{C}$ (for some $0 \le \sigma < \pi$).

(ii) $\int\limits_{-\infty}^{\infty} \frac{\ln_+ |g(x)|}{1+x^2}\,\mathrm{d}x < \infty$.

Then, $\sum\limits_{n=-\infty}^{\infty} g(n)e^{inx} \in \beta(T)$. Moreover, $\sum\limits_{n=-\infty}^{\infty} g(n)e^{inx} = 0$ on $\sigma < |x| < 2\pi - \sigma$.

For example, the Mittag-Leffler function $E_\alpha(z) = \sum\limits_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n+1)}$ (where $\alpha > 0$ and $\Gamma(x) = \int\limits_0^\infty e^{-t}t^{x-1}\,\mathrm{d}t$) is an entire function of order $1/\alpha$.

Thus, for $\alpha > 1$, $\sum\limits_{n=-\infty}^{\infty} E_\alpha(n)e^{inx} \in \beta(T)$ and $\sum\limits_{n=-\infty}^{\infty} E_\alpha(n)e^{inx} = 0$ on $0 < |x| < 2\pi$.

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DENSITY OF SMOOTH BOOLEAN FUNCTIONS

Joel Ratsaby

The Sauer-Shelah lemma has been instrumental in the analysis of algorithms in many areas including learning theory, combinatorial geometry, graph theory. Algorithms over discrete structures, for instance, sets of Boolean functions, often involve a search over a constrained subset which satisfies some properties. In this paper we study the complexity of classes of functions h of finite VC-dimension which satisfy a local "smoothness" property expressed as having long repeated values around elements of a given sample. A tight upper bound is obtained on the density of such classes. It is shown to possess a sharp threshold with respect to the smoothness parameter.

1. INTRODUCTION

Let $[n] = \{1, \ldots, n\}$ and denote by $2^{[n]}$ the class of all 2^n functions $h : [n] \to \{0, 1\}$. Let $\mathcal{H}$ be a class of functions and for a set $A = \{x_1, \ldots, x_k\} \subseteq [n]$ denote by $h_{|A} = [h(x_1), \ldots, h(x_k)]$. The *trace* of $\mathcal{H}$ on A is defined as $\mathrm{tr}_A(\mathcal{H}) = \{h_{|A} : h \in \mathcal{H}\}$. Define the *density function* $\rho_{\mathcal{H}}(k)$ of $\mathcal{H}$ as

$$\rho_{\mathcal{H}}(k) = \max_{A \subseteq [n]:|A|=k} \frac{|\mathrm{tr}_A(\mathcal{H})|}{2^k}.$$

The Vapnik-Chervonenkis dimension of $\mathcal{H}$, denoted as $\mathrm{VC}(\mathcal{H})$, is defined as the largest k such that $\rho_{\mathcal{H}}(k) = 1$. The following well known result obtained by [**15, 12, 13**] states that if $\mathrm{VC}(\mathcal{H}) < n$, then $\rho_{\mathcal{H}}(n)$ decreases at a rate of $O\left(\dfrac{n^{VC(\mathcal{H})}}{2^n}\right)$.

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Lemma 1. *For any* $1 \le d < n$ *let*

$$\mathbb{S}(n,d) = \sum_{k=0}^{d} \binom{n}{k}.$$

Then

$$\max_{\mathcal{H} \subset 2^{[n]} : VC(\mathcal{H}) = d} \rho_{\mathcal{H}}(n) = \frac{\mathbb{S}(n,d)}{2^n}.$$

Aside of being an interesting combinatorial result (see Chapter 17 in [**4**]), Lemma 1 has been instrumental in analysis of algorithms in statistical learning theory [**14**], combinatorial geometry [**10**], graph theory [**9, 3**] and in the theory of empirical processes [**11**]. In many problems which involve the analysis of discrete classes of structures, for instance, sets of Boolean functions, a search for some optimal element (target) in this set is employed based on an algorithm which uses available partial information, for instance in the form of a sample. This information effectively induces a smaller class of possible functions. The estimation of the density of such a class is important for analyzing the accuracy and the convergence properties of the algorithm. In this paper we study the density of finite VC-dimension classes of Boolean functions which are locally-smooth, i.e., have a repeated value over subsets of consecutive elements of $[n]$. In practice, this type of property is easy to measure and is a typical form of prior knowledge about the unknown target function.

Formally, such classes may be introduced by defining the following measure: for $h : [n] \to \{0,1\}$, $x \in [n]$ and $y \in \{0,1\}$ let the *width* $\omega_h(x,y)$ of h at x with respect to y be the largest $0 \le a \le n$ such that $h(z) = y$ for all $x - a \le z \le x + a$; if no such a exists then let $\omega_h(x,y) = -1$. Denote by $\Xi = [n] \times \{0,1\}$. For a sample $\zeta_\ell = \{(x_i, y_i)\}_{i=1}^{\ell} \in \Xi^\ell$, define by $\omega_{\zeta_\ell}(h) = \min_{1 \le i \le \ell} \omega_h(x_i, y_i)$ the width of h with respect to ζ. For instance, Figure 1 displays a sample $\zeta_2 = \{(x_1, y_1), (x_2, y_2)\}$ and

y					$y_1 = 1$									$y_2 = 0$							
h_1	0	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	0	0	1	0	0
h_2	1	1	1	1	1	1	1	1	1	1	0	0	0	0	0	0	0	0	1	1	0
$[n]$	1	2	.	.	x_1	.	.	.	.	.	.	.	.	x_2	.	.	.	.	.	.	n

Figure 1: $\omega_{\zeta_2}(h_1) = \omega_{\zeta_2}(h_2) = 3$

two functions h_1, h_2 which have a width of 3 with respect to ζ_2. The classes of Boolean functions on $[n]$ which we study have a constraint on the width, i.e.,

$$\mathcal{H}_N(\zeta_\ell) = \{h \in \mathcal{H} : \omega_{\zeta_\ell}(h) > N\}, \ N \ge 0 \tag{1}$$

where $\zeta_\ell = \{(x_i, y_i)\}_{i=1}^{\ell} \in \Xi^\ell$ is a given sample. In this paper we obtain tight bounds (in the form of Lemma 1) on the density of such a class. As it turns out, the bounds have sharp thresholds with respect to the width parameter value. In subsequent sections we investigate this in detail. For a function $h : [n] \to \{0, 1\}$ let the *difference* function be defined as

$$\delta_h(x) = \begin{cases} 1 & \text{if } h(x-1) = h(x) \\ 0 & \text{otherwise} \end{cases}$$

where we assume that any h satisfies $h(0) = 0$ (see Figure 2). Define

$$\mathcal{D}_{\mathcal{H}} \equiv \{\delta_h : h \in \mathcal{H}\}, \tag{2}$$

or for brevity we write $\mathcal{D}$. It is easy to see that the class $\mathcal{D}$ is in one-to-one correspondence with $\mathcal{H}$. For $N \geq 0$ and any sample ζ, if $\omega_h(x, y) \leq N$ for $(x, y) \in \zeta$ then the corresponding δ_h has $\omega_{\delta_h}(x, 1) \leq N$. So in order to obtain estimates on

h	1	1	1	1	1	0	0	0	0	1	1	1	0	0	0	0	1	0	1	0	0	0
δ_n	0	1	1	1	1	0	1	1	1	0	1	1	0	1	1	1	0	0	0	0	1	1
$[n]$	1	2	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	.	n

Figure 2: h and the corresponding δ_h

the cardinality of classes $\mathcal{H}_N(\zeta)$, it suffices to estimate the cardinality of the corresponding difference classes $\mathcal{D}_N(\zeta_+)$, defined based on $\zeta_+ = \{(x_i, 1) : (x_i, y_i) \in \zeta, 1 \leq i \leq \ell\}$, which turns out to be simpler. We denote by $\mathrm{VC}_\Delta(\mathcal{H})$ the VC-dimension of the difference class $\mathcal{D} = \{\delta_h : h \in \mathcal{H}\}$ and use it to characterize the complexity of $\mathcal{H}$ (it is straightforward to show that $\mathrm{VC}_\Delta(\mathcal{H}) \leq c\mathrm{VC}(\mathcal{H})$ for a small positive absolute constant c). Henceforth we use d as a parameter value of $VC_\Delta(\mathcal{H})$.

The remaining parts of the paper are organized as follows: in Section 2 we state the main results, Section 3 contains the lemmas used for proving the first two results and the sketches of the proof of the remaining results.

2. MAIN RESULTS

The first result concerns classes of functions constrained by an upper bound on the width. For any class $\mathcal{H}$ of binary functions on $[n]$ define

$$\mathcal{H}_N = \{h \in \mathcal{H} : \omega_h(x, h(x)) \leq N, x \in [n]\},\ N \geq 0 \tag{3}$$

where, as for $\mathcal{H}_N(\zeta_\ell)$ in (1), the dependence of $\mathcal{H}_N$ on $\mathcal{H}$ is left implicit.

Theorem 1. *Let* $1 \leq d \leq n$ *and* $N \geq 0$. *Then*

$$\max_{\mathcal{H} \subset 2^{[n]} : VC_\Delta(\mathcal{H}) = d} \rho_{\mathcal{H}_N}(n) = \frac{\beta_d^{(N)}(n)}{2^n} \tag{4}$$

where $\beta_d^{(N)}(n)$ *is defined in Lemma* 3.

The proof follows from Lemma 4 in Section 3.1.

If we define the threshold of $\beta_d^{(N)}(n)$ as the point N^* at which $\beta_d^{(N)}(n)$ reaches half of its maximal value then $\beta_d^{(N)}(n)$ has a sharp transition at N^*; an example is displayed in Figure 3. The next result states an estimate for N^*.

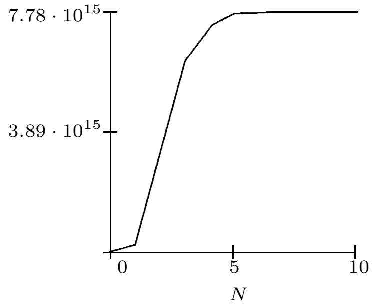

Figure 3 $\beta_{20}^{(N)}(60)$

Theorem 2. *Let $0 < \alpha < 1/2$ and $d = d_n = \alpha n$. Then for large n, N^* is approximated by $c \ln d$ for some c dependent on α.*

The proof follows from Lemma 6 in Section 3.2. The next two results concern classes of functions with a lower-bound on the width as defined in (1).

Theorem 3. *Let $1 \leq d, \ell \leq n$ and $N \geq 0$. Then*

$$\max_{\mathcal{H} \subset 2^{[n]}, \zeta \in \Xi^\ell : VC_\Delta(\mathcal{H}) = d} \rho_{\mathcal{H}_N(\zeta)}(n) = \frac{\mathbb{S}(n - \ell - 2N - 1, d)}{2^n} \tag{5}$$

which is bounded from above by $(1 + e^{-(\ell+2N+1)/n}\mathbb{S}(n, d))2^{-n}$.

The proof is in Section 3.3.

Next, consider an extremal case where the width of h is larger than N *only* on elements of ζ, for all $h \in \mathcal{H}_N(\zeta)$. In this case the class is defined as

$$\mathcal{H}_N^*(\zeta) = \{h \in \mathcal{H} : \omega_h(x, h(x)) > N \text{ iff } (x, h(x)) \in \zeta\}, \ N \geq 0.$$

This type of class arises in certain applications where given a sample ζ an algorithm obtains a solution, i.e., a binary function, which maximizes the width on ζ.

Theorem 4. *Let $1 \leq d, \ell \leq n$ and $N \geq 0$. Then*

$$\max_{\mathcal{H} \subset 2^{[n]}, \zeta \in \Xi^\ell : VC_\Delta(\mathcal{H}) = d} \rho_{\mathcal{H}_N^*(\zeta)}(n) = \frac{\beta_d^{(N)}(n - \ell - 2N - 1)}{2^n} \tag{6}$$

where

$$\beta_d^{(N)}(n - \ell - 2N - 1) \leq 3e^{-e^{-(2N+1)}} \left(1 + e^{-(\ell+2N+1)/n}\mathbb{S}(n, d)\right). \tag{7}$$

Its maximum value with respect to N is approximated by

$$N' = (\ln(n) - 1)/2. \tag{8}$$

The sketch of the proof is in Section 3.4.

Comparing (6) against (5) then N' is a critical point where, roughly, only when $N \leq N'$ the bound on the extremal class $\mathcal{H}_N^*(\zeta)$ is smaller than the bound on $\mathcal{H}_N(\zeta)$ while for $N > N'$ they are approximately equal. An example of their ratio is shown in Figure 4.

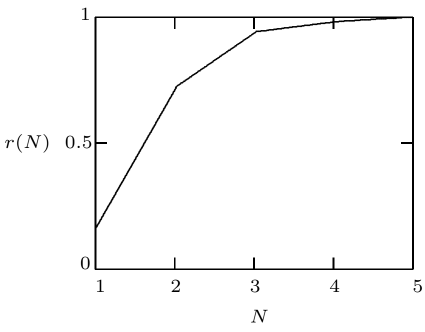

Figure 4:
$r(N) = \beta_d^{(N)}(n' - 2N)/\mathbb{S}(n' - 2N, d)$
for $n' = n - \ell - 1$, $n = 300$, $d = 20$,
$\ell = 20$, $N' \approx 3$

3. TECHNICAL WORK

We start with several lemmas used in proving the first Theorem.

3.1. LEMMAS FOR THEOREM 1

Let $\binom{n}{k}$ denote the following function

$$\binom{n}{k} = \begin{cases} n!/(k!(n-k)!) & \text{if } 0 \le k \le n \\ 0 & \text{otherwise.} \end{cases}$$

Let $\mathbb{I}(E)$ denote the indicator function which equals 1 if the expression E is true and 0 otherwise.

Lemma 2. *For any integer n, $\nu \ge 0$, $m \le n$, define the following*:

$$w_{m,\nu}(n) = \begin{cases} 0 & \textit{if } n < 0 \\ \mathbb{I}(n = 0) & \textit{if } m = 0 \textit{ or } \nu = 0 \\ \displaystyle\sum_{i=0,\nu+1,2(\nu+1),\ldots}^{n} (-1)^{i/(\nu+1)} \binom{m}{i/(\nu+1)} \binom{n-i+m-1}{n-i} & \textit{if } m \ge 1. \end{cases}$$

Then for a nonnegative integer n, the number of standard (one-dimensional) ordered partitions of n into m parts each no larger than ν is equal to $w_{m,\nu}(n)$.

Proof. The generating function (g.f.) for $w_{m,\nu}(n)$ is

$$W(x) = \sum_{n \ge 0} w_{m,\nu}(n) x^n = \left(\frac{1 - x^{\nu+1}}{1 - x}\right)^m.$$

When $m = 0$ or $\nu = 0$ the only non-zero coefficient is of x^0 and it equals 1 so $w_{m,\nu}(n) = \mathbb{I}(n = 0)$. Let $T(x) = (1 - x^{\nu+1})^m$ and $S(x) = \left(\frac{1}{1-x}\right)^m$. Then

$$T(x) = \sum_{i=0}^{m} (-1)^i \binom{m}{i} x^{i(\nu+1)}$$

which generates the sequence $t_\nu(n) = \binom{m}{n/(\nu+1)}(-1)^{n/(\nu+1)}\mathbb{I}\,(n \bmod (\nu{+}1) = 0)$.

Similarly, for $m \geq 1$, it is easy to show $S(x)$ generates $s(n) = \binom{n+m-1}{n}$. The product $W(x) = T(x)S(x)$ generates their convolution $t_\nu(n) * s(n)$, namely,

$$w_{m,\nu}(n) = \sum_{i=0,\nu+1,2(\nu+1),\ldots,}^{n} (-1)^{i/(\nu+1)} \binom{m}{i/(\nu+1)} \binom{n-i+m-1}{n-i}. \qquad \square$$

Remark 1. While our interest is in $[n] = \{1, \ldots, n\}$, we allow $w_{m,\nu}(n)$ to be defined on $n \leq 0$ for use by Lemma 3.

Remark 2. This expression may alternatively be expressed as

$$w_{m,\nu}(n) = \sum_{k=0}^{m} (-1)^k \binom{m}{k} \binom{n+m-1-k(\nu+1)}{m-1},$$

over $m \geq 1$.

We need two additional lemmas for proving (4) of Theorem 1.

Lemma 3. *Let the integer $N \geq 0$ and consider the class F of all binary-valued functions f on $[n]$, or equivalently, sequences $f = f(1), \ldots, f(n)$, satisfying*: (a) *f has no more than r 1's* (b) *every run of consecutive 1's in f is no longer than $2N+1$, except for a run that starts at $f(1)$ which may be of length $2(N+1)$. Then*

$$|F| = \beta_r^{(N)}(n)$$

where

$$\beta_r^{(N)}(n) \equiv \sum_{k=0}^{r} \sum_{m=1}^{n} c(k, n-k; m, N)$$

with

$$c(k, n-k; m, N) = \binom{n-k}{m-1} \Big(w_{m,2N}(k-m+1) + w_{m-1,2N}(k-m-2N) + w_{m-1,2N}(k-m-2N-1) \Big).$$

Remark 3. Note that when $r \leq 2N+1$, $\beta_r^{(N)}(n) = \mathbb{S}(n, d)$.

Proof. Consider the integer pair $[k, n-k]$, where $n \geq 1$ and $0 \leq k \leq n$. A two-dimensional ordered m-partition of $[k, n-k]$ is an ordered partition into m two-dimensional parts, $[a_j, b_j]$ where $0 \leq a_j, b_j \leq n$ but not both are zero and where $\sum_{j=1}^{m} [a_j, b_j] = [k, n-k]$. For instance, $[2,1] = [0,1]+[2,0] = [1,1]+[1,0] = [2,0]+[0,1]$ are three partitions of $[2,1]$ into two parts (for more examples see [**1**]).

Suppose we add the constraint that only a_1 or b_m may be zero while all remaining

$$a_j, b_k \geq 1, \quad 2 \leq j \leq m,\ 1 \leq k \leq m-1. \tag{9}$$

Denote any partition that satisfies this as *valid*. For instance, let $k = 2$, $m = 3$ then the m-partitions of $[k, n-k]$ are: $\{[0,1][1,1][1,n-4]\}$,$\{[0,1][1,2][1,n-5]\}$, $\ldots$,$\{[0,1][1,n-3][1,0]\}$, $\{[0,2][1,1][1,n-5]\}$, $\{[0,2][1,2][1,n-6]\}, \ldots$, $\{[0,2][1,n-4][1,0]\}$, $\ldots$, $\{[0,n-3][1,1][1,0]\}$. For $[k, n-k]$, let $\mathcal{P}_{n,k}$ be the collection of all valid partitions of $[k, n-k]$.

Let F_k denote all binary functions on $[n]$ which take the value 1 over exactly k elements of $[n]$. Define the mapping $\Pi : F_k \to \mathcal{P}_{n,k}$ where for any $f \in F_k$ the partition $\Pi(f)$ is defined by the following procedure: Start from the first element of $[n]$, i.e., 1. If f takes the value 1 on it then let a_1 be the length of the constant 1-segment, i.e., the set of all elements starting from 1 on which f takes the constant value 1. Otherwise if f takes the value 0 let $a_1 = 0$. Then let b_1 be the length of the subsequent 0-segment on which f takes the value 0. Let $[a_1, b_1]$ be the first part of $\Pi(f)$. Next, repeat the following: if there is at least one more element of $[n]$ which has not been included in the preceding segment, then let a_j be the length of the next 1-segment and b_j the length of the subsequent 0-segment. Let $[a_j, b_j]$, $j = 1, \ldots, m$, be the resulting sequence of parts where m is the total number of parts. Only the last part may have a zero valued b_m since the function may take the value 1 on the last element n of $[n]$ while all other parts, $[a_j, b_j]$, $2 \leq j \leq m-1$, must have $a_j, b_j \geq 1$. The result is a valid partition of $[k, n-k]$ into m parts.

Clearly, every $f \in F_k$ has a unique partition. Therefore Π is a bijection. Moreover, we may divide $\mathcal{P}_{n,k}$ into mutually exclusive subsets V_m consisting of all valid partitions of $[k, n-k]$ having exactly m parts, where $1 \leq m \leq n$. Thus

$$|F_k| = \sum_{m=1}^{n} |V_m|.$$

Consider the following constraint on components of parts:

$$a_i \leq \begin{cases} 2N+1 & \text{if } 2 \leq i \leq m \\ 2(N+1) & \text{if } i = 1. \end{cases} \tag{10}$$

Denote by $V_{m,N} \subset \mathcal{P}_{n,k}$ the collection of valid partitions of $[k, n-k]$ into m parts each of which satisfies this constraint.

Let $F_{k,N} = F \cap F_k$ consist of all functions satisfying the run-constraint in the statement of the lemma and having exactly k ones. If f has no run of consecutive 1's starting at $f(i)$ of length larger than $2N+1$ then there does not exist a segment a_i of length larger than $2N+1$, $i \geq 2$ (and similarly with a run of size $2(N+1)$ starting at $f(1)$). Hence the parts of $\Pi(f)$ satisfy (10) and for any $f \in F_{k,N}$, its unique valid partition $\Pi(f)$ must be in $V_{m,N}$. We therefore have

$$|F_{k,N}| = \sum_{m=1}^{n} |V_{m,N}|. \tag{11}$$

By definition of F it follows that

$$|F| = \sum_{k=0}^{r} |F_{k,N}|. \tag{12}$$

Let us denote by

$$c(k, n-k; m, N) \equiv |V_{m,N}| \tag{13}$$

the number of valid partitions of $[k, n-k]$ into exactly m parts whose components satisfy (10). In order to determine $|F|$ it therefore suffices to determine $c(k, n-k; m, N)$.

We next construct the generating function

$$G(t_1, t_2) = \sum_{\alpha_1 \geq 0} \sum_{\alpha_2 \geq 0} c(\alpha_1, \alpha_2; m, N) t_1^{\alpha_1} t_2^{\alpha_2}. \tag{14}$$

For $m \geq 1$,

$$\begin{aligned} G(t_1, t_2) = (t_1^0 + t_1^1 + \cdots + t_1^{2N+2})(t_2^1 + t_2^2 + \cdots)^{\mathbb{I}(m\geq 2)} \times \\ \times \big((t_1^1 + \cdots + t_1^{2N+1})(t_2^1 + t_2^2 + \cdots)\big)^{(m-2)_+} \times \\ \times (t_1^1 + \cdots + t_1^{2N+1})^{\mathbb{I}(m\geq 2)}(t_2^0 + t_2^1 + \cdots) \end{aligned} \tag{15}$$

where the values of the exponents of all terms in the first and second factors represent the possible values for a_1 and b_1, respectively. The values of the exponents in the middle $m-2$ factors are for the values of a_j, b_j, $2 \leq j \leq m-1$ and those in the factor before last and last are for a_m and b_m, respectively. Equating this to (15) implies the coefficient of $t_1^{\alpha_1} t_2^{\alpha_2}$ equals $c(\alpha_1, \alpha_2; m, N)$ which we seek.

The right side of (15) equals

$$t_1^{m-1} t_2^{m-1} \left(\frac{1}{1-t_2}\right)^m \left(\left(\frac{1-t_1^{2N+1}}{1-t_1}\right)^m + t_1^{2N+1}(1+t_1)\left(\frac{1-t_1^{2N+1}}{1-t_1}\right)^{m-1}\right). \tag{16}$$

Let $W(x) = \left(\dfrac{1-x^{2N+1}}{1-x}\right)^{m-1}$ generate $w_{m-1,2N}(n)$ which is defined in Lemma 2 and denote by $s(n) = \binom{n+m-1}{n}$. So (16) becomes

$$\begin{aligned} \sum_{\alpha_1, \alpha_2 \geq 0} s(\alpha_2) t_2^{\alpha_2+m-1} \Big(& w_{m,2N}(\alpha_1) t_1^{\alpha_1+m-1} \\ & + w_{m-1,2N}(\alpha_1) t_1^{\alpha_1+m+2N}(1+t_1)\Big). \end{aligned} \tag{17}$$

Equating the coefficients of $t_1^{\alpha_1'} t_2^{\alpha_2'}$ in (14) and (17) yields

$$\begin{aligned} c(\alpha_1', \alpha_2'; m, N) = s(\alpha_2' - m + 1)\Big(& w_{m,2N}(\alpha_1' - m + 1) \\ & + w_{m-1,2N}(\alpha_1' - m - 2N) + w_{m-1,2N}(\alpha_1' - m - 2N - 1)\Big). \end{aligned}$$

Replacing $s(\alpha'_2 - m + 1)$ by $\binom{\alpha'_2}{m-1}$, substituting k for α'_1, $n-k$ for α'_2 and combining (11), (12) and (13) yields the result. $\square$

The next lemma extends the result of Lemma 3 to the class $\mathcal{H}_N$ defined in (3).

Lema 4. *Let* $1 \leq d \leq n$ *and* $N \geq 0$. *For any class* $\mathcal{H}$ *with* $VC_\Delta(\mathcal{H}) = d$, *the cardinality of the corresponding class* $\mathcal{H}_N$ *defined in* (3) *is no larger than* $\beta_d^{(N)}(n)$. *This bound is tight.*

Proof. Denote by $\mathcal{D}_N = \{\delta_h : h \in \mathcal{H}_N\}$. Clearly, $|\mathcal{D}_N| = |\mathcal{H}_N|$. Consider any $h \in \mathcal{H}_N$. Since for all $x \in [n]$, $\omega_h(x, h(x)) \leq N$ then the corresponding δ_h in $\mathcal{D}_N$ satisfies the following: every run of consecutive 1's is of length no larger than $2N+1$, except for a run which starts at $x = 1$ whose length may be as large as $2(N+1)$. Let $\mathcal{F}_N$ be the set system corresponding to the class $\mathcal{D}_N$ which is defined as follows:

$$\mathcal{F}_N = \{A_\delta : \delta \in \mathcal{D}_N\},\ A_\delta = \{x \in [n] : \delta(x) = 1\}.$$

Clearly, $|\mathcal{F}_N| = |\mathcal{D}_N|$. Note that the above constraint on δ translates to A_δ possessing the property P_N defined as having every subset $E \subseteq A_\delta$ which consists of consecutive elements $E = \{i, i+1, \ldots, j-1, j\}$ be of cardinality $|E| \leq 2N+1$, except for such an E that contains the element $\{1\}$ which may have cardinality as large as $2(N+1)$. Hence for every element $A \in \mathcal{F}_N$, A satisfies P_N. This is denoted by $A \models P_N$. Let $G_{\mathcal{F}}(k) \equiv \max\{|\{A \cap E : A \in \mathcal{F}_N\}| : E \subseteq [n], |E| = k\}$. The corresponding notion of VC-dimension for a class $\mathcal{F}_N$ of sets is the the so-called *trace number* ([**4**], p.131) and is defined as $tr(\mathcal{F}_N) = \max\{m : G_{\mathcal{F}_N}(m) = 2^m\}$. Clearly, $tr(\mathcal{F}_N) = VC(\mathcal{D}_N) \leq \mathrm{VC}(\mathcal{D}) \equiv \mathrm{VC}_\Delta(\mathcal{H}) = d$ (where $\mathcal{D}$ is defined in (2)).

The proof proceeds as in the proof of Lemma 1 (for instance [**2**], Theorem 3.6) which is based on the shifting method (see [**4**], Ch. 17, Theorem 1 & 4 and also [**8, 6, 5**]). The idea is to transform $\mathcal{F}_N$ into an *ideal* family $\mathcal{F}'_N$ of sets E, i.e., if $E \in \mathcal{F}'_N$ then $S \in \mathcal{F}'_N$ for every $S \subset E$, and such that $|\mathcal{F}_N| = |\mathcal{F}'_N| \leq \beta_d^{(N)}(n)$.

Start by defining the operator T_x on $\mathcal{F}_N$ which removes an element $x \in [n]$ from every set $A \in \mathcal{F}_N$ provided that this does not duplicate any existing set. It is defined as follows:

$$T_x(\mathcal{F}_N) = \{A \setminus \{x\} : A \in \mathcal{F}_N\} \cup \{A \in \mathcal{F}_N : A \setminus \{x\} \in \mathcal{F}_N\}.$$

Consider now

$$\mathcal{F}'_N = T_1\big(T_2(\cdots T_n(\mathcal{F}_N)\cdots)\big)$$

and denote the corresponding function class by $\mathcal{D}'_N$. Clearly, $|\mathcal{D}'_N| = |\mathcal{F}'_N|$.

We have $|\mathcal{F}'_N| = |\mathcal{F}_N|$ since the only time that the operator T_x changes an element A into a different set $A^* = T_x(A)$ is when A^* does not already exist in the class so no additional element in the new class can be created. It is also clear that for all $x \in [n]$, $T_x(\mathcal{F}'_N) = \mathcal{F}'_N$ since for each $E \in \mathcal{F}'_N$ there exists a G that differs from it on exactly one element hence it is not possible to remove any element $x \in [n]$ from all sets without creating a duplicate. Applying this repeatedly implies that $\mathcal{F}'_N$ is an ideal. Furthermore, since for all $A \in \mathcal{F}_N$, $A \models P_N$ then removing an element x from A still leaves $A \setminus \{x\} \models P_N$. Hence for all $E \in \mathcal{F}'_N$ we have $E \models P_N$.

From Lemma 3 ([**4**], p.133) we have $G_{\mathcal{F}'_N}(k) \leq G_{\mathcal{F}_N}(k)$, for all $1 \leq k \leq n$. Since $tr(\mathcal{F}_N) \leq d$ then $tr(\mathcal{F}'_N) \leq d$. Together with $\mathcal{F}'_N$ being an ideal it follows that for all $E \in \mathcal{F}'_N$, $|E| \leq d$. For all $E \in \mathcal{F}'_N$, $E \models P_N$ hence the corresponding class $\mathcal{D}'_N$ satisfies the following: for all $\delta \in \mathcal{D}'_N$, δ has at most d 1's and every run of consecutive 1's is of length no larger than $2N+1$ except possibly for a run which starts at $x = 1$ which may be as large as $2(N+1)$. By Lemma 3 above, we therefore have $|\mathcal{D}'_N| \leq \beta_d^{(N)}(n)$. We conclude that $|\mathcal{H}_N| = |\mathcal{D}_N| = |\mathcal{F}_N| = |\mathcal{F}'_N| = |\mathcal{D}'_N|$ and hence $|\mathcal{H}_N| \leq \beta_d^{(N)}(n)$. This bound is tight since consider $\mathcal{H}^*$ whose corresponding class $\mathcal{D}^*$ has *all* functions on $[n]$ with at most d 1's. Clearly, $\mathrm{VC}_\Delta(\mathcal{H}^*) = \mathrm{VC}(\mathcal{D}^*) = d$. The cardinality of $\mathcal{H}^*_N$ equals that of $\mathcal{D}^*_N$ which consists of all $\delta \in \mathcal{D}^*$ that satisfy the above condition on runs of 1's. Clearly, $|\mathcal{D}^*_N| = \beta_d^{(N)}(n)$. □

REMARK 4. As indicated in Remark 3, when N is greater than $(d-1)/2$ the bound $\beta_d^{(N)}(n)$ is as in Lemma 1 and hence the effect of N is void. It turns out that this starts to happen at a much smaller value of N (see Remark 5).

In the following section we study the function $\beta_d^{(N)}(n)$ with respect to N.

3.2. LEMMAS FOR THEOREM 2

We start with a lemma that estimates $c(k, n-k; m, N)$ (defined in (9) which is the number of two-dimensional valid ordered m-partitions of $[k, n-k]$ satisfying (10) where a valid partition is defined according to (9).

Lemma 5. *For $n \geq k \geq m-1 \geq 1$ we have*

$$c(k, n-k; m, N) = b_1 \,(1 + b_2 \alpha)\,(1-\alpha)^m \binom{k}{m-1}\binom{n-k}{m-1}$$

for some absolute positive constants $b_1 \leq 1$, $b_2 \leq 2$ and $\alpha = \alpha(N, m, k) \equiv e^{-(2N+1)(m-1)/k}$.

Proof sketch. By definition, from (9) the quantity $c(k, n-k; m, N)$ involves a sum of three terms, $w_{m,2N}(k-m+1)$, $w_{m-1,2N}(k-m-2N-1)$ and $w_{m-1,2N}(k-m-2N)$. Using Remark 2 the first equals

$$w_{m,2N}(k-m+1) = \sum_{l=0}^{m} (-1)^\ell \binom{m}{\ell}\binom{k-\ell(2N+1)}{m-1}. \tag{18}$$

By Lemma 2 we have $w_{m-1,2N}(k-m-2N) \leq w_{m,2N}(k-m-2N)$ and $w_{m-1,2N}(k-m-2N-1) \leq w_{m,2N}(k-m-2N-1)$. We have

$$w_{m,2N}(k-m-2N) = \sum_{\ell=0}^{m} (-1)^\ell \binom{m}{\ell} k - \ell\,(2N+1) - (2N+1)m - 1$$

and similarly for $w_{m,2N}(k-m-2N-1)$. Hence

$$c(k,n-k;m,N) = \binom{n-k}{m-1}\sum_{\ell=0}^{m}(-1)^{\ell}\binom{m}{\ell}\binom{k-\ell(2N+1)}{m-1}\big(1+\epsilon(m,k,N,l)\big)$$

where

$$0 < \epsilon(m,k,N,\ell) \leq \frac{\binom{k-\ell(2N+1)-(2N+1)}{m-1}}{\binom{k-\ell\,(2N+1)}{m-1}} + \frac{\binom{k-\ell\,(2N+1)-2(N+1)}{m-1}}{\binom{k-\ell\,(2N+1)}{m-1}}$$

which for all $0 \leq l \leq m$ is bounded from above by

$$\frac{\binom{k-(2N+1)}{m-1}}{\binom{k}{m-1}} + \frac{\binom{k-2(N+1)}{m-1}}{\binom{k}{m-1}}. \tag{19}$$

Using a standard combinatoric identity it is easy to show that both terms of (19) are bounded from above by $\alpha = \alpha(m,k,N) = \exp\big(-(2N+1)(m-1)/k\big)$. The same argument applied on $\binom{k-\ell\,(2N+1)}{m-1}$ completes the stated result. $\square$

Lemma 6. *Let N^* be the value at which the function $\beta_d^{(N)}(n)$ reaches half of its maximum value. Assume $1 \leq d < n/2$ and denote by $t = 1 + d(n-d)/n$ then N^* is approximated by*

$$\frac{n}{2(n-d)}\ln\left(\frac{2b_2t}{b_2 - t + \sqrt{(b_2+t)^2 - 2tb_2/b_1}}\right)$$

for some absolute positive constants $b_1 \leq 1$, $b_2 \leq 2$.

Remark 5. It follows that for $0 < \alpha < 1/2$, $d = d_n = \alpha n$ then for large n, N^* is approximated by $c \ln d$ for some $c > 0$ dependent on α.

Proof sketch. We seek the solution N^* of the equation

$$\sum_{k=0}^{d}\sum_{m=1}^{n} c(k,n-k;m,N) = \frac{1}{2}\sum_{k=0}^{d}\binom{n}{k}$$

which, using Lemma 5 and a common identity (see [**7**], (5.23)), can be approximated by the solution of

$$\sum_{k=0}^{d}\sum_{m=1}^{n}\binom{k}{m-1}\binom{n-k}{m-1}\left(f(m) - \frac{1}{2}\right) = 0$$

where

$$f(m) = b_1 \left(1 + b_2 e^{-(2N+1)(m-1)/k}\right) \left(1 - e^{-(2N+1)(m-1)/k}\right)^m,$$

$0 < b_1, b_2 \leq 2$. The first sum is approximated as

$$\sum_{m=1}^{n} \binom{k}{m-1}\binom{n-k}{m-1} f(m) \approx f(m^*) \sum_{m=1}^{n} \binom{k}{m-1}\binom{n-k}{m-1} = f(m^*)\binom{n}{k},$$

where $\binom{k}{m-1}\binom{n-k}{m-1}$ peaks at $m = m^* \equiv 1 + k(n-k)/n$. Hence the solution may be approximated by solving

$$\sum_{k=0}^{d} \binom{n}{k} \left(b_1 \left(1 + b_2 e^{-(2N+1)(m^*-1)/k}\right) \left(1 - e^{-(2N+1)(m^*-1)/k}\right)^{m^*} - \frac{1}{2}\right) = 0$$

for N. For $1 \leq d < n/2$, the dominant term is $k = d$. Simple calculus then yields the result. $\square$

3.3. SKETCH OF PROOF OF THEOREM 3

Fix any $(x, y) \in \zeta$. The condition $\omega_h(x, y) > N$ implies that h must have a constant value of y over all elements z, $x - N - 1 \leq z \leq x + N + 1$. For this x, the uniquely corresponding δ_h has a constant value of 1 over the interval $I_N(x) \equiv \{z : x - N \leq z \leq x + N + 1\}$. By definition of $\mathcal{H}_N(\zeta)$ this holds for any $(x, y) \in \zeta$. Denote by $\mathcal{D}_N(\zeta_+) = \{\delta_h : h \in \mathcal{H}_N(\zeta)\}$ where $\zeta_+ = \{x_i : (x_i, y_i) \in \zeta, 1 \leq i \leq \ell\}$. Clearly, $|\mathcal{D}_N(\zeta_+)| = |\mathcal{H}_N(\zeta)|$. Hence we seek an upper bound on $|\mathcal{D}_N(\zeta_+)|$ for any ζ_+ and $\mathcal{H}$ with $\mathrm{VC}_\Delta(\mathcal{H}) = d$.

Let $R(\zeta_+) = \bigcup_{x \in \zeta_+} I_N(x)$. Since for every $\delta \in \mathcal{D}_N(\zeta_+)$, $\delta(z) = 1$ for all $z \in R(\zeta_+)$ then the cardinality of the restriction $\mathcal{D}_N(\zeta_+)_{|R(\zeta_+)}$ of the class $\mathcal{D}_N(\zeta_+)$ on the set $R(\zeta_+)$ equals one. Denote by $R^c(\zeta_+) \equiv [n] \setminus R(\zeta_+)$ then we have

$$|\mathcal{D}_N(\zeta_+)| = |\mathcal{D}_N(\zeta_+)_{|R^c(\zeta_+)}|.$$

Since $\mathrm{VC}(\mathcal{D}_N(\zeta_+)) \leq \mathrm{VC}_\Delta(\mathcal{H}) = d$ then by Lemma 1 it follows that

$$|\mathcal{D}_N(\zeta_+)_{|R^c(\zeta_+)}| \leq \mathbb{S}(|R^c(\zeta_+)|, d). \tag{20}$$

We also have

$$\max\{|R^c(S)| : S \subset [n], |S| = \ell\} = n - \ell - 2N - 1 \tag{21}$$

which is achieved for instance by a set $S' = \{N+2, \ldots, N+\ell+1\}$ with $R(S') = \{2, \ldots, 2(N+1)+\ell\}$. Hence for any ζ_+ as above we have

$$|\mathcal{D}_N(\zeta_+)| \leq \mathbb{S}(n - 2N - \ell - 1, d). \tag{22}$$

Since the bound of Lemma 1 is tight then there exists a class $\mathcal{D}_N(\zeta_+)$ (with a corresponding class $\mathcal{H}_N(\zeta)$) of this size. The first claim of Theorem 3 follows. The right side of (22) may be bounded as in the statement of the theorem using a similar argument as in the proof of Lemma 5. □

3.4. SKETCH OF PROOF OF THEOREM 4

The proof follows that of Theorem 3 up to (20) with $\mathcal{H}^*_N(\zeta)$ instead of $\mathcal{H}_N(\zeta)$. By Theorem 1 we have

$$|\mathcal{D}^*_N(\zeta_+)_{|R^c(\zeta_+)}| \le \beta_d^{(N)}\big(|R^c(\zeta_+)|\big)$$

and from (22) the statement of (6) follows. By the tightness of the bound in Theorem 1 there exists a class $\mathcal{D}^*_N(\zeta_+)$ and hence $\mathcal{H}^*_N(\zeta)$ of this size. We now sketch the proof of the approximation statement of the theorem. Using Lemma 5 we have

$$\text{(23)}\quad \beta_d^{(N)}(n-\ell-2N-1) \le 3\sum_{k=0}^{d}\sum_{m=1}^{n'}\binom{k}{m-1}\binom{n'-k}{m-1}\left(1-e^{-\frac{(2N+1)(m-1)}{k}}\right)^m$$

where $n' = n-\ell-2N-1$. Denote by

$$\mathbb{P}(m) = \frac{\binom{k}{m-1}\binom{n'-k}{m-1}}{\sum_{m=1}^{n'}\binom{k}{m-1}\binom{n'-k}{m-1}}$$

and consider bounding from above the quantity

$$\mathbb{E}\left(1-e^{-\frac{(m-1)(2N+1)}{k}}\right)^m$$

where expectation is taken with respect to $\mathbb{P}$. Using Jensen's inequality, this leads to the following bound on the right side of (23),

$$\text{(24)}\qquad 3\sum_{k=0}^{d}\binom{n'}{k}\left(1-e^{-\frac{(\mu-1)(2N+1)}{k}}\right)$$

where μ is the mean of a random variable with probability distribution $\mathbb{P}$. Solving for the generating function of the sequence $f(n) \equiv \sum_{m\ge 1} m\binom{k}{m-1}\binom{n-k}{m-1}$ we obtain that $f(n) = k\binom{n-1}{k} + \binom{n}{k}$ which then yields

$$\mu = \frac{k(n-k)}{n} + 1.$$

Replacing n by n' above, substituting this for μ in (24) and using the inequality $1-a \le e^{-a}$ which holds for all $a \in \mathbb{R}$ gives (7). Using this estimate of $\beta_d^{(N)}(n-\ell-2N-1))$ we solve for the N' at which it is maximized. Simple calculus yields (8). $\square$

4. CONCLUSIONS

Letting the width of a binary function at x denote the degree to which it is smooth, i.e., constant around x, the paper extends the classical Sauer's lemma to Vapnik-Chervonenkis classes of binary functions which are smooth at elements of a sample. Using a novel approach based on a bijection between a class of such functions and integer partitions, the cardinality of such a class is computed. Tight upper bounds with a dependence on the width parameter N are obtained and shown to exhibit a sharp threshold with respect to N.

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ON ITERATIVE COMBINATION OF BERNSTEIN–DURRMEYER POLYNOMIALS

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The Bernstein–Durrmeyer polynomials

$$M_n(f;t) = (n+1)\sum_{k=0}^{n} p_{n,k}(t)\int_0^1 p_{n,k}(u)f(u)\,\mathrm{d}u,$$

where $p_{n,k}(t) = \binom{n}{k} t^k(1-t)^{n-k}$, $0 \le t \le 1$, defined on $L_B[0,1]$, the space of bounded and integrable functions on $[0,1]$ were introduced by Durrmeyer and extensively studied by Derriennic and several other researchers . It turns out that the order of approximation by these operators is, at best $O(n^{-1})$, however smooth the function may be. In order to improve this rate of approximation we consider an iterative combination $T_{n,k}(f;t)$ of the operators $M_n(f;t)$. This technique of improving the rate of convergence was given by Micchelli who first used it to improve the order of approximation by Bernstein polynomials $B_n(f;t)$. The object of this paper is to study direct theorems in ordinary as well as in simultaneous approximation by the operators $T_{n,k}(f;t)$. We prove that the order of approximation by these operators is $O(n^{-k})$ for sufficiently smooth functions.

1. INTRODUCTION

For $f \in L_B[0,1]$ the operators $M_n(f;t)$ can be expressed as

$$M_n(f;t) = \int_0^1 W_n(u,t)f(u)\,\mathrm{d}u, \tag{1.1}$$

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where $W_n(u,t) = (n+1)\sum\limits_{k=0}^{n} p_{n,k}(t)p_{n,k}(u)$ is the kernel of the operators.

For $m \in \mathbb{N}_0$ (the set of non-negative integers), the m-th order moment for the operators M_n is defined as

$$\mu_{n,m}(t) = M_n\big((u-t)^m;t\big). \tag{1.2}$$

The iterative combination $T_{n,k} : L_B[0,1] \to C^\infty[0,1]$ of the operators $M_n(f;t)$ is defined as

$$T_{n,k}(f;t) = \big(I - (I-M_n)^k\big)(f;t) = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}M_n^r(f;t), \quad k \in \mathbb{N}, \tag{1.3}$$

where $M_n^0 = I$, and $M_n^r = M_n(M_n^{r-1})$ for $r \in \mathbb{N}$.

In Section 2 of this paper we give some definitions and auxiliary results which will be needed to prove the main results. In Section 3 first we establish a Voronovskaja type asymptotic formula and then find the degree of approximation for functions of a given smoothness in ordinary approximation. Subsequently in Section 4 first we show that the operators $T_{n,k}$ possess simultaneous approximation property i.e. the property that the derivatives of the operators $T_{n,k}$ converge to the corresponding order derivatives of $f(x)$ and then extend the results of Section 3 to the case of simultaneous approximation.

2. PRELIMINARIES

In the sequel we shall require the following results:

Lemma 1 [2]. *For the function $\mu_{n,m}(t)$, we have $\mu_{n,0}(t) = 1$, $\mu_{n,1}(t) = \dfrac{1-2t}{n+2}$, and there holds the recurrence relation*

$$\begin{aligned}(n+m+2)\mu_{n,m+1}(t) = t(1-t)\big(\mu'_{n,m}(t) + 2m\mu_{n,m-1}(t)\big)\\ + (m+1)(1-2t)\mu_{n,m}(t), \quad \textit{for } m \geq 1.\end{aligned}$$

Consequently, we have

(i) $\mu_{n,m}(t)$ are polynomials in t of degree m;

(ii) for every $t \in [0,1]$, $\mu_{n,m}(t) = O\Big(n^{-[(m+1)/2]}\Big)$, where $[\beta]$ is the integer part of β.

The m-th order moment for the operator M_n^p is defined as

$$\mu_{n,m}^{[p]}(t) = M_n^p\big((u-t)^m;t\big),$$

$p \in \mathbb{N}$ (the set of natural numbers). We denote $\mu_{n,m}^{[1]}(t)$ by $\mu_{n,m}(t)$.

Lemma 2 [7]. *For the function $p_{n,k}(t)$, there holds the result*

$$t^r(1-t)^r D^r p_{n,k}(t) = \sum_{\substack{2i+j\le r\\ i,j\ge 0}} n^i(k-nt)^j q_{i,j,r}(t)p_{n,k}(t), \tag{2.1}$$

where D^r stands for $\dfrac{\mathrm{d}^r}{\mathrm{d}t^r}$ and $q_{i,j,r}(t)$ are certain polynomials in t independent of n and k.

Lemma 3. *There holds the recurrence relation*

$$\mu_{n,m}^{[p+1]}(t) = \sum_{j=0}^{m}\sum_{i=0}^{m-j}\binom{m}{j}\frac{1}{i!}\,D^i\big(\mu_{n,m-j}^{[p]}(t)\big)\mu_{n,i+j}(t). \tag{2.2}$$

Proof. We can write

$$\begin{aligned}\mu_{n,m}^{[p+1]}(t) &= M_n^{p+1}\big((u-t)^m;t\big)\\ &= M_n\Big(M_n^p\big((u-t)^m;x\big);t\Big) = M_n\Big(M_n^p\big((u-x+x-t)^m;x\big);t\Big)\\ &= \sum_{j=0}^{m}\binom{m}{j}M_n\Big((x-t)^j M_n^p\big((u-x)^{m-j};x\big);t\Big).\end{aligned} \tag{2.3}$$

Since $M_n^p\big((u-x)^{m-j};x\big)$ is a polynomial in x of degree $m-j$, by TAYLOR's expansion, we can write as

$$M_n^p\big((u-x)^{m-j};x\big) = \sum_{i=0}^{m-j}\frac{(x-t)^i}{i!}\,D^i\big(\mu_{n,m-j}^{[p]}(t)\big). \tag{2.4}$$

From (2.3) and (2.4) we get the required result.

Lemma 4. *For $k,\ell\in\mathbb{N}$, there holds $T_{n,k}\big((u-t)^\ell;t\big) = O(n^{-k})$.*

Proof. We apply induction on k. For $k=1$, the result follows from Lemma 1. Assume that it is true for a certain k, then by the definition of $T_{n,k}$ we get

$$\begin{aligned}T_{n,k+1}\big((u-t)^\ell;t\big) &= \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k+1}{r}M_n^r\big((u-t)^\ell;t\big)\\ &= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}M_n^r\big((u-t)^\ell;t\big)\\ &\qquad + \sum_{r=1}^{k+1}(-1)^{r+1}\binom{k}{r-1}M_n^r\big((u-t)^\ell;t\big)\\ &= I_1+I_2,\ \text{say}.\end{aligned}$$

We can write I_1 as

$$I_1 = T_{n,k}\big((u-t)^\ell;t\big). \tag{2.5}$$

Next, by Lemma 3

$$\begin{aligned}
I_2 &= \sum_{r=0}^{k}(-1)^r\binom{k}{r}\mu_{n,\ell}^{[r+1]}(t)\\
&= \mu_{n,\ell}(t) - \sum_{j=1}^{\ell}\sum_{i=0}^{\ell-j}\binom{\ell}{j}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell-j};t\big)\Big)\mu_{n,i+j}(t)\\
&\qquad\qquad - \sum_{i=0}^{\ell}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell};t\big)\Big)\mu_{n,i}(t)\\
&= \mu_{n,\ell}(t) - \sum_{j=1}^{\ell}\sum_{i=0}^{\ell-j}\binom{\ell}{j}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell-j};t\big)\Big)\mu_{n,i+j}(t)\\
&\qquad\qquad - \sum_{i=1}^{\ell}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell};t\big)\Big)\mu_{n,i}(t) - T_{n,k}\big((u-t)^\ell;t\big),
\end{aligned}$$

$$\begin{aligned}
I_2 &= -\sum_{j=1}^{\ell-1}\sum_{i=0}^{\ell-j}\binom{\ell}{j}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell-j};t\big)\Big)\mu_{n,i+j}(t)\\
&\qquad\qquad - \sum_{i=1}^{\ell}\frac{1}{i!}\Big(D^iT_{n,k}\big((u-t)^{\ell};t\big)\Big)\mu_{n,i}(t) - T_{n,k}\big((u-t)^\ell;t\big)
\end{aligned} \tag{2.6}$$

From Lemma 1, (2.5) and (2.6) we get $T_{n,k+1}\big((u-t)^\ell;t\big) = O\big(n^{-(k+1)}\big)$. Thus, the result is proved for all $k \in \mathbb{N}$.

Lemma 5. *For $p \in \mathbb{N}$, $m \in \mathbb{N}_0$ and $t \in [0,1]$, we have*

(2.7). $$\mu_{n,m}^{[p]}(t) = O\big(n^{-[(m+1)/2]}\big).$$

Proof. For $p = 1$, the result follows from Lemma 1. Suppose (2.7) is true for a certain p. Then $\mu_{n,m-j}^{[p]}(t) = O\big(n^{-[(m-j+1)/2]}\big)$, $0 \le j \le m$. Also $\mu_{n,m-j}^{[p]}(t)$ is a polynomial in t of degree $m-j$, therefore, we have

$$D^i\big(\mu_{n,m-j}^{[p]}(t)\big) = O\big(n^{-[(m-j+1)/2]}\big)\ \forall\ 0 \le i \le m-j.$$

Now, applying Lemma 3,

$$\mu_{n,m}^{[p+1]}(t) = \sum_{j=0}^{m}\sum_{i=0}^{m-j} O\big(n^{-[(m-j+1)/2]}\big)\cdot O\big(n^{-[(i+j+1)/2]}\big) = O\big(n^{-[(m+1)/2]}\big).$$

Hence, the lemma is proved by induction on p.

3. ORDINARY APPROXIMATION

Theorem 1. (Voronovskaja type asymptotic formula). *Let* $f \in L_B[0,1]$ *admitting a derivative of order* $2k$ *at a point* $t \in [0,1]$. *Then*

$$\lim_{n\to\infty} n^k\big(T_{n,k}(f;t)-f(t)\big)=\sum_{v=1}^{2k}\frac{f^{(v)}(t)}{v!}\,Q(v,k,t) \tag{3.1}$$

and

$$\lim_{n\to\infty} n^k\big(T_{n,k+1}(f;t)-f(t)\big)=0, \tag{3.2}$$

where $Q(v,k,t)$ *are certain polynomials in* t *of degree* v. *Further, the limits in* (3.1) *and* (3.2) *hold uniformly in* $[0,1]$ *if* $f^{(2k)}(t)$ *is continuous in* $[0,1]$.

Proof. Since $f^{(2k)}(t)$ exists, we can write an expansion of f as:

$$f(u)=\sum_{v=0}^{2k}\frac{f^{(v)}(t)}{v!}\,(u-t)^v+\varepsilon(u,t)(u-t)^{2k}, \tag{3.3}$$

where $\varepsilon(u,t)\to 0$ as $u\to t$ and is bounded and integrable in $[0,1]$. The proof is as follows:

Let $\varepsilon(u,t)=\dfrac{f(u)-\sum\limits_{i=0}^{2k}\dfrac{f^{(i)}(t)}{i!}\,(u-t)^i}{(u-t)^{2k}}$. Then,

$$\begin{aligned}
\lim_{u\to t}\varepsilon(u,t) &= \lim_{u\to t}\frac{f(u)-\left(f(t)+(u-t)f'(t)+\cdots+\dfrac{(u-t)^{2k}}{(2k)!}\,f^{(2k)}(t)\right)}{(u-t)^{2k}}\\
&= \lim_{u\to t}\frac{f^{(2k-1)}(u)-\big(f^{(2k-1)}(t)+(u-t)f^{(2k)}(t)\big)}{2k!(u-t)}\\
&\qquad\qquad\text{(applying L'Hospital's rule successively } (2k-1) \text{ times)}\\
&= \frac{1}{2k!}\lim_{u\to t}\frac{f^{(2k-1)}(u)-f^{(2k-1)}(t)}{u-t}-\frac{f^{(2k)}(t)}{2k!}\\
&= 0.
\end{aligned}$$

Operating by $T_{n,k}$ on both sides of (3.3) we get

$$\begin{aligned}
n^k\big(T_{n,k}(f;t)-f(t)\big) &= n^k\sum_{v=1}^{2k}\frac{f^{(v)}(t)}{v!}\,T_{n,k}\big((u-t)^v;t\big)+n^kT_{n,k}\big(\varepsilon(u,t)(u-t)^{2k};t\big).\\
&= I_1+I_2,\quad \text{say.}
\end{aligned}$$

Making use of Lemma 4, we obtain

$$I_1 = \sum_{v=1}^{2k} \frac{f^{(v)}(t)}{v!} Q(v,k,t) + o(1),$$

where $Q(v,k,t)$ is the coefficient of n^{-k} in $T_{n,k}\big((u-t)^v;t\big)$.

Since $\varepsilon(u,t) \to 0$ as $u \to t$, for a given $\varepsilon' > 0$ we can find a $\delta > 0$ such that $|\varepsilon(u,t)| < \varepsilon'$ whenever $0 < |u-t| < \delta$ and for $|u-t| \geq \delta$, $|\varepsilon(u,t)| \leq K$ for some $K > 0$. Suppose $\chi(u)$ is the characteristic function of the interval $(t-\delta, t+\delta)$, then

$$\begin{aligned}|I_2| &= n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r\big(|\varepsilon(u,t)|(u-t)^{2k}\chi(u);t\big)\\ &\qquad + n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r\Big(|\varepsilon(u,t)|(u-t)^{2k}\big(1-\chi(u)\big);t\Big)\\ &= I_3 + I_4, \text{ say.}\end{aligned}$$

In view of Lemma 5,

$$I_3 = \varepsilon' O(1).$$

Now, applying Lemma 5, we have for any integer $s > k$,

$$\begin{aligned}I_4 &\leq n^k \sum_{r=1}^{k} \binom{k}{r} M_n^r\big(K(u-t)^{2s}/\delta^{2s-2k};t\big) = O(n^{k-s}) \text{ for any integer } s > k.\\ &= o(1).\end{aligned}$$

Due to arbitrariness of ε' it follows that $|I_2| = o(1)$.

Combining the estimates of I_1 and I_2, we obtain (3.1). Similarly, the assertion (3.2) follows from the fact $T_{n,k+1}\big((u-t)^\ell;t\big) = O(n^{-k-1})$ for all $\ell \in \mathbb{N}$.

The uniformity assertion follows du to the uniform continuity of $f^{(2k)}$ on $[0,1]$ which enables δ to become independent of t and the uniformness of the term $o(1)$ in the estimate of I_1.

In our next result we obtain an estimate of the degree of approximation of a function with specified smoothness.

Theorem 2. *Let $1 \leq p \leq 2k$ be an integer and $f^{(p)} \in C[0,1]$. Then, for sufficiently large n there holds*

$$\big\|T_{n,k}(f;t) - f(t)\big\| \leq \max\big\{C_1 n^{-p/2}\omega(f^{(p)};n^{-1/2}), C_2 n^{-k}\big\}, \tag{3.4}$$

where $C_1 = C_1(k,p)$, $C_2 = C_2(k,p,f)$, $\|\cdot\|$ is sup-norm on $[0,1]$ and $\omega(f^{(p)};\delta)$ is the modulus of continuity of $f^{(p)}$ on $[0,1]$.

Proof. By TAYLOR's expansion, we can write

$$f(u) - f(t) = \sum_{i=1}^{p} \frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{f^{(p)}(\xi) - f^{(p)}(t)}{p!}(u-t)^p, \tag{3.5}$$

where ξ lies between u and t.

Operating by $T_{n,k}$ on both sides of (3.5) and breaking the right hand side into two parts I_1 and I_2 say, corresponding to two terms on the right hand side of (3.5), we get

$$T_{n,k}(f;t) - f(t) = I_1 + I_2, \text{ say.}$$

In view of Lemma 4,

$$I_1 = \sum_{i=1}^{p} \frac{f^{(i)}(t)}{i!} T_{n,k}\big((u-t)^i;t\big) = O(n^{-k}), \text{ uniformly for every } t \in [0,1].$$

Since $f^{(p)} \in C[0,1]$, we have

$$\big|f^{(p)}(\xi) - f^{(p)}(t)\big| \le \omega\big(f^{(p)}; |\xi - t|\big) \le \big(1 + |u-t|/\delta\big)\omega(f^{(p)};\delta), \text{for any } \delta > 0.$$

Hence, using Schwarz inequality and Lemma 5,

$$|I_2| \le \frac{\omega(f^{(p)};\delta)}{p!} \sum_{r=1}^{k} \binom{k}{r} M_n^r \Big(\big|(u-t)\big|^p \big(1 + |u-t|/\delta\big);t\Big).$$

Choosing $\delta = n^{-1/2}$, we get

$$|I_2| \le \omega(f^{(p)};n^{-1/2}) O(n^{-p/2}), \text{ uniformly in } [0,1].$$

Combining the estimates of I_1 and I_2, the theorem follows.

4. SIMULTANEOUS APPROXIMATION

In this section we discuss simultaneous approximation property of the operators $T_{n,k}$. First we prove that $T_{n,k}^{(p)}$ is an approximation process for $f^{(p)}$, $p = 1,2,3,\dots$.

Theorem 3. *Let $f \in L_B[0,1]$ admitting a derivative of order p at a fixed point $t \in (0,1)$. Then*

$$\lim_{n\to\infty} T_{n,k}^{(p)}(f;t) = f^{(p)}(t). \tag{4.1}$$

Further, if $f^{(p)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,1)$, $\eta > 0$, then (4.1) *holds uniformly in $t \in [a,b]$.*

Proof. We can expand $f(u)$ as

$$f(u) = \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!} (u-t)^i + \varepsilon(u,t)(u-t)^p,$$

where $\varepsilon(u,t) \to 0$ as $u \to t$ and is bounded and integrable on $[0,1]$.

In order to prove (4.1), it is sufficient to show that $\lim\limits_{n\to\infty} D^p\big(M_n^r(f;t)\big) = f^{(p)}(t)$. Therefore, from the above expansion of f and the definition of M_n^r

$$\begin{aligned}
D^p M_n^r(f;t) &= \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!} \int_0^1 W_n^{(p)}(s,t) M_n^{r-1}\big((u-t)^i;s\big)\,\mathrm{d}s \\
&\qquad + \int_0^1 W_n^{(p)}(s,t) M_n^{r-1}\big(\varepsilon(u,t)(u-t)^p;s\big)\,\mathrm{d}s \\
&= I_1 + I_2, \text{ say.}
\end{aligned}$$

Now

$$\begin{aligned}
I_1 &= \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-t)^{i-j} \int_0^1 W_n^{(p)}(s,t) M_n^{r-1}(u^j;s)\,\mathrm{d}s \\
&= \sum_{i=0}^{p} \frac{f^{(i)}(t)}{i!} \sum_{j=0}^{i} \binom{i}{j} (-t)^{i-j} D^p M_n^r(u^j;t).
\end{aligned}$$

Since $M_n^r(u^j;t)$ is a polynomial in t of degree j and the coefficient of t^j is equal to $\prod\limits_{i=1}^{j} \big((n-i+1)/(n+i+1)\big)^r$, which tends to 1 as $n\to\infty$, it follows that $I_1 \to f^{(p)}(t)$ as $n\to\infty$. Since $\varepsilon(u,t)\to 0$ as $u\to t$, for a given $\varepsilon'>0$ we can find a $\delta>0$ such that $|\varepsilon(u,t)|<\varepsilon'$ whenever $0<|u-t|<\delta$, $\varepsilon(u,t)$ is bounded by some $K>0$, say. Suppose $\chi(u)$ is the characteristic function of the interval $(t-\delta,t+\delta)$, then in view of Lemma 2

$$\begin{aligned}
I_2 &= (n+1)\sum_{k=0}^{n} \int_0^1 \Big(D^p\big(p_{n,k}(t)\big)\Big) p_{n,k}(s) M_n^{r-1}\big(\varepsilon(u,t)(u-t)^p;s\big)\,\mathrm{d}s \\
&= (n+1)\sum_{k=0}^{n} \sum_{\substack{2i+j\le p\\ i,j\ge 0}} \frac{n^i(k-nt)^j}{t^p(1-t)^p}\, q_{i,j,p}(t) p_{n,k}(t) \times \\
&\qquad \times\Bigg(\int_0^1 p_{n,k}(s) M_n^{r-1}\big(\varepsilon(u,t)(u-t)^p\chi(u);s\big)\,\mathrm{d}s \\
&\qquad\qquad + \int_0^1 p_{n,k}(s) M_n^{r-1}\big(\varepsilon(u,t)(u-t)^p(1-\chi(u));s\big)\,\mathrm{d}s\Bigg) \\
&= I_3 + I_4, \text{ say.}
\end{aligned}$$

Let $C_1 = \sup\limits_{\substack{2i+j\le p\\ i,j\ge 0}} \big|q_{i,j,p}(t)/(t^p(1-t)^p)\big|$, applying SCHWARZ inequality three times we get

$$|I_3| \le \varepsilon' C_1 \sum_{\substack{2i+j\le p\\ i,j\ge 0}} n^i \Big(\sum_{k=0}^{n} p_{n,k}(t)(k-nt)^{2j} \Big)^{1/2} \times$$
$$\times \bigg((n+1) \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 p_{n,k}(s) M_n^{r-1}\big((u-t)^{2p}; s\big)\,\mathrm{d}s \bigg)^{1/2}.$$

Now, it is known [**3**] that for $0 \le t \le 1$ and $m \in \mathbb{N}_0$,

$$\sum_{k=0}^{n} p_{n,k}(t)(k-nt)^{2j} = O(n^j). \tag{4.2}$$

Therefore, using Lemma 5 we get

$$I_3 = \varepsilon' O(1). \tag{4.3}$$

Again,

$$|I_4| \le \sum_{k=0}^{n} (n+1) \binom{k}{r} \sum_{\substack{2i+j\le p\\ i,j\ge 0}} C_1 n^i p_{n,k}(t) \big|(k-nt)\big|^j \times$$
$$\times \int_0^1 p_{n,k}(s) M_n^{r-1}\Big(|\varepsilon(u,t)|\ |(u-t)|^p \big(1-\chi(u)\big); s \Big)\,\mathrm{d}s.$$

Using Schwarz inequality, (4.2) and Lemma 5, for any integer $s > p$ we obtain

$$|I_4| \le C_1 O(n^{p/2}) K \delta^{-s+p} \times$$
$$\times \bigg((n+1) \sum_{k=0}^{n} p_{n,k}(t) \int_0^1 p_{n,k}(s) M_n^{r-1}\Big((u-t)^{2s}\big(1-\chi(u)\big); s\Big) \mathrm{d}s \bigg)^{1/2}$$
$$\le K'(n^{(p-s)/2}).$$

Therefore we have

$$I_4 = o(1). \tag{4.4}$$

As $\varepsilon' > 0$ is arbitrary, from (4.3) and (4.4) we see that $I_2 = o(1)$. Hence (4.1) follows from the estimates of I_1 and I_2. The second assertion follows due to the fact that $\delta(\varepsilon')$ can be chosen independent of $t \in [a,b]$ and all the other estimates hold uniformly in $[a,b]$.

In our next theorem we study an asymptotic result for $T_{n,k}$ in simultaneous approximation.

Theorem 4. *Let $f \in L_B[0,1]$. If $f^{(2k+p)}(t)$ exists at the point $t \in (0,1)$, then we have*

$$\lim_{n\to\infty} n^k\big(T_{n,k}^{(p)}(f;t) - f^{(p)}(t)\big) = \sum_{j=p}^{2k+p} Q_1(j,k,p,t) f^{(j)}(t), \tag{4.5}$$

where $Q_1(j,k,p,t)$ are certain polynomials in t. Further, if $f^{(2k+p)}$ is continuous in $(a-\eta, b+\eta) \subset (0,1)$, $\eta > 0$, then (4.5) holds uniformly in $[a,b]$.

Proof. By our hypothesis we can write

$$\begin{aligned} T_{n,k}^{(p)}(f;t) &= \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\int_0^1 W_n^{(p)}(s,t)M_n^{r-1}\Bigg(\sum_{i=0}^{2k+p}\frac{f^{(i)}(t)}{i!}(u-t)^i \\ &\qquad + \varepsilon(u,t)(u-t)^{2k+p};s\Bigg)\mathrm{d}s \\ &= I_1 + I_2, \text{ say,} \end{aligned}$$

where $\varepsilon(u,t) \to 0$ as $u \to t$ and is bounded and integrable on $[0,1]$.

On an application of Lemma 1 and Theorem 1 we obtain

$$\begin{aligned} I_1 &= \sum_{i=p}^{2k+p}\frac{f^{(i)}(t)}{i!}\sum_{\ell=0}^{i}\binom{i}{\ell}(-t)^{i-\ell}T_{n,k}^{(p)}(u^\ell;t) \\ &= \sum_{i=p}^{2k+p}\frac{f^{(i)}(t)}{i!}\sum_{\ell=0}^{i}\binom{i}{\ell}(-t)^{i-\ell}\Bigg(D^p t^\ell + n^{-k}\sum_{j=1}^{2k}D^p\Big(\frac{Q(j,k,t)}{j!}D^j t^\ell\Big) + o(n^{-k})\Bigg) \\ &= f^{(p)}(t) + \sum_{i=p}^{2k+p} n^{-k}\sum_{\ell=0}^{i}\binom{i}{\ell}(-t)^{i-\ell}\frac{f^{(i)}(t)}{i!}\Bigg(\sum_{j=1}^{2k}D^p\Big(\frac{Q(j,k,t)}{j!}D^j t^\ell\Big)\Bigg) + o(n^{-k}) \\ &= f^{(p)}(t) + n^{-k}\sum_{j=p}^{2k+p}Q_1(j,k,p,t)f^{(j)}(t) + o(n^{-k}), \end{aligned}$$

where we used the identities $\displaystyle\sum_{\ell=0}^{i}(-1)^\ell\binom{i}{\ell}\binom{\ell}{p} = \begin{cases} 0, & i > p \\ (-1)^p, & i = p. \end{cases}$

To estimate $\displaystyle I_2 = \sum_{r=1}^{k}(-1)^{r+1}\binom{k}{r}\int_0^1 W_n^{(p)}(s,t)M_n^{r-1}\big(\varepsilon(u,t)(u-t)^{2k+p};s\big)\,\mathrm{d}s,$ proceeding as in the estimate of I_4 in Theorem 3, it follows that $n^k I_2 \to 0$ as $n \to \infty$. Hence, combining the estimates of I_1 and I_2, (4.5) is established. The uniformity assertion follows as in Theorem 3.

Theorem 5. *Let $p, q \in \mathbb{N}$, $p \le q \le 2k+p$ and $f \in L_B[0,1]$. If $f^{(q)}$ exists and is continuous on $(a-\eta, b+\eta) \subset (0,1)$, for some $\eta > 0$ then*

$$\big\|T_{n,k}^{(p)}(f;t) - f^{(p)}(t)\big\| \le \max\big\{C_1 n^{-(q-p)/2}\omega(f^{(q)}; n^{-1/2}), C_2 n^{-k}\big\}, \tag{4.6}$$

where $C_1 = C_1(k,p)$, $C_2 = C_2(k,p,f)$, $\|\cdot\|$ is the sup-norm on $[a,b]$ and the modulus of continuity of $f^{(q)}$ on $(a-\eta, b+\eta)$ is $\omega(f^{(q)}; n^{-1/2})$.

Proof. By our hypothesis, we may write for all $u \in [0,1]$ and $t \in [a,b]$

$$f(u) = \sum_{i=0}^{q} \frac{f^{(i)}(t)}{i!}(u-t)^i + \frac{f^{(q)}(\xi) - f^{(q)}(t)}{q!}(u-t)^q \chi(u) + F(u,t)\big(1-\chi(u)\big), \tag{4.7}$$

where $\chi(u)$ is the characteristic function of $(a-\eta, b+\eta)$, ξ lies between u and t and $F(u,t)$ is defined as $F(u,t) = f(u) - \sum\limits_{i=0}^{q} \frac{f^{(i)}(t)}{i!}(u-t)^i$, $\forall u \in [0,1]$ and $t \in [a,b]$.

The function $F(u,t)$ is bounded by $M|u-t|^q$ for $t \in [a,b]$ and M is some positive number. Now operating by $T_{n,k}^{(p)}$ on both sides of (4.7) and breaking the right hand side into three parts I_1, I_2 and I_3 say, corresponding to the three terms on the right hand side of (4.7), we get

$$T_{n,k}^{(p)}(f;t) - f^{(p)}(t) = I_1 + I_2 + I_3, \text{ say.}$$

Now,

$$I_1 = \sum_{i=1}^{q}\sum_{j=0}^{i}(-t)^{i-j}\binom{i}{j}\frac{f^{(i)}(t)}{i!}T_{n,k}^{(p)}(u^j;t).$$

Proceeding as in the estimate of I_1 of Theorem 4

$$\begin{aligned} I_1 &= \sum_{i=1}^{q}\sum_{j=0}^{i}(-t)^{i-j}\binom{i}{j}\frac{f^{(i)}(t)}{i!}D^p\left(t^j + n^{-k}\left(\sum_{r=1}^{2k} D^p\Big(\frac{Q(r,k,t)}{r!}D^r t^j\Big) + o(n^{-k})\right)\right) \\ &= O(n^{-k}), \text{ uniformly in } t \in [a,b]. \end{aligned}$$

Next, applying Lemma 2,

$$\begin{aligned} |I_2| \le \frac{\omega(f^{(q)};\delta)}{q!}\sum_{r=1}^{k}\binom{k}{r}(n+1)\sum_{v=0}^{n}\sum_{\substack{2i+j\le p\\ i,j\ge 0}} n^i p_{n,v}(t)\frac{|q_{i,j,p}(t)|}{t^p(1-t)^p}|v-nt|^j \times \\ \times \int_0^1 p_{n,v}(s) M_n^{r-1}\Big(|u-t|^q\big(1+|u-t|/\delta\big);s\Big)\,\mathrm{d}s. \end{aligned}$$

Let $C' = \sup\limits_{\substack{2i+j\le p\\ i,j\ge 0,\ t\in[a,b]}} \big|q_{i,j,p}(t)/\big(t^p(1-t)^p\big)\big|$. Using Schwarz inequality, (4.2) and Lemma 5 we obtain $|I_2| \le C'\omega(f^{(q)};\delta)\big(O(n^{-(q-p)/2}) + O(n^{-(q+1-p)/2})\big)$, uniformly in $t \in [a,b]$.

Choosing $\delta = n^{-1/2}$, it follows that $I_2 = \omega(f^{(q)}; n^{-1/2})O(n^{-(q-p)/2})$ uniformly in $t \in [a, b]$. Lastly, to estimate $I_3 = T_{n,k}^{(p)}\big(F(u,t)(1-\chi(u)); t\big)$, proceeding in a manner similar to the estimate of I_4 in Theorem 3, it follows that $I_3 = O(n^{(p-s)/2})$, where s is an integer greater than $2k+p+2$. Thus $I_3 = o(n^{-(k+1)})$, uniformly in $t \in [a, b]$.

Combining the estimates of I_1, I_2 and I_3, (4.6) is established. This completes the proof.

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NOTE ON ASYMPTOTIC CONTRACTIONS

Ivan D. Aranđelović

In 2003 W. A. Kirk introduced the notion of asymptotic contractions. In this paper we present one fixed point theorem of Kirk's type unifying and generalizing recent results of W. A. Kirk , J. Jachymski, I. Jóźwik and Y.-Z. Chen.

1. INTRODUCTION AND PRELIMINARIES

W. A. Kirk [**8**] introduced the notion of asymptotic contractions and proved fixed point theorem for this class of mappings. In note [**1**] we present a new short and simple proof of Kirk's theorem. Further results on this class of mappings was obtained by: J. Jachymski, I. Jóźwik [**7**], Y.-Z. Chen [**4**], P. Gerhardy [**5**], [**6**], T. Suzuki [**9**], H. K. Xu [**12**], M. Arav, F. E. C. Santos, S. Reich, A. Zaslavski [**2**] and K. Włodarczyk, D. Klim, R. Plebaniak [10], [11]. The papers [10] and [11] presents some ideas for application of the theory of asymptotic contractions in the analysis of set-valued dynamic systems.

In this paper we present one fixed point theorem of Kirk's type unifying and generalizing recent results of W. A. Kirk [8], J. Jachymski, I. Jóźwik [7] and Y.-Z. Chen [4].

Let X be a nonempty set and $f : X \to X$ arbitrary mapping. $x \in X$ is a fixed point for f if $x = f(x)$. If $x_0 \in X$, we say that a sequence (x_n) defined by $x_n = f^n(x_0)$ is a sequence of Picard iterates of f at point x_0 or that (x_n) is the orbit of f at point x_0.

In [2] M. Arav, F. E. C. Santos, S. Reich and A. Zaslavski proved the following result:

Proposition 1. *Let (X, d) be a metric space, $f : X \to X$ continuous function and (φ_i) sequence of functions such that $\varphi_i : [0, \infty) \to [0, \infty)$ and for each $x, y \in X$*

$$d\big(f^i(x), f^i(y)\big) \le \varphi_i\big(d(x, y)\big).$$

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Assume also that there exists upper semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ such that for any $r > 0$ $\varphi(r) < r$, $\varphi(0) = 0$ and $\varphi_i \to \varphi$ uniformly on any bounded interval $[0,b]$. If there exists $y \in X$ such that $y = f(y)$ then all sequences of Picard iterates defined by f converge to y, uniformly on each bounded subset of X.

2. MAIN RESULT'S

Now we present our results.

Theorem 1. *Let (X,d) be a complete metric space, $f : X \to X$ continuous function and (φ_i) sequence of functions such that $\varphi_i : [0,\infty) \to [0,\infty)$ and for each $x, y \in X$*

$$d\big(f^i(x), f^i(y)\big) \le \varphi_i\big(d(x,y)\big).$$

Assume also that there exists upper semicontinuous function $\varphi : [0,\infty) \to [0,\infty)$ such that for any $r > 0$ $\varphi(r) < r$, $\varphi(0) = 0$ and $\varphi_i \to \varphi$ uniformly on any bounded interval$[0,b]$. If one of the following conditions is satisfying:

1) *there exists $x \in X$ such that orbit of f at x is bounded*; *or*
2) $\underline{\lim}_{t\to\infty} \big(t - \varphi(t)\big) > 0$; *or*
3) $\overline{\lim}_{t\to\infty} \dfrac{\varphi(t)}{t} < 1$.

then f has an unique fixed point $y \in X$ and all sequences of Picard iterates defined by f converge to y, uniformly on each bounded subset of X.

Proof. For any $x, y \in X$, $x \neq y$, we have:

$$\overline{\lim}\, d\big(f^n(x), f^n(y)\big) \le \overline{\lim}\, \varphi_n\big(d(x,y)\big) = \varphi\big(d(x,y)\big) < d(x,y).$$

Suppose that there exist $x, y \in X$ and $\varepsilon > 0$ such that $\overline{\lim}\, d\big(f^n(x), f^n(y)\big) = \varepsilon$. Then there exists sequence of integers (m_j), such that

$$\lim d(x_{m_j}, y_{m_j}) = \overline{\lim}\, d(x_m, y_m).$$

If $\overline{\lim}\, d\big(f^k(x), f^k(y)\big) \ge \varepsilon$, for each $k \in (m_j)$, then from upper semicontinuity of φ follows $\overline{\lim}\, d(x_{m_j}, y_{m_j}) \le \varphi(\varepsilon) < \varepsilon$, which is a contradiction. So there exists $k \in (m_j)$ such that

$$\varphi\Big(d\big(f^k(x), f^k(y)\big)\Big) < \varepsilon.$$

This implies that

$$\begin{aligned}\overline{\lim}\, d\big(f^n(x), f^n(y)\big) &= \overline{\lim}_n\, d\Big(f^n\big(f^k(x)\big), f^n\big(f^k(y)\big)\Big)\\ &\le \overline{\lim}_n\, \varphi_n\Big(d\big(f^k(x), f^k(y)\big)\Big) = \varphi\Big(d\big(f^k(x), f^k(y)\big)\Big) < \varepsilon,\end{aligned}$$

which is a contradiction. So we obtain that

$$\lim d\big(f^n(x), f^n(y)\big) = 0, \tag{1}$$

for any $x, y \in X$, which implies that all sequences of Picard iterates defined by f, are equiconvergent.

Now let $a \in X$ be arbitrary, (a_n) be a sequence of Picard iterates of f at point a, $Y = \overline{(a_n)}$ and $F_n = \{x \in Y : d\big(x, f^k(x)\big) \le 1/n \ \ (k = 1, \dots, n)\}$. From (1) follows that F_n is nonempty and since f is continuous F_n is closed, for any n. Also, we have $F_{n+1} \subseteq F_n$. Let (x_n) and (y_n) be arbitrary sequences, such that $x_n, y_n \in F_n$. Let (n_j) be a sequence of integers, such that $\lim d(x_{n_j}, y_{n_j}) = \overline{\lim}\, d(x_n, y_n)$.

For any $\varepsilon > 0$ there exists positive integer k such that

$$\varphi(t) + \varepsilon \ge \varphi_k(t)$$

for all $t \in [0, +\infty)$ and $m \ge k$, because $\varphi_n \to \varphi$ uniformly on the rang of d. Now we have:

$$\begin{aligned} \lim d(x_{n_j}, y_{n_j}) &\le \overline{\lim}\Big(d\big(x_{n_j}, f^{n_j}(x_{n_j})\big) + d\big(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})\big) \\ &\qquad + d\big(y_{n_j}, f^{n_j}(y_{n_j})\big)\Big) = \overline{\lim}\, d\big(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})\big) \\ &\le \overline{\lim}\, \varphi_{n_j}\big(d(x_{n_j}, y_{n_j})\big) \le \varepsilon + \overline{\lim}\, \varphi\big(d(x_{n_j}, y_{n_j})\big) \\ &\le \varepsilon + \varphi\big(\lim d(x_{n_j}, y_{n_j})\big), \end{aligned}$$

for $n_j \ge k$ and so $\lim d(x_{n_j}, y_{n_j}) = \varphi\big(\lim d(x_{n_j}, y_{n_j})\big) \Rightarrow \lim d(x_{n_j}, y_{n_j}) \in \{0, +\infty\}$. Now we have following three cases:

A) Let $\lim d(x_{n_j}, y_{n_j}) = 0$. Thus $\overline{\lim}\, d(x_n, y_n) = 0$ and so $\lim d(x_n, y_n) = 0$. This implies that $\lim \operatorname{diam} F_n = 0$. By completeness of Y follows that there exists $z \in X$ such that

$$\bigcap_{i=1}^{\infty} F_n = \{z\}.$$

We remember that 1) $\Rightarrow$ A). Since $d\big(z, f(z)\big) \le 1/n$ for any n, we have $f(z) = z$. From (1) follows that all sequences of Picard iterates defined by f converge to z. From Proposition 1 follows that this convergence is uniform on bounded subsets of X.

B) Let $\lim d(x_{n_j}, y_{n_j}) = +\infty$ and $\underline{\lim}_{t\to\infty} \big(t - \varphi(t)\big) > 0$. Then from

$$d(x_{n_j}, y_{n_j}) \le d\big(x_{n_j}, f^{n_j}(x_{n_j})\big) + d\big(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})\big) + d\big(y_{n_j}, f^{n_j}(y_{n_j})\big)$$

follows

$$\begin{aligned} d\big(x_{n_j}, f^{n_j}(x_{n_j})\big) + d\big(y_{n_j}, f^{n_j}(y_{n_j})\big) &\ge d(x_{n_j}, y_{n_j}) - d\big(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})\big) \\ \ge d(x_{n_j}, y_{n_j}) - \varphi_{n_j}\big(d(x_{n_j}, y_{n_j})\big) &\ge d(x_{n_j}, y_{n_j}) - \varphi\big(d(x_{n_j}, y_{n_j})\big) - \varepsilon, \end{aligned}$$

for $n_j \geq k$. Thus
$$\underline{\lim}_{n\to\infty} \big(t - \varphi(t)\big) < 0,$$
which is a contradiction.

C) Let $\lim d(x_{n_j}, y_{n_j}) = +\infty$ and $\overline{\lim}_{t\to\infty} \dfrac{\varphi(t)}{t} < 1$. Then from
$$d(x_{n_j}, y_{n_j}) \leq d\big(x_{n_j}, f^{n_j}(x_{n_j})\big) + d\big(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})\big) + d\big(y_{n_j}, f^{n_j}(y_{n_j})\big)$$
follows
$$\begin{aligned} 1 &\leq \frac{d\big(x_{n_j}, f^{n_j}(x_{n_j})\big) + d\big(f^{n_j}(x_{n_j}), f^{n_j}(y_{n_j})\big) + d\big(y_{n_j}, f^{n_j}(y_{n_j})\big)}{d(x_{n_j}, y_{n_j})} \\ &\leq \frac{d\big(x_{n_j}, f^{n_j}(x_{n_j})\big) + \varphi_{n_j}\big(d(x_{n_j}, y_{n_j})\big) + d\big(y_{n_j}, f^{n_j}(y_{n_j})\big)}{d(x_{n_j}, y_{n_j})} \\ &\leq \frac{d\big(x_{n_j}, f^{n_j}(x_{n_j})\big) + \varphi\big(d(x_{n_j}, y_{n_j})\big) + \varepsilon + d\big(y_{n_j}, f^{n_j}(y_{n_j})\big)}{d(x_{n_j}, y_{n_j})}, \end{aligned}$$
for $n_j \geq k$. Thus
$$1 \leq \overline{\lim} \frac{\varphi\big(d(x_{n_j}, y_{n_j})\big)}{d(x_{n_j}, y_{n_j})} < 1$$
which is a contradiction.

3. COMMENTS AND REMARKS

The statement of W. A. KIRK [**8**] – Theorem 2.1 has additional assumptions that all φ_i are continuous, and so KIRK's result is include in our Theorem 1.1), as theorem Y.-Z. CHEN [**4**]- Theorem 2.2 which has additional assumptions that one of (φ_i) is upper semicontinuous.

The statement of J. JACHYMSKI, I. JÓŹWIK [7] – Theorem 2 has additional assumptions "f is uniformly continuous" and condition
$$\lim_{t\to\infty} \big(t - \varphi(t)\big) = +\infty$$
which is stronger then our condition
$$\underline{\lim}_{t\to\infty} \big(t - \varphi(t)\big) > 0.$$
Thus this result is include in our Theorem 1.2.

The statement of Y.-Z. CHEN [**4**] – Corollary 2.4 has additional assumptions that one of (φ_i) is upper semicontinuous, and so it is include in our Theorem 1.3.

In the statements of W. A. KIRK [**8**] – Theorem 2.1 and Y.-Z. CHEN [4] – Theorem 2.2, the assumption "f is continuous" was inadvertently left out, but it was used in the proofs of theorems. J. Jachymski, I. Jóźwik [**7**], give the following example for necessity of this condition.

Example 1. Let $X = [0, 1]$ an $f : X \to X$ defined by

$$f(x) = \begin{cases} 1, & x = 0, \\ x/2, & x \neq 0. \end{cases}$$

So $f(X) \subseteq (0, 1)$ which implies $f^n(X) \subseteq (0, 1/2^{n-1})$. A sequence of functions $\varphi_n = 1/2^{n-1}$ satisfies the conditions of Theorem 2.1 because $\varphi_n \to 0$ uniformly, but f is fixed point free.

Now we give the following example for necessity of uniformly convergence of sequence (φ_n).

Example 2. Let (x_n) be an arbitrary sequence, $X = \{x_n\}$, and $d : X^2 \to [0, +\infty)$ mapping defined by

$$d(x_n, x_{n+k}) = \frac{k}{n+k} + \frac{1}{nk}\,,$$

$n \in \{1, 2, 3, \dots\}$, $k \in \{0, 1, 2, \dots\}$. (X, d) is complete metric space, because each ball which radius is less then 1 contains only a finite number elements of X. d is discrete metric and all nonzero distance are distinct. Also we have:

$$\lim_k d(x_n, x_{n+k}) = 1$$

and

$$\lim_n d(x_n, x_{n+k}) = 0.$$

Now define $f : X \to X$ by $f(x_n) = x_{n+1}$. Let $\varphi(t) = t/2$ and

$$\varphi_n\big(d(x_i, x_j)\big) = \max\Big\{d(x_{i+n}, x_{j+n}), \frac{d(x_i, x_j)}{2}\Big\},$$

for $i \neq j$. The function φ_n is well defined because the number $d(x_i, x_j)$ occurs only once in the rang of d. From

$$\lim_n d(x_{i+n}, x_{j+n}) = 0$$

follows $\varphi_n \to \varphi$ on the rang of d. The inequality

$$d\big(f^n(x_i), f^n(x_j)\big) \leq \varphi_n\big(d(x_i, x_j)\big)$$

is also satisfied, but f is fixed point free.

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ON THE NEWTON-LIKE METHOD FOR THE INCLUSION OF A POLYNOMIAL ZERO*

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One of the most important problems in solving nonlinear equations is the construction of such initial conditions which provide both the guaranteed and fast convergence of the considered numerical algorithms. In this paper we study an iterative method of Newton's type for the inclusion of isolated complex zero of a given polynomial. We state computationally verifiable initial condition for the convergence of the considered interval method, which depends only on attainable data, and prove the quadratic convergence of this method. A numerical example is given.

1. INTRODUCTION

During the last forty years various techniques for *a posteriori* error estimates for the approximation of polynomial complex zeros were developed, see [**6**]. A quite different and efficient approach to error estimates uses complex circular arithmetic, as pointed out by Gargantini and Henrici [**3**]. Iterative methods realized in circular interval arithmetic produce resulting disks that contain complex zeros of a polynomial. In this manner, not only very close approximations to the zeros (given by the centers of disks) but also the upper error bounds for the zeros (given by the radii of disks) are provided. More details about interval methods for the inclusion of polynomial zeros can be found in [**1**], [**6**], [**7**], [**10**] and references cited there.

The aim of this paper is to present an iterative method of Newton's type for the inclusion of an isolated complex zero of a polynomial. A special attention will be paid to the construction of computationally verifiable initial condition that provides the guaranteed convergence of the proposed method. Let us stress that this subject, known in literature as a composite part of "point estimation theory"

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(Smale [**11**]), is one of the most important problems in the theory and practice of iterative processes which attracts a great attention in recent time (for more details see [**8**]).

To state the interval method for the inclusion of polynomial complex zeros, we need the basic properties of circular interval arithmetic. A disk Z with center $\operatorname{mid} Z = c$ and radius $\operatorname{rad} Z = r$, that is $Z := \{z : |z - c| \le r\}$, will be denoted briefly by the parametric notation $Z = \{c; r\}$. The set of all complex circular intervals (disks) is denoted by $K(\mathbb{C})$. The basic circular arithmetic operations are defined as follows:

$$\begin{aligned}
&\alpha\{c; r\} = \{\alpha c; |\alpha| r\} \quad (\alpha \in \mathbb{C}),\\
&\{c_1; r_1\} \pm \{c_2; r_2\} = \{c_1 \pm c_2; r_1 + r_2\},\\
&\{c_1; r_1\} \cdot \{c_2; r_2\} = \{c_1 c_2; |c_1| r_2 + |c_2| r_1 + r_1 r_2\}.
\end{aligned}$$

The inversion of a disk $Z = \{c; r\}$ which does not contain the origin (that is, $|c| > r$ holds) is defined by the Möbius transformation,

$$Z^{-1} = \Big\{\frac{1}{z} :\ z \in \{c; r\}\Big\} = \Big\{\frac{\bar{c}}{|c|^2 - r^2}; \frac{r}{|c|^2 - r^2}\Big\}. \tag{1}$$

Following the introduced inversion, division is defined as

$$Z_1 : Z_2 = Z_1 \cdot Z_2^{-1} \quad (0 \notin Z_2).$$

For two disks $Z_1 = \{c_1; r_1\}$ and $Z_2 = \{c_2; r_2\}$ the following is valid:

$$\{c_1; r_1\} \cap \{c_2; r_2\} = \emptyset \quad \Leftrightarrow \quad |c_1 - c_2| > r_1 + r_2, \tag{2}$$

$$|c_1 - c_2| \le r_1 - r_2 \quad \Leftrightarrow \quad \{c_1; r_1\} \subseteq \{c_2; r_2\}. \tag{3}$$

A fundamental property of interval computation is the *inclusion isotonicity* which forms the basis for almost all applications of interval arithmetic. Let f be a complex function over a given disk $Z \in K(\mathbb{C})$. The complex-valued set $\{f(z) |\ z \in Z\}$ is not a disk in general. To deal with disks, we introduce an *circular extension* F of f, defined on a subset $D \subseteq K(\mathbb{C})$ such that

$$\begin{aligned}
&F(Z) \supseteq \{f(z) |\ z \in Z\} \quad \text{for all } Z \in D \quad (\textit{inclusion}),\\
&F(z) = f(z) \quad \text{for all } z \in Z \quad (\textit{complex restriction}).
\end{aligned}$$

We shall say that the complex interval extension F is *inclusion isotone* if the implication

$$Z_1 \subseteq Z_2 \ \Rightarrow\ F(Z_1) \subseteq F(Z_2)$$

is satisfied for all $Z_1, Z_2 \in D$. In particular, we have

$$z \in Z \ \Rightarrow\ f(z) = F(z) \in F(Z). \tag{4}$$

Let us note that the four basic operations in circular complex arithmetic are inclusion isotone ([**1**, Ch. 5]).

2. NEWTON-LIKE INTERVAL METHOD

Let

$$P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = \prod_{j=1}^{n}(z-\zeta_j) \quad (a_i \in \mathbb{C})$$

be a monic polynomial of degree n with simple complex zeros $\zeta_1, \ldots, \zeta_n$. Let us assume that we have found a disk $A := \{z : \ |z-a| \le \eta\}$, shortly denoted by parametric notation $\{a;\eta\}$ ($a = \operatorname{mid} A,\ \eta = \operatorname{rad} A$), that contains only one zero of P. All other zeros are supposed to lie in the region $W = \{w : \ |w-a| > \eta\} = \operatorname{ext} A$, that is, in the exterior of the disk $A = \{a;\eta\}$ (see Fig. 1). There is a lot of results for the localization of polynomial zeros (see, e.g., Marden [**5**], Henrici [**4**]) and we will not discuss this subject in this paper. We also note that a useful computational test for the existence of polynomial zero can be found in [**9**]. Without loss of generality, we will adopt that the sought zero is denoted by ζ_1; moreover, we write ζ instead of ζ_1. In our study we always assume that $n \ge 3$.

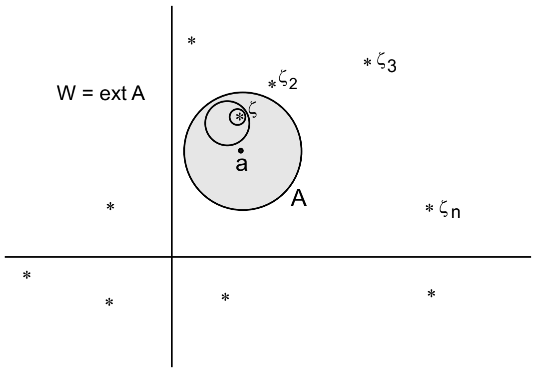

Figure 1.

Using the logarithmic derivative, we find

$$\frac{P'(z)}{P(z)} = \sum_{j=1}^{n} \frac{1}{z-\zeta_j} = \frac{1}{z-\zeta} + \sum_{j=2}^{n} \frac{1}{z-\zeta_j}. \tag{5}$$

We single out the zero $\zeta_1 = \zeta$ from (5) and obtain the fixed point relation

$$\zeta = z - \frac{1}{\dfrac{P'(z)}{P(z)} - \displaystyle\sum_{j=2}^{n}(z-\zeta_j)^{-1}}. \tag{6}$$

According to the inclusion isotonicity property (4) we have $(z-\zeta_j)^{-1} \in (z-W)^{-1}$ for any $z \in \{a;\eta\}$ and $j = 2, 3, \ldots, n$. Since $z \notin W$, that is, $|z-a| < \eta$, the inversion of the open region $z - W$ is a closed interior of a circle given by

$$H = (z - W)^{-1} = \left\{ w : \ \left| w + \frac{\bar{z} - \bar{a}}{\eta^2 - |z - a|^2} \right| \le \frac{\eta}{\eta^2 - |z - a|^2} \right\} = \{h; d\},$$

where

$$h = \operatorname{mid} H = \frac{\bar{a} - \bar{z}}{\eta^2 - |z - a|^2}, \quad d = \frac{\eta}{\eta^2 - |z - a|^2}.$$

(see Gargantini [**2**]). Taking into account that $\zeta_j \in W$, we return to (6) and obtain

$$\zeta \in z - \frac{1}{\dfrac{P'(z)}{P(z)} - \displaystyle\sum_{j=2}^{n} (z - W)^{-1}} = z - \frac{1}{\dfrac{P'(z)}{P(z)} - (n-1)\{h; d\}}. \tag{7}$$

The relation (7) suggests the construction of an iterative method for the inclusion of isolated zero of the polynomial P. Let $Z^{(m)} = \{z^{(m)}; r^{(m)}\}$ be a disk with center $z^{(m)} = \operatorname{mid} Z^{(m)}$ and radius $r^{(m)} = \operatorname{rad} Z^{(m)}$ $(m = 0, 1, \ldots)$. For an initial inclusion disk we take $Z^{(0)} = \{a; \eta\} = A$, that is, $z^{(0)} = a$ and $r^{(0)} = \eta$. We will use the following abbreviations:

$$H^{(m)} = \{h^{(m)}; d^{(m)}\}, \quad h^{(m)} = \frac{\bar{a} - \bar{z}^{(m)}}{\eta^2 - |z^{(m)} - a|^2}, \quad d^{(m)} = \frac{\eta}{\eta^2 - |z^{(m)} - a|^2}.$$

Starting from (7) we can construct the following iterative method for the inclusion of one zero of the given polynomial P :

$$Z^{(m+1)} = z^{(m)} - \frac{1}{\dfrac{P'(z^{(m)})}{P(z^{(m)})} - (n-1)\{h^{(m)}; d^{(m)}\}} \quad (m = 0, 1, \ldots), \tag{8}$$

where $z^{(m)} = \operatorname{mid} Z^{(m)}$.

Remark 1. The iterative formula (8), written in the form,

$$Z^{(m+1)} = z^{(m)} - \frac{P(z^{(m)})}{P'(z^{(m)})} \cdot \frac{1}{1 - \dfrac{P(z^{(m)})}{P'(z^{(m)})}(n-1)\{h^{(m)}; d^{(m)}\}} \quad (m = 0, 1, \ldots),$$

obviously resembles the Newton method. For this reason, we will refer (8) to as the Newton-like interval method.

Remark 2. Regarding the role of the disk $H^{(m)}$ in (8), we can conclude that it does not have any influence to the convergence rate of the iterative method (8). However, the *main* role of the interval $H^{(m)} = \{h^{(m)}; d^{(m)}\}$ is to provide the inclusion of the zero ζ within the disk $Z^{(m+1)}$.

REMARK 3. The interval method (8) is a modification of the GARGANTINI-HENRICI method [**3**] of the third order

$$Z_i^{(m+1)} = z_i^{(m)} - \cfrac{1}{\cfrac{P'(z_i^{(m)})}{P'(z_i^{(m)})} - \sum\limits_{\substack{j=1\\ j\neq i}}^{n} \Big(z_i^{(m)} - Z_j^{(m)}\Big)^{-1}} \qquad (i = 1, \ldots, n;\ m = 0, 1, \ldots).$$

for the simultaneous inclusion of all simple zeros of the polynomial P.

3. CONVERGENCE ANALYSIS

In this section we will study the convergence behavior of the NEWTON-like method (8). Our main goal is to state computationally verifiable initial condition that enables the guaranteed convergence of this method. As a consequence, we prove that the NEWTON-like method (8) has the quadratic convergence.

Assume that we have found an initial disk $Z^{(0)} = \{a; \eta\}$ that contains one and only one zero ζ of P and let the inequality

$$\Big|\frac{P(a)}{P'(a)}\Big| < \frac{\eta}{3(n-1)} \tag{9}$$

hold. We note that this inequality involves only known data: the center and radius of the initial inclusion disk $Z^{(0)} = A = \{a; \eta\}$ and the polynomial degree n. We will show later that, if the inequality (9) is valid, then the iterative method (8) is convergent with the quadratic convergence. Let us stress that the initial condition of the form (9) is of great practical importance since it depends only on attainable data.

Let us introduce the quantity

$$\rho^{(m)} = \eta - |z^{(m)} - a| \quad (m = 0, 1, \ldots).$$

Lemma 1. *The following implication is valid*:

$$\Big|\frac{P(a)}{P'(a)}\Big| < \frac{\eta}{3(n-1)} \quad \Rightarrow \quad \rho^{(1)} > 5(n-1)r^{(1)}.$$

Proof. The inversion of the open region $T = \{z :\ |z - t| > R\}$, when $0 \notin T$ (that is, $|t| < R$), is given by

$$T^{-1} = \left\{\frac{-\bar{t}}{R^2 - |q|^2}; \frac{R}{R^2 - |q|^2}\right\}. \tag{10}$$

By using (10) we obtain

$$\frac{n-1}{z^{(0)} - W} = (n-1)\Big\{0; \frac{1}{\eta}\Big\} = \Big\{0; \frac{n-1}{\eta}\Big\} \quad (z^{(0)} = a).$$

The iterative formula (8) for $m = 0$ has a simple form

$$Z^{(1)} = a - \frac{1}{\left\{ \dfrac{P'(a)}{P(a)}; \dfrac{n-1}{\eta} \right\}}.$$

Having in mind the inequality (9) we find

$$\left| \frac{P'(z^{(0)})}{P(z^{(0)})} \right| = \left| \frac{P'(a)}{P(a)} \right| > \frac{3(n-1)}{\eta} > \frac{n-1}{\eta} = \operatorname{rad} \frac{n-1}{z^{(0)} - W}.$$

Therefore, according to (2), the disk in the denominator of (8) does not contain 0 when $m = 1$, which means that $Z^{(1)}$ is also a disk.

We apply (1) and (9) and estimate

$$(11) \qquad r^{(1)} = \operatorname{rad} Z^{(1)} = \operatorname{rad} \frac{1}{\left\{ \dfrac{P'(a)}{P(a)}; \dfrac{n-1}{\eta} \right\}} = \frac{\dfrac{n-1}{\eta}}{\left| \dfrac{P'(a)}{P(a)} \right|^2 - \dfrac{(n-1)^2}{\eta^2}}$$

$$< \frac{\dfrac{n-1}{\eta}}{\left[\dfrac{3(n-1)}{\eta} \right]^2 - \dfrac{(n-1)^2}{\eta^2}} = \frac{\eta}{8(n-1)} \leq \frac{\eta}{16}.$$

In the similar way, using (9) we find

$$(12) \qquad |z^{(1)} - z^{(0)}| = |z^{(1)} - a| = \frac{\left| \dfrac{P'(a)}{P(a)} \right|}{\left| \dfrac{P'(a)}{P(a)} \right|^2 - \dfrac{(n-1)^2}{\eta^2}}$$

$$< \frac{\dfrac{3(n-1)}{\eta}}{\left[\dfrac{3(n-1)}{\eta} \right]^2 - \dfrac{(n-1)^2}{\eta^2}} = \frac{3\eta}{8(n-1)}.$$

Using (11) and (12) we obtain

$$(13) \qquad \rho^{(1)} = \eta - |z^{(1)} - a| > \eta - \frac{3\eta}{8(n-1)} = \frac{8n-11}{8(n-1)}\eta \geq \frac{13}{16}\eta > 5(n-1)r^{(1)}.$$

In this way we have proved that the initial condition (9) implies the inequality

$$(14) \qquad \rho^{(1)} > 5(n-1)r^{(1)},$$

which has the important role in the convergence analysis. □

Let us note that $\zeta \in Z^{(1)}$ according to the inclusion isotonicity property. We will now consider the iterative method (8) for $m \geq 1$ starting with the inclusion disk $Z^{(1)}$ and the condition (14). This condition holds if (9) is valid, which will not be further particularly cited. Using circular arithmetic operations, the iterative formula (8) can be rewritten in the form

$$Z^{(m+1)} = z^{(m)} - \left\{c^{(m)}; d^{(m)}\right\}^{-1} \quad (m = 0, 1, \ldots), \tag{15}$$

where

$$c^{(m)} = \frac{P'(z^{(m)})}{P(z^{(m)})} + \frac{(n-1)(\bar{z}^{(m)} - \bar{a})}{\eta^2 - |z^{(m)} - a|^2}, \quad d^{(m)} = \frac{(n-1)\eta}{\eta^2 - |z^{(m)} - a|^2}.$$

For simplicity, in what follows we will omit the iteration index always when there is no possibility of a confusion.

Lemma 2. *If the inequality*

$$\rho > 5(n-1)r \tag{16}$$

is valid, then $0 \notin \{c; d\}$ *and*

$$\frac{|c|}{|c|^2 - d^2} < \frac{15}{8} r. \tag{17}$$

Proof. Using (5) and (16) we find for $m \geq 2$

$$\begin{aligned} |c| &> \frac{1}{|z - \zeta|} - \sum_{j=2}^{n} \frac{1}{|z - \zeta_j|} - \frac{(n-1)|z-a|}{\eta^2 - |z-a|^2} \\ &> \frac{1}{r} - \frac{n-1}{\rho} - \frac{(n-1)(\eta - \rho)}{\eta^2 - (\eta - \rho)^2} > \frac{3}{5r}, \end{aligned} \tag{18}$$

and

$$d = \frac{(n-1)\eta}{\eta^2 - |z-a|} = \frac{(n-1)\eta}{\eta^2 - (\eta - \rho)^2} < \frac{n-1}{\rho}. \tag{19}$$

Since

$$|c| > \frac{3}{5r} > \frac{3}{5} \cdot \frac{5(n-1)}{\rho} > \frac{n-1}{\rho} > d,$$

according to (2) it follows that $0 \notin \{c; d\}$.

To prove the second assertion of the lemma, we use (16), (18) and (19) to find

$$\frac{|c|}{|c|^2 - d^2} < \frac{\frac{3}{5r}}{\left(\frac{3}{5r}\right)^2 - \frac{(n-1)^2}{\rho^2}} < \frac{\frac{3}{5r}}{\left(\frac{3}{5r}\right)^2 - \frac{1}{5r^2}} < \frac{15}{8} r. \qquad \square$$

Theorem 1. *Let the sequence of disks* $\{Z^{(m)}\}$ *be defined by the iterative method* (8), *assuming that the initial disk* $Z^{(0)} = \{a;\eta\}$ *is chosen so that the condition* (9) *is satisfied. Then, the Newton-like method* (8) *is convergent, and the following is true in each iterative step:*

1° $\zeta \in Z^{(m)}$;

2° $r^{(m+1)} < \dfrac{25(n-1)}{4\eta}\big[r^{(m)}\big]^2.$

Proof. The proof of the assertion 1° follows from the construction of the method (8), based on the inclusion isotonicity and the relation (7), and the fact that $z^{(m)} \in \{a;\eta\}$ for each $m = 0, 1, \ldots$, which is obvious because of $\eta - |z^{(m)} - a| = \rho^{(m)} > 5(n-1)r^{(m)} > 0$.

We recall that the initial condition (9) implies the inequality (14), which will be used in the convergence analysis. From (15) we obtain

$$r^{(2)} = \operatorname{rad} Z^{(2)} = \frac{d^{(1)}}{|c^{(1)}|^2 - \big[d^{(1)}\big]^2}.$$

Hence, using (14), (18) and (19), we find

$$\begin{aligned} r^{(2)} &< \frac{n-1}{\rho^{(1)}\Big[\Big(\dfrac{3}{5r^{(1)}}\Big)^2 - \Big(\dfrac{n-1}{\rho^{(1)}}\Big)^2\Big]} < \frac{n-1}{\rho^{(1)}\Big[\Big(\dfrac{3}{5r^{(1)}}\Big)^2 - \Big(\dfrac{1}{5r^{(1)}}\Big)^2\Big]} \\ &< \frac{25(n-1)\big[r^{(1)}\big]^2}{8\rho^{(1)}}. \end{aligned} \tag{20}$$

Starting from (20), we find by (14)

$$r^{(2)} < \frac{25(n-1)r^{(1)}}{8\rho^{(1)}/r^{(1)}} < \frac{5}{8}r^{(1)}, \tag{21}$$

which means that the disk $Z^{(2)}$ is contracted compared to $Z^{(1)}$. Using Lemma 2 we obtain

$$\rho^{(2)} > \eta - |z^{(2)} - a| = \eta - \left| z^{(1)} - a - \frac{\bar{c}^{(1)}}{|c^{(1)}|^2 - \big[d^{(1)}\big]^2} \right| > \rho^{(1)} - \frac{|\bar{c}^{(1)}|}{|c^{(1)}|^2 - \big[d^{(1)}\big]^2},$$

that is,

$$\rho^{(2)} > \rho^{(1)} - \frac{15}{8}r^{(1)}. \tag{22}$$

Taking into account (14), (21) and (22), we find

$$\rho^{(2)} > \rho^{(1)} - \frac{15}{8}r^{(1)} > 5(n-1)r^{(1)} - \frac{15}{8}r^{(1)} > \frac{8}{5}r^{(2)}\Big[5(n-1) - \frac{15}{8}\Big],$$

and whence, for $n \geq 3$,

$$\rho^{(2)} > 5(n-1)r^{(2)}. \tag{23}$$

Therefore, we have proved the implications

$$\left|\frac{P(a)}{P'(a)}\right| < \frac{\eta}{3(n-1)} \quad \Rightarrow \quad \rho^{(1)} > 5(n-1)r^{(1)} \quad \Rightarrow \quad \rho^{(2)} > 5(n-1)r^{(2)}.$$

This chain of implications has the key role in the proof by induction. Besides, since (23) holds, then it follows $0 \notin \{c^{(2)}; d^{(2)}\}$ and the inclusion method (15) is well defined for $m = 2$.

Assume that for $m \geq 2$ the following is true:

$$r^{(m)} < \frac{25(n-1)\left[r^{(m-1)}\right]^2}{8\rho^{(m-1)}}, \tag{24}$$

$$r^{(m)} < \frac{5}{8}r^{(m-1)}, \tag{25}$$

$$\rho^{(m)} > 5(n-1)r^{(m)}, \tag{26}$$

$$\rho^{(m)} > \rho^{(m-1)} - \frac{8}{5}r^{(m-1)}. \tag{27}$$

These inequalities have already been proved for $m = 2$. We will prove that they are valid for the index $m + 1$.

Applying the above consideration for $m = 2$ and (26), we obtain

$$r^{(m+1)} < \frac{25(n-1)\left[r^{(m)}\right]^2}{8\rho^{(m)}} < \frac{5}{8}r^{(m)}. \tag{28}$$

In the similar way as for $m = 1$, it is easy to show that

$$\rho^{(m+1)} > 5(n-1)r^{(m+1)} \quad \text{and} \quad \rho^{(m+1)} > \rho^{(m)} - \frac{8}{5}r^{(m)}.$$

By the successive application of (25) and (27), we find

$$\begin{aligned}
\rho^{(m)} &> \rho^{(m-1)} - \frac{8}{5}r^{(m-1)} > \rho^{(m-2)} - \frac{8}{5}r^{(m-2)} - \frac{8}{5}r^{(m-1)} \\
&> \rho^{(m-2)} - \frac{8}{5}r^{(m-2)} - \frac{8}{5}\cdot\frac{5}{8}r^{(m-2)} = \rho^{(m-2)} - \frac{8}{5}r^{(m-2)}\Big(1 + \frac{5}{8}\Big) \\
&\;\vdots \\
&> \rho^{(1)} - \frac{8}{5}r^{(1)}\Big(1 + \frac{5}{8} + \Big(\frac{5}{8}\Big)^2 + \cdots\Big) > \rho^{(1)} - \frac{8}{5}\cdot\frac{1}{1-\frac{5}{8}}r^{(1)} > \rho^{(1)} - 5r^{(1)}.
\end{aligned}$$

Using (11) and (13) we estimate

$$\rho^{(m)} > \rho^{(1)} - 5r^{(1)} > \frac{13}{16}\eta - 5\cdot\frac{\eta}{16} = \frac{\eta}{2}.$$

Substituting this bound in (28) we obtain

$$r^{(m+1)} < \frac{25(n-1)\big[r^{(m)}\big]^2}{8\rho^{(m)}} < \frac{25(n-1)\big[r^{(m)}\big]^2}{8(\eta/2)} = \frac{25(n-1)}{4\eta}\big[r^{(m)}\big]^2,$$

and the assertion 2° is proved. The last relations point to the quadratic convergence of the NEWTON-like method (8).

The interval method (15) is well defined in each iterative step since $0 \notin \{c^{(m)}; d^{(m)}\}$ (according to Lemma 2). The convergence of the NEWTON-like method (8) follows according to (25); indeed, the sequence of radii $\{r^{(m)}\}$ converges to 0.

Finally, let us note that $Z^{(m)} \subset A = \{a;\eta\}$ for each m, thus, there is no possibility that any disk $Z^{(m)}$ includes zeros lying outside the initial disk A (see Fig. 1). Indeed, since $\rho^{(m)} > 5(n-1)r^{(m)} > r^{(m)}$, we have

$$\rho^{(m)} = \eta - |z^{(m)} - a| > r^{(m)}, \quad \text{that is,} \quad |z^{(m)} - a| < \eta - r^{(m)}.$$

Hence, according to (3), it follows $Z^{(m)} = \left\{z^{(m)}; r^{(m)}\right\} \subset \{a;\eta\} = A$. $\square$

4. NUMERICAL EXAMPLE

We have applied the NEWTON-like method (8) to the algebraic polynomial

$$P(z) = z^7 + z^5 - 10z^4 - z^3 - z + 10$$

with the zeros 2, ± 1, $\pm i$, $-1 \pm 2\,i$. Applying methods for the existence of zeros and their localization, and the proximity test for detecting zeros, we have found that the disk $A = \{0.1 + 0.9\,i; 1.5\}$ contains only one zero of P. Inclusion disks obtained by (8) are given below:

$$\begin{aligned}
Z^{(1)} &= \{0.14083 + 1.07444\,i; 0.1976\},\\
Z^{(2)} &= \{0.010327 + 0.974089\,i; 0.08369\},\\
Z^{(3)} &= \{-0.001419 + 1.001066\,i; 0.00344\},\\
Z^{(4)} &= \{-4.88 \times 10^{-6} + 0.99999699\,i; 1.27 \times 10^{-5}\},\\
Z^{(5)} &= \{5.88 \times 10^{-11} + 0.999999999987\,i; 1.32 \times 10^{-10}\},\\
Z^{(6)} &= \{-1.04 \times 10^{-21} + 0.9999999999999999999934\,i; 1.46 \times 10^{-20}\}.
\end{aligned}$$

All presented disks contain the exact zero $\zeta = i$.

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ON UNICYCLIC REFLEXIVE GRAPHS

Zoran Radosavljević

A simple graph is said to be reflexive if the second largest eigenvalue of its $(0,1)$-adjacency matrix does not exceed 2. Based on some recent results on reflexive graphs with more cycles and some new observations, we construct in this paper several classes of maximal unicyclic reflexive graphs.

1. INTRODUCTION

If G is a simple graph (a non-oriented graph without loops or multiple edges), its $(0,1)$-adjacency matrix A is symmetric and roots of the *characteristic polynomial* $P_G(\lambda) = \det(\lambda I - A)$ (the *eigenvalues* of G, making up its *spectrum*) are all real numbers, for which we assume their non-increasing order: $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$. In a connected graph for the largest eigenvalue λ_1 (the *index* of the graph) $\lambda_1 > \lambda_2$ holds, which need not take place otherwise, since the spectrum of a disconnected graph is the union of spectra of its components. The interrelation between the spectra of a graph and its induced subgraphs is established by the *interlacing theorem*:

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a graph G and $\mu_1 \geq \mu_2 \geq ... \geq \mu_m$ eigenvalues of its induced subgraph H. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, \ldots, m)$ hold.

Thus e.g. if $m = n-1$, $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2, \ldots$, and also $\lambda_1 > \mu_1$ if G is connected.

Reflexive graphs are graphs having $\lambda_2 \leq 2$. They correspond to some sets of vectors in the Lorentz space $R^{p,1}$ and have some applications to the construction and classification of reflection groups [**7**]. Reflexive graphs that have been investigated so far are trees [**4**], [**6**], some classes of bicyclic graphs [**10**], [**13**] (see also [**8**]) and various classes of cactuses with more than two cycles [**5**], [**9**], [**11**], [**12**].

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A *cactus*, or a *treelike graph*, is a graph in which any two cycles have at most one common vertex, i.e. are edge-disjoint. A vertex of a cycle in a cactus is said to be *loaded* if its degree is greater than 2. A cycle of a cactus is a *free cycle* if it has only one vertex of degree greater than 2.

In this paper we consider unicyclic reflexive graphs. According to the interlacing theorem, for any given number A, any graphic property $\lambda_i \leq A$ is a *hereditary* one, i.e. all induced subgraphs preserve this property, and that is why it is natural to present reflexive graphs through sets of *maximal* (connected) graphs, of course inside the considered class.

Graph G is a maximal reflexive graph inside a given class of graphs C if G is reflexive and any extension $G+v$ that belongs to C has $\lambda_2 > 2$.

Some important general and auxiliary facts, which are essential for further investigations, are given in Section 2. The rest of the article is devoted to its aim - the construction of classes of maximal unicyclic reflexive graphs.

2. SOME FORMER, GENERAL AND AUXILIARY RESULTS

Connected graphs that have $\lambda_1 = 2$ are known as *Smith graphs.*

Lemma 1 ([**15**]). *For a simple graph G $\lambda_1(G) \leq 2$ (resp. $\lambda_1(G) < 2$) if and only if each component of G is an induced subgraph (resp. proper induced subgraph) of one of the graphs of Fig. 1, all of which have index equal to 2.*

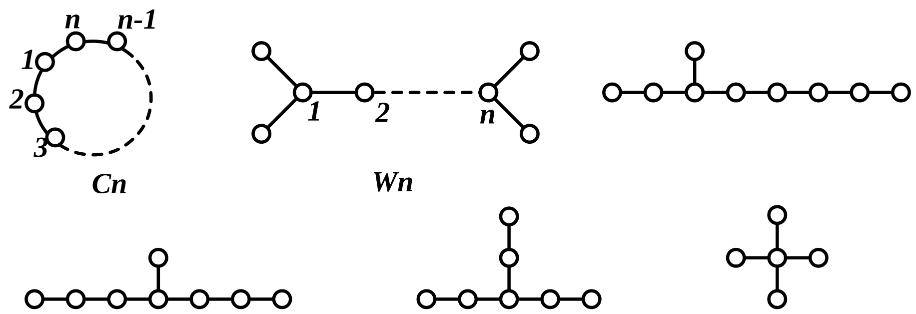

Figure 1.

(In what follows, when saying "subgraph" we will always understand "induced subgraph".)

Lemma 2. ([**14**]). *Given a graph G, let $C(v)$ $(C(uv))$ denote the set of all cycles containing a vertex v and an edge uv of G, respectively. Then*

(i) $P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum\limits_{u\in \mathrm{Adj}(v)} P_{G-v-u}(\lambda) - 2\sum\limits_{C\in \mathcal{C}(v)} P_{G-V(C)}(\lambda),$

(ii) $P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2\sum\limits_{C\in \mathcal{C}(uv)} P_{G-V(C)}(\lambda),$

where $Adj(v)$ denotes the set of neighbors of v, while $G-V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.

These relations have the following consequences (see, e.g. [**1**], p. 59).

Corollary 1. *Let G be a graph obtained by joining a vertex v_1 of a graph G_1 to a vertex v_2 of a graph G_2 by an edge. Let G_1' (G_2') be the subgraph of G_1 (G_2) obtained by deleting the vertex v_1 (v_2) from G_1 (resp. G_2). Then*

$$P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G_1'}(\lambda)P_{G_2'}(\lambda).$$

Corollary 2. *Let G be a graph with a pendant edge v_1v_2, v_1 being of degree 1. Then*

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda),$$

where G_1 (G_2) is the graph obtained from G (resp. G_1) by deleting the vertex v_1 (resp. v_2).

A list of values of $P_G(2)$ for some small graphs is a useful tool in any search for reflexive graphs.

Lemma 3 [**13**]. *Let $G_1, \ldots, G_4$ be the graphs depicted in Fig. 2. Then*

1. $P_{G_1}(2) = k + 2$;
2. $P_{G_2}(2) = 4$;
3. $P_{G_3}(2) = -k\ell m + k + \ell + m + 2$;
4. $P_{G_4}(2) = 4(1 - k\ell)$;

(k, ℓ, m are lengths of corresponding paths).

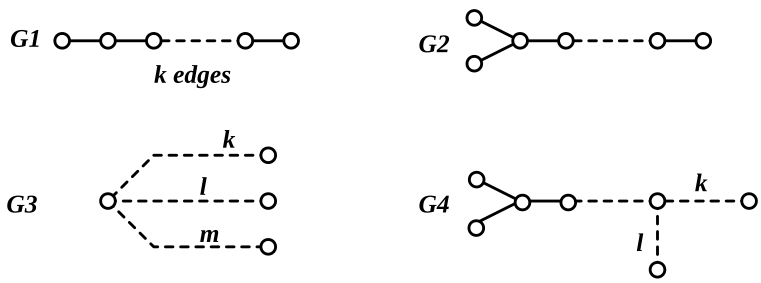

Figure 2.

First supergraphs of Smith graphs have the following property.

Lemma 4 ([**13**]). *Let G be a graph obtained by extending any of Smith graphs by a vertex of arbitrary positive degree. Then $P_G(2) < 0$ (i.e. $\lambda_2(G) < 2 < \lambda_1(G)$).*

The next general theorem can be used to detect a lot of reflexive graphs.

Theorem RS ([**13**]). *Let G be a graph with cut-vertex u.*

(i) *If at least two components of $G-u$ are supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then $\lambda_2(G) > 2$.*

(ii) *If at least two components of* $G-u$ *are Smith graphs, and the rest are subgraphs of Smith graphs, then* $\lambda_2(G)=2$.

(iii) *If at most one component of* $G-u$ *is a Smith graph, and the rest are proper subgraphs of Smith graphs, then* $\lambda_2(G)<2$.

This theorem can be applied to a wide class of graphs with a cut-vertex, but if it comes about that $G-u$ consists of one proper supergraph and the rest of proper subgraphs of Smith graphs, it cannot answer whether the graph is reflexive or not and such cases will be called *RS-indefinite*. In our current investigations we always presuppose that maximal reflexive graphs we are looking for are RS-indefinite.

It turns out that a free cycle in a maximal reflexive cactus can be replaced under some conditions by an arbitrary Smith tree.

Theorem R (The theorem of replacement) [**10**]. *Suppose that a graph of the form shown in Fig.* 3(a) *is a maximal reflexive cactus for which* $P(2)=0$ *and* $P_G(2)<0$ *and for any extension* G_1 *formed by attaching to* G *a pendant edge at any vertex* $P_{G_1}(2)-2P_{G_1-v}(2)>0$ *holds. If the free cycle* C *(of arbitrary length) is replaced by an arbitrary Smith tree* S*, attached to the vertex* v *in an arbitrary way (i.e. at an arbitrary vertex of* S*), then the resulting graph (Fig.* 3(b)) *is again a maximal reflexive cactus.*

If we form a tree T by identifying vertices u_1 and u_2 ($u_1=u_2=u$) of two (rooted) trees T_1 and T_2, respectively (the *coalescence* $T_1\cdot T_2$ of T_1 and T_2), we usually say that T can be *split* at its vertex u into T_1 and T_2 (Fig. 4(a)). Of course, splitting at a given vertex is not determined uniquely if its degree is greater than 2.

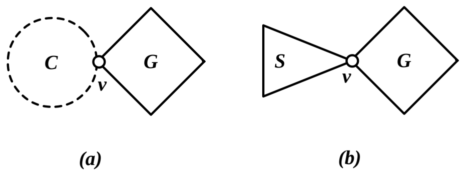

Figure 3.

If we split a tree T at all its vertices in all possible ways, and in each case attach the parts at vertices of splitting u_1 and u_2 to some vertices v_1 and v_2 of a graph G (i.e. lean the parts on G by identifying u_1 with v_1 and u_2 with v_2, and vice versa), we shall say that in the obtained family of graphs the tree T *pours* between v_1 and v_2 (Fig. 4(b)). Of course, this includes attachment of the complete tree T, rooted at any vertex v, to v_1 and v_2. Pouring of Smith trees turns out to be a very important tool in describing some classes of maximal reflexive graphs ([**5**], [**9**], [**10**], [**12**], [**13**]).

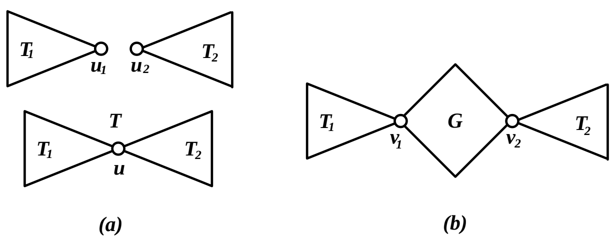

Figure 4.

The result of the next Lemma was already used in [**10**]. Now, we shall formalize the statement.

Lemma 5. *Let a Smith tree S be split at vertex u* $(\deg u > 1)$ *into its subtrees S_1 and S_2 and let us introduce the notation*

$$P_{S_1-u}(2) = p_1, P_{S_2-u}(2) = p_2;$$

$$\sum_{v\in Adj(u)\cap S_1} P_{S_1-u-v}(2) = \Sigma_1, \qquad \sum_{v\in Adj(u)\cap S_2} P_{S_2-u-v}(2) = \Sigma_2.$$

Then $\Sigma_1 = \alpha p_1$, $\Sigma_2 = (2-\alpha)\, p_2$, for six different possible values of α.

Proof. If we split W_n into two analogous parts, then $p_1 = \Sigma_1 = p_2 = \Sigma_2 = 4$ and $\alpha = 1$. In the remaining cases one of the two parts S_1, S_2 must be a path, and let it be S_1. We see by an easy calculation based on the application of Lemma 3 that $\Sigma_1 = \alpha p_1$ for $\alpha = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}$, depending on whether the path is of length $1, 2, 3, 4, 5$, respectively (and assuming that for a void graph (with no vertices) $P(2) = 1$). Then for all these values of α, $\Sigma_2 = (2-\alpha)\, p_2$ holds, which completes the proof.

3. CLASSES OF MAXIMAL UNICYCLIC REFLEXIVE GRAPHS

Thus far, unicyclic reflexive graphs have not been the subject of any consideration and there are no published results about them. The general problem, to find or describe all such maximal graphs, seems intractable. It is sufficient to have a look at Theorem RS to realize that these graphs can have an arbitrary number of vertices, the cycle can be of arbitrary length, they can have a vertex of arbitrary degree and, after its removal, the remaining graph can have an arbitrary number of components. This means that the investigations should be directed towards specified classes and recent considerations of reflexive graphs with more cycles lead to a number of such classes.

3.1. GRAPHS GENERATED BY MAXIMAL BICYCLIC REFLEXIVE GRAPHS WITH THE BRIDGE BETWEEN THE CYCLES

All maximal reflexive bicyclic graphs whose two cycles are connected by a bridge c_1c_2 were determined in [**13**]. For some practical reasons, in this result a distinction has been made between black vertices of the cycles (those being adjacent to c_1 or c_2) and white vertices (neither black ones nor c_1 or c_2).

Now, based on this result and by applying Theorem R, we can obtain a class of maximal unicyclic reflexive graphs. By inspection of all resulting graphs of [**13**], we see that among those of them that have exactly one free cycle there are those with $\lambda_2 = 2$ as well as others having $\lambda_2 < 2$. One can make sure that the conditions of Theorem R are satisfied always when $\lambda_2 = 2$, and the corresponding unicyclic graphs are displayed in Fig. 5 (all cases with a loaded white vertex) and Fig. 6 (a loaded black vertex , including a case with two loaded black vertices). Clearly, in all these graphs S is an arbitrary Smith tree, rooted at an arbitrary vertex.

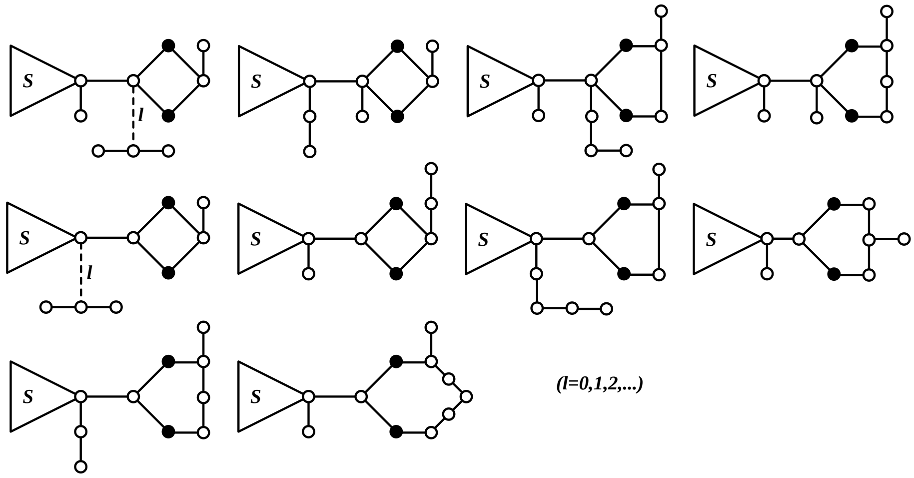

Figure 5.

3.2. POURING OF PAIRS OF SMITH TREES (1)

If we apply Theorem RS to the vertex c_1 of the tricyclic (family of) graphs of Fig. 7(a), we see that $\lambda_2 = 2$, and these graphs are not maximal reflexive graphs since they can be extended at vertices of the free cycle attached to c_1 (bounds of such extensions are just determined by Theorem RS). If we move the other free cycle, e.g. from c_3, to c_2 (Fig. 7(b)), again $\lambda_2 = 2$, but now we have a family of maximal tricyclic reflexive graphs [**13**]. Also, if a Smith tree pours between c_2 and c_3 (Fig. 7(c)), all such graphs are maximal reflexive graphs inside the class of bicyclic graphs with a bridge between the cycles [**13**].

Now, consider the (family of) unicyclic graphs, displayed in Fig. 7(d), where two Smith trees, $S_1 \cdot S_1^{'}$ and $S_2 \cdot S_2^{'}$, pour between the vertices c_2 and c_3. It was established in [**5**] (Lemma 5) that such a graph has $\lambda_2 = 2$ and that any extension by a pendant edge at any vertex of S_1, S_2, $S_1^{'}$ or $S_2^{'}$ implies $\lambda_2 > 2$. In order to construct a class of maximal unicyclic reflexive graphs, we should only examine the possibilities of extension at the vertices of the free cycle attached to c_1.

Theorem 1. *A graph of Fig.* 7(d) *is a maximal unicyclic reflexive graph if and only if it is RS-indefinite, i.e. if and only if the two coalescences at* c_2 *and* c_3 $(S_1 \cdot S_2$ *and* $S_1^{'} \cdot S_2^{\prime})$ *are not Smith trees.*

Proof. If $S_1 \cdot S_2$, and then consequently $S_1^{'} \cdot S_2^{'}$, are Smith trees, Theorem RS gives $\lambda_2 = 2$ and an extension at the vertices of the cycle is possible up to the boundaries when we get the third Smith tree.

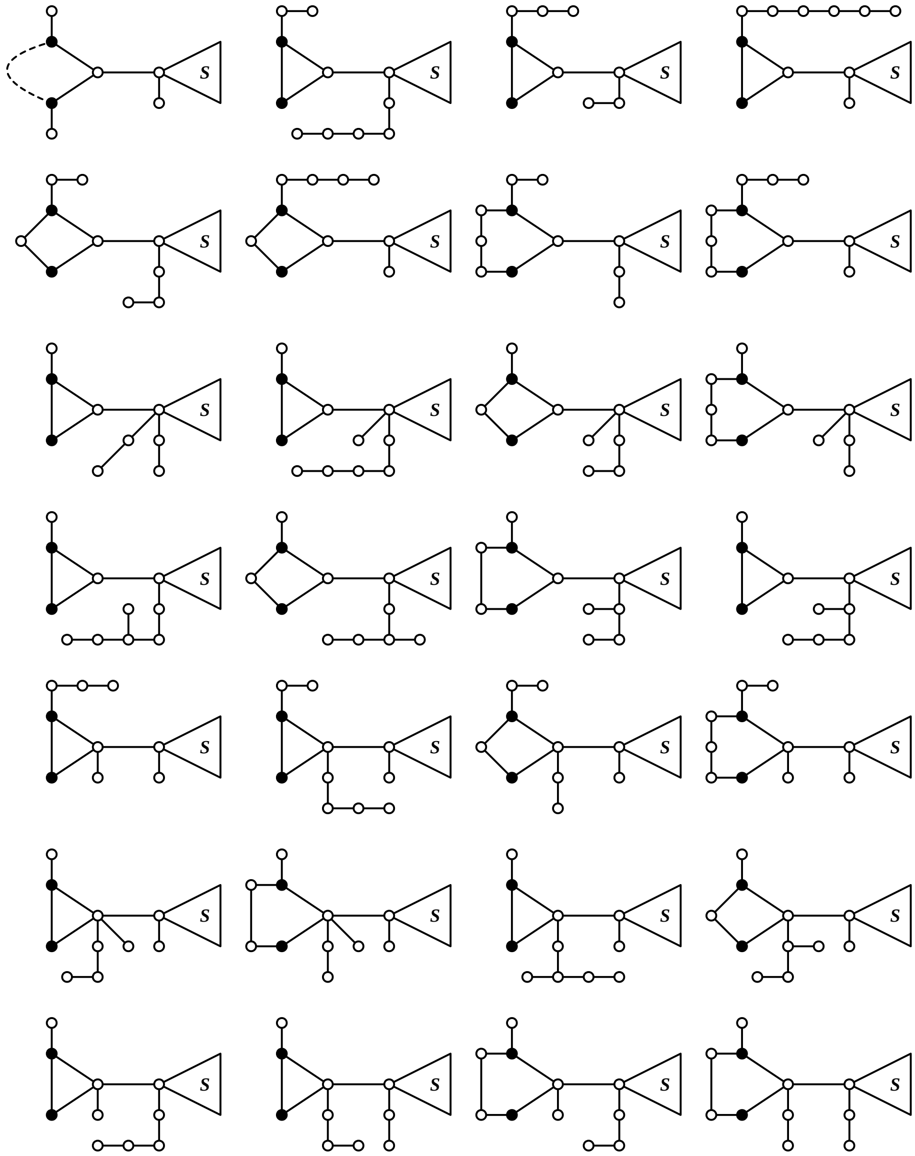

Figure 6.

Suppose now that $S_1 \cdot S_2$ and $S_1^{'} \cdot S_2^{'}$ are not Smith trees, i.e. that one of them, say $S_1 \cdot S_2$ is a proper subgraph of a Smith tree (then $P_{S_1 \cdot S_2}(2) > 0$ holds), and the other a proper supergraph ($P_{S_1' \cdot S_2'}(2) < 0$). If we extend the graph by a pendant edge at the vertex c_1, apply Corollary 2 to this edge and use Lemma 3(1), we get

$$P(2) = 0 - nP_{S_1 \cdot S_2}(2)\, P_{S_1' \cdot S_2'}(2) > 0,$$

which means that such a graph is no more reflexive.

Let us consider now the general case of extension (Fig. 7(e)) and let us introduce the following notation:

$$P_{S_i - c_2}(2) = p_i, P_{S_i^{'} - c_3}(2) = p_i^{'};$$

$$\sum_{v \in S_i \cap Adj(c_2)} P_{S_i - c_2 - v}(2) = \Sigma_i, \sum_{v \in S_i^{'} \cap Adj(c_3)} P_{S_i^{'} - c_3 - v}(2) = \Sigma_i^{'},$$

where $i = 1, 2$. According to Lemma 2.(i), for the two coalescences at c_2 and c_3 the following relations hold:

$$\begin{aligned} P_{S_1 \cdot S_2}(2) &= 2p_1 p_2 - p_1 \Sigma_2 - p_2 \Sigma_1, \\ P_{S_1^{'} \cdot S_2^{'}}(2) &= 2p_1^{'} p_2^{'} - p_1^{'} \Sigma_2^{'} - p_2^{'} \Sigma_1^{'}. \end{aligned}$$

Applying now Corollary 2, and then Lemma 2.(i) to (the vertex c_1 of) the remaining graphs, and using also Lemma 3.(1), we obtain:

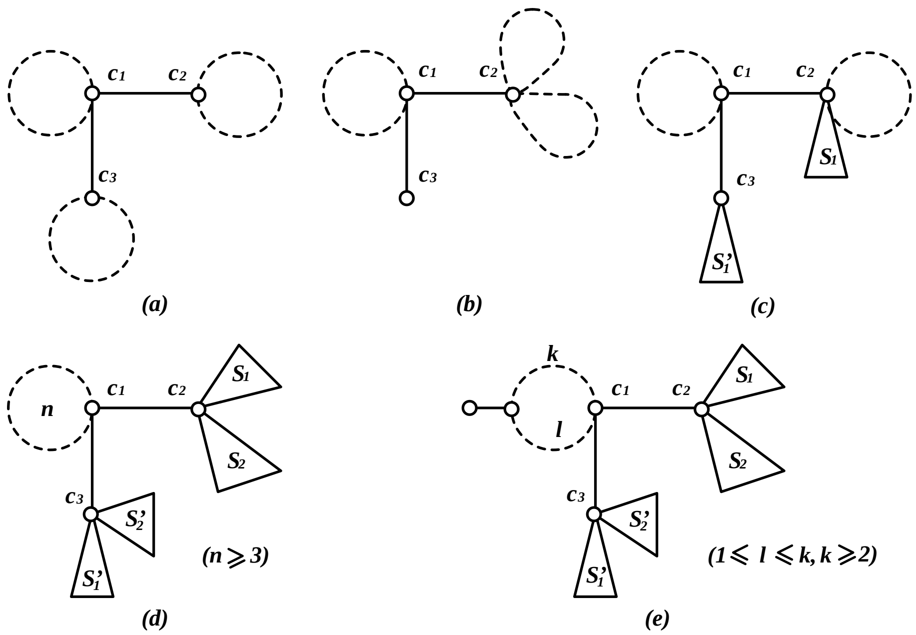

Figure 7.

$$\begin{aligned} P(2) &= 0 - (2k\ell - k(\ell - 1) - (k-1)\ell)\,(2p_1 p_2 - p_1 \Sigma_2 - p_2 \Sigma_1) \cdot \\ &\cdot \Big(2p_1^{'} p_2^{'} - p^{'} \Sigma_2^{'} - p_2^{'} \Sigma_1^{'}\Big) + k\ell \Big(p_1 p_2 \,(2p_1^{\prime} p_2^{\prime} - p_1^{\prime} \Sigma_2^{\prime} - p_2^{\prime} \Sigma_1^{\prime}) \\ &\quad + p_1^{\prime} p_2^{\prime}\,(2p_1 p_2 - p_1 \Sigma_2 - p_2 \Sigma_1)\Big). \end{aligned}$$

Now, according to Lemma 5, $\Sigma_1 = \alpha p_1$, $\Sigma_1' = (2-\alpha)\, p_1'$ and also $\Sigma_2 = \beta p_2$, $\Sigma_2' = (2-\beta)\, p_2'$, which gives

$$P\,(2) = (k+l)\, p_1 p_2 p_1' p_2' \,(\alpha+\beta-2)^2 \geq 0.$$

Since S_1, S_2, $S_1^{'}$ and $S_2^{'}$ all are parts of SMITH trees, it is clear from the list of numbers that appear in the proof of Lemma 5 that, if $S_1 \cdot S_2$ and $S_1' \cdot S_2'$ are not SMITH trees, then $\beta \neq 2-\alpha$ holds. But since $P\,(2) > 0$ means $\lambda_2 > 2$, the extension is not possible and the proof is complete.

3.3. POURING OF PAIRS OF SMITH TREES (2)

If we introduce a new vertex c_4 to the graph of Fig. 7(d), join it to c_2 and c_3, and then attach to c_4 a free cycle (Fig. 8(a)), such a graph still has $\lambda_2 = 2$ and is a maximal tricyclic reflexive graph. It cannot be extended at vertices of two pouring SMITH trees because of (the already mentioned) Lemma 5 of [**5**], while any extension at vertices of the free cycles is impossible because the removal of c_2 and application of Theorem RS to c_3 would give $\lambda_2 > 2$.

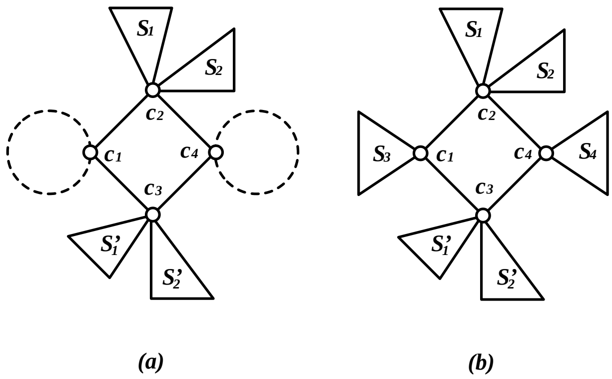

Figure 8.

Can the two free cycles be replaced by two SMITH trees S_3 and S_4? In this case Theorem R cannot be applied ($P_G\,(2) < 0$ does not hold).

In the same way as it was done in [**5**], we can verify the fact that a graph obtained by removing c_4 from the case (b) also allows no extension at the vertices of S_i and $S_i^{'}$ $(i = 1, 2)$. As for vertices of S_3 and S_4, no extension is possible for the same reason as at free cycles in the case (a). Thus, if the graph of Fig. 8(b) has $\lambda_2 = 2$, it is a maximal unicyclic reflexive graph. On the other hand, no counter-example ($\lambda_2 > 2$) is known, but the case has to be verified by a computer.

Conjecture. *All graphs of the form of Fig.* 8(b) *have* $\lambda_2 = 2$ *and therefore all of them are maximal unicyclic reflexive graphs.*

3.4. POURING OF TRIPLES OF SMITH TREES

Consider the family of bicyclic graphs in Fig. 9(a): a free cycle is attached to a vertex c_1 of a triangle, while three SMITH trees pour between c_2 and c_3. Let p_i and Σ_i $(i = 1, 2, 3)$ have the same meaning as in Theorem 1. According to a result of [**10**], such graphs are maximal bicyclic reflexive graphs, with the following three exceptions:

1) two complete SMITH trees are attached to c_2 and c_3, respectively, while the third (pouring) tree is W_n, split into two analogous parts;

2) a complete SMITH tree S_1 is attached to, say, c_2, while each of two remaining SMITH trees is split into K_2, attached to c_2, and $S_i^{'}$ $(i = 2, 3)$, attached to c_3;

3) for one of the two coalescences of three parts of three pouring SMITH trees, say S_1, S_2, S_3, there exist corresponding parts $\bar{S}_1$, $\bar{S}_2$, $\bar{S}_3$ such that S_i and $\bar{S}_i$ $(i = 1, 2, 3)$ have the same values p_i and Σ_i (i.e. belong to the same one of the six classes described in Lemma 5), which, of course, includes the possibility $S_i = \bar{S}_i$ for some i, and such that the analogous coalescence generated by $\bar{S}_1$, $\bar{S}_2$ and $\bar{S}_3$ consists of a complete SMITH tree and two additional pendant edges (as in case(2)).

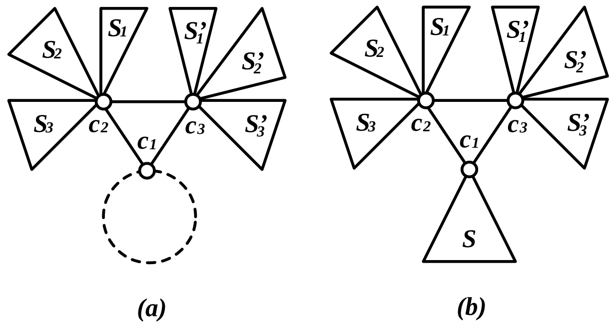

Figure 9.

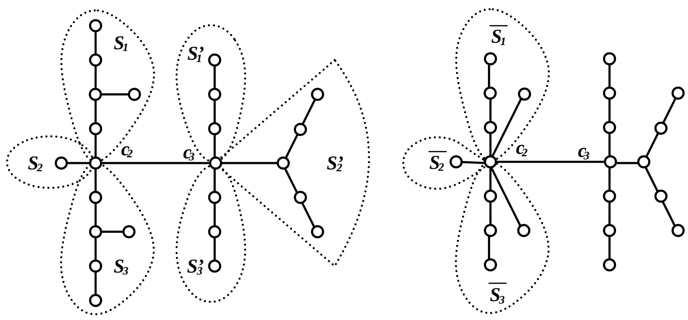

Figure 10.

The graphs of Fig. 10 illustrate the description of case (3).

In these three cases graphs of Fig. 9(a) are not maximal and can be extended at some vertices of the cycle attached to c_1; the resulting maximal graphs are also found in [**10**], and the same family of exceptional maximal graphs appears in all three exceptional cases described above.

Based on the analysis which led to this result of [**10**] (the removal of c_1 and the application of Corollary 1 to the bridge c_2c_3 may give $P(2) < 0$ or $P(2) = 0$), one can make sure that the theorem of replacement (Theorem R) can apply to graphs of Fig. 9(a) and generate those of Fig 9(b) except exactly in the three exceptional cases. On the other hand, if some of these three cases occurs, replacement of the free cycle by Smith trees gives graphs that are not maximal and allow further extensions. Finding maximal graphs in these cases requires additional investigation.

Theorem 2. *Let a graph G consist of a triangle, a Smith tree S attached (in an arbitrary way) to its vertex c_1, and let a triple of Smith trees pour between the remaining two vertices c_2 and c_3 (Fig. 9(b)). If G is none of the three exceptional cases, described above, then G is a maximal unicyclic reflexive graph.*

3.5. MAXIMUM NUMBER OF LOADED VERTICES

As we have seen, the cycle of a maximal unicyclic reflexive graph need not have more than one loaded vertex. We are going now to examine the case of maximum number of loaded vertices.

Theorem 3. *The cycle of unicyclic reflexive graph of length greater than 8 cannot have more than 7 loaded vertices.*

Proof. In C_9 (the cycle of length 9) one can verify by direct calculation that there cannot be 8 loaded vertices.

Let now the length of the cycle be at least 10 and suppose that it has two vertices, u and v, such that, after deleting them, each component contains at least 4 loaded vertices. In this case each such component is a proper supergraph of W_n, and, according to Theorem RS, $\lambda_2 > 2$.

If such vertices u and v do not exist, then on the cycle there must be $8, 7, 6$ or 5 consecutive loaded vertices, or loaded vertices must be grouped in 3 sets of consecutive vertices of the form $3 + 3 + 2$.

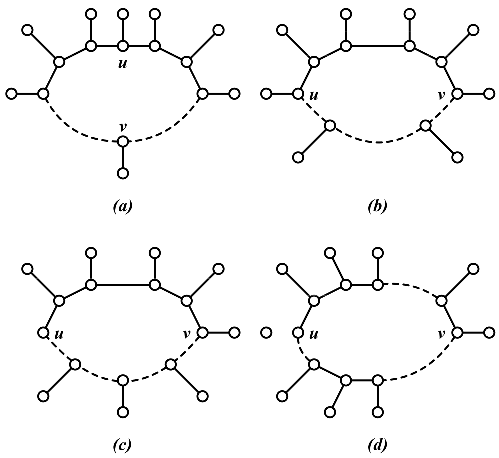

Figure 11.

Case 1 (8 consecutive vertices): if we delete all other (non-loaded) vertices of the cycle, the remaining graph has $\lambda_2 > 2$.

Case 2 (7 vertices): after deleting the two vertices u and v in Fig. 11(a), we obtain two proper supergraphs of W_n.

Case 3 (6 vertices): the deletion of the two vertices u and v in Fig. 11(b) gives rise to two supergraphs of W_n, at least one of them being proper.

Case 4 (5 vertices): as in previous cases, we get two proper supergraphs of W_n (Fig. 11(c)).

Case 5 $(3+3+2)$: now the cycle is at least C_{11} and we again have two proper supergraphs of W_n (Fig. 11(d)).

The proof is complete.

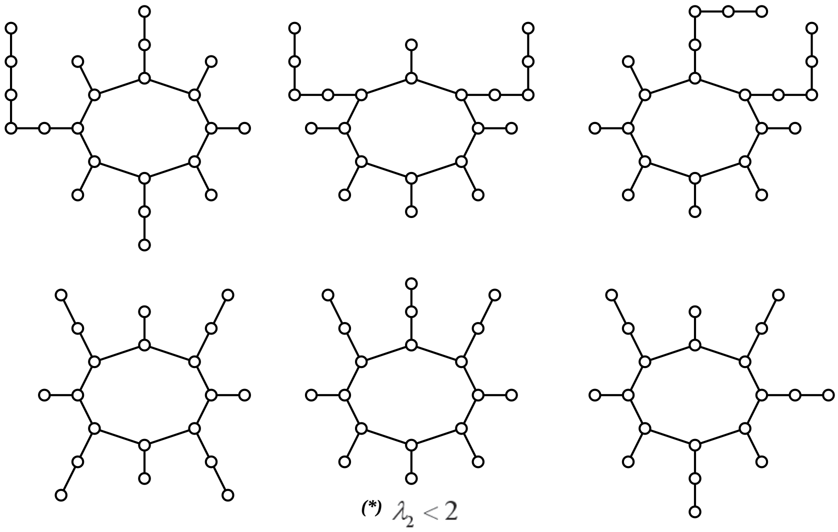

Figure 12.

However, C_8 can have all the vertices loaded and direct checking shows that there are six such cases.

Theorem 4. *The maximum number of loaded vertices of the cycle of a maximal unicyclic reflexive graph is* 8. *There are six such graphs and they are displayed in Fig.* 12.

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ON A CLASS OF MAXIMAL REFLEXIVE θ-GRAPHS GENERATED BY SMITH GRAPHS

Marija Rašajski

A simple graph is said to be reflexive if its second largest eigenvalue does not exceed 2. The property $\lambda_2 \leq 2$ is a hereditary one, i.e. any induced subgraph of a reflexive graph preserves this property and that is why reflexive graphs are usually represented by maximal graphs within a given class. Bicyclic graphs whose two cycles have a common path are called θ-graphs. We consider classes of maximal reflexive θ-graphs arising from a Smith tree and a cycle attached to it in a specified way.

1. INTRODUCTION

Let $P_G(\lambda) = \det(\lambda I - A)$ be the *characteristic polynomial* of the $(0,1)$- adjacency matrix of a simple graph G (an undirected graph without loops or multiple edges). The roots of $P_G(\lambda)$ are the *eigenvalues* of G. The family of these roots forms the *spectrum* of G. The eigenvalues of a simple graph are real, and we assume their non-increasing order: $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$. The relation between the spectrum of a graph and the spectra of its induced subgraphs is established by the *interlacing theorem*:

Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$ be the eigenvalues of a graph G and $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_m$ eigenvalues of its induced subgraph H. Then the inequalities $\lambda_{n-m+i} \leq \mu_i \leq \lambda_i$ $(i = 1, \ldots, m)$ hold.

Thus, for example, if $m = n-1$, $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2$. Also, $\lambda_1 > \mu_1$ if G is connected.

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Graphs with the property $\lambda_2 \leq 2$ are called *reflexive graphs* and if $\lambda_2 \leq 2 \leq \lambda_1$ they are also called *hyperbolic graphs,* ([**7**], [**8**]).

The terminology concerning graph spectra follows [**2**], while for general graph theoretic concepts one can see [**4**].

Since the graphic property $\lambda_2 \leq 2$ is hereditary (every induced subgraph maintains the property), the result is expressed through the set of maximal graphs within a given class.

Bicyclic graphs whose two cycles have a common path are called θ-*graphs*.

Smith graphs are connected graphs with the property $\lambda_1 = 2$. Smith graphs are widely present in the sets of maximal reflexive graphs investigated so far. Many families of such graphs can be described completely or almost completely by Smith graphs.

So far various classes of reflexive graphs have been studied such as: reflexive trees ([**5**], [6]), bicyclic reflexive graphs with a bridge between the cycles [**13**], treelike reflexive graphs with three or more cycles ([**9**], [**11**], [**12**], [**14**], [**15**]), some classes of bicyclic reflexive graphs [**9**], [**10**], and there are some preliminary results on θ-graphs [**10**], [**14**], [**16**].

In this paper we construct a class of maximal reflexive θ-graphs using Smith graphs.

Some general and auxiliary results to be used in our investigations are presented in the next section. At some stages the work has been supported by using the expert system GRAPH ([**1**], [**3**]).

2. PRELIMINARIES

The following theorem gives useful interrelations between the characteristic polynomial of a graph and its induced subgraphs.

Lemma 1. (Schwenk [**17**]). *Given a graph G, let $C(v)$ $(C(uv))$ denote the set of all cycles containing a vertex v and an edge uv of G, respectively. Then*

(i) $P_G(\lambda) = \lambda P_{G-v}(\lambda) - \sum\limits_{u \in \mathrm{Adj}(v)} P_{G-v-u}(\lambda) - 2 \sum\limits_{C \in \mathcal{C}(v)} P_{G-V(C)}(\lambda)$,

(ii) $P_G(\lambda) = P_{G-uv}(\lambda) - P_{G-v-u}(\lambda) - 2 \sum\limits_{C \in \mathcal{C}(uv)} P_{G-V(C)}(\lambda)$,

where $\mathrm{Adj}(v)$ *denotes the set of neighbors of v, while $G-V(C)$ is the graph obtained from G by removing the vertices belonging to the cycle C.*

These relations have the following consequences (see, e.g. [**2**], p. 59).

Corollary 1. *Let G be a graph obtained by joining a vertex v_1 of a graph G_1 to a vertex v_2 of a graph G_2 by an edge. Let G_1' $\left(G_2'\right)$ be the subgraph of G_1 (G_2) obtained by deleting the vertex v_1 (v_2) from G_1 (resp. G_2). Then*

$$P_G(\lambda) = P_{G_1}(\lambda)P_{G_2}(\lambda) - P_{G_1'}(\lambda)P_{G_2'}(\lambda).$$

Corollary 2. *Let G be a graph with a pendant edge v_1v_2, v_1 being of degree* 1. *Then*

$$P_G(\lambda) = \lambda P_{G_1}(\lambda) - P_{G_2}(\lambda),$$

where $G_1\,(G_2)$ is the graph obtained from G (resp. G_1) by deleting the vertex v_1 (resp. v_2).

3. SMITH GRAPHS

The set of connected graphs for which $\lambda_1 = 2$ is depicted in Fig. 1. These graphs are known as SMITH graphs. The set contains cycles of all possible lengths, a family W_n of trees of arbitrary diameter and four small trees, one of which is actually W_0 but sometimes it is convenient to be treated separately. Proper induced subgraphs of SMITH graphs all have $\lambda_1 < 2$ (they are also known as COXETER-DYNKIN graphs).

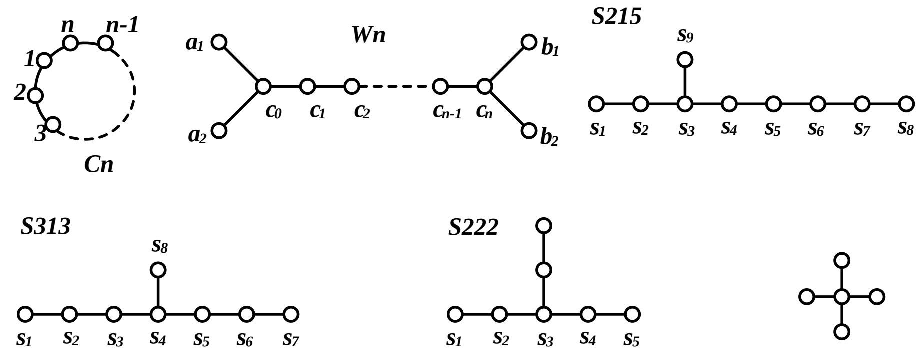

Figure 1.

Theorem S. (SMITH [**18**], see also [**2**, p.79]) $\lambda_1(G) \le 2$ (*resp.* $\lambda_1(G) < 2$) *if and only if each component of graph G is a subgraph (resp. proper subgraph) of one of the graphs of Fig.* 1, *all of which have index equal to* 2.

Any connected graph is either an induced subgraph or an induced supergraph of some SMITH graphs.

Lemma 2. (RADOSAVLJEVIĆ and SIMIĆ, [**13**]). *Let G be a graph obtained by extending any of Smith graphs by a vertex of arbitrary positive degree. Then* $P_G(2) < 0$ (*i.e.* $\lambda_2(G) < 2 < \lambda_1(G)$).

Theorem RS. (RADOSAVLJEVIĆ and SIMIĆ, [**13**]). *Let G be a graph with cut-vertex u.*

(i) *If at least two components of $G - u$ are induced supergraphs of Smith graphs, and if at least one of them is a proper supergraph, then* $\lambda_2(G) > 2$.

(ii) *If at least two components of* $G - u$ *are Smith graphs, and the rest are induced subgraphs of Smith graphs, then* $\lambda_2(G) = 2$.

(iii) *If at most one component of* $G - u$ *is a Smith graph, and the rest are proper induced subgraphs of Smith graphs, then* $\lambda_2(G) < 2$.

If $G - u$ (u being a cut-vertex) has one proper supergraph and the remaining components are proper (induced) subgraphs of SMITH graphs, Theorem RS is not applicable and these cases are interesting for further investigations.

4. A CLASS OF θ-GRAPHS

If two cycles of a bicyclic graph have a common path, we shall say that they form a θ-graph (Figure 2) and the same name will be used for any bicyclic graph with such cyclic structure.

Research on maximal reflexive θ-graphs is the first step in the area of new classes of reflexive graphs which are not trees or treelike graphs. In previous investigations on maximal reflexive treelike graphs we have noticed a constant presence of SMITH trees.

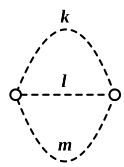

Figure 2.

Therefore, it is obvious that SMITH graphs have an important role in forming of maximal reflexive graphs. This is the reason why we are making first steps in this area by constructing θ-graphs from SMITH graphs.

Consider the SMITH tree S depicted in Figure 3. $P_S(2) = 0$. Let us introduce the notation: $U_S = P_{S-u}(2)$, $V_S = P_{S-v}(2)$, $C = P_{S-p}(2)$, where p is the unique path connecting vertices u and v (within the SMITH tree).

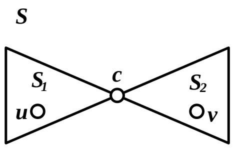

Figure 3.

Consider θ-graph in Figure 4. This graph is formed from a SMITH tree (S) and a cycle (length n). SMITH tree is connected with the cycle by two paths of length 2 (one starting at u and ending at u_1, and another one starting at v and ending at v_1). Lengths of paths connecting vertices u_1 and v_1 within the cycle are and n_1 and n_2, $(n_1 + n_2 = n, n_1, n_2 \geq 4)$.

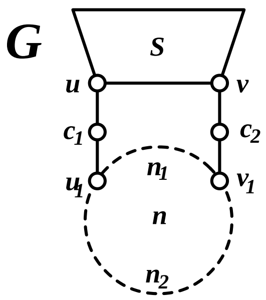

Figure 4.

Proposition 1. *Let* G *be the graph in Figure* 4. *Then* $P_G(2) = n\,(U_S + V_S - 2C)$.

Proof. Let us remove the vertex c_1 from graph G, and then vertices u_1 and u. We get graphs H_1, H_2 and H_3 of Figure 5, respectively. Applying Theorem RS we get $P_{H_1}(2) = 0$. Application of Lemma 1 to the graph H_2 at the vertex c_2 gives the following result:

$$P_{H_2}(2) = 2P_{H_2-c_2}(2) - P_{H_2-c_2-v_1}(2) - P_{H_2-c_2-v}(2) = -nV_S.$$

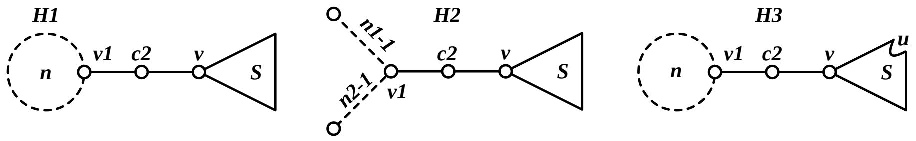

Figure 5.

Applying Lemma 1 to the graph H_3 at the vertex c_2 we get:

$$P_{H_3}(2) = 2P_{H_3-c_2}(2) - P_{H_3-c_2-v_1}(2) - P_{H_3-c_2-v}(2) = -nU_S.$$

Finally, we use these results to get $P_G(2)$.

$$\begin{aligned} P_G(2) &= 2P_{H_1}(2) - P_{H_2}(2) - P_{H_3}(2) - 2n_1C - 2n_2C \\ &= nU_S + nV_S - 2(n_1+n_2)C = nU_S + nV_S - 2nC. \end{aligned}$$

We see that $P_G(2) = n\,(U_S + V_S - 2C)$ and this completes the proof.

The next step is to go through all Smith trees and find all cases in which $U_S + V_S - 2C = 0$ holds, because in Proposition 1 we proved that then 2 belongs to the spectrum of the corresponding θ-graph.

5. ANALYSIS OF SMITH TREES

5.1 SMITH TREE $S215$

We can find now all pairs of vertices (u, v) of $S215$ (Fig. 1), for which $U_S + V_S - 2C = 0$ holds. They are:

$$(u, v) \in \{(s_1, s_7), (s_7, s_1), (s_2, s_5), (s_5, s_2), (s_6, s_9), (s_9, s_6)\}.$$

Those θ-graphs corresponding to these pairs are shown in Fig. 6. They are all maximal reflexive graphs in their class, and $n_1 = n_2 = 4$. In all three cases $\lambda_2 = \lambda_3 = 2$ holds.

In these proofs the expert system GRAPH is used in the final stages to check wether the graph is maximal (wether it could be extended at some vertices) and determine the limits of the lengths n_1 and n_2 of the given cycle (for larger n_1 and n_2 2 would still belong to the spectrum, but it would no longer be λ_2, but λ_3 or λ_4, etc.).

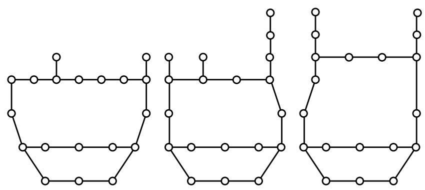

Figure 6.

5.2 SMITH TREE S313

All pairs of vertices (u, v) of $S313$ (Fig. 1) for which $U_S + V_S - 2C = 0$ holds are:

$$(u, v) \in \{(s_1, s_7), (s_2, s_6), (s_2, s_8), (s_8, s_2), (s_3, s_5)\}.$$

Consider the pair (s_1, s_7). The corresponding maximal reflexive θ-graphs are shown in Figure 7.

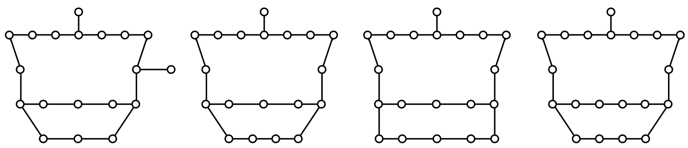

Figure 7.

Application of Theorem RS gives the explanation why the extension of the starting graph is possible only at vertices c_1 and c_2.

For the remaining pairs (u, v), the corresponding maximal θ-graphs are shown in Fig. 8.

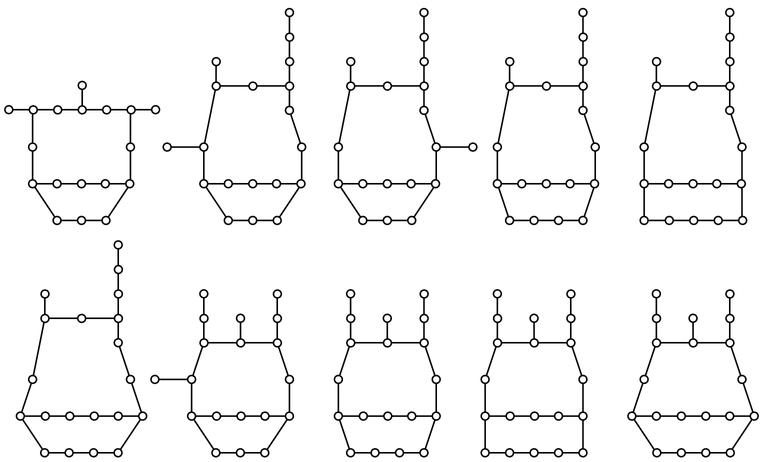

Figure 8

5.3 SMITH TREE $S222$

All pairs of vertices (u, v) of $S222$ (Fig. 1) for which $U_S + V_S - 2C = 0$ holds are:

$(u, v) \in \{(s_1, s_5), (s_2, s_4)\}$. Corresponding maximal reflexive θ-graphs are shown in Fig. 9.

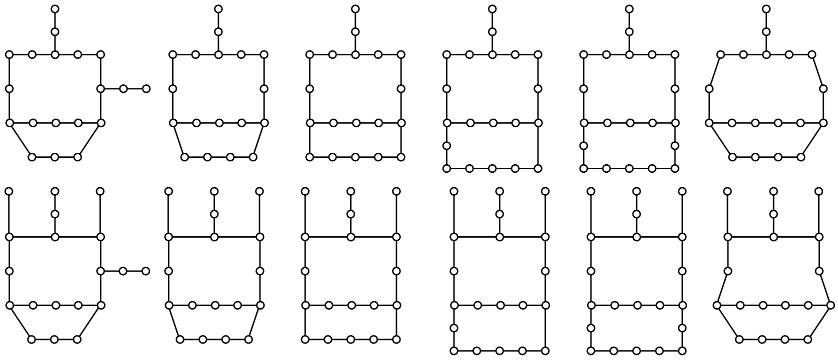

Figure 9.

5.4 SMITH TREE $W n$

All pairs of vertices (u, v) of the Smith graph W_n (Fig. 1) for which $U_S + V_S - 2C = 0$ holds are: $(u, v) = (a_1, a_2), (u, v) = (c_k, c_{k+l}), k \in \{0, 1, \ldots, n-1\}, k + l \in \{1, \ldots, n\}$ and $(u, v) = (a_1, b_1)$.

Maximal reflexive θ-graphs corresponding to the pair (a_1, a_2) are shown in Fig 10.

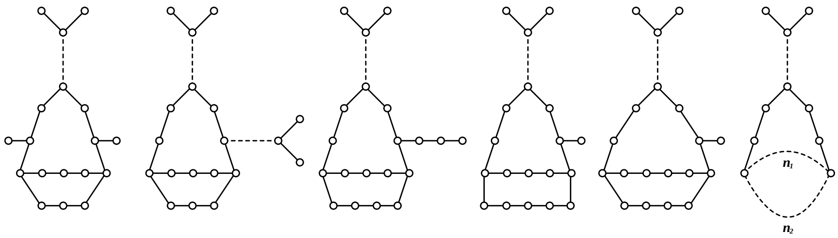

Figure 10.

The values n_1 and n_2 for the last graph in Figure 10 are:

$$(n_1, n_2) \in \{(4,7), (4,8), (4,9), (4,10), (4,11), (4,12), (5,6), (5,7), (6,6)\}.$$

To the pairs $(u, v) = (c_k, c_{k+\ell},\ k \in \{0, 1, 2, \ldots, n-1\},\ k + \ell \in \{1, 2, \ldots, n\}$ there corresponds θ-graph in Figure 11(a)

For $\ell \geq 3$ it holds $\lambda_2 > 2$ $(\lambda_3 = 2, \ldots)$. For $\ell = 2$ we get maximal reflexive θ-graph shown in Figure 11(b).

For $\ell = 1$ we get maximal reflexive θ-graph shown in Figure 12.

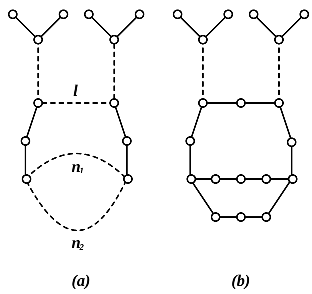

Figure 11.

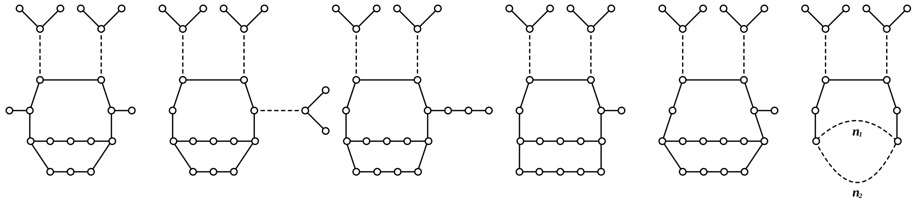

Figure 12.

$$(n_1, n_2) \in \{(4,7), (4,8), (4,9), (4,10), (4,11), (4,12), (5,6), (5,7), (6,6)\}.$$

From the pair $(u, v) = (a_1, b_1)$ we get θ-graphs in Figure 13.

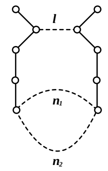

Figure 13.

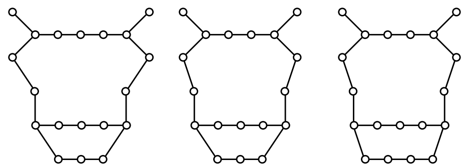

Figure 14.

For $\ell > 4$ corresponding θ -graphs are not reflexive, $\lambda_2 > 2$ ($\lambda_3 = 2, \ldots$). For $\ell = 4$ and $\ell = 3$ the corresponding maximal reflexive θ-graphs are shown in Figure 14.

For $\ell = 2$ the corresponding maximal reflexive θ-graphs are shown in Figure 15.

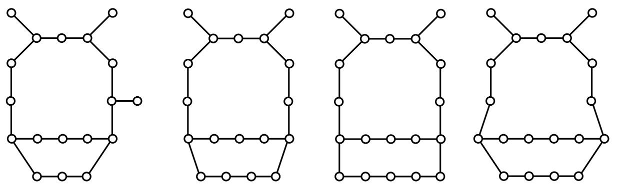

Figure 15.

For $\ell = 1$ the corresponding maximal reflexive θ-graphs are shown in Figure 16.

$$(n_1, n_2) \in \{(4,6), (4,7), (4,8), (5,5), (5,6)\}.$$

Based on previously shown results we have proven the following theorem.

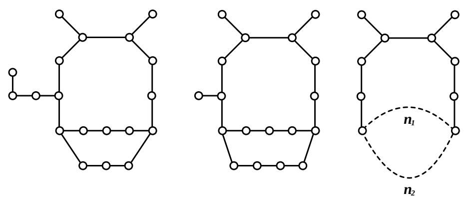

Figure 16.

Theorem 1. *Consider the graph with the cyclic structure of graph G in Figure* 4. *Then, graph is maximal reflexive θ-graph if and only if it is one of the* 72 *graphs in Figures* $6 - 10$, 11(b), 12 *and* $14 - 16$.

6. CONCLUSION

This is only one of many cases where the presence of SMITH graphs is noticed when investigating reflexive graphs. Currently we are working on determining all maximal reflexive θ-graphs for various values of parameters k, l, m (Figure 3). Various forms of presence of SMITH graphs are noticed in most of the resulting maximal reflexive graphs and this is one of the areas for us to focus on in the future work.

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ON A CORRELATION BETWEEN DIFFERENTIAL EQUATIONS AND THEIR CHARACTERISTIC EQUATIONS

Boro M. Piperevski

Abstract: The aim of this paper is to derive the dependence of the nature of a solution of a class of differential equations of n-th order with polynomial coefficients on the solutions of the corresponding characteristic algebraic equation of n-th degree.

1. INTRODUCTION

Many theoretical and practical problems from theory and practice acquire resolving trough plenty of concrete process characteristics. Problems of this type are eigenvalues problems, boundary value problems, optimal values at variation calculation, of polynomials existence as special functions with determined characteristic.

Classical theory of partial differential equations from mathematical physics is related with the functions of Lame, Mathieu, classical orthogonal polynomials, polynomials of Appell, polynomials of Stieltjes, etc.

Heine problem for the number of linear differential equations with polynomial coefficients that have polynomial solution, connected with the practical problem of equilibrium (Stieltjes [**10**]), is known [**6**, **7**].

The connection between roots of characteristic algebraic equation with the form of solutions of homogenous linear differential equation with constant coefficients, is already known in classical theory of differential equations.

The similar result for existence of polynomial solution of linear homogenous differential equation with polynomial coefficients is obtained [**8**].

1991 Mathematics Subject Classification. 34-A05.
Key Words and Phrases. Differential equations, polynomial solutions, root of characteristic equation.

2. NOTIONS, DEFINITIONS AND KNOWN RESULTS

Here we will try to generalize already known results.

Let us consider differential equation:

$$Ay'' + By' + Cy = 0,$$

where

$$A = a_2x^2 + a_1x + a_0, \quad B = b_1x + b_0, \quad C = c_0, \quad a_2 \neq 0.$$

Its characteristic equation is

$$\frac{t(t-1)}{2}\, A'' + tB' + C = 0.$$

It is known that if one root of characteristic equation is positive integer, then the solution of the differential equation is polynomial function. This solution is given by known RODRIGUES's formula:

$$y = A \exp\Big(-\int \frac{B}{A}\,\mathrm{d}x\Big) \frac{\mathrm{d}^n}{\mathrm{d}x^n}\Big(A^{n-1} \exp\Big(\int \frac{B}{A}\mathrm{d}x\Big)\Big).$$

This problem for existence of polynomial solutions of differential equation is closely related with operators eigenvalues and eigenfunctions. LEGENDRE's and other classical orthogonal polynomials are well known solution of differential equations [**2, 3, 4**].

We can show that differential equation

$$(x^2-1)y'' + 2xy' - n(n+1)y = 0,$$

$n-$positive integer, having LEGENDRE polynomial solution given by

$$y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big((x^2-1)^n\big),$$

has characteristic equation whose roots n and $-(n+1)$ are integer numbers.

We also know that if characteristic algebraic equation

$$\binom{t}{0}P_0(x) + \binom{t}{1}P_1'(x) + \binom{t}{2}P_2''(x) + \cdots + \binom{t}{m}P_m^{(m)}(x) = 0, \tag{1}$$

of degree m with respect to t of differential equation

$$P_m(x)y^{(m)} + P_{m-1}(x)y^{(m-1)} + \cdots + P_0(x)y = 0, \tag{2}$$

where $P_i(x), i = \overline{0,m}$ are polynomial of i-th degree, has positive integer root, then this equation has polynomial solution [**8**].

When the roots of characteristic equation (1) are successive positive integers, then the result is known [**1,9**].

Here, we will try to find a solution of differential equation (2) in case when the roots of characteristic equation (1) are successive negative integers.

3. MAIN RESULT

Theorem. *The linear differential equation*

$$P_m(x)y^{(m)} + P_{m-1}(x)y^{(m-1)} + \cdots + P_0(x)y = 0,$$

where $P_i(x), i = \overline{0,m}$, are polynomials of i-th degree and $P_m^{(m)}(x) \neq 0$, is solvable if the following conditions are satisfied:

a) *The roots of characteristic equation* (1) *are a $m-1$ negative integer $t_1 = -(n+1), t_2 = -(n+2), \ldots, t_{m-1} = -(n+m-1), n$-positive integer. If t_m is negative integer then $|t_m| > |t_i|, i = \overline{1, m-1}$.*

b) *The polynomial coefficients are satisfying the following conditions*:

$$\begin{aligned} P_{k-1}(x) - \binom{n+m-1}{1}P_k'(x) + \binom{n+m}{2}P_{k+1}''(x) + \cdots \\ + (-1)^{m-k+1}\binom{n+2m-k-1}{m}P_m^{(m-k+1)}(x) = 0, \quad k = \overline{1, m-1}. \end{aligned} \tag{3}$$

In that case, the general solution can be given by

$$\begin{aligned} y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\Bigg(P_m^{n+m-1}\exp\Big(-\int\frac{P_{m-1}}{P_m}\,\mathrm{d}x\Big)\Big(C_1 \\ + \int(C_2 + C_3x + \cdots + C_mx^{m-2})P_m^{-(n+m)}\exp\Big(\int\frac{P_{m-1}}{P_m}\,\mathrm{d}x\Big)\mathrm{d}x\Big)\Bigg), \end{aligned} \tag{4}$$

where $C_1, C_2, \ldots, C_m$ are arbitrary constants.

Proof. Let suppose that the conditions of theorem are satisfied. We consider linear differential equation of the first order

$$P_m(x)z' + \big(P_{m-1}(x) - (n+m-1)P_m'(x)\big)z = C_2 + C_3x + C_4x^2 + ... + C_mx^{m-2},$$

where $C_2, \ldots, C_m$ are arbitrary constants.

Differentiating it $n+m-1$ times, we obtain the equation (2) where $z^{(n)} = y$ and condition (3) is used.

With this differentiation procedure, we get a formula for general solution (4) using formula for the general solution of the linear differential equation of the first order .

4. ANALYSIS OF THE NATURE OF THE SOLUTION OF THE DIFFERENTIAL EQUATION (2)

We consider special case, when m-th root

$$t_m = (m+n)(m-1) - \frac{P_{m-1}^{(m-1)}}{(m-1)!}$$

is positive integer. Then, equation (2) has unique polynomial solution of degree $(m+n)(m-1) - \frac{P_{m-1}^{(m-1)}}{(m-1)!}$.

Example 1. For differential equation

$$x(x-1)(x-2)y''' + (x+1)(x+2)y'' + (-52x+72)y' - 108y = 0$$

we get $t_1 = -3, t_2 = -4, t_3 = 9, n = 2$, and general solution is obtained by formula

$$y = \frac{\mathrm{d}^2}{\mathrm{d}x^2}\left(\frac{x^3(x-1)^{10}}{(x-2)^2}\left(C_1 + \int (C_2 + C_3 x)\frac{x-2}{x^4(x-1)^{11}}\,\mathrm{d}x\right)\right).$$

Since $t_3 = 9$, equation has only one polynomial solution of *nine*-th degree.

Particularly, if $P_{m-1}(x) = kP_m'(x)$, and if the m-th root $t_m = (m+n)(m-1) - km$ -positive integer, then there exists a polynomial solution, given by the formula

$$y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(P_m^{n+m-1-k}(x)\big).$$

Remark 1. For $m = 3$, equation (2) has a form

$$Ay''' + By'' + Cy' + Dy = 0,$$

where

$$A = \alpha_3 x^3 + \alpha_2 x^2 + \alpha_1 x + \alpha_0, B = \beta_2 x^2 + \beta_1 x + \beta_0, C = \gamma_1 x + \gamma_0, D = \delta, \alpha_3 \neq 0.$$

If conditions of the theorem are fulfilled, the general solution is

$$y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\left(A^{n+2}\exp\Big(-\int\frac{B}{A}\,\mathrm{d}x\Big)\left(C_1 + \int (C_2 + C_3 x)A^{-(n+3)}\exp\Big(\int\frac{B}{A}\,\mathrm{d}x\Big)\mathrm{d}x\right)\right),$$

and the equation has a form

$$Ay''' + By'' + (n+2)\Big(B' - \frac{n+3}{2}A''\Big)y' + (n+1)(n+2)\Big(\frac{1}{2}B'' - \frac{n+3}{3}A'''\Big)y = 0,$$

$n-$ positive integer .

Example 2. For the differential equation

$$x(x-1)(x-2)\,y''' + 5(x+1)(x+2)\,y'' + (-6x+81)\,y' - 18y = 0$$

we get $t_1 = -2, t_2 = -3, t_3 = 3, n = 1$ and polynomial solution of third degree is obtained by

$$y = x^3 + \frac{63}{4}x^2 + 129x + 598.$$

Particularly, if $B = kA'$, than differential equation has a form

$$Ay''' + kA'y'' + (n+2)\Big(k - \frac{n+3}{2}\Big)A''y' + (n+2)\Big(\frac{n+1}{2}\,k - \frac{(n+1)(n+3)}{3}\Big)A'''y = 0,$$

and the general solution is given by the formula

$$y = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\bigg(A^{n-k+2}\Big(C_1 + \textstyle\int (C_2 + C_3x)A^{k-n-3}\,\mathrm{d}x\Big)\bigg).$$

If $t_3 = 2n - 3k + 6$ is positive integer, then the polynomial solution of degree $2n - 3k + 6$ is given by the formula

$$L_{n,k} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}(A^{n-k+2}).$$

For $k = 2$ and $A = x(1 - x^2)$ we have special differential equation

$$x(1-x^2)\,y''' + 2(1-3x^2)\,y'' - 3(n-1)(n+2)xy' + 2n(n+1)(n+2)y = 0$$

and the polynomial solution of $2n$-th degree is

$$L_{2n} = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(x^n(1-x^2)^n\big).$$

These polynomials are Appell's polynomials or generalized Legendre polynomials [**5**].

Remark 2. For $m = 4$ the equation (2) with conditions given above, has a form

$$\begin{aligned} P_4(x)y^{(IV)} &+ P_3(x)y''' + (n+3)\Big(P_3'(x) - \frac{n+4}{2}P_4''(x)\Big)y'' \\ &+ (n+2)(n+3)\Big(\frac{1}{2}\,P_3''(x) - \frac{n+4}{3}\,P_4'''(x)\Big)y' \\ &+ \quad (n+1)(n+2)(n+3)\Big(\frac{1}{6}\,P_3'''(x) - \frac{n+4}{8}\,P_4^{(IV)}(x)\Big)y = 0, \end{aligned}$$

$n-$ positive integer .

If $3n + 12 - \dfrac{P_3'''(x)}{3!}$ is positive integer then there will be only one polynomial solution for the equation.

Particularly, for $P_3(x) = kP_4'(x)$, if $3n + 12 - 4k$ is positive integer than the polynomial solution will be obtained.

For $k = 3$, we have polynomial solution of $3n$-th degree given by the formula

$$P_{3n}(x) = \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(P_4^n(x)\big)$$

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ON THE MATRIX EQUATION $XA - AX = \tau(X)$

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We study the matrix equation $XA - AX = \tau(X)$ in $M_n(K)$, where τ is an automorphism of a field K of finite order k. A criterion under which this equation has a nontrivial solution is given. In case when $k = 1$ that criterion boils down to an already known result.

1. INTRODUCTION

The main purpose of this paper is to develop the connection between the eigenvalues of a class of pseudo-linear transformation over a field K and the eigenvalues of a certain linear transformation. The use of linear transformations enables us to use Cayley- Hamilton theorem which in pseudo-linear setting does not hold.

This work was directly inspired by the paper [**2**] for $p = 1$. In this case we get linear matrix equation $XA - AX = X$. We went one step further by introducing an automorphism τ of a field K of finite order k, $XA - AX = \tau(X)$. Since it does not remain linear matrix equation anymore, the classical methods can not be used. By equivalent transformations this equation can be viewed in another form $\tau^{-1}(X)\tau^{-1}(A) - \tau^{-1}(X)\tau^{-1}(A) = X$. The left hand side of the equation is a pseudo-linear transformation of $M_n(K)$, $T(X) = \tau^{-1}(X)\tau^{-1}(A) - \tau^{-1}(A)\tau^{-1}(X)$. In fact, in order to find out if the equation has nontrivial solutions we will investigate whether $\lambda = 1$ is the eigenvalue of T or, equivalently, of linear transformation T^k. In case $k = 1$ we get an already known criterion.

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Key Words and Phrases. Skew polynomials, pseudo-linear transformations, matrix equation.

2. RECAPITULATION

Let K be a field and $\sigma \in \mathrm{Aut}\,(K)$. A skew polynomial ring (also called Ore extension) $K[t;\sigma]$ consists of polynomials $\sum\limits_{i=0}^{n} a_i t^i$, $a_i \in K$ which are added in the usual way but are multiplied according to the following rule

$$ta = \sigma(a)t, \qquad a \in K.$$

The evaluation $f(a)$ of a polynomial $f(t) \in K[t;\sigma]$ at some element $a \in K$ is the remainder one gets when $f(t) = \sum\limits_{i=0}^{n} a_i t^i$ is divided on the right by $t - a$. It is easy to show by induction that

$$f(a) = \sum_{i=0}^{n} a_i N_i(a)$$

where the maps N_i are defined by induction in the following way. For any $a \in K$

$$N_0(a) = 1 \text{ and } N_{i+1}(a) = \sigma\big(N_i(a)\big)a,$$

which leads to

$$N_k(a) = \sigma^{k-1}(a)\sigma^{k-2}(a)\cdots\sigma(a)a \qquad (k \in \mathbb{N}).$$

We define $f(A)$ for $A \in M_n(K)$ similarly:

$$f(A) = \sum_{i=0}^{n} a_i N_i(A)$$

where σ has been extended to $M_n(K)$ in the natural way.

Let V be a vector space over K. A $\sigma-$pseudo-linear transformation of V is an additive map $T : V \to V$ such that

$$T(\alpha v) = \sigma(\alpha)T(v), \quad \alpha \in K.$$

We will use the abbreviation σ-PLT for a pseudo-linear transformation with respect to the automorphism σ. A vector $v \in V \setminus \{0\}$ is an eigenvector of the $\sigma - PLT\ T$ with the corresponding eigenvalue $\lambda \in K$ if and only if

$$T(v) = \lambda v.$$

An important feature of σ-PLT is the absence of a Cayley-Hamilton theorem. In addition to that, unlike the classical linear transformations of a finite dimensional vector space over a commutative field, a pseudo-linear transformation need not be algebraic.

If V is finite-dimensional and $e = [e_1, \ldots, e_n]$ is a basis of V, let us write $T(e_i) = \sum_{j=1}^{n} a_{ij} e_j$, $a_{ij} \in K$ or, in the matrix notation $Te = Ae$, where $A = [a_{ij}] \in M_n(K)$. The matrix A will be denoted by $[T]_e$. The equality

$$[f(T)]_e = f([T]_e)$$

holds for any polynomial $f(t) \in K[t, \sigma]$ as well. If v is an eigenvector of the σ-PLT T with an eigenvalue $\lambda \in K$ then

$$\sigma(v_e)[T]_e = \lambda v_e$$

where v_e denotes coordinates of the vector v with respect to the basis e ([**6**]).

If T is an algebraic σ-PLT on V and $\mu_T \in K[t; \sigma]$ is its minimal polynomial than $\lambda \in K$ is an eigenvalue for T if and only if $t - \lambda$ divides on the right (left) the polynomial μ_T in $K[t; \sigma]$ (Proposition 4.5. [**6**]).

We will also use the notion of a Wedderburn polynomial. For $f \in K[t; \sigma]$, let

$$V(f) := \{a \in K \mid f(a) = 0\}.$$

A (monic) polynomial is said to be Wedderburn if $f = \mu_{V(f)}$ i.e. f is equal to the minimal polynomial of $V(f)$-set of its roots ([**5**]).

3. GENERAL RESULTS

Let K be a field, $\sigma \in \mathrm{Aut}\,(K)$ of order k, i.e. $\sigma \neq id_K$ and k is the least nonnegative integer such that $\sigma^k = id_K$. If T is σ-PLT on a vector space V over K then T^k is a linear transformation of V since it is additive and

$$T^k(\alpha v) = \sigma^k(\alpha) T^k(v) = \alpha T^k(v), \quad \alpha \in K.$$

Therefore, if V is a finite-dimensional vector space, there exist $m \in \mathbb{N}$, $a_0, \ldots, a_m \in K$, $a_m \neq 0$, such that

$$a_m (T^k)^m + \cdots + a_1 T^k + a_0 I = 0,$$

which means that σ-PLT T is algebraic. We will denote its minimal polynomial by μ_T. This polynomial is invariant in $K[t; \sigma]$ and it is also the right factor of the polynomial $\varphi_{T^k}(t^k)$, where φ_{T^k} denotes the characteristic polynomial of T^k. What we want is to find relations between eigenvalues of the linear transformation T^k and σ-PLT T.

Theorem 1. *Let T be σ-PLT on a finite dimensional vector space V over a field K and $\sigma \in \mathrm{Aut}\,(K)$ of order k. An element $\lambda \in K$ is the eigenvalue of T if and only if $N_k(\lambda)$ is an eigenvalue of T^k.*

Proof. Let $v \in V \setminus \{0\}$ be such that $T(v) = \lambda v$. Then

$$\begin{aligned} T^k(v) &= T^{k-1}(\lambda v) = \sigma^{k-1}(\lambda)T^{k-1}(v) \\ &\vdots \\ &= \sigma^{k-1}(\lambda)\cdots\sigma(\lambda)\lambda v = N_k(\lambda)v. \end{aligned}$$

The polynomial $h(t) = t^k - N_k(\lambda)$ is a WEDDERBURN polynomial, since it is the minimal polynomial of the set

$$\Gamma = \{\sigma(c)\lambda c^{-1} \mid c \in K^*\}.$$

For any $c \in K^*$, we have

$$N_k(\sigma(c)\lambda c^{-1}) = \sigma^k(c)N_k(\lambda)c^{-1} = N_k(\lambda).$$

The above shows that h vanishes on Γ. Let $f(t) = \sum\limits_{i=1}^{m} a_i t^i$ be the monic minimal polynomial of Γ. Then $m = \deg f \leqslant k$, and the constant term $a_0 \neq 0$. Let $d \in K^*$. For any $e \in \Gamma$, we have $0 = \sum\limits_{i=0}^{m} a_i\sigma^i(d)N_i(e)d^{-1}$. Thus, Γ satisfies the polynomial $\sum\limits_{i=0}^{m} a_i\sigma^i(d)t^i$. By the uniqueness of the minimal polynomial, we must have $\sigma^m(d)a_i = a_i\sigma^i(d)$ for every i. Since $a_0 \neq 0$, this implies that $\sigma^m = id_K$. Therefore, we have $m = k$ and $f(t) = t^k - N_k(\lambda)$.

We can write $t^k - N_k(\lambda) = (t-\lambda_k)(t-\lambda_{k-1})\cdots(t-\lambda_1)$ where $\lambda_1, \dots, \lambda_k$ are $\sigma-$conjugated to λ (Theorem 5.1. [**5**]). This gives us

$$T^k - N_k(\lambda)id_K = (T - \lambda_k id_K)(T - \lambda_{k-1}id_K)\cdots(T - \lambda_1 id_K).$$

Now it is easy to conclude that if there exists $0 \neq v \in V$ such that $(T^k - N_k(\lambda)id_K)(v) = 0$, then there exist $l \in \{1, \dots, k\}$ and $0 \neq u \in V$ such that $(T - \lambda_l id_K)(u) = 0$. Since λ_l is $\sigma-$conjugated to λ, there exists $a \in K^*$ such that $\lambda_l = \sigma(a)\lambda a^{-1}$. Then for $u_0 = a^{-1}u$ we obtain

$$T(u_0) = T(a^{-1}u) = \sigma(a^{-1})T(u) = \sigma(a^{-1})\sigma(a)\lambda a^{-1}u = \lambda u_0$$

i.e. λ is an eigenvalue for T, as desired. $\square$

4. APPLICATIONS

Let K be a field, $\tau \in \mathrm{Aut}\,(K)$ of order k and $A \in M_n(K)$. What we want is to find all solutions of the matrix equation

$$XA - AX = \tau(X). \tag{4.1}$$

Instead of this equation we will consider the equivalent equation

$$\sigma(X)B - B\sigma(X) = X \tag{4.2}$$

where $\sigma = \tau^{-1}$ and $B = \tau^{-1}(A)$. This equation always has a solution, for any given B, namely $X = 0$. The mapping $T : M_n(K) \to M_n(K)$,

$$T(X) = \sigma(X)B - B\sigma(X)$$

is σ-PLT. Relative to the basis $e = [E_{ij}, 1 \leqslant i, j \leqslant n]$ of $M_n(K)$ T has the matrix:

$$\mathcal{B} = [T]_e = E \times B - B^T \times E$$

where $\times$ denotes KRONECKER product of the matrices. The matrix equation (4.2) has a nontrivial solution if and only if σ-PLT T has the eigenvalue $\lambda = 1$. By Theorem 1 this is equivalent to the fact that linear transformation T^k also has the eigenvalue $N_k(1) = 1$. Since $[T^k]_e = N_k(\mathcal{B})$, in order to find out if the equation (4.2) has nontrivial solutions or not we will examine if 1 is a zero of the characteristic polynomial φ_{T^k} of linear operator T^k or not.

We will assume in the majority of cases that $k \geqslant 2$. If $k = 1$ we obtain the linear matrix equation $XA - AX = X$ which is a special case of the SYLVESTER matrix equation $AX + XB = C$. Let $L : M_n(K) \to M_n(K)$, with $L(X) = AX + XB$ be the SYLVESTER operator. It is well known that when K is an algebraically closed field the linear operator L is singular if and only if A and $-B$ have a common eigenvalue. For $B = E - A$ we obtain the following result.

Proposition 2. *The matrix equation $XA - AX = X$ has a nonzero solution if and only if A and $A - E$ have a common eigenvalue.*

This proposition is equivalent to the fact that the matrix equation $XA - AX = X$ has nonzero solutions if and only if 1 is an eigenvalue of the matrix $E \times A - A^T \times E$. Since the eigenvalues of $C \times E + E \times D$ are all of the form $\lambda + \mu$ where λ and μ are eigenvalues of C and D respectively, 1 is the eigenvalue of $E \times A - A^T \times E$ if and only if $1 = \lambda - \mu$ for some eigenvalues λ and μ of A. This means that λ and $\lambda - 1$ are two different eigenvalues of A which is equivalent to the fact that A and $A - E$ have a common eigenvalue.

EXAMPLE 1. Let

$$A = \begin{bmatrix} -i+1 & 1 \\ -1 & i \end{bmatrix} \in M_2(\mathbb{C})$$

and $\sigma \in \operatorname{Aut}(\mathbb{C})$, $\tau(x) = \bar{x}$, the complex conjugation. We are looking for all nonzero solutions of the equation

$$XA - AX = \bar{X}, \tag{4.3}$$

or the equivalent equation

$$\bar{X}\bar{A} - \bar{A}\bar{X} = X. \tag{4.4}$$

In this case, τ is the automorphism of $\mathbb{C}$ of order $k = 2$. Therefore $\tau^{-1} = \tau$. First, for $B = \bar{A}$, we determine the matrix $P = E \times B - B^T \times E$,

$$P = \begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & -2i-1 & 0 & 1 \\ -1 & 0 & 2i+1 & 1 \\ 0 & -1 & -1 & 0 \end{bmatrix},$$

then the matrix

$$N_2(P) = \bar{P}P = \begin{bmatrix} -2 & -2i-1 & 2i+1 & 2 \\ -2i+1 & 3 & -2 & 2i-1 \\ 2i-1 & -2 & 3 & -2i+1 \\ 2 & 2i+1 & -2i-1 & -2 \end{bmatrix}.$$

The matrix $N_2(P)$ can be calculated using the following formula as well:

$$N_2(P) = E \times N_2(\bar{A}) - A^T \times \bar{A} - \bar{A}^T \times A + N_2(\bar{A}^T) \times E.$$

Next, we calculate the characteristic polynomial $\varphi_{N_2(P)}$ and check whether 1 is its root or not. In this case we have

$$\varphi_{N_2(P)}(t) = t^2(t-1)^2.$$

Since $\varphi_{N_2(P)}(1) = 0$, we can conclude that the our matrix equation has nonzero solutions.

In this case, we go one step further. We are going to determine all non zero solutions of the equation (4.3). Since $\mu_{N_2(P)}(t) = t(t-1)$,

$$M_2(\mathbb{C}) = \ker T^2 \oplus \ker(T^2 - id_K),$$

where $T : M_2(\mathbb{C}) \to M_2(\mathbb{C}), \quad T(X) = \bar{X}\bar{A} - \bar{A}\bar{X}$.

All solutions of the equation (4.3) belong to the set $U = \ker(T^2 - id_K)$ which has the basis $[C, D]$, where

$$C = \begin{bmatrix} -1 & -1-2i \\ 0 & 1 \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

System $[D, T(D)]$ is one basis of U as well, since $T(D) \neq 0$. So, if $X \in M_2(\mathbb{C})$ satisfies (4.4), then $X = \alpha D + \beta T(D)$ for uniquely determined $\alpha, \beta \in \mathbb{C}$. From $T(X) = X$ it follows

$$\bar{\alpha}T(D) + \bar{\beta}D = \alpha D + \beta T(D),$$

which is valid for any $\alpha \in \mathbb{C}$ and $\beta = \bar{\alpha}$. Finally,

$$X = \alpha D + \bar{\alpha}T(D) = \alpha \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + \bar{\alpha} \begin{bmatrix} -2 & -1-2i \\ 1+2i & 2 \end{bmatrix}$$

i.e.

$$X = \begin{bmatrix} -2\bar{\alpha} & \alpha - \bar{\alpha}(1+2i) \\ \alpha + \bar{\alpha}(1+2i) & 2\bar{\alpha} \end{bmatrix}, \ \alpha \in \mathbb{C}.$$

So, the set of solutions is

$$\left\{ \begin{bmatrix} -2\bar{\alpha} & \alpha - \bar{\alpha}(1+2i) \\ \alpha + \bar{\alpha}(1+2i) & 2\bar{\alpha} \end{bmatrix} \mid \alpha \in \mathbb{C} \right\}.$$

In general for

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \in M_2(\mathbb{C})$$

the characteristic polynomial of the matrix $N_2(P)$ is

$$\varphi_{N_2(P)} = t^2 \left(t - (|a-d|^2 + 2(\bar{b}c + b\bar{c}) \right)^2.$$

So, the equation $XA - AX = \bar{X}$ has a nontrivial solution if and only if

$$|a-d|^2 + 2(\bar{b}c + b\bar{c}) = 1.$$

Example 2. Let

$$A = J(n,\lambda) = \begin{bmatrix} \lambda & 1 & \cdots & 0 & 0 \\ 0 & \lambda & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda & 1 \\ 0 & 0 & \cdots & 0 & \lambda \end{bmatrix} \in M_n(\mathbb{C})$$

and take $\tau \in \operatorname{Aut}(\mathbb{C})$, $\tau(x) = \bar{x}$ to be the complex conjugation. The equation

$$XA - AX = \bar{X},$$

has only the trivial solution $X = 0$, since in this case

$$\operatorname{rank}\big(N_2(P) - E\big) = n^2.$$

In the end, we state some basic properties of the solutions of (4.1).

1. If X is a solution then $\operatorname{tr} X = 0$.

2. If X is a solution then so is cX for any $c \in K_0$, where

$$K_0 = \{a \in K \mid \tau(a) = a\}$$

i.e. the set of all solutions is one K_0 vector subspace of $M_n(K)$.

3. *Let* $A, X \in M_n(K)$ *and* $A_1 = SAS^{-1}, X_1 = SXS^{-1}$, $S \in Gl_n(K_0)$. *Then*

$$XA - AX = \tau(X) \quad \Leftrightarrow \quad X_1A_1 - A_1X_1 = \tau(X_1).$$

Proof. The equation $\tau(X) = XA - AX$ is equivalent to

$$\begin{aligned}\tau(X_1) = \tau(SXS^{-1}) &= S\tau(X)S^{-1} = S(XA - AX)S^{-1} \\ &= (SXS^{-1})(SAS^{-1}) - (SAS^{-1})(SXS^{-1}) \\ &= X_1A_1 - A_1X_1.\end{aligned}$$ □

Having applied the previous property with $A_1 = SAS^{-1} = A$ where $S \in Gl_n(K_0)$ we obtain the following.

4. *If* X_0 *is a matrix solution of* $XA - AX = \tau(X)$ *then so is* $X = SX_0S^{-1}$, *for any* $S \in C(A) \cap Gl_n(K_0)$, *where* $C(A) = \{S \in M_n(K) \mid SA = AS\}$ *is the centralizer of* A. □

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OPTIMAL VENTCEL GRAPHS, MINIMAL COST SPANNING TREES AND ASYMPTOTIC PROBABILITIES

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For each $\epsilon > 0$, let $\{X_n^\epsilon\}$ be an irreducible, time-homogeneous Markov chain with a finite state space S and transition function $p^\epsilon(i,j) = p_{i,j}\epsilon^{U(i,j)}(1 + o(1))$ where $0 \le U(i,j) \le \infty$ is a cost function. (We assume $p_{i,j} = 0$ iff $U(i,j) = \infty$.) It has been shown [**2**] that independent of the initial distribution, there are constants $h(i) \ge 0$ and $\beta_i > 0$ such that $\lim\limits_{\epsilon\downarrow 0} \mu^\epsilon(i)/\epsilon^{h(i)} = \beta_i$ for any $i \in S$, where μ^ϵ is the invariant distribution of $\{X_n^\epsilon\}$. Let $\underline{S} = \{i \in S : h(i) = 0\}$, which is called the global minimum set. Various asymptotic probabilities related to $\underline{S}$ have been established in [3]. Among others, starting with the uniform or invariant distribution, the expected hitting time $E^\epsilon T$ of $\underline{S}$ is of order $\epsilon^{-\delta}$ and the constants δ and $h(i)$ above can be expressed in terms of a complicated hierarchy of "cycles" related to the cost function U. In this paper, we shall express these constants in terms of Ventcel graphs (minimum cost spanning trees) to simplify the concept and computation of these constants. We also establish some new properties of optimal Ventcel graphs.

1. INTRODUCTION

Let S be a finite set and $U : S \times S \to [0,\infty]$ be a cost function, where $U(i,j)$ is interpreted as the cost from the state i to a different state j. Consider a family of irreducible, time-homogeneous Markov chains $\{X_n^\epsilon\}$ defined on S with transition probability

$$p^\epsilon(i,j) = p_{i,j} \cdot \epsilon^{U(i,j)}(1 + o(1)) \text{ for all } i \ne j, i,j \in S. \tag{1.1}$$

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Here we assume ϵ is a small parameter and $p_{i,j} = 0$ iff $U(i,j) = \infty$. Note that $U(i,i)$ plays no role in (1.1) because $\sum_j p^\epsilon(i,j) = 1$. The purpose of running such Markov chains is to find the smallest set $\underline{S} \subseteq S$ such that for small ϵ, $P(X_n^\epsilon \in \underline{S}) \approx 1$ as $n \to \infty$ and the order estimates for the expected time of hitting $\underline{S}$. The set $\underline{S}$ is referred to as the global minimum set of the cost function U. In many physical models, $U(i,j) = (u(j) - u(i))^+$ if j is a neighbor of i and is ∞ otherwise, where u is a potential function on S. It turns out that in this case, $\underline{S} = \{i \in S : u(i) = \min_S u\}$ as expected. However, it takes some efforts to determine $\underline{S}$ for a general cost function U.

Instead of running a family of Markov chains, one can have a single but time-inhomogeneous Markov chain. This is called simulated annealing process and readers are referred to [**7, 8**] for details.

Various properties related to $\{X_n^\epsilon\}$ have been obtained in [**3**]. Let μ^ϵ be the invariant distribution of $\{X_n^\epsilon\}$. In this paper we shall be concerned with the following issues :

(1) For any state $i \in S$, the invariant distribution μ^ϵ satisfies $\mu^\epsilon(i) \approx \epsilon^{h(i)}$ for ϵ small.
Hence, $\underline{S} = \{i \in S : h(i) = 0\}$. Note that $\sum_{j \in S} \mu^\epsilon(j) = 1$.

(2) Starting from μ^ϵ, $E^\epsilon T \approx \epsilon^{-\delta_h}$ for ϵ small, where T is the hitting time of $\underline{S}$.

(3) Furthermore, $E^\epsilon T_{i_0} \approx \epsilon^{-\delta_v}$ where T_{i_0} is the hitting time of any fixed $i_0 \in \underline{S}$.
The constants $h(i), \delta_h$ and δ_v are defined in [**2**] through a hierarchy of the so called "cycles". While conceptually it is easy to comprehend these constants, it is hard to actually compute them even through computers. The quantity $\mu^\epsilon(i)$ in (1) already appeared in [**6,10**] by solving linear equations. Related problems of (2) and (3) have been studied in [**4,5**].

Our aim of this paper is first to define these constants $h(i), \delta_h$ and δ_v in terms of optimal Ventcel graphs [**6,10**] and then simplify their computation by using minimum cost spanning trees. Indeed, optimal Ventcel graphs will be viewed as a kind of minimum cost spanning trees with pre-assigned roots.

One example is the potential case of the spin glass model. In this model, $S = \{-1, 1\}^{\mathbf{D}_n}$, where $\mathbf{D}_n$ is the 2-dim lattice of size $n \times n$. For each state $i \in S$, its nearest-neighbor potential energy is defined as

$$u(i) = -\sum_{|x-y|=1} J_{x,y} \cdot i(x) \cdot i(y),$$

where the real number $J_{x,y}$ denotes the interaction strength between two neighboring sites x, y in $\mathbf{D}_n$. Let $\mathrm{N}(i) = \{j \in S : i(x) = j(x)$ for all sites $x \in \mathbf{D}_n$ except one $\}$ be the neighborhood of state i. Then the transition probability in (1.1) is given by $p^\epsilon(i,j) = \frac{1}{|\mathrm{N}(i)|}\,\epsilon^{U(i,j)}$, where $U(i,j) = \big(u(j) - u(i)\big)^+$ if $j \in \mathrm{N}(i)$ and ∞ otherwise. The purpose of running the Markov chains with transition probability $p^\epsilon(i,j)$ is to find the states with the smallest potential energy in the spin-glass model.

Here is an example with non-potential cost U. In the well-known 2-person prisoner's dilemma game, each prisoner has to play a strategy from $\{C, N\}$. Here C and N stand for "confess" and "not-confess" respectively. The unique Nash equilibrium of the game requires each prisoner to play C. However, it is to both prisoners' favor if they both play N [**9**]. It is always interesting to see how one can overcome such a dilemma. Recently, it was shown possible [**1**] in some evolutionary prisoner's dilemma games with local interaction, imitation and mutation. Instead of two players, there are N players sitting around a circle in such a model. At each time period, players first meet with each of their two neighbors once to play the prisoner's dilemma game, then imitate their neighbors or themselves whoever have the highest payoffs and finally, can make mistake independently with a small positive probability ϵ to choose the other strategy instead of the rational one. The dynamical process can be described by the above Markov chain $\{X_n^\epsilon\}$ with the state space $S = \{C, N\}^{\mathrm{N}}$ and the cost function $U(i, j)$ in (1.1) counts the number of player x who makes mistake at the final stage by adopting the non-rational strategy $j(x)$. Since $|S| = 2^{\mathrm{N}}$ can be very huge, it will usually take a great effort to get $\underline{S}$ by penetrating the hierarchy of cycles. Besides $\underline{S}$, the order estimate like $E^\epsilon T \approx \epsilon^{-\delta_h}$ is important in applications. In this model, using Ventcel graphs turns out to be most efficient to get $\underline{S}$, δ_h and so on.

We now review the concepts of cycles and Ventcel W-graphs. One example is given at the end of this section to illustrate the process. For a subset $W \subseteq S$, a W-graph is a function g from $S \setminus W$ to S with no cycles, i.e., for any $i \in S \setminus W$, there exist $i_0 = i, i_1, \ldots, i_m$ in $S \setminus W$ such that $g(i_k) = i_{k+1}$ for $0 \le k < m$, but $g(i_m) \in W$. For a Ventcel W-graph g, the cost of g is defined as follow :

$$V(g) = \sum_{i \in S \setminus W} U(i, g(i)).$$

A W-graph g is called W-optimal if

$$V(g) = v(W) \stackrel{def}{=} \min\{V(h) : h \text{ is a W-graph}\}. \tag{1.2}$$

Let G(k) be the set of all W-graphs with $\mid W \mid = k$. Define

$$v_k = \min\{V(g) \,:\, g \in G(k)\} \text{ for } k \ge 1. \tag{1.3}$$

A W-graph g is said k-optimal if $V(g) = v_k$ and $\mid W \mid = k$. We shall characterize optimal W-graphs, optimal k-graphs and v_k in Sections 2, 3 and 4 respectively. We next define cycles. For $i \in S$, let

$$V(i) = \min\{U(i, j) \,:\, j \in S \text{ and } j \neq i\}$$

be the minimum cost for reaching out from i. For any two states $i, j \in S$, we say that $i \ge j$ if there exist $i_0 = i, i_1, \ldots, i_m = j$ such that $U(i_k, i_{k+1}) = V(i_k)$ for each k. This simply means there is a path from i to j such that each intermediate step has its minimum cost. A state i is said minimal if $i \geq j$ implies $j \geq i$ for any other $j \in S$. Two different states i, j are said equivalent ($i \sim j$) if

$$\text{(i)} \quad i \text{ is minimal}, \qquad \text{(ii)} \quad i \geq j \text{ and } j \geq i.$$

We always assume $i \sim i$ and thus "$\sim$" is an equivalence relation. The equivalent classes under "$\sim$" will be called cycles. A hierarchy of cycles can be established as follows. First, let $S^0 = S, U^0 = U$ and $V^0 = V$. Having defined S^{n-1}, U^{n-1} and V^{n-1}, let $S^n = \{\text{cycles of } S^{n-1}\}$. Hence if $C^n \in S^n$ then $C^n = \{C_i^{n-1}\}_i$ where $C_i^{n-1} \in S^{n-1}$ for each i and $\{C_i^{n-1}\}_i$ forms a cycle under U^{n-1}. The depth of C^n is defined as

$$d^{n-1}(C^n) = \max\{V^{n-1}(C_i^{n-1}) : C_i^{n-1} \in C^n\}.$$

For any two different states $C^n = \{C_i^{n-1}\}$ and $\bar{C}^n = \{\bar{C}_j^{n-1}\}$ in S^n, we now define

$$U^n(C^n, \bar{C}^n) = d^{n-1}(C^n) + \min_{i,j} \{U^{n-1}(C_i^{n-1}, \bar{C}_j^{n-1}) - V^{n-1}(C_i^{n-1})\} \tag{1.4}$$

and

$$V^n(C^n) = \min\{U^n(C^n, \bar{C}^n) : \bar{C}^n \in S^n \text{ and } \bar{C}^n \neq C^n\}.$$

This process will terminate first at some N, i.e., $| S^{N+1} |= 1$. For each state $i \in S$ we can find a unique sequence of cycles $i = C^0 \in C^1 \in \cdots \in C^{n-1} \in C^n \in \cdots \in C^N \in C^{N+1} = S^{N+1}$. Such a sequence will be referred to as the family tree of i. We shall abuse the notation a bit by saying that $C^k \in C^n$ if there are $C^j \in S^j$ for $k < j < n$ such that $C^k \in C^{k+1} \in \cdots \in C^n$ is part of some family tree. Finally, for a W-graph g, let

$$V(g; C^n) = \sum_{i \in C^n \setminus W} U(i, g(i)) \tag{1.5}$$

be the cost of g restricted to the cycle C^n. If i has the family tree $i = C^0 \in C^1 \in \cdots \in C^n \in \cdots \in C^{N+1} = S^{N+1}$, then the global minimum set $\underline{S}$, the constants $h(i), \delta_h$ and δ_v are characterized in [2] as follows:

$$\begin{cases} \underline{S} = \{i \in S : h(i) = 0\} \text{ where } h(i) = \sum_{n=0}^{N} (d^n(C^{n+1}) - V^n(C^n)), \\ \delta_h = \max\{V^k(C^k) : \text{ all cycles } C^k \in S^k \text{ with } C^k \cap \underline{S} = \emptyset\}, \\ \delta_v = \max\{V^k(C^k) : \text{ all cycles } C^k \in S^k \text{ with } i_0 \notin C^k\}, \end{cases} \tag{1.6}$$

where $i_0 \in \underline{S}$ is fixed.

Note that δ_v above is in fact independent of the choice of state $i_0 \in \underline{S}$.

The main purpose of this paper is to represent the above constants in terms of Ventcel W-graphs. The following will be proved in Section 4.

Main Theorem. *For any $i \in S$, we have $h(i) = v(\{i\}) - v_1$, $\delta_v = v_1 - v_2$ and $\delta_h = v_{k_0-1} - v_{k_0}$, where $k_0 = \inf\{k \geq 2 : \exists$ an optimal k-graph W with $W \not\subseteq \underline{S}\}$.*

EXAMPLE. Let $S = S^0 = \{1, 2, 3\}$ with the cost function $U = U^0$ on $S \times S$ given by $U(1, 2) = U(3, 1) = 4$, $U(1, 3) = U(2, 1) = 3$, $U(2, 3) = 1$ and $U(3, 2) = 0$. Note that the value of $U(i, i)$ is unimportant, but serves to make $\sum_{j=1}^{3} p^\epsilon(i, j) = 1$ in (1.1).

States 2 and 3 form a cycle and state 1 is itself a cycle in S^1. Thus $S^1 = \{\{1\}, \{2, 3\}\}$. A simple computation via (1.4) shows $U^1(\{1\}, \{2, 3\}) = U^1(\{2, 3\}, \{1\}) = 3$. Naturally, $\{1\}$ and $\{2, 3\}$ form a cycle in S^2 and the process terminates.

Based on (1.2), one can easily compute that

$v(\{1\}) = 3$ and the $\{1\}$-optimal graph is $g(3) = 2$ and $g(2) = 1$,

$v(\{2\}) = 3$ and the $\{2\}$-optimal graph is $g(1) = 3$ and $g(3) = 2$.

Similarly, $v(\{2\}) = 4$ and the $\{2\}$-optimal graph is $g(1) = 3$ and $g(3) = 2$. Thus, $v_1 = v(\{1\}) = v(\{2\}) = 3$ by (1.3). From the Main Theorem we have $h(1) = h(2) = 0$ and $h(3) = 1$. By (1.6), the global minimum set $\underline{S} = \{1, 2\}$. Obviously, $v_2 = 0$ and the 2-optimal graph is a $\{1, 2\}$-graph with $g(3) = 2$. Since $|S| = 3$ and $|\underline{S}| = 2$, $k_0 = 3$ in the Main Theorem and thus $\delta_v = v_1 - v_2 = 3$ and $\delta_h = v_2 - v_3 = 0 - 0 = 0$.

2. CONSTRUCTION OF OPTIMAL W-GRAPHS.

In this section, we shall identify the optimal Ventcel graphs for a fixed subset $W \in S$.

Definition 2.1. *Let $C^k, \bar{C}^k \in S^k$. For a W-graph h we say $h \in (C^k \to \bar{C}^k)$ if there exist $i \in C^k$, $j \in \bar{C}^k$ such that $h(i) = j$. In the case that $\bar{C}^k$ satisfies $U^k(C^k, \bar{C}^k) = V^k(C^k)$, we simply write $h \in (C^k \to)$.*

For two cycles C^k and $\bar{C}^k$ in S^k, we define the minimal cost on C^k of W-graphs in $(C^k \to \bar{C}^k)$ as follows. For $k = 1$, $C^1 \cap W = \emptyset$ and any $\bar{C}^1 \neq C^1$, let

$$V_W(C^1 \to \bar{C}^1) = \sum_{i \in C^1} V(i) + U^1(C^1, \bar{C}^1) - d^0(C^1). \tag{2.1}$$

Note that $V_W(C^1 \to \bar{C}^1)$ is undefined if $C^1 \cap W \neq \emptyset$. For $C^1 \cap W \neq \emptyset$ we let

$$V_W(C^1 \to C^1) = \sum_{i \in C^1 \setminus W} V(i) \tag{2.2}$$

and $V_W(C^1 \to C^1)$ remains undefined if $C^1 \cap W = \emptyset$. We write $V_W(C^1 \to)$ for $V_W(C^1 \to \bar{C}^1)$ in (2.1) if $U^1(C^1, \bar{C}^1) = V^1(C^1)$. Suppose we have defined $V_W(C^{k-1} \to \bar{C}^{k-1})$ for any $C^{k-1}, \bar{C}^{k-1} \in S^{k-1}$ as in (2.1) and (2.2). Then for any $C^k = \{C_i^{k-1}\} \neq \bar{C}^k$ in S^k, let

$$V_W(C^k \to \bar{C}^k) = \sum_i V_W(C_i^{k-1} \to) + U^k(C^k, \bar{C}^k) - d^{k-1}(C^k) \text{ if } \quad C^k \cap W = \emptyset. \tag{2.3}$$

Note that $V_W(C^k \to \bar{C}^k)$ is undefined if $C^k \cap W \neq \emptyset$. For $C^k \cap W \neq \emptyset$ we let

$$V_W(C^k \to C^k) = \sum_{C_i^{k-1} \cap W = \emptyset} V_W(C_i^{k-1} \to) + \sum_{C_i^{k-1} \cap W \neq \emptyset} V_W(C_i^{k-1} \to C_i^{k-1}) \tag{2.4}$$

and $V_W(C^k \to C^k)$ remains undefined if $C^k \cap W = \emptyset$. Similarly, we shall write $V_W(C^k \to)$ for $V_W(C^k \to \bar{C}^k)$ if $U^k(C^k, \bar{C}^k) = V^k(C^k)$ and $C^k \cap W = \emptyset$. Then by definition (2.3),

$$(2.5)\quad V_W(C^k \to \bar{C}^k) - V_W(C^k \to) = U^k(C^k, \bar{C}^k) - V^k(C^k) \geq 0 \text{ for } \bar{C}^k \neq C^k \in S^k.$$

Theorem 2.2. *Let $W \subseteq S$ and $C^k, \bar{C}^k \in S^k$ be different. Then for any W-graph h,*

$$V(h; C^k) \geq \begin{cases} V_W(C^k \to C^k) & \text{if } C^k \cap W \neq \emptyset, \\ V_W(C^k \to \bar{C}^k) & \text{if } C^k \cap W = \emptyset \text{ and } h \in (C^k \to \bar{C}^k). \end{cases}$$

Proof. We first consider $k = 1$. Let $C^1, \bar{C}^1 \in S^1$. If $C^1 \cap W \neq \emptyset$ then by (2.2),

$$V(h; C^1) = \sum_{i \in C^1 \setminus W} U(i, h(i)) \geq \sum_{i \in C^1 \setminus W} V(i) = V_W(C^1 \to C^1).$$

If $C^1 \cap W = \emptyset$ and $h \in (C^1 \to \bar{C}^1)$, then there are $i_0 \in C^1$ and $j \in \bar{C}^1$ such that $h(i_0) = j$. By definition (1.4) for $U^1(C^1, \bar{C}^1)$ and (2.1), we have

$$\begin{aligned} V(h; C^1) &= \sum_{i \in C^1} U\big(i, h(i)\big) \geq \sum_{i \in C^1 \setminus \{i_0\}} V(i) + U(i_0, j) \\ &\geq \sum_{i \in C^1 \setminus \{i_0\}} V(i) + U^1(C^1, \bar{C}^1) - d^0(C^1) + V(i_0) \\ &= \sum_{i \in C^1} V(i) + U^1(C^1, \bar{C}^1) - d^0(C^1) = V_W(C^1 \to \bar{C}^1) \end{aligned}$$

and the equalities hold iff $U\big(i, h(i)\big) = V(i)$ for all $i \neq i_0$ and $U\big(i_0, h(i_0)\big) = U(C^1, \bar{C}^1) - d^0(C^1) + V(i_0)$. Suppose we have proved the theorem up to k. Let $C^{k+1} = \{C_i^k\}$. If $C^{k+1} \cap W \neq \emptyset$ then by (1.5), (2.5) and the induction hypothesis,

$$\begin{aligned} V(h; C^{k+1}) &= \sum_{C_i^k \cap W \neq \emptyset} V(h; C_i^k) + \sum_{C_i^k \cap W = \emptyset} V(h; C_i^k) \\ &\geq \sum_{C_i^k \cap W \neq \emptyset} V_W(C_i^k \to C_i^k) + \sum_{C_i^k \cap W = \emptyset} V_W(C_i^k \to) = V_W(C^{k+1} \to C^{k+1}). \end{aligned}$$

If $C^{k+1} \cap W = \emptyset$ and $h \in (C^{k+1}, \bar{C}^{k+1})$, then there exist $C_{i_0}^k \in C^{k+1}$ and $\bar{C}^k \in \bar{C}^{k+1}$ such that $h \in (C_{i_0}^k \to \bar{C}^k)$. Using (2.5), (2.3) and the induction hypothesis again,

$$\begin{aligned} V(h; C^{k+1}) &= \sum_{i \neq i_0} V(h; C_i^k) + V(h; C_{i_0}^k) \geq \sum_{i \neq i_0} V_W(C_i^k \to) + V_W(C_{i_0}^k \to \bar{C}^k) \\ &\geq \sum_{i \neq i_0} V_W(C_i^k \to) + V_W(C_{i_0}^k \to) + U^k(C_{i_0}^k, \bar{C}^k) - V^k(C_{i_0}^k) \\ &\geq \sum_i V_W(C_i^k \to) + U^{k+1}(C^{k+1}, \bar{C}^{k+1}) - d^k(C^{k+1}) \\ &= V_W(C^{k+1} \to \bar{C}^{k+1}), \end{aligned}$$

where (1.4) is used in the last inequality. This completes the proof by induction.

REMARK 2.3. Theorem 2.2 actually describes all the possible ways to construct optimal W-graphs. Indeed, if $C^k \cap W = \emptyset$ and h is a W-graph, then it is obvious that $h \in (C^k \to \bar{C}^k)$ for some other $\bar{C}^k \in S^k$ and thus $h \in (C_i^{k-1} \to \bar{C}_j^{k-1})$ for some $C_i^{k-1} \in C^{k-1}$ and $\bar{C}_j^{k-1} \in \bar{C}^{k-1}$. Theorem 2.2 then dictates that C_i^{k-1} and $\bar{C}_j^{k-1}$ must satisfy

$$U^k(C^k, \bar{C}^k) = U^{k-1}(C_i^{k-1}, \bar{C}_j^{k-1}) + d^{k-1}(C^k) - U^{k-1}(C_i^{k-1}) \tag{2.6}$$

in order for h to be optimal, which simply means that the minimum in (1.4) is attained at the pair $(C_i^{k-1}, \bar{C}_j^{k-1})$. For the other $C_r^{k-1} \in C^k$, Theorem 2.2 forces h to be in $(C_r^{k-1} \to)$. Different pairs $(C_i^{k-1}, \bar{C}_j^{k-1})$ satisfying (2.6) provide different W-graphs but they all have the same cost on C^k and thus are optimal. Obviously this is the only option we have in constructing optimal W-graphs h on C^k. If $C^k \cap W \neq \emptyset$ then Theorem 2.2 implies that $h \in (C^k \to C^k)$ and for each $C_i^{k-1} \in C^k$, $h \in (C_i^{k-1} \to)$ or $h \in (C_i^{k-1} \to C_i^{k-1})$ depending on $C_i^{k-1} \cap W = \emptyset$ or not. Since obviously $C^{N+1} \cap W \neq \emptyset$, an induction procedure can be initiated to construct all optimal W-graphs.

3. CONSTRUCTION OF r-OPTIMAL GRAPHS.

In this section we shall construct r-optimal graphs for any $1 \leq r \leq |S|$. We first make some notations. Recall that $N \geq 0$ is the first number that $|S^{N+1}| = |\{C^{N+1}\}| = 1$. For any $r \geq 1$ and $C^k \in S^k$ with $k \leq N+1$, let

$$V_r(C^k \to C^k) = \inf\{V_W(C^k \to C^k) : |W \cap C^k| = r\}. \tag{3.1}$$

For $k \leq N$ and $C^k \neq \bar{C}^k$, let

$$V_0(C^k \to \bar{C}^k) = V_W(C^k \to \bar{C}^k) \text{ for any } W \text{ with } \quad W \cap C^k = \emptyset. \tag{3.2}$$

Note that the right hand side of (3.2) is independent of W as long as $W \cap C^k = \emptyset$. We use $V_0(C^k \to)$ for $V_0(C^k \to \bar{C}^k)$ if $U^k(C^k, \bar{C}^k) = V^k(C^k)$. Finally, for $C^k = \{C_i^{k-1}\}$ we define $V_0(C^k \to C^k)$ for $1 \leq k \leq N+1$ as follows :

$$V_0(C^k \to C^k) = \Sigma_i V_0(C_i^{k-1} \to). \tag{3.3}$$

In particular, (3.3) for $k = 1$ can be written as $V_0(C^1 \to C^1) = \sum_{i \in C^1} V(i)$.

For any $C^k \in S^k$, let $\mathrm{P}(C^k) = \{\text{all cycles } \ C^i \in C^k\}$. A sequence $C^0 \in C^1 \in \cdots \in C^i \in \cdots \in C^k$ is called a principal sequence of C^k if it has the property that $V^i(C^i) = d^i(C^{i+1})$ for each $i \leq k-1$. We use the notation $\mathrm{PS}(C^k)$ to denote such a sequence. Principal sequences of C^k may not be unique. Finally, let

$$m_1(C^k) = \max\{V^i(C^i) : C^i \in \mathrm{P}(C^k) \backslash \mathrm{PS}(C^k)\}.$$

Here $\mathrm{P}(C^k) \setminus \mathrm{PS}(C^k)$ denote the collection of cycles in C^k except one and any principal sequence of C^k. It is easy to see that $m_1(C^k)$ is independent of the choice of such a principal sequence. If $m_1(C^k)$ is attained at some $C^{i_1} \in \mathrm{P}(C^k) \setminus \mathrm{PS}(C^k)$, i.e., $m_1(C^k) = V^{i_1}(C^{i_1})$, let

$$m_2(C^k) = \max\{V^i(C^i) : C^i \in \mathrm{P}(C^k) \setminus \mathrm{PS}(C^k) \cup \mathrm{PS}(C^{i_1})\}.$$

Similarly, we can define $m_r(C^k)$ until $\mathrm{P}(C^k)\setminus\mathrm{PS}(C^k)\cup\mathrm{PS}(C^{i_1})\cup\cdots\cup\mathrm{PS}(C^{i_{r-1}}) = \emptyset$. We now prove the main result of this section.

Theorem 3.1. *For any different $C^k, \bar{C}^k \in S^k$ and $r \geq 1$, we have*

$$V_0(C^k \to \bar{C}^k) - V_r(C^k \to C^k) = U^k(C^k, \bar{C}^k) + \sum_{i=1}^{r-1} m_i(C^k) \text{ for } 1 \leq k \leq N \tag{3.4}$$

and

$$V_0(C^k \to C^k) - V_r(C^k \to C^k) = d^{k-1}(C^k) + \sum_{i=1}^{r-1} m_i(C^k) \text{ for } 1 \leq k \leq N+1. \tag{3.5}$$

Proof. We first prove (3.4) by induction on k. Let $k = 1$ and $C^1 \neq \bar{C}^1 \in S^1$. By (3.2) and (2.1), $V_0(C^1 \to \bar{C}^1) = \sum_{i \in C^1} V(i) + U^1(C^1, \bar{C}^1) - d^0(C^1)$. By using (3.1), (2.2) and the definitions of $m_i(C^1)$, $V_r(C^1 \to C^1) = \sum_{i \in C^1} V(i) - \sum_{i=1}^{r-1} m_i(C^1) - d^0(C^1)$. A simple arithmetic verifies (3.6) for $k = 1$. Suppose (3.6) holds true up to $k-1 \leq N-1$. For $k \leq N$ and any different $C^k = \{C_i^{k-1}\}$, $\bar{C}_k \in S^k$, (3.2) and (2.5) imply that for any $W \cap C^k = \emptyset$,

$$\begin{aligned} V_0(C^k \to \bar{C}^k) &= \sum_i V_W(C_i^{k-1} \to) + U^k(C^k, \bar{C}^k) - d^{k-1}(C^k) \\ &= \sum_i V_0(C_i^{k-1} \to) + U^k(C^k, \bar{C}^k) - d^{k-1}(C^k). \end{aligned}$$

For some W fulfilling (3.1) with $|W \cap C^k| = r$, (2.4) and the induction hypothesis imply that

$$\begin{aligned} &V_r(C^k \to C^k) = \sum_{C_i^k \cap W = \emptyset} V_W(C_i^{k-1} \to) + \sum_{C_i^k \cap W \neq \emptyset} V_W(C_i^{k-1} \to C_i^{k-1}) \\ &= \sum_{C_i^k \cap W = \emptyset} V_0(C_i^{k-1} \to) + \sum_{|C_i^k \cap W| = r_i} V_{r_i}(C_i^{k-1} \to C_i^{k-1}) \quad \text{where } \sum_i r_i = r \\ &= \sum_{C_i^k \cap W = \emptyset} V_0(C_i^{k-1} \to) + \sum_{|C_i^k \cap W| = r_i} \Big(V_0(C_i^{k-1} \to) - V^{k-1}(C_i^{k-1}) - \sum_{j=1}^{r_i - 1} m_j(C_i^{k-1}) \Big) \\ &= \sum_i V_0(C_i^{k-1} \to) - \sum_{|C_i^{k-1} \cap W| = r_i} \Big(V^{k-1}(C_i^{k-1}) + \sum_{j=1}^{r_i - 1} m_j(C_i^{k-1}) \Big). \end{aligned}$$

Taking the difference of the two equations above, (3.4) follows as

$$U^k(C^k,\bar{C}^k) - d^{k-1}(C^k) + \sum_{|C_i^{k-1}\cap W|=r_i} \Big(V^{k-1}(C_i^{k-1}) + \sum_{j=1}^{r_i-1} m_j(C_i^{k-1})\Big) = U^k(C^k,\bar{C}^k) + \sum_{i=1}^{r-1} m_i(C^k)$$

by the definitions of of $m_i(C^k)$. The proof of (3.5) is similar and thus omitted.

4. PROOF OF THE MAIN THEOREM

The proof is done in three parts. We first consider δ_v.

Part (i). $\delta_v = v_1 - v_2$.

The proof is almost obvious. Let i_0 be a state in $\underline{S}$ and $i_0 = C^0(i_0) \in C^1(i_0) \in \cdots \in C^N(i_0) \in C^{N+1}$ be the family tree of i_0. It is a principal sequence of C^{N+1} because $i_0 \in \underline{S}$. By (1.3), (3.1) and (3.5), $v_1 - v_2 = V_1(C^{N+1} \to C^{N+1}) - V_2(C^{N+1} \to C^{N+1}) = m_1(C^{N+1}) = \max\{V^k(C^k) : C^k \in S^k \text{ but } C^k \neq C^k(i_0)\} = \delta_v$ in view of its definition in (1.6).

Part (ii). $\delta_h = v_{k_0-1} - v_{k_0}$, where

$$k_0 = \inf\{k \geq 2 : \exists \text{ an optimal } k\text{-graph W with } W \not\subseteq \underline{S}\}.$$

By (1.3), (3.1) and (3.5) again,

$$\begin{aligned} &v_{k_0-1} - v_{k_0} = m_{k_0-1}(C^{N+1}) \\ &\quad = \max\{V^k(C^k) : C^k \in \mathrm{P}(C^{N+1}) \setminus \mathrm{PS}(C^{i_1}) \setminus \mathrm{PS}(C^{i_2}) \setminus \cdots \setminus \mathrm{PS}(C^{i_{k_0-1}})\}. \end{aligned}$$

Since every $\mathrm{PS}(C^{i_r})$ is a part of the family tree of a state in $\underline{S}$, we obviously have

$$\{C^k : C^k \in \mathrm{P}(C^{N+1}) \setminus \mathrm{PS}(C^{i_1}) \setminus \mathrm{PS}(C^{i_2}) \setminus \cdots \setminus \mathrm{PS}(C^{i_{k_0-1}})\} \supseteq \{C^k : C^k \cap \underline{S} = \emptyset\}.$$

Thus $v_{k_0-1} - v_{k_0} \geq \delta_h$ by its definition in (1.6). On the other hand, if $v_{k_0-1} - v_{k_0} = m_{k_0-1}(C^{N+1}) > \max\{V^k(C^k) : C^k \cap \underline{S} = \emptyset\}$ then $m_{k_0-1} = V^k(C^k)$ for some C^k with $C^k \cap \underline{S} \neq \emptyset$. This implies that for any W-optimal graph with $|W| = k_0$, we must have $W \subseteq \underline{S}$ which contradicts the definition of k_0. The proof of Part (ii) is completed.

Part (iii). For any $i \in S$, $h(i) = v(\{i\}) - v_1$.

For $i \in S$, let $i = C^0 \in C^1(i) \in \cdots \in C^k(i) \in \cdots \in C^N(i) \in C^{N+1}$ be the family tree of i. Suppose temporarily that the following holds.

Lemma 4.1. *Let $W=\{i\}$. Then for $1 \le k \le N+1$,*

$$V_W\big(C^k(i) \to C^k(i)\big) - V_1\big(C^k(i) \to C^k(i)\big) = \sum_{r=1}^{k} \Big(d^{r-1}\big(C^r(i)\big) - V^{r-1}\big(C^{r-1}(i)\big)\Big). \tag{3.6}$$

The conclusion follows then from (1.6) *and the lemma with $k = N+1$ as shown below*:

$$\begin{aligned} h(i) &= \sum_{r=1}^{N+1} d^{r-1}\big(C^r(i)\big) - V^{r-1}\big(C^{r-1}(i)\big) \\ &= V_W\big(C^{N+1}(i) \to C^{N+1}(i)\big) - V_1\big(C^{N+1}(i) \to C^{N+1}(i)\big) = v(\{i\}) - v_1. \end{aligned}$$

It remains to verify Lemma 4.1, which is done by induction on k.

Proof of Lemma 4.1. For $k=1$, we have from (2.2), (3.5) and (3.3) that

$$\begin{aligned} V_W\big(C^1(i) \to C^1(i)\big) = \sum_{j \in C^1(i)\setminus\{i\}} V(j) \text{ and } V_1\big(C^1(i) \to C^1(i)\big) \\ = \sum_{j \in C^1(i)} V(j) - d^0\big(C^1(i)\big). \end{aligned}$$

By taking the difference, (3.6) for $k=1$ is verified. Suppose the lemma is proved up to $k-1$. Let $C^k(i) = \{C_j^{k-1}\}$. By (2.4), the induction hypothesis and (3.5),

$$\begin{aligned} V_W\big(C^k(i) \to C^k(i)\big) &= \sum_{C_j^{k-1} \neq C^{k-1}(i)} V_0(C_j^{k-1} \to) + V_W\big(C^{k-1}(i) \to C^{k-1}(i)\big) \\ &= \sum_j V_0(C_j^{k-1} \to) - V_0\big(C^{k-1}(i) \to \big) + V_1\big(C^{k-1}(i) \to C^{k-1}(i)\big) \\ &\quad + \sum_{r=1}^{k-1} d^{r-1}\big(C^r(i)\big) - V^{r-1}\big(C^{r-1}(i)\big) \\ &= \sum_j V_0(C_j^{k-1} \to) - V^{k-1}\big(C^{k-1}(i)\big) + \sum_{r=1}^{k-1} d^{r-1}\big(C^r(i)\big) - V^{r-1}\big(C^{r-1}(i)\big). \end{aligned} \tag{3.7}$$

But $V_1\big(C^k(i) \to C^k(i)\big) = \sum_j V_0(C_j^{k-1} \to) - d^{k-1}\big(C^k(i)\big)$ by (3.5) and (3.3). By subtracting it from (3.7), the proof of (3.6) is completed by induction and thus so does Part (iii).

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ON DIFFERENTIAL EQUATIONS WITH NONSTANDARD COEFFICIENTS

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In the frame of Colombeau generalized functions we prove the existence and uniqueness of the solution of a system of linear differential equations with given initial data. The obtained result is applied to the Riccati equation.

1. INTRODUCTION

The classical distribution theory turns out to be insufficient for treating certain differential equations involving nonlinear operations and distributions. This kind of problems appear, e.g., in a rather simple dynamical system which describes the evolution of the population densities of predators and preys.

In the theory of distributions there are two complementary points of view:

(1) A distribution $f \in \mathcal{D}'(\mathbb{R}^n)$ is a continuous linear functional on the space $\mathcal{D}(\mathbb{R}^n)$ of compactly supported smooth functions with an appropriate convergence, see, [**8**]. Here we have a linear action

$$\varphi \to \langle f, \varphi \rangle$$

of f on a test function φ.

(2) If $\{\varphi_n\}$ is a sequence of smooth functions converging to the Dirac δ function, a family of regularizations $\{f_n\}$ can be produced by the convolution,

$$f_n(x) = f * \varphi_n = \langle f(y), \varphi_n(x-y) \rangle,$$

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since it converges weakly to the original distribution $f \in \mathcal{D}'(\mathbb{R}^n)$. Identifying two sequences $\{f_n\}$ and $\{\overline{f}_n\}$ if they have the same limit, we obtain a sequential representation of the space of distributions. Other authors use the equivalence classes of nets of regularization. The delta-net $\{\varphi_\epsilon\}_{\epsilon>0}$ is defined by $\varphi_\epsilon(x) = \epsilon^{-n}\varphi\left(x/\epsilon\right)$. This point of view is due to P. Antosik, J. Mikusiński and R. Sikorski, see [**1**]. But, when working with regularizations, the nonlinear structure is lost by identifying sequences (nets) with the same limit. Furthermore, in associative algebras of generalized functions multiplication and differentiation cannot simultaneously extend the corresponding classical operations unrestrictedly. One, therefore, has to reduce requirements on the multiplication.

The actual construction of differential algebras enjoying these optimal properties is due to J. F. Colombeau, see [**2**]. Colombeau theory of algebras of generalized functions offers the possibility of applying large classes of nonlinear operations to distributional objects. Colombeau algebra, denoted by $\mathcal{G}$, is an associative differential algebra with distributions linearly embedded in it. In Section 2, we give only few notions from Colombeau's theory, in fact those that will be used in the paper. Besides [**2**], on the theory of Colombeau algebras one can see also the monograph [**6**], and the papers [**5**], [**3**], [**4**] and [**7**].

In Section 3 we analyze the following system of differential equations

$$z_1'(t) = z_2(t),\ z_2'(t) = z_3(t), \ldots, z_n'(t) = f(t)\, z_1(t), \tag{1}$$

with initial conditions

$$z_1(0) = z_{10},\ z_2(0) = z_{20}, \ldots, z_n(0) = z_{n0}, \tag{2}$$

where f and z_i, $i = 1, 2, \ldots, n$ are elements from the Colombeau algebra $\mathcal{G}(\mathbb{R})$, and z_{10}, z_{20}, $\ldots$, z_{n0} are given elements from the Colombeau algebra $\overline{\mathcal{C}}$ of generalized complex numbers. Note that products like fz_1, appearing in (1), have sense when both f and z_1 are in $\mathcal{G}$.

We prove that under certain conditions the system (1) has a unique solution in $\mathcal{G}(\mathbb{R})$. Note that the system (1) corresponds to $z^{(n)} = f(t)z$ (with the corresponding initial conditions), where f is an arbitrary Colombeau generalized function with an appropriate condition given for a representative f_ε of f.

Next we turn to the case $n = 2$ and introduce a function $x(t)$ by $x(t) = z'(t)/z(t)$. It turns out that x is the solution of the Riccati differential equation

$$x'(t) + x^2(t) = f(x), \quad x(0) = x_0,$$

where x_0 is a Colombeau generalized number. In fact, using the relation between x_0 and the data $(z_1(0), z_2(0))$, we solve the system (1) and thus obtain the solution of the Ricati equation.

2. COLOMBEAU ALGEBRAS

We start with the basic notions of Colombeau theory. The main idea of this theory in its simplest form is that of embedding the space of distributions into a factor algebra. We use the following spaces.

$$\begin{aligned}\mathcal{A}_0(\mathbb{R}^n) &= \left\{\varphi \in \mathcal{D}(\mathbb{R}^n) \ : \ \int \varphi(x)dx = 1\right\}, \ \text{ and}\\ \mathcal{A}_q(\mathbb{R}^n) &= \left\{\varphi \in \mathcal{A}_0(\mathbb{R}^n) \ : \ \int x^\alpha \varphi(x)dx = 0 \ \text{ for } \ 1 \leq |\alpha| \leq q\right\}.\end{aligned}$$

Let $\mathcal{E}(\mathbb{R}^n)$ be the algebra of functions

$$u(\varphi, x) : \mathcal{A}_0(\mathbb{R}^n) \times \mathbb{R}^n \to \mathbb{C},$$

where $u(\varphi, x)$ is required to be infinitely differentiable by a fixed "parameter" φ. Thus $\mathcal{E}(\mathbb{R}^n)$ denotes an algebra of complex valued functions having appropriate smoothness properties on a suitable domain.

Now let $\varphi_\epsilon(x) = \epsilon^{-n}\varphi(x/\epsilon)$ for $\varphi \in \mathcal{A}_0(\mathbb{R}^n)$. The sequence $(u * \varphi_\epsilon)_{\epsilon>0}$ converges to u in $\mathcal{D}'(\mathbb{R}^n)$. Taking this sequence as a representative of u, we obtain an embedding of $\mathcal{D}'(\mathbb{R}^n)$ into the algebra $\mathcal{E}(\mathbb{R}^n)$. However, embedding $\mathcal{C}^\infty(\mathbb{R}^n) \subset \mathcal{D}'(\mathbb{R}^n)$ into this algebra via convolution as above will not yield a subalgebra because, in general,

$$(f * \varphi_\epsilon)(g * \varphi_\epsilon) - (fg) * \varphi_\epsilon \neq 0. \tag{3}$$

The idea, therefore, is to find an ideal $\mathcal{N}(\mathbb{R}^n)$ such that the difference on the left-hand side of (3) vanishes in the resulting quotient. In fact, for the construction of $\mathcal{N}(\mathbb{R}^n)$ it is sufficient to find an ideal containing all the differences of the form $((f * \varphi_\epsilon) - f)_{\epsilon>0}$.

The Taylor expansion of $(f * \varphi_\epsilon) - f$ shows that this term will vanish faster that any power of ϵ, uniformly on compact sets, in all derivatives. The set of all such sequences is not an ideal in $\mathcal{E}(\mathbb{R}^n)$, so we consider the set of moderate sequences $\mathcal{E}_M(\mathbb{R}^n)$ whose every derivative is bounded uniformly on compact sets by some negative power of ϵ. Then the generalized functions of Colombeau are elements of the quotient algebra

$$\mathcal{G} \equiv \mathcal{G}(\mathbb{R}^n) = \mathcal{E}_M(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n).$$

Here the moderate functionals $\mathcal{E}_M(\mathbb{R}^n)$ are defined by the property:

$$\begin{aligned}&\forall K \subset\subset \mathbb{R}^n \ \forall \alpha \in \mathbb{N}_0^n \ \exists p \geq 0 \text{ such that } \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n),\\ &\qquad \sup_{x\in K} |\partial^\alpha u(\varphi_\epsilon, x)| = \mathcal{O}(\epsilon^{-p}) \text{ as } \epsilon \to 0,\end{aligned}$$

and the null functionals $\mathbb{N}(\mathbb{R}^n)$ are defined by the property:

$$\begin{aligned}&\forall K \subset\subset \mathbb{R}^n \ \forall \alpha \in \mathbb{N}_0^n \ \exists p \geq 0 \text{ such that } \forall q \geq p \text{ and } \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n),\\ &\qquad \sup_{x\in K} |\partial^\alpha u(\varphi_\epsilon, x)| = \mathcal{O}(\epsilon^{q-p}) \text{ as } \epsilon \to 0.\end{aligned}$$

In words, moderate functionals satisfy a locally uniform polynomial estimate as $\epsilon \to 0$ when acting on φ_ϵ, together with all derivatives, while null functionals vanish faster than any power of ϵ in the same situation. The null functionals form a differential ideal in the collection of moderate functionals.

We define the space of functions $u : \mathcal{A}_0 \to \mathcal{C}$ and denote it by $\mathcal{E}_0(\mathbb{R}^n)$. It is a subalgebra in $\mathcal{E}(\mathbb{R}^n)$. Moderate functionals, denoted by $\mathcal{E}_{0M}(\mathbb{R}^n)$, are defined by the property:

$$\exists p \geq 0, \text{ such that } \forall \varphi \in \mathcal{A}_p(\mathbb{R}^n), \quad |u\,(\varphi_\epsilon)\,| = \mathcal{O}(\epsilon^{-p}) \text{ as } \epsilon \to 0.$$

Null functionals, denoted by $\mathcal{N}_0(\mathbb{R}^n)$, are defined by the property:

$$\exists p \geq 0 \text{ such that } \forall q \geq p \text{ and } \forall \varphi \in \mathcal{A}_q(\mathbb{R}^n), \quad |u\,(\varphi_\epsilon)\,| = \mathcal{O}(\epsilon^{q-p}) \text{ as } \epsilon \to 0.$$

Now the space of generalized complex numbers and of generalized real numbers is respectively the factor algebra defined as

$$\overline{\mathbb{C}} = \mathcal{E}_{0M}(\mathbb{C})/\mathcal{N}_0(\mathbb{C}), \quad \overline{\mathbb{R}} = \mathcal{E}_{0M}(\mathbb{R})/\mathcal{N}_0(\mathbb{R}).$$

The algebra $\mathcal{G}$ contains the space of distributions $\mathcal{D}'(\mathbb{R}^n)$ on $\mathbb{R}^n$ embedded by the map

$$i \;:\; \mathcal{D}'(\mathbb{R}^n) \to \mathcal{G}(\mathbb{R}^n), \;\; i(u) = \tilde{u} = \text{class}\left[(u * \varphi)(x) : \varphi \in \mathcal{A}_q(\mathbb{R}^n)\right].$$

Equivalence classes of sequences $(u_\epsilon)_{\epsilon>0}$ in $\mathcal{G}(\mathbb{R}^n)$ will be denoted by

$$U = \text{class}\left[(u_\epsilon)_{\epsilon>0}\right].$$

Definition 1. *A generalized function $F \in \mathcal{G}(\mathbb{R}^n)$ is said to admit some $u \in \mathcal{D}'(\mathbb{R}^n)$ as “associated distribution”, denoted $F \approx u$, if for some representative $f(\varphi_\epsilon, x)$ of F and any $\psi(x) \in D(\mathbb{R}^n)$ there is a $q \in \mathbb{N}_0$ such that, for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R}^n)$,*

$$\lim_{\epsilon\to 0} \int f(\varphi_\epsilon, x)\psi(x)\,\mathrm{d}x = \langle u, \psi\rangle.$$

This definition is independent of the representatives and the association is a faithful generalization of the equality of distributions.

Definition 2. *Let $F, G \in \mathcal{G}(\mathbb{R}^n)$. Then they are associated generalized functions, denoted $G \approx F$, if there exist representatives $g(\varphi_\epsilon, x)$ and $f(\varphi_\epsilon, x)$ of G and F respectively, and for any $\psi(x) \in D(\mathbb{R}^n)$ there is a $q \in \mathbb{N}_0$ such that, for any $\varphi(x) \in \mathcal{A}_q(\mathbb{R}^n)$*

$$\lim_{\epsilon\to 0} \int \big(g(\varphi_\epsilon, x) - f(\varphi_\epsilon, x)\big)\psi(x)\,\mathrm{d}x = 0\,.$$

3. COLOMBEAU ALGEBRA AND ORDINARY DIFFERENTIAL EQUATIONS

Now we will use the Colombeau algebra for solving the system of ordinary differential equations from (1). with initial conditions (2). As noted in the introduction, f and $z_i,\ i=1,2,\dots,n$ are elements from Colombeau algebra $\mathcal{G}(\mathbb{R})$, while $z_{10},\ z_{20},\dots,z_{n0}$ are given elements from Colombeau algebra $\overline{\mathcal{C}}$ of generalized complex numbers.

The system (1) has its equivalent matrix form:

$$Z'(t)=A(t)Z(t)\,,\quad Z(0)=Z_0, \tag{4}$$

where $Z=(z_1,z_2,\dots,z_n)\,,\,Z_0=(z_{10},z_{20},\dots,z_{n0})$ and

$$A(t)=(A_{ij}(t))=\begin{pmatrix} 0 & 1 & 0 & \cdots & 0\\ 0 & 0 & 1 & \cdots & 0\\ \vdots & \vdots & & \dots & \vdots\\ 0 & 0 & 0 & \cdots & 1\\ f(t) & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

Now, let K be a compact set in $\mathbb{R}$, $A(t)=\text{class}\left[a\left(\varphi_\epsilon,t\right)_{\epsilon>0}\right]$ an element from $\mathcal{G}^{n\times n}(\mathbb{R})$, and $\|a\left(\varphi_\epsilon,t\right)\|_K=\sup_{t\in K}\|a\left(\varphi_\epsilon,t\right)\|$. We will prove the following theorem.

Theorem 1. *Let $A\in\mathcal{G}^{n\times n}(\mathbb{R})$. Assume that there exists a representative $a\left(\varphi_\epsilon,t\right)$ of A such that for every compact set K in $\mathbb{R}$, and for every $\varphi\in\mathcal{A}_q$ ($q\in\mathbb{N}$ large enough) there is a constant $c>0$ satisfying the following condition:*

$$\exp\left\{\int\limits_0^t\|a(\varphi_\epsilon,x)\|_K\,\mathrm{d}x\right\}\le\frac{c}{\epsilon^q}.$$

Then the matrix differential equation $Z'(t)=A(t)Z(t)$, with the initial condition $Z_{|t=0}=Z_0=(z_{10},z_{20},\cdots,z_{n0})\in\overline{\mathcal{C}}^n$, has a unique solution $Z=(z_1,z_2,\dots,z_n)\in\mathcal{G}^n(\mathbb{R})$.

Proof. The proof consists of two parts. First, we will build up a solution that belongs to the Colombeau algebra. Second, existence and uniqueness in $\mathcal{G}^n(\mathbb{R})$ will be proved by showing that every solution of the above equation is in the same class of the Colombeau algebra.

We have that $A(t)=(A_{ij}(t))\,,\ i,j=1,2,\dots,n$, where $A_{ij}\in\mathcal{G}(\mathbb{R})$ for $i,j=1,2,\dots,n$. Now we are examining the problem

$$\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_n\end{pmatrix}'(t)=\begin{pmatrix} 0 & 1(\varphi_\epsilon,t) & 0 & \cdots & 0\\ 0 & 0 & 1(\varphi_\epsilon,t) & \cdots & 0\\ \vdots & \vdots & & \dots & \vdots\\ 0 & 0 & 0 & \cdots & 1(\varphi_\epsilon,t)\\ f(\varphi_\epsilon,t) & 0 & 0 & \cdots & 0\end{pmatrix}\begin{pmatrix} z_1\\ z_2\\ \vdots\\ z_n\end{pmatrix}(t)\,.$$

Integrating with respect to t and inserting initial conditions, we get an equivalent system of integral equations:

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix}(t) = \begin{pmatrix} z_{10} \\ z_{20} \\ \vdots \\ z_{n0} \end{pmatrix} + \int\limits_0^t \begin{pmatrix} 0 & 1(\varphi_\epsilon,\tau) & 0 & \cdots & 0 \\ 0 & 0 & 1(\varphi_\epsilon,\tau) & \cdots & 0 \\ \vdots & \vdots & & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 1(\varphi_\epsilon,\tau) \\ f(\varphi_\epsilon,\tau) & 0 & 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} z_1(\tau) \\ z_2(\tau) \\ \vdots \\ z_n(\tau) \end{pmatrix} \mathrm{d}\tau.$$

Let $Z = (z_1, z_2, \ldots, z_n)$ and $\|Z(\varphi_\epsilon, t)\|_K = \max_{j=1,2,\ldots,n} \sup_{t\in K} |z_j(\varphi_\epsilon, t)|$. Using Gronwall's inequality, we have

$$\|Z(\varphi_\epsilon, t)\|_K \le \|Z_0(\varphi_\epsilon, t_0)\| \exp\left\{\int\limits_0^t \|a(\varphi_\epsilon, x)\|_K \,\mathrm{d}x\right\} \le \frac{c*}{\epsilon^q},$$

for $\epsilon > 0$ and $\varphi \in \mathcal{A}_q$. Also, there is a sufficiently large $q_r \in \mathbb{N}$, such that for $\varphi \in \mathcal{A}_{q_r}$ and $\epsilon > 0$ we have

$$\|\partial^r \left(Z(\varphi_\epsilon, t)\right)\|_K \le \frac{c_r}{\epsilon^{q_r}}.$$

The last relation implies that $Z(\varphi_\epsilon, t) \in \mathcal{E}^n_M(\mathbb{R})$, so we have the solution Z of (4) in $\mathcal{G}^n$ and that solution is the class in $\mathcal{G}^n$ containing the element $Z(\varphi_\epsilon, t)$.

Concerning uniqueness, let us suppose that W =class$[w(\varphi_\epsilon, t)]$ is another solution of (4) in $\mathcal{G}^n(\mathbb{R})$ different from Z. This means that for a representative $w(\varphi_\epsilon, t)$ of W we have $w'(\varphi_\epsilon, t) = a(\varphi_\epsilon, t)w(\varphi_\epsilon, t) + n(\varphi_\epsilon, t)$ where $n(\varphi_\epsilon, t)$ belongs in $\mathcal{N}^n(\mathbb{R})$. Similarly as before, now we have:

$$\begin{aligned}\|Z(\varphi_\epsilon, t) - W(\varphi_\epsilon, t)\|_K &\le \|Z(\varphi_\epsilon, t_0) - W(\varphi_\epsilon, t_0)\| \\ &\quad + \|n(\varphi_\epsilon, t)\|_K \exp\left\{\int\limits_0^t \|a(\varphi_\epsilon, x)\|_K \,\mathrm{d}x\right\} \le O(\epsilon^q)\end{aligned}$$

and we get that $Z = W$ in $\mathcal{G}^n(\mathbb{R})$, which completes the proof of the theorem.

Now we will show that the system (1) is equivalent to a nonlinear differential equation. In fact, differentiating the first equation (1), we obtain $z_1''(t) = z_2'(t) = z_3(t)$. Repeating this method we finally obtain

$$z_1^{(n)}(t) = z_n'(t) = f(t)z_1(t).$$

Now if z_1 is a solution of the equation

$$z_1'(t) = x(t)z_1(t), \tag{5}$$

where x is an element from Colombeau algebra, then using the Leibniz formula, we have

$$z_1^{(n)}(t) = f(t)z_1(t)\,, \quad (x(t)z_1(t))^{(n-1)} = f(t)z_1(t)\,,$$
$$\sum_{k_1=0}^{n-1}\binom{n-1}{k_1}x^{(n-k_1-1)}(t)z_1^{(k_1)}(t) = f(t)z_1(t)\,, \quad \text{and}$$
$$\sum_{k_1=0}^{n-1}\binom{n-1}{k_1}x^{(n-k_1-1)}(t)\left[\sum_{k_2=0}^{k_1-1}\binom{k_1-1}{k_2}x^{(k_1-k_2-1)}(t)\left(\sum \ldots z_1^{(k_n)}(t)\right)\right] = f(t)z_1(t).$$

From the fact that $0 \le k_1 \le n-1,\ 0 \le k_2 \le n-2, \ldots, 0 \le k_{n-1} \le 1$, we have that $k_n = 0$, i. e.,

$$\sum_{k_1=0}^{n-1}\binom{n-1}{k_1}x^{(n-k_1-1)}(t)\left[\sum_{k_2=0}^{k_1-1}\binom{k_1-1}{k_2}x^{(k_1-k_2-1)}(t)\cdots z_1(t)\right] = f(t)z_1(t),$$

which is a nonlinear differential equation of the form:

$$x^{(n-1)}(t) + nx(t)x^{(n-2)}(t) + \cdots + x^n(t) = f(t). \tag{6}$$

Putting $n = 2$ in (6) we get the RICCATI nonlinear differential equation

$$x'(t) + x^2(t) = f(t), \tag{7}$$

where f and x are elements from COLOMBEAU algebra $\mathcal{G}(\mathbb{R})$. Let, moreover

$$x(0) = x_0, \tag{8}$$

where x_0 is a known element from $\overline{\mathbb{C}}$. In view of the previous analysis, the solution of the problem (8) is equivalent to the solution of the following system of differential equations:

$$z_1'(t) = z_2(t), \qquad z_2'(t) = f(t) \cdot z_1(t),$$

where z_1 is solution of the equation (5), satisfying the conditions $z_1(0) = 1$ and $z_1'(0) = z_2(0) = x_0$.

The equivalent matrix form of this system is

$$Z'(t) = A(t)Z(t), \quad Z(0) = Z_0\,,$$

where $Z = (z_1, z_2)$, $Z_0 = (z_1(0), z_2(0)) = (1, x_0)$, and $A(t) = \begin{pmatrix} 0 & 1 \\ f(t) & 0 \end{pmatrix}$.

As a result we have the following corollary:

Corolary 1. *Let $A \in \mathcal{G}^{2\times 2}(\mathbb{R})$. Assume that there exists a representative $a(\varphi_\epsilon, t)$ of A exists such that for every K compact set in $\mathbb{R}$, and for every $\varphi \in \mathcal{A}_q$ ($q \in \mathbb{N}$ large enough) there is a constant $c > 0$ satisfying the following condition:*

$$\exp\left\{\int_0^t \|a(\varphi_\epsilon, x)\|_K \,\mathrm{d}x\right\} \le \frac{c}{\epsilon^q}.$$

Then the Ricati differential equation (7) *with the initial condition* (8) *has a unique solution in* $\mathcal{G}(\mathbb{R})$.

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NEW CONCEPTS AND RESULTS ON THE AVERAGE DEGREE OF A GRAPH

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The idea of equilibrium of a graph G, initially applied to maximal outerplanar graphs (mops), was conceived to observe how the vertex degree distribution affects the average degree of the graph, $d(G)$. In this work, we formally extend the concept to graphs in general. From $d(G)$, two new parameters are introduced - the *top* and the *gap* of G, sustaining the definitions of tuner set, balanced and non-balanced graphs. We show properties of the new concepts when applied to particular families of graphs as trees and unicyclic graphs. We also establish bounds to the top of non-balanced graphs with integer average degree and we characterize their tuner sets.

1. INTRODUCTION

The idea of equilibrium of a graph G, initially applied to maximal outerplanar graphs (mops) in Rodrigues, Abreu and Markenzon [**3**], was conceived to observe how the vertex degree distribution affects the average degree of the graph, $d(G)$. In this work, we extend the concept to graphs in general.

We introduce two new parameters of a graph G – the *top* and the *gap* – both given as a function of the number of vertices and the average degree of G. We formalize *balanced* and *non-balanced* graphs and we show how particular families behave when the new concepts are applied to them.

The definition of *tuner sets*, a subset of vertices of G with degree lower than $d(G)$ that are able to compensate the presence of vertices with degree greater than $d(G)$ is presented. We show that a graph can have distinct tuner sets. An important

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result is the characterization of tuner sets for graphs that have the average degree as an integer. For these graphs, the following property holds: *the tuner set is unique and it is composed of all vertices with degree lower than d(G).*

Throughout this paper, G is a simple and connected graph, V its vertex set and E its edge set; $|V| = n$ and $|E| = m$. In Section 2 basic concepts are reviewed and we define the *top*, $\mu(G)$, and the *gap*, $h(G)$, of G. In Section 3, properties concerning the new parameters are shown, particularly of trees and unicyclic graphs. We determine bounds to $\mu(G)$ and, for a given feasible value of $\mu(G)$, a graph G is built. In Section 4, it is proved a characterization of graphs which have the average degree as an integer number.

2. BASIC NOTIONS

In this section, basic concepts already known are reviewed (see, for instance, Diestel [**1**] and Gross and Yellen [**2**]) and new definitions are presented. Two new parameters are defined, both related to the average degree of the graph: $\mu(G)$, the *top*, and $h(G)$, the *gap* of a graph G.

Let $d(v_i)$ be the *degree* of vertex v_i and $\Delta(G) = \max\limits_{1\leq i\leq n} d(v_i)$ be the *maximal degree* of G. The *average degree of* G is

$$d(G) = \frac{\sum\limits_{1\leq i\leq n} d(v_i)}{n}. \tag{1}$$

So, $0 \leq d(G) \leq n-1$, and it is not necessarily an integer. We define the *top* of graph G, $\mu(G)$, as $\lceil d(G) \rceil$.

For all $1 \leq i \leq n$, $0 \leq d(v_i) \leq n-1$, there is j, $0 \leq j \leq n-1$, such that $d(v_i) = j$. Let $Y \subseteq V(G)$. The *frequency of the degree* v_i is the cardinality of the set $\{v \in Y : d(v) = j\}$, denoted $\omega_Y(j)$. When $Y = V(G)$, we simply use $\omega(j)$.

As the sum of the degrees of all vertices is twice the number of edges in G, we have

$$\sum_{i=1,\ldots,n} i\,\omega(i) = 2m. \tag{2}$$

The *gap of* G is $h(G) = n(\mu(G) - d(G))$. If the gap of G is zero, then $d(G) = \mu(G)$ and we say that G is a *graph with zero gap.* Consequently, if $d(G) \in \mathbb{N}$, $h(G) = 0$.

For instance, if G is a k-regular graph, $d(G) = k$ and G is a graph with zero gap. However, non-regular graphs can also have this property. Figure 1 displays a non-regular graph with zero gap.

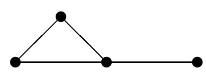

Figure 1. A non-regular graph G with $d(G) = \mu(G) = 2$.

From (1) and (2), we get

$$h(G) = \mu(G)\, n - \sum_{i=1,\dots,n} i\,\omega(i). \tag{3}$$

According to (3), $h(G) \in \mathbb{N}$.

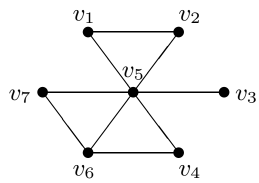

Figure 2. A graph G with $d(G) \approx 2.571$ and $\mu(G) = h(G) = 3$

Figure 2 displays a graph G with $d(G) \approx 2.571$. Then, $\mu(G) = 3$ and, from (3), we have $h(G) = 3$.

We call $B_G = \{v \in V(G) : d(v) = \mu(G)\}$ the *balanced vertex set*; $U_G = \{v \in V(G) : d(v) > \mu(G)\}$, the *upper vertex set* and $L_G = \{v \in V(G) : d(v) < \mu(G)\}$ the *lower vertex set* of G. Consequently, $|B_G| + |U_G| + |L_G| = n$ and if the subsets are different from the empty set, they determine a partition of $V(G)$. We simply use B, U and L, when it is not necessary to specify which graph we refer to. If $U = \emptyset$, G is said to be a *balanced graph*. If not, G is a *non-balanced graph*. For example, the graph G in Figure 2 is a non-balanced graph because $U = \{v_5\}$. It also has $B = \{v_6\}$ and $L = \{v_1, v_2, v_3, v_4, v_7\}$ as balanced and lower vertex sets, respectively.

3. PROPERTIES CONCERNING THE NEW PARAMETERS

In this section we present some properties of graphs concerning the new concepts and parameters presented in Section 2. Their proofs come straightforward from the definitions.

Basic Properties:

(1) For every graph with zero gap, $m = \dfrac{d(G)}{2}n = \dfrac{\mu(G)}{2}n$. Since $d(G) = \mu(G)$, if n is odd, $\mu(G)$ must be even.

(2) Every regular graph is balanced and it is a graph with zero gap. Of course $U = L = \emptyset$.

(3) There are balanced and non-regular graphs for which $L \neq \emptyset$. As an example, the graph $G = C_n + \{e\}$ has $\mu(G) = 3$ and L is constituted by all vertices of degree 2, being C_n a cycle with length n.

(4) Let G be a connected graph with $\mu(G) \leq n-2$. If G has at least one universal vertex v, $d(v) = n-1$, then G is a non-balanced graph. This property gives enough condition for the existence of non-balanced graphs with zero gap. A necessary condition is given by Proposition (5).

(5) Every graph G such that $\mu(G) = n-1$ is a balanced graph.

The new concepts turn out to be particularly interesting when dealing with well known families such as trees and unicyclic graphs. The following three simple results are stated here without proof.

Proposition 1. *For every tree $T \neq K_2$, T is not a graph with zero gap. Moreover, $\mu(T) = h(T) = 2$.*

Proposition 2. *For $n > 2$, P_n is a non-regular balanced graph.*

Proposition 3. *Let T be a tree with $n > 2$. T is a non-balanced graph if and only if $T \neq P_n$.*

Let G be a unicyclic graph. As $d(G) = 2$, G is a graph with zero gap. Proposition 4 gives a more general property for members of this family.

Proposition 4. *Let G be an unicyclic graph with n vertices. Then, G is a non-balanced graph with zero gap if and only if $G \neq C_n$.*

Proof. Suppose that G is an unicyclic non-balanced graph with n vertices. So, $m = n$, $d(G) = 2$ and G is a graph with zero gap. Since G is a non-balanced graph, $U \neq \emptyset$. Then, there is a vertex u in G such that $d(u) > 2$. Consequently, $G \neq C_n$. The reciprocal proof is equivalent. □

Proposition 5. *Let G with n vertices be a connected non-balanced graph. Then, $2 \leq \mu(G) \leq n - 2$. Moreover, the upper bound is achieved if and only if n is even.*

Proof. From Propositions 1 and 3, if G is a tree $T \neq P_n$, T satisfies all the hypotheses above and $\mu(T) = 2$.

Let G be a graph with $\mu(G) = n - 1$. From Basic Property (5), $U = \emptyset$ and G is a balanced graph. Consequently, if G is a non-balanced graph, $\mu(G) \leq n - 2$. If n is odd, $\mu(G) = n - 2$ is odd too. This disagrees with Basic Property (1). □

We are particularly interested in the behavior of the top of graphs with zero gap. The next result characterizes non-balanced graphs for which the top attains the upper bound given by Proposition 5.

Proposition 6. *Let n be even. G with n vertices is a connected non-balanced graph with zero gap and $\mu(G) = n - 2$ if and only if G has $\dfrac{n(n-2)}{2}$ edges and it has at least one universal vertex.*

Proof. Let n be even and G be a graph with n vertices. G is a graph with zero gap and $\mu(G) = n - 2$ if and only if $d(G) = \mu(G) = n - 2$. Besides, $d(G) = \dfrac{2m}{n}$. So, $m = \dfrac{n(n-2)}{2}$.

The following sentences are equivalent:

1. G is a non-balanced graph $\Leftrightarrow$
2. $U_G \neq \emptyset \Leftrightarrow$
3. There is a vertex u such that $d(u) > n - 2 \Leftrightarrow$
4. $d(u) = n - 1$.

Therefore u is an universal vertex. □

Now, the following question can be raised: For every $q \in \mathbb{N}$, $2 \leq q \leq n-2$, is there a non-balanced graph with zero gap G such that $\mu(G) = q$? For n even, the answer is affirmative and the proof is presented in the next theorem. For n odd, the subject will be handled later.

Theorem 1. *Let $n > 2$ be even. For every $q \in \mathbb{N}$, $2 \leq q \leq n-2$, there is a connected non-balanced graph with zero gap G of order n such that $\mu(G) = q$.*

Proof. The lower bound $\mu(G) = 2$ is achieved by unicyclic graphs, excluding C_n. The upper bound is proved in Proposition 6.

Let $n > 4$ and $q \in \mathbb{N}$, $2 < q < n-2$. Let us build a non-balanced graph G with zero gap and $\mu(G) = q$. Let T be a tree with n vertices. In order to obtain the desired graph, we must add t new edges to T such that

1. T is a spanning tree of G $(V(T) = V(G))$;
2. $\mu(G) = d(G) = q$;
3. there is $v \in V(G)$ such that $d(v) > \mu(G)$.

We know that, for a graph with zero gap, $\mu(G) = d(G) = \dfrac{2m}{n}$. So, $q = \dfrac{2m}{n}$ and $m = \dfrac{nq}{2}$. As $t = m - (n-1)$, being m the number of edges of G, we have

$$t = \left(\frac{q}{2} - 1\right) n + 1.$$

As n is even and $q > 2$, $t \in \mathbb{N}$ and $t > 1$. G will have $n - 1 + t$ edges; by construction it is a graph with zero gap.

If $\Delta(T) > q$, the t edges can be inserted randomly, and G, the resultant graph, is non-balanced. If $\Delta(T) \leq q$, we need to make sure that at least one vertex v of G attains $d(v) > q$. Let v be a vertex of T with $d(v) = \Delta(T)$. Adding $q - \Delta(T) + 1$ edges $\{v, w\}$ such that $w \in V(T), w \neq v$ and $\{v, w\} \notin E(T)$ the degree of v becomes exactly $q + 1$; this ensures that G is a non-balanced graph. The remaining $t - q + \Delta(T) - 1$ edges can be inserted randomly. □

Corollary 1. *Let G and G' be non-balanced graphs with zero gap of order n with m and m' edges, respectively. If n is even and $\mu(G') = \mu(G) + 1$ then $m' = m + \dfrac{n}{2}$.*

Proof. Let n be even. From Theorem 1, in order to obtain G and G' such that $\mu(G) = q$ and $\mu(G') = q + 1$, we must add $t = \left(\dfrac{q}{2} - 1\right) n + 1$ and $t' = \left(\dfrac{q+1}{2} - 1\right) n + 1$, respectively, to a tree T. So, $m' - m = t' - t = \dfrac{n}{2}$ and $m' = m + \dfrac{n}{2}$. □

Theorem 1 and Corollary 1 establish the theoretical foundation for the systematic generation of graphs with zero gap. The proof of Theorem 1 even provides the outline of an efficient algorithm to perform this task.

Actually, for a fixed even n, the algorithm presented below produces a whole sequence of graphs $G_2, G_3, \ldots, G_{n-2}$, being $\mu(G_i) = i$ and such that $G_{i-1} \subset G_i, 3 \leq i \leq n-2$.

Algorithm *GEN.*

Input: n, even.

Initial Step: Generate a random tree $T \neq P_n$; compute $T + \{e\}$, obtaining G_2, an unicyclic graph. G_2 is a graph with zero gap and $\mu(G_2) = 2$.

The following steps are repeated $n-4$ times, generating $G_3, G_4, \ldots G_{n-2}$.

Step 1: Let G_i be the last graph generated (for the first time, $i = 2$). Find a vertex v such that $d(v)$ is maximum in G_i. If $d(v) \leq i+1$ then add $m' = i + 2 - d(v)$ edges adjacent to vertex v; else m' is equal to zero.

Step2: Add $(n/2 - m')$ edges to G_i obtaining G_{i+1}.

For n even, Algorithm *GEN* generates $n-3$ graphs. From Basic Property (1), when n is odd, $\mu(G)$ must be even. So, for n odd, Theorem 1 holds if and only if $q = 2k, k \in \mathbb{N}$. The proof for this affirmative is similar to the one of Theorem 1. Algorithm *GEN* can also be slightly modified to obey the convenient conditions. Therefore, it will be possible to obtain a sequence of $\dfrac{n-3}{2}$ graphs $G_2, G_4, \ldots, G_{n-3}$ with n vertices, zero gap and $\mu(G_i) = i$.

4. TUNER SETS FOR GRAPHS

In this section, we introduce the concept of *tuner set* for a graph G, a subset of vertices of G with degree lower than $d(G)$ that are able to compensate the presence of vertices with degree greater than $d(G)$. We show that a graph can have distinct tuner sets. Finally, we characterize tuner sets for graphs with zero gap (see Theorem 2).

Let L be the lower vertex set and U be the upper vertex set of G such that $U \neq \emptyset$. If there is $\Psi \subseteq L$ for which the following equality holds:

$$\mu(G) = \frac{\sum\limits_{t \in \Psi} d(t) + \sum\limits_{u \in U} d(u)}{|\Psi| + |U|}\,, \tag{4}$$

we say that G has a *tuner set* Ψ *determined by* U or that U *determines a tuner set* Ψ *in* G. If $|\Psi| = k$, $k \leq |L|$. When $|L| = k$, $\Psi = L$ is unique and it is called the *full tuner set* of G *determined by* U. When $k < |L|$, Ψ is strictly contained in L and Ψ is called a *proper tuner set* of G. In general, proper tuner sets are not unique of G. Moreover, G can have more than one proper tuner set of the same order k or G

cannot have any tuner sets. For instance, there are not any tuner sets for regular graphs.

As an example, the graph G showed in Figure 2 (Section 2) has $\mu(G) = 3$, $U = \{v_5\}$, $B = \{v_6\}$ and $L = \{v_1, v_2, v_3, v_4, v_7\}$. From (4), $\Psi_1 = \{v_1, v_3\}$ is a tuner set of G determined by U. Observe that $\Psi_2 = \{v_6\}$ is another tuner set of G determined by U. However, there are subsets of L which are not tuner sets. In this example, $\{v_1, v_3, v_7\}$ is a subset of L but it is not a tuner set of G. Figure 3 displays a graph which does not have any tuner set. If we enumerate all subsets of L we can verify that none of them satisfies the equality (4).

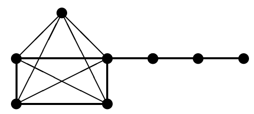

Figure 3. G does not have any tuner set.

The next theorem characterizes all graphs with zero gap as a function of their respective tuner sets. Observe that a graph with zero gap has an unique tuner set.

Theorem. *Let G be a graph and $L \neq \emptyset$ its lower vertex set. G has the full tuner set $\Psi = L$ if and only if $h(G) = 0$.*

Proof. ($\Leftarrow$) Let $h(G) = 0$. From (3) we have

$$\mu(G) = \frac{\sum\limits_{i=1,\dots,n} i\,\omega(i)}{n}.$$

First of all, it is necessary to know if G has a tuner set Ψ. For this, we need to find a subset of L satisfying the equality (4). From the definitions of the subsets U, B and L which determine a partition of $V(G)$, we obtain

$$\sum_{i=1,\dots,n} i\,\omega(i) = \sum_{u\in U} d(u) + \sum_{b\in B} d(b) + \sum_{\ell\in L} d(\ell).$$

If $B = \emptyset$, from (4), $\Psi = L$. Suppose $B \neq \emptyset$. As B is the balanced set, the equality below holds.

$$\mu(G) = \frac{\sum\limits_{b\in B} d(b)}{|B|}. \tag{5}$$

From (5), we get

$$\frac{\sum\limits_{b\in B} d(b)}{|B|} = \frac{\sum\limits_{u\in U} d(u) + \sum\limits_{b\in B} d(b) + \sum\limits_{\ell\in L} d(\ell)}{|U| + |B| + |L|}. \tag{6}$$

Since $h(G) = 0$, from (5) and using some algebraic manipulation on (6), we find

$$\mu(G) = \frac{\sum\limits_{u\in U} d(u) + \sum\limits_{\ell\in L} d(\ell)}{|U| + |L|}. \tag{7}$$

The equality (7) shows that $\Psi = L$. So, the full tuner set is the only subset of L that satisfies (4).

($\Rightarrow$) Let us suppose that G has the full tuner set $\Psi = L$. From (4),

$$\mu(G) = \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{|U| + |L|}. \tag{8}$$

If B is an empty set, $|B| = 0$ and from (8), $\mu(G) = d(G)$. So, $h(G) = 0$.

Now, suppose $B \neq \emptyset$. Then, $|B| \neq 0$. Let us consider $h(G) \neq 0$. So, $\mu(G) - d(G) > 0$. As B is the balanced set, $\mu(G) = \frac{\sum_{b\in B} d(b)}{|B|}$ and

$$\frac{\sum_{b\in B} d(b)}{|B|} - d(G) > 0. \tag{9}$$

We know that $\sum_{u\in U} d(u) + \sum_{b\in B} d(b) + \sum_{\ell\in L} d(\ell) = 2m$. From (9), we have

$$\frac{\sum_{b\in B} d(b)}{|B|} - \frac{\sum_{u\in U} d(u) + \sum_{b\in B} d(b) + \sum_{\ell\in L} d(\ell)}{n} > 0. \tag{10}$$

From (10), we have

$$\frac{\sum_{b\in B} d(b)}{|B|} - \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{n} - \frac{\sum_{b\in B} d(b)}{n} > 0.$$

After some algebraic manipulations we get

$$\frac{(n - |B|) \sum_{b\in B} d(b)}{|B|} > \sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)$$

and so,

$$\frac{\sum_{b\in B} d(b)}{|B|} > \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{n - |B|}. \tag{11}$$

Applying (5) to inequality (11) we have

$$\mu(G) > \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{n - |B|}.$$

But, $n - |B| = |U| + |L|$. So,

$$\mu(G) > \frac{\sum_{u\in U} d(u) + \sum_{\ell\in L} d(\ell)}{|U| + |L|}. \tag{12}$$

Since the inequality (12) is incompatible with the equality (7), our admission $h(G) \neq 0$ does not hold. Consequently, for every graph G such that $\Psi = L$, G is a graph with zero gap. □

Corollary 2. *Let G be a graph with zero gap. If G is a non-balanced graph then the lower vertex set $L \neq \emptyset$.*

Proof. Let G be a graph with zero gap. Suppose that G is a non-balanced graph. So, $U \neq \emptyset$ and from Theorem 2, $\Psi = L$. Besides, from (4) we have

$$\mu(G) = \frac{\sum\limits_{u \in U} d(u) + \sum\limits_{t \in \Psi} d(t)}{|U| + |\Psi|}.$$

If $L = \emptyset$ and as $\Psi \subseteq L$, the equality above becomes $\mu(G) = \frac{\sum\limits_{u \in U} d(u)}{|U|}$. But it is impossible, since each vertex $u \in U$ has $d(u) > \mu(u)$. Then, $L \neq \emptyset$. □

Finally, it is interesting to observe that Algorithm *GEN* presented in Section 3 can also be applied to generate graphs with full tuner sets.

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SOME RESULTS ON STARLIKE AND CONVEX FUNCTIONS

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Let $\mathcal{A}$ be the class of analytic functions in the unit disk that are normalized with $f(0) = f'(0) - 1 = 0$. In this paper we give sharp sufficient conditions on the expression

$$\frac{1 - \alpha + \alpha z f''(z)/f'(z)}{z f'(z)/f(z)}$$

that implies starlikeness and convexity of function f.

1. INTRODUCTION AND PRELIMINARIES

Let $\mathcal{A}$ denotes the class of functions $f(z)$ that are analytic in the unit disk $\mathcal{U} = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$.

Further, let $f, g \in \mathcal{A}$. Then we say that $f(z)$ is *subordinate* to $g(z)$, and we write $f(z) \prec g(z)$, if there exists a function $\omega(z)$, analytic in the unit disk $\mathcal{U}$, such that $\omega(0) = 0$, $|\omega(z)| < 1$ and $f(z) = g(\omega(z))$ for all $z \in \mathcal{U}$. Specially, if $g(z)$ is univalent in $\mathcal{U}$ then $f(z) \prec g(z)$ if and only if $f(0) = g(0)$ and $f(\mathcal{U}) \subseteq g(\mathcal{U})$.

If $-1 \le B < A \le 1$ then an important class is defined by

$$S^*[A, B] = \left\{ f \in \mathcal{A} : \frac{z f'(z)}{f(z)} \prec \frac{1 + Az}{1 + Bz} \right\}.$$

Geometrically, this means that the image of $\mathcal{U}$ by $z f'(z)/f(z)$ is inside the open disk centered on the real axis with diameter end points $(1 - A)/(1 - B)$ and $(1 + A)/(1 + B)$. Special selection of A and B lead us to the following classes:

- $S^*[1, -1] \equiv S^*$ is the *class of starlike functions*;

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- $S^*[1-2\alpha,-1] \equiv S^*(\alpha)$, $0 \le \alpha < 1$, is the *class of starlike functions of order* α.

Also, $K^*(\alpha)$, $0 \le \alpha < 1$, is the class of *convex functions of order* α, defined by $f(z) \in K(\alpha)$ if and only if $zf'(z) \in S^*(\alpha)$, i.e., $\mathrm{Re}\left(1+\dfrac{zf''(z)}{f'(z)}\right) > \alpha$, $z \in \mathcal{U}$.

In this paper we will study the class

$$G_{\lambda,\alpha} = \left\{ f \in \mathcal{A} : \left| \frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)} - (1-\alpha) \right| < \lambda, z \in \mathcal{U} \right\},$$

$0 < \alpha \le 1$, $\lambda > 0$, and give sufficient conditions that embed it into the classes $S^*[A,B]$ and $K(\delta)$, $0 \le \delta < 1$. Comparison with previous known results will be done.

In that purpose from the theory of first-order differential subordinations we will make use of the following lemma.

Lema 1 ([**1**]). *Let $q(z)$ be univalent in the unit disk $\mathcal{U}$, and let $\theta(\omega)$ and $\phi(\omega)$ be analytic in a domain D containing $q(\mathcal{U})$, with $\phi(\omega) \neq 0$ when $\omega \in q(\mathcal{U})$. Set $Q(z) = zq'(z)\phi\big(q(z)\big)$, $h(z) = \theta(q(z)) + Q(z)$, and suppose that*

i) $Q(z) \in S^*$; *and*

ii) $\mathrm{Re}\,\dfrac{zh'(z)}{Q(z)} = \mathrm{Re}\left\{\dfrac{\theta'(q(z))}{\phi(q(z))} + \dfrac{zQ'(z)}{Q(z)}\right\} > 0$, $z \in \mathcal{U}$.

If $p(z)$ is analytic in $\mathcal{U}$, with $p(0) = q(0)$, $p(\mathcal{U}) \subseteq D$ and

$$\theta(p(z)) + zp'(z)\phi(p(z)) \prec \theta(q(z)) + zq'(z)\phi(q(z)) = h(z), \tag{1}$$

then $p(z) \prec q(z)$, and $q(z)$ is the best dominant of (1)

2. MAIN RESULTS AND CONSEQUENCES

In the beginning, using Lemma 1 we will prove the following result.

Theorem 1. *Let $f \in \mathcal{A}$, $-1 \le B < A \le 1$ and $\dfrac{1+|A|}{3+|A|} \le \alpha \le 1$. If*

$$\frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)} \prec \alpha + (1-2\alpha)\frac{1+Bz}{1+Az} + \frac{\alpha z(A-B)}{(1+Az)^2} \equiv h(z)$$

then $f \in S^*[A,B]$. This result is sharp.

Proof. We choose $p(z) = \dfrac{f(z)}{zf'(z)}$, $q(z) = \dfrac{1+Bz}{1+Az}$, $\theta(\omega) = (1-2\alpha)\omega + \alpha$ and $\phi(\omega) = -\alpha$. Then $q(z)$ is convex, thus univalent, because $1 + zq''(z)/q'(z) =$

$(1-Az)/(1+Az)$; $\theta(\omega)$ and $\phi(\omega)$ are analytic in the domain $D = \mathcal{C}$ which contains $q(\mathcal{U})$ and $\phi(\omega)$ when $\omega \in q(\mathcal{U})$. Further,

$$Q(z) = zq'(z)\phi(q(z)) = \frac{\alpha(A-B)z}{(1+Az)^2}$$

is starlike because $\dfrac{zQ'(z)}{Q(z)} = \dfrac{1-Az}{1+Az}$. Further,

$$h(z) = \theta(q(z)) + Q(z) = \alpha + (1-2\alpha)\frac{1+Bz}{1+Az} + \frac{\alpha z(A-B)}{(1+Az)^2}$$

and

$$\operatorname{Re}\frac{zh'(z)}{Q(z)} = \operatorname{Re}\left(1 - \frac{1}{\alpha} + \frac{2}{1+Az}\right) > 1 - \frac{1}{\alpha} + \frac{2}{1+|A|},$$

$z \in \mathcal{U}$, which is greater or equal to zero if and only if $\alpha \geq \dfrac{1+|A|}{3+|A|}$. Therefore from Lemma 1 follows that $p(z) \prec q(z)$, i.e., $f \in S^*[A,B]$.

The result is sharp as the functions ze^{Az} and $z(1+Bz)^{A/B}$ show in the cases $B = 0$ and $B \neq 0$, respectively. □

Remark 1. According to the definition of subordination, the sharpness of the result of Theorem 1 means that $h(\mathcal{U})$ is the greatest region in the complex plane with the property that if

$$\frac{1-\alpha+\alpha zf''(z)/f'(z)}{zf'(z)/f(z)} \in h(\mathcal{U})$$

for all $z \in \mathcal{U}$ then $f(z) \in S^*[A,B]$.

The following corollary embeds $G_{\lambda,\alpha}$ into $S^*[A,B]$.

Corollary. *$G_{\lambda,\alpha} \subseteq S^*[A,B]$ when $\dfrac{1+|A|}{3+|A|} \leq \alpha \leq 1$ and*

$$\lambda = (A-B)\cdot\frac{(1-2\alpha)|A| - (1-3\alpha)}{(1+|A|)^2}.$$

This result is sharp, i.e., given λ is the greatest so that inclusion holds.

Proof. In order to prove this corollary, due to Theorem 1 it is enough to show that $\lambda = \min\{|h(z)-(1-\alpha)| : |z| = 1\} \equiv \widehat{\lambda}$, where $h(z)$ is defined as in the statement of the theorem and

$$h(z) - (1-\alpha) = -z(A-B)\cdot\frac{A(1-2\alpha)z + 1 - 3\alpha}{(1+Az)^2}.$$

Further, let

$$\begin{aligned}\psi(t) &\equiv \left|h(e^{i\gamma\pi/2}) - (1-\alpha)\right|^2 \\ &= (A-B)^2\cdot\frac{\left((1-2\alpha)^2A^2 + 2(1-3\alpha)(1-2\alpha)At + (1-3\alpha)^2\right)}{(1+2At+A^2)^2},\end{aligned}$$

$t = \cos(\gamma\pi/2) \in [-1, 1]$. Thus $\widehat{\lambda} = \min\{\sqrt{\psi(t)} : -1 \leq t \leq 1\}$.

If $\alpha \leq 1/2$ then $1 - 2\alpha \geq 0$ and having in mind that $1 - 3\alpha \leq -\frac{2|A|}{3+|A|} \leq 0$ we receive that $\psi(t)$ is a monotone function and

$$\widehat{\lambda} = \min\{\sqrt{\psi(-1)}, \sqrt{\psi(1)}\} = \min\{|h(-1) - (1-\alpha)|, |h(1) - (1-\alpha)|\} = \lambda.$$

The last equality holds because $1 - 3\alpha + A(1-2\alpha)z \geq 0$ is equivalent to $\alpha \geq \frac{1+|A|}{3+|A|} \geq \frac{1-|A|}{3-2|A|}$.

If $\alpha > 1/2$ we have the following analysis. Equation $\psi_t'(t) = 0$ has unique solution

$$t_* = -\frac{A^2(1-\alpha)(1-2\alpha) + (1-3\alpha)(1-4\alpha)}{2A(1-2\alpha)(1-3\alpha)}.$$

It can be verified that $|t_*| > 1$ is equivalent to

$$\varphi(A,\alpha) \equiv A^2(1-\alpha)(1-2\alpha) - 2|A|(1-2\alpha)(1-3\alpha) + (1-3\alpha)(1-4\alpha) > 0.$$

Now, $\varphi(A,\alpha)$ is decreasing function of $|A| \in [0,1]$ which implies $\varphi(A,\alpha) \geq \varphi(1,\alpha) = 2\alpha^2 > 0$. Thus, $|t_*| > 1$ which implies that $\psi(t)$ is a monotone function on $[-1,1]$ leading to $\widehat{\lambda} = \min\{\sqrt{\psi(t)} : -1 \leq t \leq 1\} = \min\{\sqrt{\psi(-1)}, \sqrt{\psi(1)}\} = \min\{|h(-1) - (1-\alpha)|, |h(1) - (1-\alpha)|\}$. At the end, the function

$$\eta(A,\alpha) \equiv |h(1) - (1-\alpha)| - |h(-1) - (1-\alpha)| = 2A \cdot \frac{1 - A^2 - 2\alpha(2-A^2)}{(1+A)^2(1-A)^2}$$

has the opposite sign of the sign of coefficient A. Therefore,

$$\widehat{\lambda} = \left\{ \begin{array}{ll} |h(1) - (1-\alpha)|, & A \geq 0 \\ |h(-1) - (1-\alpha)|, & A < 0 \end{array} \right\} = \lambda.$$

Sharpness of the result follows from the sharpness of Theorem **??** (see Remark 1) and the fact that obtained λ is the greatest which embeds the disc $|\omega - (1-\alpha)| < \lambda$ in $h(\mathcal{U})$. $\square$

The following example exhibits some concrete conclusions that can be obtained from the results of the previous section by specifying the values α, A, B.

Example 1. Let $-1 \leq B < A \leq 1$.

i) $G_{\lambda,1/2} \subseteq S^*[A,B]$ when $\lambda = \frac{A-B}{2(1+|A|)^2}$.

ii) $G_{\lambda,1} \subseteq S^*[A,B]$ when $\lambda = (A-B) \cdot \frac{2-|A|}{2(1+|A|)^2}$.

iii) $G_{\lambda,1/(2-\gamma)} \subseteq S^*[A,B]$ when $\gamma \geq -\frac{1-|A|}{1+|A|}$ and $\lambda = (A-B) \cdot \frac{1+\gamma-\gamma|A|}{2(1+|A|)^2}$.

iv) $G_{\lambda,\alpha} \subseteq S^*$ when $1/2 \leq \alpha \leq 1$ and $\lambda = \alpha/2$.

v) $G_{\lambda,\alpha} \subseteq S^*[0,B] \subset S^*(1/(1-B))$ when $1/3 \le \alpha \le 1$, $-1 \le B < 0$ and $\lambda = B(1-3\alpha)$.

Given λ is the greatest so that inclusions hold.

Remark 2. The result from Example 1 (i) is the same as in Corollary 2.6 in [**5**]. Also, for $\alpha = 1/2$ in Example 1 (v) we receive the same result as in Theorem 1 from [**2**]. Finally, for $\alpha = 1$ and $B = -1$ in Example 1(v) we receive the same result as in Corollary 2 from [**3**].

Next theorem studies connection between $G_{\lambda,\alpha}$ and the class of convex functions of some order.

Theorem 2. $G_{\lambda,\alpha} \subseteq K\left(2-\dfrac{1}{\alpha}\right)$ *when* $\dfrac{1}{2} \le \alpha < 1$ *and* $\lambda = \dfrac{(1-\alpha)(3\alpha-1)}{\sqrt{2(5\alpha^2-4\alpha+1)}}$.

Proof. Let $f \in G_{\lambda,\alpha}$ and $B = \dfrac{\lambda}{(1-3\alpha)}$. Then, by Example 1 (v) we have $f \in S^*[0,B]$, i.e., $\left|\dfrac{f(z)}{zf'(z)} - 1\right| < B$, $z \in \mathcal{U}$. Further,

$$1 + \frac{zf''(z)}{f'(z)} - \left(2 - \frac{1}{\alpha}\right) = \frac{zf'(z)}{\alpha f(z)} \cdot \frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)}.$$

and for all $z \in \mathcal{U}$ we obtain

$$\begin{aligned}\left|\arg\left(1 + \frac{zf''(z)}{f'(z)} - 2 + \frac{1}{\alpha}\right)\right| &\le \left|\arg\frac{zf'(z)}{f(z)}\right| + \left|\arg\frac{1-\alpha+\alpha z f''(z)/f'(z)}{zf'(z)/f(z)}\right| \\ &\le \arcsin|B| + \arcsin\frac{\lambda}{1-\alpha} \\ &= \arcsin\left(\frac{\lambda}{1-\alpha}\cdot\sqrt{1-B^2} + |B|\cdot\sqrt{1-\frac{\lambda^2}{(1-\alpha)^2}}\right) \\ &= \arcsin 1 = \frac{\pi}{2},\end{aligned}$$

i.e., $f \in K\left(2-\dfrac{1}{\alpha}\right)$. □

Example 2. For $\alpha = 1/2$ and $\alpha = 1/(2-\gamma)$ in the previous theorem we receive

i) $G_{\lambda,1/2} \subseteq K$ when $\lambda = \sqrt{2}/4$.

ii) $G_{\lambda,1/(2-\gamma)} \subseteq K(\gamma)$ when $0 \le \gamma < 1$ and $\lambda = \dfrac{1-\gamma^2}{(2-\gamma)\sqrt{2(1+\gamma^2)}}$.

Remark 3. By putting $\alpha = \dfrac{1}{2-\gamma}$, $0 \le \gamma < 1$, we receive the result from Theorem 2 in [**4**].

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POTENTIAL THEORY FOR BOUNDARY VALUE PROBLEMS ON FINITE NETWORKS

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We aim here at analyzing self-adjoint boundary value problems on finite networks associated with positive semi-definite Schrödinger operators. In addition, we study the existence and uniqueness of solutions and its variational formulation. Moreover, we will tackle a well-known problem in the framework of Potential Theory, the so-called condenser principle. Then, we generalize of the concept of effective resistance between two vertices of the network and we characterize the Green function of some BVP in terms of effective resistances.

1. INTRODUCTION

In this paper we analyze self-adjoint boundary value problems on finite networks associated with positive semi-definite Schrödinger operators. Among others, we treat general mixed boundary value problems that include the well-known Dirichlet and Neumann problems and also the Poisson equation. In the last years, these problems have deserved the attention of many researchers, see for instance [**1, 3, 4, 5**]. The first of that papers is concerned with the general analysis of self-adjoint boundary value problems associated with non-negative perturbations of the combinatorial Laplacian and its associated Green functions from a Potential Theory point of view. The two last ones are mainly concerned with the inverse problem of identifying the conductivity function of the network, in terms of the boundary data.

A Schrödinger operator on a finite network is a linear operator of the form $\mathcal{L}_q = \mathcal{L} + q$, where $\mathcal{L}$ is the combinatorial Laplacian of the network and q is a function on the vertex set. That function is usually known as ground-state since it represent that each vertex of the network is connected with a conductor medium

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with null potential. So, a Schrödinger operator can be seen as a perturbation of the combinatorial Laplacian. It is well-known that the energy associated with this operator is a Dirichlet form if and only if the ground state is non-negative, [**7**]. Some of the authors obtained in [**3**] a generalization of this result, when the ground state takes negative values, which was applied to the study of Dirichlet problems and Poisson equations. Here we extend the above results to the energy associated with general self-adjoint BVP. In particular, we show that any BVP has a unique solution provided that its associated energy is positive definite and we characterize when this happens in terms of the ground state. Moreover, we tackle the variational treatment of the self-adjoint BVP and we obtain the general version of the celebrated Dirichlet Principle.

In addition, we are concerned with the Condenser Principle, a classic topic in the framework of the Potential Theory associated with BVP. We extend the situation treated in [**2**], where only the case in which the ground state is null and a part of the boundary is insulated was considered. For that, we first tackle the natural extension, namely when the ground state is associated with a weight; which allows us to define the effective resistance with respect to this weight. As byproducts we obtain the Generalized Foster's Theorem that relates the total amount of the ratios between the conductances of the network and the effective conductances, see [**9**] for its usual formulation, and the expression of the Green function for the problem in which a single vertex is grounded in terms of the effective resistances. In its classical statement this expression is known as the inverse resistive problem and it has been considered for several author. The problem is the following: Let $(c(x,y))_{x,y\in V}$ denote the edge conductances of an electrical network, so that there is a resistor of $r_{xy} = 1/c(x,y)$ ohms between nodes x and y. This uniquely determines the matrix $(R_{xy})_{x,y\in V}$ of effective resistances, defined such that if a potential of 1 V is applied across nodes x and y, a current of $1/R_{xy}$ A will flow. Matrix $(c(x,y))_{x,y\in V}$ is called the resistive inverse of $(R_{xy})_{x,y\in V}$. Coppersmith et al. [**6**] gave a simple but obscure four-step algorithm for computing the resistive inverse. After Ponzio gave a self-contained combinatorial explanation of this algorithm, [**8**]. In this work we prove an analogous result when more general cases are considered. To do that we consider the effective resistances, which can be obtained from the solution of condenser problems. Next we determine the Green function for the problem in terms of the effective resistances. Therefore, to obtain the inverse resistive it will suffice to invert the Green function and to complete this inverse so that it be the Laplacian of the network.

Finally, we study the case in which the energy is positive definite and we show that the Green function for the corresponding Robin problem can be also obtained as an inverse resistive of a suitable network.

2. PRELIMINARIES

Along the paper, $\Gamma = (V,E)$ denotes a simple, finite and connected graph without loops, with vertex set V and edge set E. Two different vertices, $x, y \in V$, are called *adjacent*, which will be represented by $x \sim y$, if $\{x,y\} \in E$. Given

$x, y \in V$, if $d(x,y)$ is the length of the shortest path joining x and y it is well-known that d defines a distance on the graph.

Given a vertex subset $F \subset V$, we denote by F^c its complementary in V and we call *boundary* and *closure of* F, the sets $\delta(F) = \{x \in V : d(x,F) = 1\}$ and $\overline{F} = F \cup \delta(F)$, respectively. Clearly, $\overline{F} = \{x \in V : d(x,F) \le 1\}$.

The sets of functions and non-negative functions on V are denoted by $\mathcal{C}(V)$ and $\mathcal{C}^+(V)$ respectively. If $u \in \mathcal{C}(V)$, its *support* is given by $\mathsf{supp}(u) = \{x \in V : u(x) \neq 0\}$. Moreover, if F is a non empty subset of V, its characteristic function is denoted by $\chi_{_F}$ and we can consider the sets $\mathcal{C}(F) = \{u \in \mathcal{C}(V) : \mathsf{supp}(u) \subset F\}$ and $\mathcal{C}^+(F) = \mathcal{C}(F) \cap \mathcal{C}^+(V)$. For any $u \in \mathcal{C}(F)$, we denote by $\int_F u(x)\,\mathrm{d}x$ or simply by $\int_F u\,\mathrm{d}x$ the value $\sum\limits_{x\in F} u(x)$. We call *weight on* F any function $\sigma \in \mathcal{C}^+(F)$ such that $\mathsf{supp}(\sigma) = F$. The set of weights on F is denoted by $\mathcal{C}^*(F)$.

We call *conductance on* Γ a function $c\colon V \times V \to \mathbb{R}^+$ such that $c(x,y) > 0$ iff $x \sim y$. We call *network* any pair (Γ, c), where c is a conductance on Γ. In what follows we consider fixed the network (Γ, c) and we refer to it simply by Γ.

The *combinatorial Laplacian* or simply the *Laplacian* of the network Γ is the linear operator $\mathcal{L} : \mathcal{C}(V) \to \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function

$$\mathcal{L}(x) = \int_V c(x,y)\,\big(u(x) - u(y)\big)\,\mathrm{d}y, \quad x \in V. \tag{1}$$

If F is a proper subset of V, for each $u \in \mathcal{C}(\overline{F})$ we define the *normal derivative of* u as the function in $\mathcal{C}\big(\delta(F)\big)$ given by

$$\left(\frac{\partial u}{\partial \mathsf{n_F}}\right)(x) = \int_F c(x,y)\,\big(u(x) - u(y)\big)\,\mathrm{d}y, \text{ for any } x \in \delta(F). \tag{2}$$

The relation between the values of the Laplacian on F and the values of the normal derivative at $\delta(F)$ is given by the *First Green Identity*, proved in [**1**]

$$\int_F v\,\mathcal{L}(u)\,\mathrm{d}x = \frac{1}{2}\int_{\overline{F}}\int_{\overline{F}} c_F(x,y)\big(u(x)-u(y)\big)\big(v(x)-v(y)\big)\,\mathrm{d}x\mathrm{d}y - \int_{\delta(F)} v\,\frac{\partial u}{\partial \mathsf{n_F}}\,\mathrm{d}x,$$

where $u, v \in \mathcal{C}(\overline{F})$ and $c_F = c \cdot \chi_{(F\times F)\setminus(\delta(F)\times\delta(F))}$. A direct consequence of the above identity is the so-called *Second Green Identity*

$$\int_F \Big(v\,\mathcal{L}(u) - u\,\mathcal{L}(v)\Big)\,\mathrm{d}x = \int_{\delta(F)} \left(u\,\frac{\partial v}{\partial \mathsf{n_F}} - v\,\frac{\partial u}{\partial \mathsf{n_F}}\right)\,dx, \text{ for all } u, v \in \mathcal{C}(\overline{F}).$$

When $F = V$ the above identity tell us that the combinatorial Laplacian is a self-adjoint operator and that $\int_V \mathcal{L}(u)\,\mathrm{d}x = 0$ for any $u \in \mathcal{C}(V)$. Moreover, since Γ is connected $\mathcal{L}(u) = 0$ iff u is a constant function.

Given $q \in \mathcal{C}(V)$ the *Schrödinger operator* on Γ with *ground state* q is the linear operator $\mathcal{L}_q : \mathcal{C}(V) \to \mathcal{C}(V)$ that assigns to each $u \in \mathcal{C}(V)$ the function $\mathcal{L}_q(u) = \mathcal{L}(u) + qu$.

3. SELF-ADJOINT BOUNDARY VALUE PROBLEMS

In this section we study different type of boundary value problems associated with the SCHRÖDINGER operator with ground state q. Given a non-empty subset $F \subset V$, $\delta(F) = H_1 \cup H_2$, where $H_1 \cap H_2 = \emptyset$ and functions $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$, $g_2 \in \mathcal{C}(H_2)$, a *boundary value problem on* F consists on finding $u \in \mathcal{C}(\overline{F})$ such that

$$\mathcal{L}_q(u) = g \ \text{ on } \ F, \quad \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} + qu = g_1 \ \text{ on } \ H_1 \quad \text{ and } \quad u = g_2 \ \text{ on } \ H_2. \tag{3}$$

In addition, *the associated homogeneous boundary value problem* consists on finding $u \in \mathcal{C}(\overline{F})$ such that $\mathcal{L}_q(u) = 0$ on F, $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} + qu = 0$ on H_1 and $u = 0$ on H_2.

The GREEN Identity implies that the boundary value problem (3) is self-adjoint in the sense that $\int_F v\, \mathcal{L}_q(u)\, \mathrm{d}x = \int_F u\, \mathcal{L}_q(v)\, \mathrm{d}x$ for all $u, v \in \mathcal{C}(F \cup H_1)$ verifying that $\dfrac{\partial u}{\partial \mathsf{n}_\mathsf{F}} + qu = \dfrac{\partial v}{\partial \mathsf{n}_\mathsf{F}} + qv = 0$ on H_1.

Problem (3) is generically known as a *mixed Dirichlet-Robin problem* and summarizes the different boundary value problems that appear in the literature with the following proper names:

(i) *Dirichlet problem*: $\emptyset \neq H_2 = \delta(F)$ and hence $H_1 = \emptyset$.

(ii) *Robin problem*: $\emptyset \neq H_1 = \delta(F)$ and $q \neq 0$ on H_1.

(iii) *Neumann problem*: $\emptyset \neq H_1 = \delta(F)$ and $q = 0$ on H_1.

(iv) *Mixed Dirichlet-Neumann problem*: $H_1, H_2 \neq \emptyset$ and $q = 0$ on H_1.

(v) *Poisson equation* on V: $F = V$.

The study of the boundary value problem (3) when $q \in \mathcal{C}^+(V)$ has been extensively treated, see for instance [**1, 4, 5**], where the existence and uniqueness of solutions was established, whereas the analysis for DIRICHLET Problem and POISSON equation in the case in which when q can take negative value has been developed in [**3**]. In this work we extend the above results for the self-adjoint boundary value problem (3).

Proposition 3.1. (FREDHOLM Alternative) *Given $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$, $g_2 \in \mathcal{C}(H_2)$, the boundary value problem*

$$\mathcal{L}_q(u) = g \ \text{ on } \ F, \quad \frac{\partial u}{\partial \mathsf{n}_\mathsf{F}} + qu = g_1 \ \text{ on } \ H_1 \quad \text{ and } \quad u = g_2 \ \text{ on } \ H_2$$

has solution iff for any $v \in \mathcal{C}(\overline{F})$ solution of the homogeneous problem it is verified

$$\int_F gv\, \mathrm{d}x + \int_{H_1} g_1 v\, \mathrm{d}x = \int_{H_2} g_2 \frac{\partial v}{\partial \mathsf{n}_\mathsf{F}}\, \mathrm{d}x.$$

In addition, when the above condition holds, then there exists a unique $u \in \mathcal{C}(\overline{F})$ solution of the boundary value problem such that $\int_{\overline{F}} uv\,\mathrm{d}x = 0$, for any $v \in \mathcal{C}(\overline{F})$ solution of the homogeneous problem.

Proof. First observe that problem (3) is equivalent to the boundary value problem

$$\mathcal{L}_q(u) = g - \mathcal{L}_q(g_2) \quad \text{on} \quad F, \quad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = g_1 \quad \text{on} \quad H_1 \quad \text{and} \quad u = 0 \quad \text{on} \quad H_2$$

in the sense that u is a solution of this problem iff $u + g_2$ is a solution of (3).

Consider now the linear operator $\mathcal{F}\colon \mathcal{C}(F \cup H_1) \to \mathcal{C}(F \cup H_1)$ defined as $\mathcal{F}(u) = \mathcal{L}_q(u)$ on F and $\mathcal{F}(u) = \dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu$ on H_1. If $\mathcal{V}$ denotes the space of solutions of the homogeneous problem, then $\ker \mathcal{F} = \mathcal{V}$. Moreover, from the Second Green Identity, we get that $\int_{F\cup H_1} v\mathcal{F}(u)\,\mathrm{d}x = \int_{F\cup H_1} u\mathcal{F}(v)\,\mathrm{d}x$; that is, $\mathcal{F}$ is self-adjoint and hence $\operatorname{Img} \mathcal{F} = \mathcal{V}^{\perp}$, using the classical Fredholm Alternative. Consequently problem (3) has a solution iff the function $\widetilde{g} \in \mathcal{C}(F \cup H_1)$ given by $\widetilde{g} = g - \mathcal{L}_q(g_2)$ on F and $\widetilde{g} = g_1$ on H_1 verifies that

$$\begin{aligned} 0 = \int_{F\cup H_1} \widetilde{g}v\,\mathrm{d}x &= \int_F gv\,\mathrm{d}x + \int_{H_1} g_1 v\,\mathrm{d}x - \int_F v\mathcal{L}_q(g_2)\,\mathrm{d}x \\ &= \int_F gv\,\mathrm{d}x + \int_{H_1} g_1 v\,\mathrm{d}x - \int_{H_2} g_2 \frac{\partial v}{\partial \mathsf{n}_{\mathsf{F}}}\,\mathrm{d}x, \end{aligned}$$

for any $v \in \mathcal{V}$. Finally, the Fredholm Alternative also establishes that when the necessary and sufficient condition are attained there exists a unique $w \in \mathcal{V}^{\perp}$ such that $\mathcal{F}(w) = \widetilde{g}$. Therefore, $u = w + g_2$ is the unique solution of problem (3) such that for any $v \in \mathcal{V}$

$$\int_{\overline{F}} uv\,\mathrm{d}\nu = \int_{F\cup H_1} uv\,\mathrm{d}\nu = \int_{F\cup H_1} wv\,\mathrm{d}\nu = 0,$$

since $v = 0$ on H_2 and $g_2 = 0$ on $F \cup H_1$. $\square$

Fredholm Alternative establishes that the existence of solution of problem (3) for any data g, g_1 and g_2 is equivalent to the uniqueness of solution and hence it is equivalent to the fact that the homogeneous problem has $v = 0$ as its unique solution. So, applying the First Green Identity, if $v \in \mathcal{V}$

$$0 = \int_F v\mathcal{L}_q(v)\,\mathrm{d}x = \frac{1}{2}\int_{\overline{F}}\int_{\overline{F}} c_{F}(x,y)\big(v(x) - v(y)\big)^2 \mathrm{d}x\,\mathrm{d}y + \int_{\overline{F}} q\,v^2 \mathrm{d}x$$

and hence uniqueness is equivalent to be $v = 0$ the unique solution of the above equality.

The above equality leads to define the *energy* associated with Problem (3) as the symmetric bilinear form $\mathcal{E}_q^F : \mathcal{C}(\overline{F}) \times \mathcal{C}(\overline{F}) \to \mathbb{R}$ given for any $u, v \in \mathcal{C}(\overline{F})$ by

$$(4) \quad \mathcal{E}_q^F(u,v) = \frac{1}{2}\int_{\overline{F}}\int_{\overline{F}} c_{F}(x,y)\big(u(x) - u(y)\big)\big(v(x) - v(y)\big)\,\mathrm{d}x\,\mathrm{d}y + \int_{\overline{F}} q\,u\,v\,\mathrm{d}x.$$

A sufficient condition so that the homogeneous problem associated with (3) have $v = 0$ as its unique solution is that the energy be positive definite. Next, we characterize when this property is achieved. To do this, it will be useful to introduce for any weight σ on $\overline{F}$, the so-called *ground state associated with* σ as $q_\sigma = -\dfrac{1}{\sigma}\,\mathcal{L}(\sigma)$ on F, $q_\sigma = -\dfrac{1}{\sigma}\,\dfrac{\partial\sigma}{\partial \mathsf{n}_F}$ on $\delta(F)$ and $q_\sigma = q$ otherwise. Clearly, if $\sigma \in \mathcal{C}^*(\overline{F})$ then for any $a > 0$, $\mu = a\sigma \in \mathcal{C}^*(\overline{F})$ and moreover $q_\mu = q_\sigma$.

Observe that $q_\sigma = 0$ iff $\sigma = a\chi_{\overline{F}}$, with $a > 0$. More generally, if $\sigma \in \mathcal{C}^*(\overline{F})$, then tacking $v = \chi_{\overline{F}}$ in the Second Green Identity we obtain that $\int_{\overline{F}} \sigma\, q_\sigma = 0$, which implies that q_σ must take positive and negative values, except when $\sigma = a\chi_{\overline{F}}$, $a > 0$. Moreover, in [**3**] it was proved that $-\int_{\overline{F}} c_F(x,y)\,\mathrm{d}y < q_\sigma(x)$ for any $x \in \overline{F}$ and also that when $H_2 \neq \emptyset$, then it is possible to choose $\sigma \in \mathcal{C}^*(\overline{F})$ such that $q_\sigma(x) < 0$ for any $x \in F \cup H_1$.

Proposition 3.2. *The Energy $\mathcal{E}_q^F$ is positive semi-definite iff there exists $\sigma \in \mathcal{C}^*(\overline{F})$ such that $q \geq q_\sigma$. Moreover, it is not strictly definite iff $q = q_\sigma$, in which case $\mathcal{E}_q^F(v,v) = 0$ iff $v = a\sigma$, $a \in \mathbb{R}$.*

Proof. Consider the network $\Gamma_F = (\overline{F}, \overline{E}, c_F)$, where $\overline{E} = \{(x,y) \in E : c_F(x,y) > 0\}$ and let $\overline{\mathcal{L}}$ its combinatorial Laplacian. Then, for any $u \in \mathcal{C}(\overline{F})$, $\overline{\mathcal{L}}(u) = \mathcal{L}(u)$ on F and $\overline{\mathcal{L}}(u) = \dfrac{\partial u}{\partial \mathsf{n}_F}$ on $\delta(F)$. Moreover, $\mathcal{E}_q^F(u,u) = \int_{\overline{F}} u\overline{\mathcal{L}}(u)\,\mathrm{d}x + \int_{\overline{F}} qu^2 dx$ and hence the results follow by applying Proposition 3.3 and Corollary 3.4 of [**3**]. □

The next result establishes the fundamental result about the existence and uniqueness of solution for Problem (3) and about its variational formulation.

Proposition 3.3. (Dirichlet principle) *Suppose that there exists $\sigma \in \mathcal{C}^*(\overline{F})$ such that $q \geq q_\sigma$. Given $g \in \mathcal{C}(F)$, $g_1 \in \mathcal{C}(H_1)$ and $g_2 \in \mathcal{C}(H_2)$, consider the convex set $C_{g_2} = \{v \in \mathcal{C}(\overline{F}) : v = g_2$ on $H_2\}$ and the quadratic functional $\mathcal{J}_q \colon \mathcal{C}(\overline{F}) \to \mathbb{R}$ determined by the expression*

$$\mathcal{J}_q(u) = \frac{1}{2}\int_{\overline{F}}\int_{\overline{F}} c_F(x,y)\,\big(u(x)-u(y)\big)^2 \mathrm{d}x\,\mathrm{d}y + \int_{\overline{F}} q\,u^2\,\mathrm{d}x - 2\int_F g\,u\,\mathrm{d}x - 2\int_{H_1} g_1 u\,\mathrm{d}x.$$

Then $u \in \mathcal{C}(\overline{F})$ is a solution of (3) *iff u minimizes $\mathcal{J}_q$ on C_{g_2}. Moreover, if it is not simultaneously true that $H_2 = \emptyset$ and $q = q_\sigma$, then $\mathcal{J}_q$ has a unique minimum on C_{g_2}. Otherwise, $\mathcal{J}_q$ has a minimum iff $\int_F g\sigma\,\mathrm{d}x + \int_{\delta(F)} g_1\sigma\,\mathrm{d}x = 0$. In this case, there exists a unique minimum $u \in \mathcal{C}(\overline{F})$ such that $\int_{\overline{F}} u\sigma\,\mathrm{d}x = 0$.*

Proof. Observe first that $C_{g_2} = g_2 + \mathcal{C}(F \cup H_1)$ and that for all $v \in \mathcal{C}(F \cup H_1)$ we get $\mathcal{J}_q(v) = \mathcal{E}_q^F(v,v) - 2\int_F g\,v\,\mathrm{d}x - 2\int_{H_1} g_1\,v\,\mathrm{d}x$. Keeping in mind, that $q \geq q_\sigma$, we get that $\mathcal{J}_q$ is a convex functional on $\mathcal{C}(F \cup H_1)$ and hence on C_{g_2}. Moreover, it is an strictly convex functional iff it is not simultaneously true that $H_2 = \emptyset$ and $q = q_\sigma$ and then $\mathcal{J}_q$ has a unique minimum on C_{g_2}.

On the other hand, when $H_2 = \emptyset$ and $q = q_\sigma$ simultaneously the minima of $\mathcal{J}_q$ are characterized by the *Euler identity*: $\mathcal{E}_q^F(u,v) = \int_F g\,v\,dx + \int_{H_1} g_1 v\,\mathrm{d}x$, for all $v \in \mathcal{C}(\overline{F})$. Since in this case $\mathcal{E}_q^F(u,\sigma) = 0$ for all $u \in \mathcal{C}(\overline{F})$, necessarily g and g_1 must satisfy that $\int_F g\,\sigma\,\mathrm{d}x + \int_{H_1} g_1\sigma\,\mathrm{d}x = 0$. Moreover, if this condition holds and $\mathcal{V}$ denotes the vector subspace generated by σ, then $u \in \mathcal{V}^\perp$ minimizes $\mathcal{J}_q$ on $\mathcal{V}^\perp$ iff u minimizes $\mathcal{J}_q$ on $\mathcal{C}(\overline{F})$ and the existence of minimum follows since $\mathcal{J}_q$ is strictly convex on $\mathcal{V}^\perp$. In any case, the equations described in (3) are the *Euler-Lagrange identities* for the corresponding minimization problem. □

The following result is an extension of the monotonicity property of the Schrödinger operator in the case $q \ge q_\sigma$ that was proved in [**3**].

Proposition 3.4. *Suppose that $q \ge q_\sigma$ and that it is not simultaneously true that $H_2 = \emptyset$ and $q = q_\sigma$. If $u \in \mathcal{C}(\overline{F})$ verifies that $\mathcal{L}_q(u) \ge 0$ on F, $\dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu \ge 0$ on H_1 and $u \ge 0$ on H_2, then $u \in \mathcal{C}^+(\overline{F})$.*

Proof. Consider again the network $\Gamma_F = (\overline{F}, \overline{E}, c_{\scriptscriptstyle F})$, where $\overline{E} = \{(x,y) \in E : c_{\scriptscriptstyle F}(x,y) > 0\}$ and let $\overline{\mathcal{L}}$ its combinatorial Laplacian. Then, if $u \in \mathcal{C}(\overline{F})$ verifies the hypotheses, $\overline{\mathcal{L}}(u) \ge 0$ on $F \cup H_1$ and the conclusion follows by applying Proposition 4.1 in [**3**]. □

Suppose that there exists $\sigma \in \mathcal{C}^*(\overline{F})$ such that $q \ge q_\sigma$ and it is not simultaneously true that $H_2 = \emptyset$ and $q = q_\sigma$. The *Green operator associated with Problem* (3) is the linear operator $\mathcal{G}_q^F : \mathcal{C}(F) \to \mathcal{C}(\overline{F})$ that assigns to any $g \in \mathcal{C}(F)$ the unique solution of the boundary value problem $\mathcal{L}_q(u) = g$ on F, $\dfrac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = 0$ on H_1 and $u = 0$ on H_2. Moreover, we define the *Green function associated with Problem* (3) as the function $G_q^F : \overline{F} \times F \to \mathbb{R}$ that assigns to any $y \in F$ and any $x \in \overline{F}$ the value $G_q^F(x,y) = \mathcal{G}_q^F(\varepsilon_y)(x)$, where ε_y stands for the Dirac function at y. So, for any $g \in \mathcal{C}(F)$ it is verified that $\mathcal{G}_q^F(g)(x) = \int_F G_q^F(x,y)\,g(y)\,\mathrm{d}y$. Finally, let us remark that from the above proposition $G_q^F \ge 0$ and moreover $G_q^F(x,y) = G_q^F(y,x)$ for any $x, y \in F$, since the boundary value problem (3) is self-adjoint.

4. THE CONDENSER PRINCIPLE

In this section we obtain a generalization of the well-known *Condenser Principle*. From no on we suppose that there exists $\sigma \in \mathcal{C}(\overline{F})$ such that $q \ge q_\sigma$. Given a non-empty subset $F \subset V$, suppose that $\delta(F) = H_1 \cup \{x\} \cup \{y\}$, where $x, y \notin H_1$ and $x \ne y$. The *generalized Condenser Problem* consists in the following mixed boundary value problem

$$(5)\quad \mathcal{L}_q(u) = 0 \ \text{ on } \ F, \qquad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{F}}} + qu = 0 \ \text{ on } \ H_1, \ \ u(x) = \sigma(x) \ \text{ and } \ \ u(y) = 0.$$

Proposition 4.1. (Condenser Principle) *If $u \in \mathcal{C}(\overline{F})$ is the unique solution of the Condenser Problem* (5), *then $0 \le u \le \sigma$ on V.*

Proof. The positiveness of u follows directly from Proposition 3.4. Moreover, if $v = \sigma - u$ then $\mathcal{L}_q(v) = \sigma(q - q_\sigma) \geq 0$ on F, $\dfrac{\partial v}{\partial \mathsf{n}_\mathsf{F}} + qu = \sigma(q - q_\sigma) \geq 0$ on H_1, $v(x) = 0$ and $v(y) = \sigma(y)$. Therefore, applying again Proposition 3.4, $v \geq 0$. □

Under the hypotheses of the above proposition, $\overline{F}$ is called *condenser* with *source* and *sink* x and y, respectively when H_1 is connected with a medium of conductivity q. Moreover, the above boundary value problem is called *the condenser problem corresponding to* $\overline{F}$.

Next, we introduce a concept that is closely related with the condenser problem in the case $q = q_\sigma$, namely the *effective resistance* between x and y when a subset of the network, D, is connected with a medium of conductivity q_σ. Fixed a weight $\sigma \in \mathcal{C}^*(V)$ and the set $D \subset V$, consider for any $x, y \notin D$ with $x \neq y$, the unique solution $u \in \mathcal{C}(V)$ of the boundary value problem

$$(6)\quad \mathcal{L}_{q_\sigma}(u) = 0 \text{ on } D^c \setminus \{x,y\},\ \frac{\partial u}{\partial \mathsf{n}_{\mathsf{D^c}}} + q_\sigma u = 0 \text{ on } D,\ u(x) = \sigma(x) \text{ and } u(y) = 0.$$

The *effective conductance between* x, y *relative to* D *with respect to* σ, is defined as the value $C_\sigma^D(x,y) = \mathcal{E}_{q_\sigma}^{D^c}(u,u)$. Clearly, $C_\sigma^D(x,y) > 0$, otherwise, $u = a\,\sigma$ and hence u can not verify $u(y) = 0$ and $u(x) = \sigma(x)$ simultaneously. In addition, it is verified that

$$(7)\qquad C_\sigma^D(x,y) = \sigma(x)\mathcal{L}_{q_\sigma}(u)(x) = -\sigma(y)\mathcal{L}_{q_\sigma}(u)(y).$$

The *effective resistance between* x, y *relative to* D *with respect to* σ, is defined as the value $R_\sigma^D(x,y) = C_\sigma^D(x,y)^{-1}$. The effective conductance, and hence the effective resistance, is a symmetric set function, that is, $C_\sigma^D(x,y) = C_\sigma^D(y,x)$ since $\mathcal{E}_{q_\sigma}^{D^c}(u,u) = \mathcal{E}_{q_\sigma}^{D^c}(\sigma - u, \sigma - u)$. So, it is irrelevant which vertex acts as the source and which one acts as the sink. On the other hand, applying the Dirichlet Principle we obtain that

$$C_\sigma^D(x,y) = \min\left\{\mathcal{E}_{q_\sigma}^{D^c}(v,v) : v(x) = \sigma(x) \text{ and } v(y) = 0\right\}.$$

Proposition 4.2. *If for any* $z \notin D$, $\nu_z^D \in \mathcal{C}(V)$ *denotes the unique solution of the problem*

$$\mathcal{L}_{q_\sigma}(\nu_z^D) = 1 \text{ on } D^c \setminus \{z\},\ \frac{\partial \nu_z^D}{\partial \mathsf{n}_{\mathsf{D^c}}} + q_\sigma \nu_z^D = 0 \text{ on } D \text{ and } \nu_z^D(z) = 0,$$

then the function

$$u = \frac{\sigma(x)}{\sigma(y)\nu_y^D(x) + \nu_x^D(y)\sigma(x)}\left(\sigma(y)\nu_y^D - \sigma(y)\nu_x^D + \nu_x^D(y)\sigma\right)$$

is the unique solution of the boundary value problem (6). *In addition,*

$$R_\sigma^D(x,y) = \left(\int_{D^c} \sigma\, \mathrm{d}x\right)^{-1}\left(\frac{\nu_y^D(x)}{\sigma(x)} + \frac{\nu_x^D(y)}{\sigma(y)}\right).$$

Proof. If $v = \sigma(y)\nu_y^D - \sigma(y)\nu_x^D + \nu_x^D(y)\sigma$, then a direct evaluation gives

$$\mathcal{L}_{q_\sigma}(v) = 0 \text{ on } D^c \setminus \{x,y\}, \ \frac{\partial v}{\partial \mathsf{n}_{\mathsf{D^c}}} + q_\sigma v = 0 \text{ on } D \text{ and } v(y) = 0.$$

Moreover $v(x) = \sigma(y)\nu_y^D(x) - \sigma(y)\nu_x^D(x) + \nu_x^D(y)\sigma(x) = \sigma(y)\nu_y^D(x) + \nu_x^D(y)\sigma(x)$, which implies that $u = \dfrac{\sigma(x)\, v}{\sigma(y)\nu_y^D(x) + \nu_x^D(y)\sigma(x)}$. On the other hand, applying the Identity (7), we get that

$$C_\sigma^D(x,y) = \sigma(x)\mathcal{L}_{q_\sigma}(u)(x) = \frac{\sigma(x)^2\mathcal{L}_{q_\sigma}(v)(x)}{\sigma(y)\nu_y^D(x) + \nu_x^D(y)\sigma(x)}\,.$$

Finally, tacking into account that $0 = \displaystyle\int_{D^c} \sigma\, \mathcal{L}_{q_\sigma}(\nu_x^D)\,\mathrm{d}x + \int_D \sigma \left(\frac{\partial \nu_x^D}{\partial \mathsf{n}_{\mathsf{D^c}}} + q_\sigma \nu_x^D\right)\mathrm{d}x$, we obtain that $0 = \int_{D^c} \sigma\, \mathcal{L}_{q_\sigma}(\nu_x^D)\,\mathrm{d}x = \int_{D^c}\sigma\,\mathrm{d}x - \sigma(x) + \sigma(x)\mathcal{L}_{q_\sigma}(\nu_x^D)(x)$ and hence,

$$\sigma(x)\mathcal{L}_{q_\sigma}(v)(x) = \sigma(x)\sigma(y)\mathcal{L}_{q_\sigma}(\nu_y^D)(x) - \sigma(x)\sigma(y)\mathcal{L}_{q_\sigma}(\nu_x^D)(x) = \sigma(y)\textstyle\int_{D^c}\sigma\,\mathrm{d}x,$$

which implies that

$$C_\sigma^D(x,y) = \frac{\sigma(x)\sigma(y)}{\sigma(y)\nu_y^D(x) + \nu_x^D(y)\sigma(x)} \int_{D^c} \sigma\,\mathrm{d}x$$

and the last claim follows. □

Observe that if for any $x \notin D$ we define $R_\sigma^D(x,x) = 0$, then the above formula for the effective resistance between two vertices in D^c is still valid for $y = x$. Now we can generalize a well-known result about the effective resistance.

Corollary 4.3. (Generalized FOSTER's Theorem) *The following identity holds*

$$\textstyle\int_{D^c}\int_{D^c} R_\sigma^D(x,y)c_{D^c}(x,y)\,\sigma(x)\sigma(y)\,\mathrm{d}x\,\mathrm{d}y = 2(|V| - |D| - 1).$$

Proof. From the expression of the effective resistance, we have that

$$\sigma(x)\sigma(y)R_\sigma^D(x,y) = \left(\textstyle\int_{D^c}\sigma\,\mathrm{d}x\right)^{-1}\left(\sigma(y)\nu_y^D(x) + \sigma(x)\nu_x^D(y)\right).$$

On the other hand, tacking into account the symmetry of c_{D^c} we get that

$$\textstyle\int_{D^c}\int_{D^c}\sigma(x)\nu_x^D(y)c_{D^c}(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_{D^c}\int_{D^c}\sigma(y)\nu_y^D(x)c_{D^c}(x,y)\,\mathrm{d}x\,\mathrm{d}y$$

which implies that

$$\begin{aligned}&\textstyle\int_{D^c}\int_{D^c} R_\sigma^D(x,y)c_{D^c}(x,y)\,\sigma(x)\sigma(y)\,\mathrm{d}x\,\mathrm{d}y\\ &\qquad = 2\left(\textstyle\int_{D^c}\sigma\,\mathrm{d}x\right)^{-1}\textstyle\int_{D^c}\sigma(x)\int_{D^c}\nu_x^D(y)c_{D^c}(x,y)\,\mathrm{d}y\,\mathrm{d}x.\end{aligned}$$

Finally, the result follows by keeping in main that for any $x \in D^c$

$$\sigma(x)\textstyle\int_{D^c}\nu_x^D(y)c_{D^c}(x,y)\,\mathrm{d}y = \sigma(x)\mathcal{L}_{q_\sigma}(\nu_x^D)(x) = \int_{D^c}\sigma\,\mathrm{d}x - \sigma(x).$$ □

Another well-known consequence of Proposition 4.2 establishes that when $q = q_\sigma$ for any $y \notin D$, the Green function for problem

$$\mathcal{L}_{q_\sigma}(u) = f \ \text{ on } \ D^c \setminus \{y\}, \ \ \frac{\partial u}{\partial n_{D^c}} + q_\sigma u = 0 \ \text{ on } \ D, \ \ u(y) = 0 \tag{8}$$

can be seen as an *inverse resistive*; *i.e.* can be expressed in terms of effective resistances.

Corollary 4.4. *Given $x, y, z \notin D$ it is verified that*

$$G_{q_\sigma}^{D^c\setminus\{y\}}(z,x) = \frac{1}{2}\,\sigma(x)\sigma(z)\big(R_\sigma^D(x,y) + R_\sigma^D(z,y) - R_\sigma^D(z,x)\big).$$

In particular, the effective resistance determines a distance on D^c.

Proof. First, observe that if u is the solution of Problem (6), then Identity (7) implies that $\mathcal{L}_{q_\sigma}(u) = \dfrac{C_\sigma^D(x,y)}{\sigma}(\varepsilon_x - \varepsilon_y)$ on D^c. Therefore, for any $x \notin D$ and $z \in V$ it is verified that $G_{q_\sigma}^{D^c\setminus\{y\}}(z,x) = R_\sigma^D(x,y)\,\sigma(x)\,u(z)$; that is,

$$G_{q_\sigma}^{D^c\setminus\{y\}}(z,x) = \left(\int_{D^c} \sigma\,\mathrm{d}x\right)^{-1} \sigma(x)\sigma(z)\left(\frac{\nu_y^D(z)}{\sigma(z)} - \frac{\nu_x^D(z)}{\sigma(z)} + \frac{\nu_x^D(y)}{\sigma(y)}\right).$$

In particular, when $x, z \notin D$, then

$$G_{q_\sigma}^{D^c\setminus\{y\}}(x,z) = \left(\int_{D^c} \sigma\,dx\right)^{-1} \sigma(x)\sigma(z)\left(\frac{\nu_y^D(x)}{\sigma(x)} - \frac{\nu_z^D(x)}{\sigma(x)} + \frac{\nu_z^D(y)}{\sigma(y)}\right)$$

and the expression of the Green function is a consequence of its symmetry on D^c. The last conclusion is a direct consequence of being $G_{q_\sigma}^{D^c\setminus\{y\}}$ non-negative. □

We finish this section by generalizing the above corollary to the case $q \geq q_\sigma$. Specifically, we prove that the Green function of the Robin boundary value problem

$$\mathcal{L}_q(u) = f \ \text{ on } \ D^c, \quad \frac{\partial u}{\partial \mathsf{n}_{D^c}} + qu = 0 \ \text{ on } \ D,$$

can be seen as an inverse resistive relative to a new network. To do this, consider a new vertex $\widehat{x} \notin V$, the set $\widehat{V} = V \cup \{\widehat{x}\}$ and $\widehat{\sigma} \in \mathcal{C}^*(\widehat{V})$ the weight on $\widehat{V}$ defined as $\widehat{\sigma}(x) = \sigma(x)$ when $x \in V$ and as $\sigma(\widehat{x}) = 1$.

We consider the network $\widehat{\Gamma} = (\widehat{V}, \widehat{E}, \widehat{c})$ where $\widehat{c}(x,y) = c(x,y)$ when $x, y \in V$ and $\widehat{c}(\widehat{x}, x) = \sigma(x)(q(x) - q_\sigma(x))$ for any $x \in V$. Therefore, E is a proper subset of $\widehat{E}$ and this also assures that $\widehat{\Gamma}$ is connected. In addition, we denote by $\widehat{\mathcal{L}}$ the combinatorial Laplacian of $\widehat{\Gamma}$ and by $q_{\widehat{\sigma}}$ the ground state associated with $\widehat{\mathcal{L}}$ and $\widehat{\sigma}$. The following result will be the key for our purposes.

Proposition 4.5. *For any $u \in \mathcal{C}(\widehat{V})$, it is verified that*

$$\widehat{\mathcal{L}}(u) + q_{\widehat{\sigma}} u = \mathcal{L}(u_{|_V}) + q\, u - (q - q_{\widehat{\sigma}})\, u(\widehat{x}) \quad \text{on } V$$

and

$$\frac{\partial u}{\partial \mathsf{n}_{\hat{\mathsf{V}}\setminus \mathsf{D}}} + q_{\widehat{\sigma}} u = \frac{\partial u}{\partial \mathsf{n}_{\mathsf{V}\setminus \mathsf{D}}} + q\, u - (q - q_{\widehat{\sigma}})\, u(\widehat{x}) \quad \text{on } D.$$

In particular, if $u \in \mathcal{C}(V)$, then

$$\widehat{\mathcal{L}}_{q_{\widehat{\sigma}}}(u) = \mathcal{L}_q(u) \quad \text{on } V \quad \text{and} \quad \frac{\partial u}{\partial \mathsf{n}_{\hat{\mathsf{V}}\setminus \mathsf{D}}} + q_{\widehat{\sigma}} u = \frac{\partial u}{\partial \mathsf{n}_{\mathsf{V}\setminus \mathsf{D}}} + q\, u \quad \text{on } D.$$

Proof. Given $u \in \mathcal{C}(\widehat{V})$, we get that for any $x \in V$

$$\widehat{\mathcal{L}}(u)(x) = \mathcal{L}(u_{|_V})(x) + \widehat{c}(x,\widehat{x})\big(u(x) - u(\widehat{x})\big).$$

In particular, tacking $u = \widehat{\sigma}$ it is verified that $\widehat{\mathcal{L}}(\widehat{\sigma})(x) = \mathcal{L}(\sigma)(x) + \widehat{c}\,(x,\widehat{x})\big(\sigma(x) - 1\big)$, which implies that $\widehat{c}(x,\widehat{x}) = q_\sigma(x) - q_{\widehat{\sigma}}(x) + \dfrac{\widehat{c}\,(x,\widehat{x})}{\sigma(x)} = q(x) - q_{\widehat{\sigma}}(x)$ and the result follows substituting the value of $\widehat{c}(\cdot,\widehat{x})$ in the expression of $\widehat{\mathcal{L}}(u)(x)$. The same reasoning works for the normal derivative. □

Corollary 4.6. *For all $x, y \notin D$ it is verified that*

$$G_q^{D^c}(x,y) = \frac{1}{2}\, \sigma(x)\, \sigma(y) \left(R^D_{\widehat{\sigma}}(x,\widehat{x}) + R^D_{\widehat{\sigma}}(y,\widehat{x}) - R^D_{\widehat{\sigma}}(x,y) \right),$$

where $R^D_{\widehat{\sigma}}$ is the effective resistance relative to D with respect to $\widehat{\sigma}$ in the network $\widehat{\Gamma}$.

Proof. Taking into account the above proposition, we get that $u \in \mathcal{C}(V)$ is the unique solution of the problem

$$\mathcal{L}_q(u) = f \quad \text{on} \quad D^c, \qquad \frac{\partial u}{\partial \mathsf{n}_{\mathsf{D^c}}} + qu = 0 \quad \text{on} \quad D$$

iff it is the unique solution of the mixed problem

$$\widehat{\mathcal{L}}_{q_{\widehat{\sigma}}}(u) = f \quad \text{on} \quad D^c, \qquad \frac{\partial u}{\partial \mathsf{n}_{\hat{\mathsf{V}}\setminus \mathsf{D}}} + q_{\widehat{\sigma}} u = 0 \quad \text{on} \quad D \quad \text{and} \quad u(\widehat{x}) = 0.$$

The result follows by applying Corollary 4.4 to $\widehat{\Gamma}$ and taking $y = \widehat{x}$. □

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FRACTIONAL INTEGRALS AND DERIVATIVES IN q-CALCULUS

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We generalize the notions of the fractional q-integral and q-derivative by introducing variable lower limit of integration. We discuss some properties and their relations. Finally, we give a q-Taylor-like formula which includes fractional q-derivatives of the function.

1. INTRODUCTION

In the theory of q-calculus (see [**5**] and [**7**]), for a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, we introduce a q-real number $[a]_q$ by

$$[a]_q := \frac{1-q^a}{1-q} \qquad (a \in \mathbb{R}) \ .$$

The q-analog of the Pochhammer symbol (q–shifted factorial) is defined by:

$$(a;q)_0 = 1 \ , \qquad (a;q)_k = \prod_{i=0}^{k-1} (1-aq^i) \qquad \big(k \in \mathbb{N} \cup \{\infty\}\big) \ .$$

Also, the q-analog of the power $(a-b)^k$ is

$$(a-b)^{(0)} = 1 \ , \qquad (a-b)^{(k)} = \prod_{i=0}^{k-1} (a-bq^i) \qquad \big(k \in \mathbb{N};\ a,b \in \mathbb{R}\big) \ .$$

There is the following relationship between them:

$$(a-b)^{(n)} = a^n \ (b/a;q)_n \qquad (a \neq 0) \ .$$

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Their natural expansions to the reals are

(1) $$(a-b)^{(\alpha)} = a^{\alpha}\frac{(b/a;q)_\infty}{(q^\alpha b/a;q)_\infty}\ , \qquad (a;q)_\alpha = \frac{(a;q)_\infty}{(aq^\alpha;q)_\infty} \qquad (\alpha\in\mathbb{R})\ .$$

Notice that

$$(a-b)^{(\alpha)} = a^\alpha (b/a;q)_\alpha .$$

The following formulas (see, for example, [**5**] and [**4**]) will be useful:

(2) $$(a;q)_n = \left(q^{1-n}/a;q\right)_n\ (-1)^n\ a^n\ q^{\binom{n}{2}};$$

(3) $$\frac{(aq^{-n};q)_n}{(bq^{-n};q)_n} = \frac{(q/a;q)_n}{(q/b;q)_n}\ \left(\frac{a}{b}\right)^n;$$

(4) $$(a-b)^{(\alpha)} = a^\alpha\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix}\alpha\\ k\end{bmatrix}_q q^{\binom{k}{2}}\left(\frac{b}{a}\right)^k .$$

The q-gamma function is defined by

(5) $$\Gamma_q(x) = \frac{(q;q)_\infty}{(q^x;q)_\infty}\ (1-q)^{1-x} = (1-q)^{(x-1)}\ (1-q)^{1-x},$$

where $x\in\mathbb{R}\setminus\{0,-1,-2,\ldots\}$. Obviously,

$$\Gamma_q(x+1) = [x]_q\Gamma_q(x)\ .$$

We can define q-binomial coefficients with

$$\begin{bmatrix}\alpha\\ \beta\end{bmatrix}_q = \frac{\Gamma_q(\alpha+1)}{\Gamma_q(\beta+1)\ \Gamma_q(\alpha-\beta+1)} = \frac{(q^{\beta+1};q)_\infty\ (q^{\alpha-\beta+1};q)_\infty}{(q;q)_\infty\ (q^{\alpha+1};q)_\infty}$$

$\alpha,\beta,\alpha-\beta\in\mathbb{R}\setminus\{-1,-2,\ldots\}$. Particularly,

(6) $$\begin{bmatrix}\alpha\\ k\end{bmatrix}_q = \frac{(q^{-\alpha};q)_k}{(q;q)_k}\ (-1)^k q^{\alpha k}q^{-\binom{k}{2}} \quad (k\in\mathbb{N}).$$

The q–hypergeometric function is defined as

$$ {}_2\phi_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\,\middle|\ q;x\right) = \sum_{n=0}^{\infty}\frac{(a;q)_n(b;q)_n}{(c;q)_n\ (q;q)_n}\ x^n\ .$$

The famous Heine transformation formula [**5**] is

(7) $$ {}_2\phi_1\left(\begin{matrix}a,\ b\\ c\end{matrix}\,\middle|\ q;x\right) = \frac{(abx/c;q)_\infty}{(x;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}c/a,\ c/b\\ c\end{matrix}\,\middle|\ q;abx/c\right)\ .$$

We define a q-derivative of a function $f(x)$ by

$$\big(D_qf\big)(x) = \frac{f(x)-f(qx)}{x-qx}\quad (x\neq 0)\ ,\quad \big(D_qf\big)(0) = \lim_{x\to 0}\big(D_qf\big)(x)$$

and q-derivatives of higher order:

(8) $$D_q^0 f = f\ , \qquad D_q^n f = D_q\big(D_q^{n-1} f\big) \quad (n = 1, 2, 3, \ldots)\ .$$

For an arbitrary pair of functions $u(x)$ and $v(x)$ and constants $\alpha, \beta \in \mathbb{R}$, we have linearity and product rules

$$\begin{aligned} D_q\big(\alpha\ u(x) + \beta\ v(x)\big) &= \alpha\big(D_q u\big)(x) + \beta\big(D_q v\big)(x), \\ D_q\big(u(x) \cdot v(x)\big) &= u(qx)\big(D_q v\big)(x) + v(x)\big(D_q u\big)(x)\ . \end{aligned}$$

The q-integral is defined by

$$\big(I_{q,0} f\big)(x) = \int_0^x f(t)\, \mathrm{d}_q t = x(1-q) \sum_{k=0}^{\infty} f(xq^k)\, q^k \quad (0 \le |q| < 1),$$

and

(9) $$\big(I_{q,a} f\big)(x) = \int_a^x f(t)\, \mathrm{d}_q t = \int_0^x f(t)\, \mathrm{d}_q t - \int_0^a f(t)\, \mathrm{d}_q t.$$

However, these definitions cause troubles in research as they include the points outside of the interval of integration (see [**6**] and **10**]). In the case when the lower limit of integration is $a = xq^n$, i.e., when it is determined for some choice of x, q and positive integer n, the q-integral (9) becomes

(10) $$\int_{xq^n}^x f(t)\, \mathrm{d}_q t = x(1-q) \sum_{k=0}^{n-1} f(xq^k) q^k\ .$$

As for q-derivative, we can define an operator $I_{q,a}^n$ by

$$I_{q,a}^0 f = f, \qquad I_{q,a}^n f = I_{q,a}\big(I_{q,a}^{n-1} f\big) \quad (n = 1, 2, 3, \ldots)\ .$$

For operators defined in this manner, the following is valid:

(11) $$\big(D_q I_{q,a} f\big)(x) = f(x), \qquad \big(I_{q,a} D_q f\big)(x) = f(x) - f(a).$$

The formula for q-integration by parts is

$$\int_a^b u(x)\big(D_q v\big)(x)\, d_q x = \big[u(x)v(x)\big]_a^b - \int_a^b v(qx)\big(D_q u\big)(x)\, d_q x\ .$$

W. A. Al-Salam [**2**] and R. P. Agarwal [**1**] introduced several types of fractional q-integral operators and fractional q-derivatives. Here, we will only mention the fractional q-integral with the lower limit of integration $a = 0$, defined by

$$\big(I_q^{\eta,\alpha} f\big)(x) = \frac{x^{-(\eta+\alpha)}}{\Gamma_q(\alpha)} \int_0^x (x - tq)^{(\alpha-1)} t^{\eta} f(t)\ d_q t \qquad (\eta, \alpha \in \mathbb{R}^+)\,.$$

On the other hand, the solution of nth order q-differential equation

$$\big(D_q^n y\big)(x) = f(x), \qquad \big(D_q^k y\big)(a) = 0 \quad (k = 0, 1, \ldots, n-1),$$

can be written in the form of a multiple q-integral

$$y(x) = \big(I_{q,a}^n f\big)(x) = \int_a^x \mathrm{d}_q t \int_a^t \mathrm{d}_q t_{n-1} \int_a^{t_{n-1}} \mathrm{d}_q t_{n-2} \cdots \int_a^{t_2} f(t_1)\, \mathrm{d}_q t_1.$$

The reduction of the multiple q-integral to a single one was considered by AL-SALAM [**3**]. He thought of it as a *q-analog of Cauchy's formula*:

$$\text{(12)} \qquad y(x) = \big(I_{q,a}^n f\big)(x) = \frac{1}{[n-1]_q!} \int_a^x (x - qt)^{(n-1)} f(t)\ \mathrm{d}_q t \qquad (n \in \mathbb{N}).$$

In this paper, our purpose is to consider fractional q-integrals with the parametric lower limit of integration. After preliminaries, in the third section we define the fractional q-integral in that sense. On the basis of that, the fractional q-derivative is introduced in the fourth section. Finally, in the last section, we give a q-TAYLOR-like formula using these fractional q-derivatives.

2. PRELIMINARIES

We will first specify some results which are useful in the sequel and which can be proved easily.

Lemma 1. *For $a, b, \alpha \in \mathbb{R}^+$ and $k, n \in \mathbb{N}$, the following properties are valid*:

$$\text{(13)} \qquad (a - bq^k)^{(\alpha)} = a^\alpha (1 - q^k b/a)^{(\alpha)},$$

$$\text{(14)} \qquad \frac{(a - bq^k)^{(\alpha)}}{(a - b)^{(\alpha)}} = \frac{(q^\alpha b/a; q)_k}{(b/a; q)_k},$$

$$\text{(15)} \qquad (q^n - q^k)^{(\alpha)} = 0 \quad (k \le n).$$

The next result will have an important role in proving the semigroup property of the fractional q–integral.

Lemma 2. *For $\mu, \alpha, \beta \in \mathbb{R}^+$, the following identity is valid*

$$\text{(16)} \qquad \sum_{n=0}^{\infty} \frac{(1 - \mu q^{1-n})^{(\alpha-1)}\ (1 - q^{1+n})^{(\beta-1)}}{(1 - q)^{(\alpha-1)}\ (1 - q)^{(\beta-1)}}\ q^{\alpha n} = \frac{(1 - \mu q)^{(\alpha+\beta-1)}}{(1 - q)^{(\alpha+\beta-1)}}\,.$$

Proof. According to the formulas (1) and (3), we have

$$\begin{aligned}(1 - \mu q^{1-n})^{(\alpha-1)} &= \frac{(\mu q^{1-n}; q)_\infty}{(\mu q^{\alpha-n}; q)_\infty} = \frac{(\mu q^{1-n}; q)_n\ (\mu q; q)_\infty}{(\mu q^{\alpha-n}; q)_n\ (\mu q^\alpha; q)_\infty} \\ &= (1 - \mu q)^{(\alpha-1)} \frac{(\mu^{-1}; q)_n}{(\mu^{-1} q^{1-\alpha}; q)_n}\ q^{(1-\alpha)n}\,.\end{aligned}$$

Applying the identity (14) to the expression $(1-q^{1+n})^{(\beta-1)}/(1-q)^{(\beta-1)}$, the sum on the left side of (16) can be written as

$$\begin{aligned} LS &= \frac{(1-\mu q)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}} \sum_{n=0}^{\infty} \frac{(q^{\beta};q)_n}{(q;q)_n} \frac{(\mu^{-1};q)_n}{(\mu^{-1}q^{1-\alpha};q)_n}\, q^{(1-\alpha)n} q^{\alpha n} \\ &= \frac{(1-\mu q)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}}\ {}_2\phi_1\left(\begin{matrix}\mu^{-1},\ q^{\beta}\\ \mu^{-1}q^{1-\alpha}\end{matrix}\,\middle|\, q;q\right). \end{aligned}$$

Using (7), we get

$$\begin{aligned} LS &= \frac{(1-\mu q)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}}\ \frac{(q^{\alpha+\beta};q)_\infty}{(q;q)_\infty}\ {}_2\phi_1\left(\begin{matrix}q^{1-\alpha},\ \mu^{-1}q^{1-\alpha-\beta}\\ \mu^{-1}q^{1-\alpha}\end{matrix}\,\middle|\, q;q^{\alpha+\beta}\right) \\ &= \frac{(1-\mu q)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}}\ \frac{1}{(1-q)^{(\alpha+\beta-1)}} \sum_{n=0}^{\infty} \frac{(q^{1-\alpha};q)_n\ (\mu^{-1}q^{1-\alpha-\beta};q)_n}{(q;q)_n\ (\mu^{-1}q^{1-\alpha};q)_n}\, q^{(\alpha+\beta)n}\ . \end{aligned}$$

According to (2) and (1), the following is valid:

$$\begin{aligned} \frac{(\mu^{-1}q^{1-\alpha-\beta};q)_n}{(\mu^{-1}q^{1-\alpha};q)_n} &= \frac{(\mu q^{\alpha+\beta-n};q)_n}{(\mu q^{\alpha-n};q)_n}\, q^{-\beta n} = \frac{(\mu q^{\alpha+\beta-n};q)_\infty}{(\mu q^{\alpha+\beta};q)_\infty}\ \frac{(\mu q^{\alpha};q)_\infty}{(\mu q^{\alpha-n};q)_\infty}\, q^{-\beta n} \\ &= \frac{(\mu q^{\alpha};q)_\infty}{(\mu q^{\alpha+\beta};q)_\infty}\ \frac{(\mu q^{\alpha+\beta-n};q)_\infty}{(\mu q^{\alpha-n};q)_\infty}\, q^{-\beta n} \\ &= \frac{(\mu q^{\alpha};q)_\infty}{(\mu q^{\alpha+\beta};q)_\infty}\ (1-\mu q^{\alpha+\beta-n})^{(-\beta)}\ q^{-\beta n}\, . \end{aligned}$$

Hence

$$LS = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha-1)}\ (1-q)^{(\alpha+\beta-1)}} \sum_{n=0}^{\infty} \frac{(q^{1-\alpha};q)_n}{(q;q)_n}\, q^{\alpha n}(1-\mu q^{\alpha+\beta-n})^{(-\beta)}\, .$$

If we use formulas (6) and (4) and change the order of the summation, the last sum becomes

$$\begin{aligned} &\sum_{n=0}^{\infty} \frac{(q^{1-\alpha};q)_n}{(q;q)_n}\, q^{\alpha n}(1-\mu q^{\alpha+\beta-n})^{(-\beta)} \\ &= \sum_{n=0}^{\infty} \begin{bmatrix}\alpha-1\\ n\end{bmatrix}_q (-1)^n\, q^{-(\alpha-1)n} q^{\binom{n}{2}}\, q^{\alpha n} \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix}-\beta\\ k\end{bmatrix}_q q^{\binom{k}{2}} \left(\mu q^{\alpha+\beta-n}\right)^k \\ &= \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix}-\beta\\ k\end{bmatrix}_q q^{\binom{k}{2}} \left(\mu q^{\alpha+\beta}\right)^k \sum_{n=0}^{\infty} (-1)^n \begin{bmatrix}\alpha-1\\ n\end{bmatrix}_q q^{\binom{n}{2}}\, \left(q^{1-k}\right)^n \\ &= \sum_{k=0}^{\infty} (-1)^k \begin{bmatrix}-\beta\\ k\end{bmatrix}_q q^{\binom{k}{2}} \left(\mu q^{\alpha+\beta}\right)^k \left(1-q^{1-k}\right)^{(\alpha-1)} = (1-q)^{(\alpha-1)}\ . \end{aligned}$$

The last relation is valid because of $\left(1-q^{1-k}\right)^{(\alpha-1)} = 0$ for $k = 1, 2, \dots$. Finally, the identity holds:

$$LS = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha-1)}\ (1-q)^{(\alpha+\beta-1)}}\ (1-q)^{(\alpha-1)} = \frac{(1-\mu q)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}}\, . \qquad \square$$

3. THE FRACTIONAL q-INTEGRAL

In all further considerations we assume that the functions are defined in an interval $(0,b)$ $(b>0)$, and $a\in(0,b)$ is an arbitrary fixed point. Also, the required q-derivatives and q-integrals exist and the convergence of the series mentioned in the proofs is assumed.

Generalizing the formula (12), we can define the *fractional q-integral of the Riemann-Liouville type* by

$$(I_{q,a}^{\alpha}f)(x)=\frac{1}{\Gamma_q(\alpha)}\int_a^x (x-qt)^{(\alpha-1)}f(t)\,\mathrm{d}_qt \qquad (\alpha\in\mathbb{R}^+)\,. \tag{17}$$

Using formula (4), this integral can be written as

$$(I_{q,a}^{\alpha}f)(x)=\frac{x^{\alpha-1}}{\Gamma_q(\alpha)}\sum_{k=0}^{\infty}(-1)^k\begin{bmatrix}\alpha-1\\ k\end{bmatrix}_q q^{\binom{k+1}{2}}x^{-k}\int_a^x t^kf(t)\,\mathrm{d}_qt \qquad (\alpha\in\mathbb{R}^+)\,.$$

Lemma 3. *For $\alpha\in\mathbb{R}^+$, the following is valid*:

$$(I_{q,a}^{\alpha}f)(x)=(I_{q,a}^{\alpha+1}D_qf)(x)+\frac{f(a)}{\Gamma_q(\alpha+1)}(x-a)^{(\alpha)} \qquad (0<a<x<b)\,.$$

Proof. Since the q-derivative over the variable t is

$$D_q\big((x-t)^{(\alpha)}\big)=-[\alpha]_q(x-qt)^{(\alpha-1)}\,,$$

and using the q-integration by parts, we obtain

$$\begin{aligned}(I_{q,a}^{\alpha}f)(x)&=-\frac{1}{[\alpha]_q\Gamma_q(\alpha)}\int_a^x D_q\big((x-t)^{(\alpha)}\big)f(t)\,\mathrm{d}_qt\\ &=\frac{1}{\Gamma_q(\alpha+1)}\Big((x-a)^{(\alpha)}f(a)+\int_a^x (x-qt)^{(\alpha)}\big(D_qf\big)(t)\,\mathrm{d}_qt\Big)\\ &=(I_{q,a}^{\alpha+1}D_qf)(x)+\frac{f(a)}{\Gamma_q(\alpha+1)}(x-a)^{(\alpha)}\,.\end{aligned}$$

□

Lemma 4. *For $\alpha,\beta\in\mathbb{R}^+$, the following is valid*:

$$\int_0^a (x-qt)^{(\beta-1)}\big(I_{q,a}^{\alpha}f\big)(t)\,\mathrm{d}_qt=0 \qquad (0<a<x<b)\,.$$

Proof. Using Lemma 1 and formula (10), for $n\in\mathbb{N}_0$, we have

$$\begin{aligned}(I_{q,a}^{\alpha}f)(aq^n)&=\frac{1}{\Gamma_q(\alpha)}\int_a^{aq^n}(aq^n-qu)^{(\alpha-1)}f(u)\mathrm{d}_qu\\ &=\frac{-a^{\alpha}(1-q)}{\Gamma_q(\alpha)}\sum_{j=0}^{n-1}(q^n-q^{j+1})^{(\alpha-1)}f(aq^j)q^j=0\,.\end{aligned}$$

Then, according to the definition of q-integral, it follows

$$\int_0^a (x-qt)^{(\beta-1)}\big(I_{q,a}^{\alpha} f\big)(t)\,\mathrm{d}_q t = a(1-q)\sum_{n=0}^{\infty}(x-aq^{n+1})^{(\beta-1)}\big(I_{q,a}^{\alpha} f\big)(aq^n)q^n = 0. \quad \square$$

Theorem 5. *Let $\alpha,\beta\in\mathbb{R}^+$. The q-fractional integration has the following semigroup property*

$$\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) = \big(I_{q,a}^{\alpha+\beta} f\big)(x) \qquad (0<a<x<b)\ .$$

Proof. By previous lemma, we have

$$\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) = \frac{1}{\Gamma_q(\beta)}\int_0^x (x-qt)^{(\beta-1)}\big(I_{q,a}^{\alpha} f\big)(t)\,\mathrm{d}_q t,$$

i.e.,

$$\begin{aligned}\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) &= \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x (x-qt)^{(\beta-1)}\int_0^t (t-qu)^{(\alpha-1)} f(u)\,\mathrm{d}_q u\\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x (x-qt)^{(\beta-1)}\int_0^a (t-qu)^{(\alpha-1)} f(u)\,\mathrm{d}_q u\ .\end{aligned}$$

Using the result from [**1**],

$$\big(I_{q,0}^{\beta} I_{q,0}^{\alpha} f\big)(x) = \big(I_{q,0}^{\alpha+\beta} f\big)(x),$$

we conclude that

$$\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) = \big(I_{q,0}^{\alpha+\beta} f\big)(x) - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x (x-qt)^{(\beta-1)}\int_0^a (t-qu)^{(\alpha-1)} f(u)\,\mathrm{d}_q u.$$

Furthermore, we can write

$$\begin{aligned}\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) &= \big(I_{q,a}^{\alpha+\beta} f\big)(x) + \frac{1}{\Gamma_q(\alpha+\beta)}\int_0^a (x-qt)^{(\alpha+\beta-1)} f(t)\,\mathrm{d}_q t\\ &\quad - \frac{1}{\Gamma_q(\alpha)\Gamma_q(\beta)}\int_0^x (x-qt)^{(\beta-1)}\int_0^a (t-qu)^{(\alpha-1)} f(u)\,\mathrm{d}_q u,\end{aligned}$$

wherefrom it follows

$$\big(I_{q,a}^{\beta} I_{q,a}^{\alpha} f\big)(x) = \big(I_{q,a}^{\alpha+\beta} f\big)(x) + a(1-q)\sum_{j=0}^{\infty} c_j f(aq^j)q^j,$$

with

$$c_j = \frac{(x-aq^{j+1})^{(\alpha+\beta-1)}}{\Gamma_q(\alpha+\beta)} - \frac{x(1-q)}{\Gamma_q(\alpha)\Gamma_q(\beta)} \sum_{n=0}^{\infty} (x-xq^{n+1})^{(\beta-1)}(xq^n-aq^{j+1})^{(\alpha-1)}q^n.$$

By using the formulas from Lemma 1 and (5), we get

$$\begin{aligned} c_j &= ((1-q)x)^{\alpha+\beta-1} \\ &\times \left\{ \frac{\left(1-\frac{a}{x}\,q^{j+1}\right)^{(\alpha+\beta-1)}}{(1-q)^{(\alpha+\beta-1)}} - \sum_{n=0}^{\infty} \frac{(1-q^{n+1})^{(\beta-1)}}{(1-q)^{(\beta-1)}}\, \frac{\left(1-\frac{a}{x}\,q^{j+1-n}\right)^{(\alpha-1)}}{(1-q)^{(\alpha-1)}}\, q^{n\alpha} \right\}. \end{aligned}$$

Putting $\mu = q^j a/x$ into (16), we see that $c_j = 0$ for all $j \in \mathbb{N}$, which completes the proof. $\square$

Lemma 6. *For $\alpha \in \mathbb{R}^+$, $\lambda \in (-1,\infty)$, the following is valid*

$$I^{\alpha}_{q,a}\big((x-a)^{(\lambda)}\big) = \frac{\Gamma_q(\lambda+1)}{\Gamma_q(\alpha+\lambda+1)}\,(x-a)^{(\alpha+\lambda)} \qquad (0<a<x<b)\,. \tag{18}$$

Proof. For $\lambda \neq 0$, according to the definition (17), we have

$$I^{\alpha}_{q,a}\big((x-a)^{(\lambda)}\big) = \frac{1}{\Gamma_q(\alpha)}\Big(\int_0^x (x-qt)^{(\alpha-1)}(t-a)^{(\lambda)}\,\mathrm{d}_q t - \int_0^a (x-qt)^{(\alpha-1)}(t-a)^{(\lambda)}\,\mathrm{d}_q t\Big).$$

Also, the following is valid:

$$\int_0^a (x-qt)^{(\alpha-1)}(t-a)^{(\lambda)}\,\mathrm{d}_q t = a^{\lambda+1}(1-q)\sum_{k=0}^{\infty}(x-aq^{k+1})^{(\alpha-1)}(q^k-1)^{(\lambda)}q^k = 0.$$

Therefrom, by using (16), we get

$$\begin{aligned} &\int_0^x (x-qt)^{(\alpha-1)}(t-a)^{(\lambda)}\,\mathrm{d}_q t \\ &\qquad = x^{\alpha+\lambda}(1-q)\sum_{k=0}^{\infty}(1-q^{1+k})^{(\alpha-1)}\Big(1-\frac{a}{qx}q^{1-k}\Big)^{(\lambda)}q^{(\lambda+1)k} \\ &\qquad = (1-q)\frac{(1-q)^{(\alpha-1)}(1-q)^{(\lambda)}}{(1-q)^{(\alpha+\lambda)}}(x-a)^{(\alpha+\lambda)}. \end{aligned}$$

Using (5), we obtain the required formula.

Particularly, for $\lambda = 0$, using a q-integration by parts, we have

$$\begin{aligned} (I^{\alpha}_{q,a}\mathbf{1})(x) &= \frac{1}{\Gamma_q(\alpha)}\int_a^x (x-qt)^{(\alpha-1)}\,\mathrm{d}_q t = \frac{1}{\Gamma_q(\alpha)}\int_a^x \frac{D_q\big((x-t)^{(\alpha)}\big)}{-[\alpha]_q}\,\mathrm{d}_q t \\ &= \frac{-1}{\Gamma_q(\alpha+1)}\int_a^x D_q\big((x-t)^{(\alpha)}\big)\,\mathrm{d}_q t = \frac{1}{\Gamma_q(\alpha+1)}\,(x-a)^{(\alpha)}\,. \end{aligned}$$

$\square$

4. THE FRACTIONAL q-DERIVATIVE

We define the *fractional q-derivative* by

$$\big(D_{q,a}^{\alpha} f\big)(x) = \begin{cases} \big(I_{q,a}^{-\alpha} f\big)(x), & \alpha < 0 \\ f(x), & \alpha = 0 \\ \big(D_q^{\lceil \alpha \rceil} I_{q,a}^{\lceil \alpha \rceil - \alpha} f\big)(x), & \alpha > 0, \end{cases} \tag{19}$$

where $\lceil \alpha \rceil$ denotes the smallest integer greater or equal to α.

Notice that $\big(D_{q,a}^{\alpha} f\big)(x)$ has subscript a to emphasize that it depends on the lower limit of integration used in definition (19). Since $\lceil \alpha \rceil$ is a positive integer for $\alpha \in \mathbb{R}^+$, then for $\big(D_q^{\lceil \alpha \rceil} f\big)(x)$ we apply definition (8).

Lemma 7. *For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, the following is valid*:

$$\big(D_q D_{q,a}^{\alpha} f\big)(x) = \big(D_{q,a}^{\alpha+1} f\big)(x) \qquad (0 < a < x < b)\ .$$

Proof. We will consider three cases. For $\alpha \le -1$, according to Theorem 5, we have

$$\begin{aligned} \big(D_q D_{q,a}^{\alpha} f\big)(x) &= \big(D_q I_{q,a}^{-\alpha} f\big)(x) = \big(D_q I_{q,a}^{1-\alpha-1} f\big)(x) \\ &= \big(D_q I_{q,a} I_{q,a}^{-\alpha-1} f\big)(x) = \big(I_{q,a}^{-(\alpha+1)} f\big)(x) = \big(D_{q,a}^{\alpha+1} f\big)(x). \end{aligned}$$

In the case $-1 < \alpha < 0$, i.e., $0 < \alpha + 1 < 1$, we obtain

$$\big(D_q D_{q,a}^{\alpha} f\big)(x) = \big(D_q I_{q,a}^{-\alpha} f\big)(x) = \big(D_q I_{q,a}^{1-(\alpha+1)} f\big)(x) = \big(D_{q,a}^{\alpha+1} f\big)(x).$$

For $\alpha > 0$, we get

$$(D_q D_{q,a}^{\alpha} f)(x) = (D_q D_q^{\lceil \alpha \rceil} I_{q,a}^{\lceil \alpha \rceil - \alpha} f)(x) = (D_q^{\lceil \alpha \rceil + 1} I_{q,a}^{\lceil \alpha \rceil - \alpha} f)(x) = (D_{q,a}^{\alpha+1} f)(x). \quad \square$$

Theorem 8. *For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, the following is valid*:

$$(D_q D_{q,a}^{\alpha} f)(x) - (D_{q,a}^{\alpha} D_q f)(x) = \frac{f(a)}{\Gamma_q(-\alpha)} (x-a)^{(-\alpha-1)} \quad (0 < a < x < b)\ .$$

Proof. We will use formulas (11), Theorem 5, and Lemma 6, to prove the statement. Let us consider two cases. If $\alpha < 0$, then

$$\begin{aligned} (D_q D_{q,a}^{\alpha} f)(x) &= (D_q I_{q,a}^{-\alpha} f)(x) = D_q I_{q,a}^{-\alpha} \big((I_{q,a} D_q f)(x) + f(a)\big) \\ &= (D_q I_{q,a}^{-\alpha} I_{q,a} D_q f)(x) + f(a)(D_q I_{q,a}^{-\alpha} \mathbf{1})(x) \\ &= (D_q I_{q,a}^{-\alpha+1} D_q f)(x) + f(a) D_q \Big(\frac{(x-a)^{(-\alpha)}}{\Gamma_q(-\alpha+1)}\Big) \\ &= (D_q I_{q,a} I_{q,a}^{-\alpha} D_q f)(x) + f(a) \frac{[-\alpha]_q (x-a)^{(-\alpha-1)}}{\Gamma_q(-\alpha+1)} \\ &= (D_{q,a}^{\alpha} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha)} (x-a)^{(-\alpha-1)}. \end{aligned}$$

If $\alpha > 0$, there exists $l \in \mathbb{N}_0$, such that $\alpha \in (l, l+1)$. Then, applying a similar procedure, we get

$$\begin{aligned}(D_q D_{q,a}^{\alpha} f)(x) &= (D_q D_q^{l+1} I_{q,a}^{l+1-\alpha} f)(x) \\ &= D_q^{l+2} I_{q,a}^{l+1-\alpha}\big((I_{q,a} D_q f)(x) + f(a)\big) \\ &= (D_q^{l+1} D_q I_{q,a} I_{q,a}^{l+1-\alpha} D_q f)(x) + \frac{f(a)}{\Gamma_q(l+2-\alpha)} D_q^{l+1}\big((x-a)^{(l+1-\alpha)}\big) \\ &= (D_{q,a}^{\alpha} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha)}(x-a)^{(-\alpha-1)}. \qquad \square\end{aligned}$$

5. THE FRACTIONAL q-TAYLOR-LIKE FORMULA

Many authors tried to generalize the ordinary TAYLOR formula in different manners. The use of the fractional calculus is of special interest in that area (see, for example [**11**] and [**8**]). Here, we will present one more generalization, based on the use of the fractional q-derivatives.

Lemma 9. *Let $f(x)$ be a function defined on an interval $(0,b)$ and $\alpha \in \mathbb{R}^+$. Then the following is valid*:

$$\big(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\big)(x) = f(x) \qquad (0 < a < x < b)\ .$$

Proof. For $\alpha > 0$, we have

$$\begin{aligned}\big(D_{q,a}^{\alpha} I_{q,a}^{\alpha} f\big)(x) &= \big(D_q^{\lceil\alpha\rceil} I_{q,a}^{\lceil\alpha\rceil-\alpha} I_{q,a}^{\alpha} f\big)(x) = \big(D_q^{\lceil\alpha\rceil} I_{q,a}^{\lceil\alpha\rceil-\alpha+\alpha} f\big)(x) \\ &= \big(D_q^{\lceil\alpha\rceil} I_{q,a}^{\lceil\alpha\rceil} f\big)(x) = f(x)\,. \qquad \square\end{aligned}$$

Lemma 10. *Let $\alpha \in (0,1)$. Then*

$$\big(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f\big)(x) = f(x) + K(a)(x-a)^{(\alpha-1)} \qquad (0 < a < x < b)\,,$$

where $K(a)$ does not depend on x.

Proof. Let

$$A(x) = \big(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f\big)(x) - f(x).$$

Applying $D_{q,a}^{\alpha}$ to the both sides of the above expression, and using Lemma 9, we get

$$\begin{aligned}(D_{q,a}^{\alpha} A)(x) &= \big(D_{q,a}^{\alpha} I_{q,a}^{\alpha} D_{q,a}^{\alpha} f\big)(x) - D_{q,a}^{\alpha} f(x) \\ &= \big((D_{q,a}^{\alpha} I_{q,a}^{\alpha}) D_{q,a}^{\alpha} f\big)(x) - D_{q,a}^{\alpha} f(x) = 0\,.\end{aligned}$$

On the other hand, according to Lemma 6, we obtain

$$D_{q,a}^{\alpha}\big((x-a)^{(\alpha-1)}\big) = D_q I_{q,a}^{1-\alpha}\big((x-a)^{(\alpha-1)}\big) = (D_q \mathbf{1})(x) = 0\ .$$

Hence, we conclude that $A(x)$ is a function of the form

$$A(x) = K(a)(x-a)^{(\alpha-1)}\,. \qquad \square$$

Lemma 11. *Let $0 < a \le c < x < b$ and $\alpha \in (0,1)$. Then the following is valid*:

$$(I_{q,c}^{\alpha+k} D_{q,a}^{\alpha+k} f)(x) = \frac{(x-c)^{(\alpha+k)}}{\Gamma_q(\alpha+k+1)} (D_{q,a}^{\alpha+k} f)(c) + (I_{q,c}^{\alpha+k+1} D_{q,a}^{\alpha+k+1} f)(x), \quad (k \in \mathbb{N}_0).$$

Proof. According to Lemma 3 and Lemma 4, we have

$$\begin{aligned}(I_{q,c}^{\alpha+k} D_{q,a}^{\alpha+k} f)(x) &= (I_{q,c}^{\alpha+k+1} D_q D_{q,a}^{\alpha+k} f)(x) + \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} \\ &= \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} + (I_{q,c}^{\alpha+k+1} D_{q,a}^{\alpha+k+1} f)(x)\,. \qquad \square\end{aligned}$$

Now, we are ready to prove a Taylor type formula with fractional q-derivatives, which is the main result of this section.

Theorem 12. *Let $f(x)$ be defined on $(0,b)$ and $\alpha \in (0,1)$. For $0 < a < c < x < b$, the following is true*:

$$f(x) = \sum_{k=0}^{n-1} \frac{(D_{q,a}^{\alpha+k} f)(c)}{\Gamma_q(\alpha+k+1)} (x-c)^{(\alpha+k)} + R_n(f), \tag{20}$$

with $R_n(f) = R_0(f) - K(a)(x-a)^{(\alpha-1)} + E_n(f)$, where

$$R_0(f) = \frac{1}{\Gamma_q(\alpha)} \int_a^c (x-qt)^{(\alpha-1)} (D_{q,a}^{\alpha} f)(t)\, \mathrm{d}_q t\ ,$$

and $E_n(f)$ can be represented in either of the following forms:

$$E_n(f) = (I_{q,c}^{\alpha+n} D_{q,a}^{\alpha+n} f)(x), \tag{21}$$

$$E_n(f) = \frac{(D_{q,a}^{\alpha+n} f)(\xi)}{\Gamma_q(\alpha+n+1)} (x-c)^{(\alpha+n)} \quad (c < \xi < x). \tag{22}$$

Proof. We will deduce the proof of (21) by mathematical induction. Since

$$(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f)(x) = \frac{1}{\Gamma_q(\alpha)} \int_a^c (x-qt)^{(\alpha-1)} (D_{q,a}^{\alpha} f)(t)\, \mathrm{d}_q t + (I_{q,c}^{\alpha} D_{q,a}^{\alpha} f)(x),$$

using Lemma 10, we obtain

$$f(x) = (I_{q,c}^{\alpha} D_{q,a}^{\alpha} f)(x) + R_0(f) - K(a)(x-a)^{(\alpha-1)}.$$

According to Lemma 11, for $k = 0$, we have

$$\begin{aligned}(I^{\alpha}_{q,c} D^{\alpha}_{q,a} f)(x) &= \frac{(D^{\alpha}_{q,a} f)(c)}{\Gamma_q(\alpha+1)}(x-c)^{(\alpha)} + (I^{\alpha+1}_{q,c} D^{\alpha+1}_{q,a} f)(x)\\ &= \frac{(D^{\alpha}_{q,a} f)(c)}{\Gamma_q(\alpha+1)}(x-c)^{(\alpha)} + E_1(f),\end{aligned}$$

which completes the expression for $R_1(f)$ and proves (21) for $n = 1$.
Assume that (21) is valid for any $n \in \mathbb{N}$. Then, again from Lemma 11, the following holds:

$$\begin{aligned}E_n(f) = (I^{\alpha+n}_{q,c} D^{\alpha+n}_{q,a} f)(x) &= \frac{(D^{\alpha+n}_{q,a} f)(c)}{\Gamma_q(\alpha+n+1)}(x-c)^{(\alpha+n)} + (I^{\alpha+n+1}_{q,c} D^{\alpha+n+1}_{q,a} f)(x)\\ &= \frac{(D^{\alpha+n}_{q,a} f)(c)}{\Gamma_q(\alpha+n+1)}(x-c)^{(\alpha+n)} + E_{n+1}(f).\end{aligned}$$

Hence the formula (21) is valid for $n + 1$. So, it is valid for each $n \in \mathbb{N}$.

The second form of remainder, (22), can be obtained by using a mean–value theorem for q–integrals [**9**]. Indeed, there exists $\xi \in (c, x)$, such that

$$\begin{aligned}E_n(f) = (I^{\alpha+n}_{q,c} D^{\alpha+n}_{q,a} f)(x) &= \frac{1}{\Gamma_q(\alpha+n)} \int_c^x (x-qt)^{(\alpha+n-1)} \big(D^{\alpha+n}_{q,a} f\big)(t)\, \mathrm{d}_q t\\ &= \frac{(D^{\alpha+n}_{q,a} f)(\xi)}{\Gamma_q(\alpha+n)} \int_c^x (x-qt)^{(\alpha+n-1)}\, \mathrm{d}_q t = \frac{(D^{\alpha+n}_{q,a} f)(\xi)}{\Gamma_q(\alpha+n)} (I^{\alpha+n}_{q,c} \mathbf{1})(x)\\ &= \frac{\big(D^{\alpha+n}_{q,a} f\big)(\xi)}{\Gamma_q(\alpha+n+1)}(x-c)^{(\alpha+n)}. \qquad \Box\end{aligned}$$

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