

ON SOME GENERALIZATIONS OF NONLINEAR DYNAMIC INEQUALITIES ON TIME SCALES AND THEIR APPLICATIONS

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In this article, by using Young's inequality, we prove some new nonlinear dynamic inequalities of Gronwall-Bellman-Pachpatte type on time scales. These inequalities give us the integral and discrete version, and also extend some known dynamic inequalities in certain papers. We can also say, the inequalities proved in this paper can be used as handy tools in the study of qualitative properties of some dynamic equations on time scales. Some examples are introduced to demonstrate the applications of these inequalities.

1. INTRODUCTION

We can see that nonlinear phenomena can be observed in many areas such as physics, chemistry, biology, and communication engineering. In physics precisely, nonlinearity is present in fluid dynamics, nonlinear optics, plasma physics, communication technology and so on, see for insistence [17, 52]. Physically, as we know, in some situations it is not necessary to obtain the exact solutions of the initial value problems which model the phenomena, but it is enough to have knowledge and information about the qualitative and quantitative properties of the solutions, also on the other hand, in other situations, we can not get the exact solutions for our problem, so the integral inequalities involving functions of one and more than one independent variables, which provide explicit bounds on unknown function play a fundamental role in the development of the theory of dynamic equations and dynamic integral equations, and can be used as handy tools in the study of existence,

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boundedness and other qualitative properties of the solutions of certain differential equations see [2, 4, 6, 8, 18, 19, 21, 33, 34, 48].

We know that Gronwall-Bellman inequality [10, 28], play a considerable role in the study of qualitative properties of the solutions of some certain differential equations (see Theorem 1.1), (e.g., [7, 12, 21, 51]). Many other results on its generalizations may be seen in [1, 3, 27].

Theorem 1.1. *Let Ω be a continuous function defined on the interval $D = [\alpha, \alpha+h]$ and*

$$0 \leq \Omega(t) \leq \int_{\alpha}^t [\delta\Omega(s) + \gamma] ds, \quad \text{for all } t \in D,$$

where α, γ, δ and h are nonnegative constants. Then

$$0 \leq \Omega(t) \leq \gamma h e^{\delta h}, \quad \text{for all } t \in D.$$

Richard Bellman in [10], established the fundamental inequality named after that Gronwall-Bellman's inequality which considered as a generalization for Gronwall's inequality and plays a vital role in studying stability and asymptotic behaviour of solutions of differential and integral equations.

Theorem 1.2. *Let Ω and f be continuous and nonnegative functions defined on $[a, b]$, and let Ω_0 be nonnegative constant. Then the inequality*

$$(1) \quad \Omega(t) \leq \Omega_0 + \int_a^t f(s)\Omega(s)ds, \quad \text{for all } t \in [a, b],$$

implies that

$$\Omega(t) \leq \Omega_0 \exp\left(\int_a^t f(s)ds\right), \quad \text{for all } t \in [a, b].$$

Richard Bellman in [11] proved also the following variant of the inequality (1).

Theorem 1.3. *Let Ω and f be continuous and nonnegative functions defined on $[a, b]$, and let n be a continuous, positive and nondecreasing function defined on $[a, b]$, then*

$$(2) \quad \Omega(t) \leq n(t) + \int_a^t f(s)\Omega(s)ds, \quad \text{for all } t \in [a, b],$$

implies that

$$\Omega(t) \leq n(t) \exp\left(\int_a^t f(s)ds\right), \quad \text{for all } t \in [a, b].$$

Pachpatte [44] studied the following variant of the inequality (2) in getting various generalizations of Bellman's inequality (1).

$$\Omega(t) \leq p(t) + q(t) \int_a^t [f(s)\Omega(s) + g(s)]ds, \quad \text{for all } t \in [a, b].$$

The discrete version of (2) studied by Baburao G. Pachpatte [46]. In particular, he proved that: If $\Omega(n)$, $a(n)$, $\gamma(n)$ are nonnegative sequences defined for $n \in \mathbb{N}_0$ and $a(n)$ is non-decreasing for $n \in \mathbb{N}_0$, and if

$$(3) \quad \Omega(n) \leq a(n) + \sum_{s=0}^{n-1} \gamma(s)\Omega(s), n \in \mathbb{N}_0,$$

then

$$\Omega(n) \leq a(n) \prod_{s=0}^{n-1} [1 + \gamma(s)], n \in \mathbb{N}_0.$$

Many results on its generalizations can be noticed in some examples, such as in [2, 3, 4]. For instance, Bihari's [12] in 1956, extended (1) to the nonlinear inequality.

$$(4) \quad \Omega(t) \leq \Omega_0 + \int_0^t \gamma(s)\omega(\Omega(s))ds.$$

In 1957, Ou-Iang [42] proved that: If Ω , γ are nonnegative continuous functions on \mathbb{R}_+ , $\Omega_0 \geq 0$ is a constant and

$$(5) \quad \Omega^2(t) \leq \Omega_0^2 + 2 \int_0^t \gamma(s)\Omega(s)ds, \quad t \in \mathbb{R}_+,$$

then

$$\Omega(t) \leq \Omega_0 + \int_0^t \gamma(s)ds, \quad t \in \mathbb{R}_+.$$

In 1995, Pachpatte [43] studied the following generalization of Ou-Iang inequality (5) as follows: If Ω , γ and ε are nonnegative continuous functions defined on \mathbb{R}_+ and c is a nonnegative constant, then

$$(6) \quad \Omega^2(t) \leq c^2 + 2 \int_0^t [\gamma(s)\Omega^2(s) + \varepsilon(s)\Omega(s)]ds,$$

for $t \in \mathbb{R}_+$, implies

$$\Omega(t) \leq \varepsilon(t) \exp\left(\int_0^t \gamma(s)ds\right),$$

for $t \in \mathbb{R}_+$, where

$$\varepsilon(t) = c + \int_0^t \varepsilon(s)ds, \quad \forall t \in \mathbb{R}_+.$$

Related to the inequality (5), in 2000, Pachpatte [45] introduced the following result:

$$(7) \quad \Omega^p(t) \leq a(t) + b(t) \int_0^t [\varepsilon(s)\Omega^p(s) + \epsilon(s)\Omega(s)]ds, \quad \text{for all } t \in \mathbb{R}_+,$$

where Ω , a , b , ε , ϵ , are real-valued nonnegative continuous functions defined on \mathbb{R}_+ , and $p > 1$ is a real constant.

Further, in 2000, Lipovan [40] studied the retarded case of the inequality (4) by replacing t by the delay function $\hbar(t) \leq t$ where $\hbar(t_0) = 0$:

$$(8) \quad \Omega(t) \leq \Omega_0 + \int_{\hbar(t_0)}^{\hbar(t)} f(s)\omega(\Omega(s))ds, \quad t_0 \leq t \leq t_1.$$

In 2005, Ravi Agarwal et al. generalized (8) to the general form:

$$(9) \quad \Omega(t) \leq a(t) + \sum_{i=1}^n \int_{\hbar_i(t_0)}^{\hbar_i(t)} f_i(t, s)\omega_i(\Omega(s))ds, \quad t_0 \leq t \leq t_1,$$

where a is a function and ω_i 's may be distinct.

In 2014, Hassan El-Owaidy, Abdeldaim and El-Deeb [1] proved the following new form:

$$(10) \quad \Omega(t) \leq \gamma(t) + \int_a^{\hbar_1(t)} \epsilon(s)w_1(\Omega(s))ds + \int_a^{\hbar_2(t)} \varepsilon(s)w_2(\Omega(s))ds, \text{ for all } t \in I_1 = [a, b].$$

Throughout the same paper, the authors also demonstrated the following inequality:

$$\Omega(t) \leq \gamma(t) + \int_a^{\hbar(t)} \epsilon(s)w(\Omega(s))ds + \int_a^{\hbar(t)} k(t, s)w(\Omega(s))ds, \quad \text{for all } t \in I_1,$$

where Ω , ϵ , $\varepsilon \in \mathcal{C}(I_1, \mathbb{R}_+)$, \hbar , $f \in \mathcal{C}^1(I_1, I_1)$ are non-decreasing functions, with $\hbar_i(t) \leq t$, $\hbar_i(a) = a$, $\hbar_i'(t) \geq 0$, $i = 1, 2$, and $w_i \in (\mathbb{R}_+, \mathbb{R}_+)$ non-decreasing function, and $k(t, s) \in \mathcal{C}(I_1 \times I_1, \mathbb{R}_+)$ with $\frac{\partial k}{\partial t}(t, s) \in \mathcal{C}(I_1 \times I_1, \mathbb{R}_+)$.

In 2015, Abdeldaim and El-Deeb [3] discussed the new form:

$$\Omega(t) \leq \Omega_0 + \int_0^{\hbar(t)} \gamma(s)\varphi(\Omega(s)) \left[\varphi(\Omega(s)) + \int_0^s \epsilon(\lambda)\varphi(\Omega(\lambda))d\lambda \right] ds,$$

for all $t \in \mathbb{R}_+$, where γ and $\epsilon \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and φ , φ' , $\hbar \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ are increasing functions, with $\varphi'(t) \leq k$, $\varphi > 0$, $\hbar(t) \leq t$, $\hbar(0) = 0$, for all $t \in \mathbb{R}_+$, k , u_0 are positive constants.

In the same paper, by using the composite function, Abdeldaim and El-Deeb [3] introduced a new inequality with a different kernel:

$$\varphi_1(\Omega(t)) \leq \Omega_0 + \int_0^{\hbar(t)} \gamma(s)\varphi_2(\Omega(s)) \left[\Omega(s) + \int_0^s \epsilon(\lambda)\varphi_1(\Omega(\lambda))d\lambda \right]^p ds,$$

for all $t \in \mathbb{R}_+$ where $\varphi_1, \varphi_2, \hbar \in \mathcal{C}^1(\mathbb{R}_+, \mathbb{R}_+)$ are increasing functions with $\hbar(t) \leq t$, $\varphi_i(t) > 0$, $i = 1, 2$, $\hbar(0) = 0$ and $\varphi_1'(t) = \varphi_2(t)$, $p > 1$ and Ω_0 is a positive constant.

In [2], one of the new generalizations of Gronwall-type inequality has been proved by Abdeldaim and El-Deeb, as follows:

$$\varphi_1(\Omega(t)) \leq \Omega_0 + \int_0^{\hbar(t)} \epsilon(s)\varphi_1(\Omega(s))ds + \int_0^{\hbar(t)} \epsilon(s)\varphi_2(\Omega(s))ds, \quad \text{for all } t \in I,$$

with $\hbar(t) \leq t$, $\varphi_i(t) > 0$, $i = 1, 2$, $\hbar(0) = 0$, $\varphi_1'(t) = \varphi_2(t)$, and $\varphi_1^{-1}(t)$ is a submultiplicative function and u_0 is a positive constant.

After a while, in 2014, Kender et al. [32] established the following inequality:

$$(11) \quad \Omega^p(t) \leq c(t) + \int_a^t \gamma(s)\Omega(s)ds + \int_a^b \epsilon(s)\Omega^p(s)ds, \quad \text{for all } t \in [a, b] \subseteq \mathbb{R}.$$

Recently, in 2017, El-Deeb and Ahmed [20] studied the inequality (11) with retardation $\hbar(t) \leq t$, as follows:

$$\Omega^p(t) \leq c(t) + \int_a^{\hbar(t)} \gamma(s)\Omega(s)ds + \int_a^b \epsilon(s)\Omega^p(s)ds.$$

Lately, in 2019, Zizun Li and Wu-Sheng Wang [39] established the following inequality:

$$\Omega(t) \leq a(t) + \int_{t_0}^{\hbar(t)} \gamma(s) \left[\Omega^m(s) + \int_{t_0}^s \epsilon(\tau)\Omega^n(\tau)d\tau \right]^p ds,$$

where $\Omega, a, \gamma, \epsilon \in \mathcal{C}(\mathbb{R}_+, \mathbb{R}_+)$ and \hbar is a continuous, differentiable and increasing function on $[t_0, +\infty)$ with $\hbar(t) \leq t$, $\hbar(t_0) = 0$ and $p, m, n \in (0, 1]$, are positive constants.

We know that the dynamic inequalities play a very important role in the development of the qualitative theory of dynamic equations on time scales. Also the dynamic inequalities play a very important role in the study of the oscillation of the dynamic equations see [29, 47]. The study of dynamic equations on time scales which goes back to its discoverer Stefan Hilger [30] becomes an area of mathematics and recently has received a lot of attention. The general idea is to prove a result for a dynamic equation or a dynamic inequality where the domain of the unknown function is a so called time scale \mathbb{T} , which may be an arbitrary closed subset of the real numbers \mathbb{R} see [13, 15]. The purpose of the theory of time scales is to unify continuous and discrete analysis. The three most popular examples of calculus on time scales are differential calculus, difference calculus, and quantum calculus (see [31]), i.e, when $\mathbb{T} = \mathbb{R}$, $\mathbb{T} = \mathbb{N}$ and $\mathbb{T} = \overline{q^{\mathbb{Z}}} = \{q^k : k \in \mathbb{Z}\} \cup \{0\}$, $q > 1$. The book on the subject of time scales by Bohner and Peterson [14]

summarizes and organizes much of time scale calculus. During the past decade a number of dynamic inequalities have been established by some authors which are motivated by some applications, for example, when studying the behavior of solutions of certain class of dynamic equations on a time scale \mathbb{T} . We refer the reader to [5, 8, 9, 14, 16, 22, 23, 24, 25, 26, 35, 36, 37, 38, 48, 49, 50] for contributions, and the references cited therein.

In [14], Bohner and Peterson introduced a dynamic inequality on a time scale \mathbb{T} which unifies the continuous version inequality (1) and the discrete version inequality (3) as follows: If Ω, ζ are right dense continuous functions and $\gamma \geq 0$ is regressive and right dense continuous functions, then

$$\Omega(t) \leq \zeta(t) + \int_{t_0}^t \Omega(\eta)\gamma(\eta)\Delta\eta, \quad \text{for all } t \in \mathbb{T},$$

implies

$$\Omega(t) \leq \zeta(t) + \int_{t_0}^t e_\gamma(t, \sigma(\eta))\zeta(\eta)\gamma(\eta)\Delta\eta, \quad \text{for all } t \in \mathbb{T},$$

In this paper, we investigate some Gronwall type inequalities on time scales, which extend the results in [45] and also establish a slight generalization of the celebrated Gronwall-Bellman type inequalities on time scales, which can be used more effectively in the study of the qualitative behavior of the solutions of certain classes of nonlinear dynamic equations. Some applications of some of our results are also introduced to illustrate the benefits of this work. Our main results will be proved by employing some useful inequalities which will be presented in the following section. The paper is organized as in the following: In Section 2, some basic concepts of the calculus on time scales and useful lemmas are introduced. In Section 3, we state and prove the main results. In Section 4, we present several applications to study some qualitative properties of the solutions of certain nonlinear dynamic equations. In Section 5, conclusion of our paper.

2. PRELIMINARIES ON TIME SCALES

In this section, we introduce some background on time scales. We assume throughout that \mathbb{R} denotes the set of real numbers, $\mathbb{R}_+ = [0, \infty)$ and \mathbb{Z} denotes the set of integers. \mathbb{T} has the topology that it inherits from the standard topology on the real numbers \mathbb{R} , and $\mathbb{T}_0 = [t_0, \infty) \cap \mathbb{T}$. First we define the forward jump operator $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$(12) \quad \sigma(t) := \inf\{s \in \mathbb{T} : s > t\},$$

and second, the backward jump operator $\rho : \mathbb{T} \rightarrow \mathbb{T}$ by

$$(13) \quad \rho(t) := \sup\{s \in \mathbb{T} : s < t\}.$$

In this definition, we put $\inf \emptyset = \sup \mathbb{T}$ and $\sup \emptyset = \inf \mathbb{T}$, where \emptyset is the empty set. A point $t \in \mathbb{T}$ with $\inf \mathbb{T} < t < \sup \mathbb{T}$, is said to be left-dense if $\rho(t) = t$ and is right-dense if $\sigma(t) = t$, points that are simultaneously right-dense and left-dense are said to be dense, is left-scattered if $\rho(t) < t$ and right-scattered if $\sigma(t) > t$, points that are simultaneously right-scattered and left-scattered are said to be isolated. A function $g : \mathbb{T} \rightarrow \mathbb{R}$ is said to be right-dense continuous (rd-continuous) provided g is continuous at right-dense points and at left-dense points in \mathbb{T} , left-sided limits exist and are finite. The set of all such rd-continuous functions is denoted by $C_{rd}(\mathbb{T})$. A function $f : \mathbb{T} \rightarrow \mathbb{R}$ is said to be left-dense continuous (ld-continuous) provided f is continuous at left-dense points and at right-dense points in \mathbb{T} , right-sided limits exist and are finite. The set of all such ld-continuous functions is denoted by $C_{ld}(\mathbb{T})$.

The forward and backward graininess functions μ and ν for a time scale \mathbb{T} is defined by $\mu(t) := \sigma(t) - t$, and $\nu(t) := t - \rho(t)$, respectively.

Given a time scale \mathbb{T} , we introduce the sets \mathbb{T}^κ , \mathbb{T}_κ , and \mathbb{T}_κ^k as follows. If \mathbb{T} has a left-scattered maximum t_1 , then $\mathbb{T}^\kappa = \mathbb{T} - \{t_1\}$, otherwise $\mathbb{T}^\kappa = \mathbb{T}$. If \mathbb{T} has a right-scattered minimum t_2 , then $\mathbb{T}_\kappa = \mathbb{T} - \{t_2\}$, otherwise $\mathbb{T}_\kappa = \mathbb{T}$. Finally, $\mathbb{T}_\kappa^k = \mathbb{T}^\kappa \cap \mathbb{T}_\kappa$.

Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a real-valued function on a time scale \mathbb{T} . Then for all $t \in \mathbb{T}_\kappa$, we define $f^\Delta(t)$ to be the number (if it exists) with the property that given any $\varepsilon > 0$ there is a neighborhood U of t such that

$$|[f(\sigma(t)) - f(s)] - f^\Delta(t)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

For $f : \mathbb{T} \rightarrow \mathbb{R}$, we define the function $f^\sigma : \mathbb{T} \rightarrow \mathbb{R}$ by $f^\sigma(t) = f(\sigma(t))$ for all $t \in \mathbb{T}$, that is, $f^\sigma = f \circ \sigma$. Similarly, we define the function $f^\rho : \mathbb{T} \rightarrow \mathbb{R}$ by $f^\rho(t) = f(\rho(t))$ for all $t \in \mathbb{T}$, that is, $f^\rho = f \circ \rho$. A time scale \mathbb{T} is said to be regular if the following two conditions are satisfied simultaneously: (1) $\sigma(\rho(t)) = t$ and (2) $\rho(\sigma(t)) = t$, $\forall t \in \mathbb{T}$. The product and quotient rules for the derivative of the product fg and the quotient f/g (where $g g^\sigma \neq 0$, here $g^\sigma = g \circ \sigma$) of two differentiable functions f and g , are given as the following:

$$(fg)^\Delta(t) = f^\Delta(t)g(t) + f(\sigma(t))g^\Delta(t) = f(t)g^\Delta(t) + f^\Delta(t)g(\sigma(t)),$$

and

$$\left(\frac{f}{g}\right)^\Delta(t) = \frac{f^\Delta(t)g(t) - f(t)g^\Delta(t)}{g(t)g(\sigma(t))}.$$

A function $F : \mathbb{T} \rightarrow \mathbb{R}$ is called a delta antiderivative of $f : \mathbb{T} \rightarrow \mathbb{R}$ provided that $F^\Delta(t) = f(t)$ holds for all $t \in \mathbb{T}_\kappa$, and the delta integral of f is defined by

$$\int_a^b f(t)\Delta t = F(b) - F(a).$$

We will frequently use the following useful relations between calculus on time scales \mathbb{T} and differential calculus on \mathbb{R} , difference calculus on \mathbb{Z} , and quantum calculus on $q^\mathbb{Z}$. Note that if

(i) $\mathbb{T} = \mathbb{R}$, then

$$(14) \quad \sigma(t) = t, \quad \mu(t) = 0, \quad f^\Delta(t) = f'(t), \quad \int_a^b f(t)\Delta t = \int_a^b f(t)dt;$$

(ii) if $\mathbb{T} = \mathbb{Z}$, then

$$(15) \quad \sigma(t) = t + 1, \quad \mu(t) = 1, \quad f^\Delta(t) = \Delta f(t), \quad \int_a^b f(t)\Delta t = \sum_{t=a}^{b-1} f(t).$$

It can be shown (see [14]) that if $g \in C_{rd}(\mathbb{T})$, then the Cauchy integral $G(t) := \int_{t_0}^t g(s)\Delta s$ exists, $t_0 \in \mathbb{T}$, and satisfies $G^\Delta(t) = g(t)$, $t \in \mathbb{T}$. An infinite integral is defined as

$$\int_a^\infty f(t)\Delta t = \lim_{b \rightarrow \infty} \int_a^b f(t)\Delta t.$$

Now, we will give the definition of the generalized exponential function and its derivatives. We say that $p : \mathbb{T} \rightarrow \mathbb{R}$ is regressive provided $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^\kappa$, we define the set \mathfrak{R} of all regressive and rd-continuous functions. We define the set \mathfrak{R}^+ of all positively regressive elements of \mathfrak{R} by $\mathfrak{R}^+ = \{p \in \mathfrak{R} : 1 + \mu(t)p(t) > 0, \forall t \in \mathbb{T}\}$. The set of all regressive functions on a time scale \mathbb{T} forms an Abelian group under the addition \oplus defined by $p \oplus q = p + q + \mu pq$. If $p \in \mathfrak{R}$, then we can define the exponential function by

$$e_p(t, s) = \exp\left(\int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau\right), \quad s, t \in \mathbb{T},$$

where $\xi_h(z)$ is the cylinder transformation, which is defined by

$$\xi_h(z) = \begin{cases} \frac{\text{Log}(1+hz)}{h}, & h \neq 0, \\ z, & h = 0. \end{cases}$$

3. MAIN RESULTS

In this section, we will state and prove the main results and investigate some new nonlinear dynamic inequalities of Gronwall-Bellman type on time scale.

Now, before we state and prove the main results we introduce the basic lemmas that will be needed in the proof of the main results: If $p \in \mathfrak{R}$, then $e_p(t, s)$ is real-valued and nonzero on \mathbb{T} .

Lemma 3.1 (See [34]). *If $p \in \mathfrak{R}$ and fix $t \in \mathbb{T}$, then the exponential function $e_p(t, t_0)$ is the unique solution of the following initial value problem:*

$$(16) \quad \begin{cases} y^\Delta(t) = p(t)y(t), \\ y(t_0) = 1. \end{cases}$$

Lemma 3.2 ([34]). *If $p, q \in \mathfrak{R}$ and $a, b, c \in \mathbb{T}$, then*

1. $e_p(t, t) = 1$ and $e_0(t, s) = 1$;
2. $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
3. if $p \in \mathfrak{R}^+$, then $e_p(t, t_0) > 0$, for all $t \in \mathbb{T}$;
4. $\int_a^b p(t)e_p(c, \sigma(t))\Delta t = -\int_a^b [e_p(c, \cdot)]^\Delta(t)\Delta t = e_p(c, a) - e_p(c, b)$.

Note that

- if $\mathbb{T} = \mathbb{R}$, then

$$(17) \quad e_a(t, t_0) = \exp\left(\int_{t_0}^t a(s)ds\right);$$

- if $\mathbb{T} = \mathbb{Z}$, then

$$(18) \quad e_a(t, t_0) = \prod_{s=t_0}^{t-1} (1 + a(s)).$$

Lemma 3.3 (See [34]). *Let $t_0 \in \mathbb{T}^\kappa$ and $k : \mathbb{T} \times \mathbb{T}^\kappa \rightarrow \mathbb{R}$ be continuous at (t, t) , where $t > t_0$ and $t \in \mathbb{T}^\kappa$. Assume that $k^\Delta(t, \cdot)$ is rd-continuous on $[t_0, \sigma(t)]$. If for any $\varepsilon > 0$, there exists a neighborhood U of t , independent of $\tau \in [t_0, \sigma(t)]$, such that*

$$|[k(\sigma(t), \tau) - k(s, \tau)] - k^\Delta(t, \tau)[\sigma(t) - s]| \leq \varepsilon|\sigma(t) - s|, \quad \text{for all } s \in U.$$

If k^Δ denotes the derivative of k with respect to the first variable, then

$$f(t) = \int_{t_0}^t k(t, \tau)\Delta\tau,$$

yields

$$f^\Delta(t) = \int_{t_0}^t k^\Delta(t, \tau)\Delta\tau + k(\sigma(t), t).$$

Lemma 3.4 ([34]). *Suppose $u, b \in C_{rd}$ and $a \in \mathfrak{R}^+$. Then*

$$u^\Delta(t) \leq a(t)u(t) + b(t), \quad t \geq t_0, t \in \mathbb{T}^\kappa,$$

yields

$$u(t) \leq u(t_0)e_a(t, t_0) + \int_{t_0}^t e_a(t, \sigma(\tau))b(\tau)\Delta\tau, \quad t \geq t_0, t \in \mathbb{T}^\kappa.$$

Lemma 3.5 ([41], Young's Inequality). *If $x \geq 0, y \geq 0$ and $\frac{1}{p} + \frac{1}{q} = 1$ with $p > 1$, then*

$$(19) \quad x^{\frac{1}{p}}y^{\frac{1}{q}} \leq \frac{x}{p} + \frac{y}{q}.$$

Now, we are ready to state and prove the main results in this paper.

Theorem 3.4. *Let $\omega, m, g, f, n \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$, and $k(t, s), k^\Delta(t, s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}^+)$, $p > 1$ be a constant. If*

$$(20) \quad \omega^p(t) \leq m(t) + n(t) \int_{t_0}^t \left[g(s)\omega^p(s) + f(s)\omega(s) + k(t, s) \right] \Delta s,$$

for all $t \in \mathbb{T}_0$, then

$$(21) \quad \omega(t) \leq \left[m(t) + n(t) \int_{t_0}^t e_\zeta(t, \sigma(s)) \xi(s) \Delta s \right]^{1/p},$$

for all $t \in \mathbb{T}_0$, where ζ and ξ are defined as follows:

$$(22) \quad \zeta(t) = n(t) \left(g(t) + \frac{f(t)}{p} \right),$$

and

$$(23) \quad \xi(t) = g(t)m(t) + f(t) \left(\frac{p-1}{p} + \frac{m(t)}{p} \right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s) \Delta s.$$

Proof. We define the function v by:

$$(24) \quad v(t) = \int_{t_0}^t \left[g(s)\omega^p(s) + f(s)\omega(s) + k(t, s) \right] \Delta s, \quad \text{for all } t \in \mathbb{T}_0.$$

That $v(t) \geq 0$ nondecreasing with $v(t_0) = 0$. Then we can write (20) as the following:

$$(25) \quad \omega^p(t) \leq m(t) + n(t)v(t), \quad \text{for all } t \in \mathbb{T}_0.$$

From (25) and using Lemma 3.5, we get

$$(26) \quad \begin{aligned} \omega(t) &\leq \left(m(t) + n(t)v(t) \right)^{1/p} \left(1 \right)^{(p-1)/p} \\ &\leq \frac{p-1}{p} + \frac{m(t)}{p} + \frac{n(t)}{p} v(t), \quad \text{for all } t \in \mathbb{R}_+. \end{aligned}$$

By differentiating (24) and using Lemma 3.3. From (25) and (26), we obtain

$$(27) \quad \begin{aligned} v^\Delta(t) &= g(t)\omega^p(t) + f(t)\omega(t) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s) \Delta s \\ &\leq n(t) \left(g(t) + \frac{f(t)}{p} \right) v(t) \\ &\quad + \left[g(t)m(t) + f(t) \left(\frac{p-1}{p} + \frac{m(t)}{p} \right) + k(\sigma(t), t) \right. \\ &\quad \left. + \int_{t_0}^t k^\Delta(t, s) \Delta s \right] \\ &= \zeta(t)v(t) + \xi(t), \end{aligned}$$

for all $t \in \mathbb{T}_0$, where ζ and ξ are defined as in (22) and (23) respectively. Now a suitable application of Lemma 3.4 to (27) with $v(t_0) = 0$, yields

$$(28) \quad v(t) \leq \int_{t_0}^t e_{\zeta}(t, \sigma(s)) \xi(s) \Delta s.$$

We get the required inequality (78) from (28) and (25). This completes the proof. \square

Remark 3.6. If we put $\mathbb{T} = \mathbb{R}$, and $k(t, s) = 0$ in Theorem 3.4, and using the relations (14) and (17), then we obtain [45, Theorem 1 part (a₁)].

Remark 3.7. If we put $\mathbb{T} = \mathbb{Z}$, and $k(t, s) = 0$ in Theorem 3.4, and using the relations (15) and (18), then we can get the discrete analogue [45, Theorem 3 part (c₁)].

Corollary 3.8. *In Theorem 3.4. Let $c \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ be a nondecreasing function. If*

$$(29) \quad \omega^p(t) \leq c^p(t) + n(t) \int_{t_0}^t \left[g(s) \omega^p(s) + f(s) \omega(s) + k(t, s) \right] \Delta s,$$

for all $t \in \mathbb{T}_0$, then

$$(30) \quad \omega(t) \leq c(t) \left[1 + n(t) \int_{t_0}^t e_{\zeta_1}(t, \sigma(s)) \xi_1(s) \Delta s \right]^{1/p},$$

for all $t \in \mathbb{T}_0$, where ζ_1 and ξ_1 are defined as follows:

$$(31) \quad \zeta_1(t) = n(t) \left(g(t) + \frac{c^{1-p}(t) f(t)}{p} \right),$$

and

$$(32) \quad \xi_1(t) = g(t) + c^{1-p}(t) f(t) + c^{-p}(t) k(\sigma(t), t) + \int_{t_0}^t c^{-p}(s) k^\Delta(t, s) \Delta s.$$

Proof. Since $c \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ is nondecreasing function, then from (29) we observe that

$$(33) \quad \left(\frac{\omega(t)}{c(t)} \right)^p \leq 1 + n(t) \int_{t_0}^t \left[g(s) \left(\frac{\omega(s)}{c(s)} \right)^p + f(s) c^{1-p}(s) \left(\frac{\omega(s)}{c(s)} \right) + c^{-p}(s) k(t, s) \right] \Delta s,$$

for all $t \in \mathbb{T}_0$. By using the inequality which proved in Theorem 3.4, then we get the required inequality in (30). This completes the proof. \square

Remark 3.9. If we put $\mathbb{T} = \mathbb{R}$, and $k(t, s) = 0$ in Corollary 3.8, and using the relations (14) and (17), then we obtain [45, Theorem 1 part (a₂)].

Remark 3.10. If we put $\mathbb{T} = \mathbb{Z}$, and $k(t, s) = 0$ in Corollary 3.8, and using the relations (15) and (18), then we can get the discrete analogue [45, Theorem 3 part (c₂)].

Theorem 3.5. *Let $\omega, m, g, f, n \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$, and $k(t, s), k^\Delta(t, s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}^+)$, $p > 1$ be a constant. If*

$$(34) \quad \omega^p(t) \leq m(t) + n(t) \int_{t_0}^t k(t, s) \left[g(s)\omega^p(s) + f(s)\omega(s) + h(s) \right] \Delta s,$$

for all $t \in \mathbb{T}_0$, then

$$(35) \quad \omega(t) \leq \left[m(t) + n(t) \int_{t_0}^t e_{\zeta_2}(t, \sigma(s)) \xi_2(s) \Delta s \right]^{1/p},$$

for all $t \in \mathbb{T}_0$, where ζ_2 and ξ_2 are defined as follows:

$$(36) \quad \begin{aligned} \zeta_2(t) &= k(\sigma(t), t) \left[n(t) \left[g(t) + \frac{1}{p} f(t) \right] \right. \\ &\quad \left. + \int_{t_0}^t k^\Delta(t, s) n(s) \left(g(s) + \frac{1}{p} f(s) \right) \Delta s, \right. \end{aligned}$$

and

$$(37) \quad \begin{aligned} \xi_2(t) &= k(\sigma(t), t) \left[g(t)m(t) + f(t) \left(\frac{p-1}{p} \right. \right. \\ &\quad \left. \left. + \frac{m(t)}{p} \right) + h(t) \right] \\ &\quad + \int_{t_0}^t k^\Delta(t, s) \left[g(s)m(s) + f(s) \left(\frac{p-1}{p} + \frac{m(s)}{p} \right) + h(s) \right] \Delta s, \end{aligned}$$

for all $t \in \mathbb{T}_0$.

Proof. We define the function v_1 by the following form

$$(38) \quad v_1(t) = \int_{t_0}^t k(t, s) \left[g(s)\omega^p(s) + f(s)\omega(s) + h(s) \right] \Delta s,$$

for all $t \in \mathbb{T}_0$, with $v_1(t_0) = 0$, that is $v_1(t) \geq 0$. Then as in the proof of Theorem 3.4, from (34). We see that the inequalities (25) and (26) hold for v_1 . By differentiating (38) and by using Lemma 3.3, we deduce

$$(39) \quad \begin{aligned} v_1^\Delta(t) &= k(\sigma(t), t) \left[g(t)\omega^p(t) + f(t)\omega(t) + h(t) \right] \\ &\quad + \int_{t_0}^t k^\Delta(t, s) \left[g(s)\omega^p(s) + f(s)\omega(s) + h(s) \right] \Delta s, \text{ for all } t \in \mathbb{T}_0. \end{aligned}$$

From Lemma 3.5 and using (25), (26) in (39), we get

$$\begin{aligned}
 v_1^\Delta(t) &\leq k(\sigma(t), t) \left[g(t) \left(m(t) + n(t)v_1(t) \right) \right. \\
 &\quad \left. + f(t) \left(\frac{p-1}{p} + \frac{m(t)}{p} + \frac{n(t)}{p}v_1(t) \right) + h(t) \right] \\
 &\quad + \int_{t_0}^t k^\Delta(t, s) \left[g(s) \left(m(s) + n(s)v_1(s) \right) \right. \\
 (40) \quad &\quad \left. + f(s) \left(\frac{p-1}{p} + \frac{m(s)}{p} + \frac{n(s)}{p}v_1(s) \right) + h(s) \right] \Delta s, \text{ for all } t \in \mathbb{T}_0.
 \end{aligned}$$

The inequality (40) can be written as follows:

$$\begin{aligned}
 v_1^\Delta(t) &\leq \left[k(\sigma(t), t) \left[n(t) \left[g(t) + \frac{1}{p}f(t) \right] \right. \right. \\
 &\quad \left. + \int_{t_0}^t k^\Delta(t, s) n(s) \left(g(s) + \frac{f(s)}{p} \right) \Delta s \right] v_1(t) \\
 &\quad + k(\sigma(t), t) \left[g(t)m(t) + f(t) \left(\frac{p-1}{p} + \frac{m(t)}{p} \right) \right. \\
 &\quad \left. + h(t) \right] \\
 &\quad + \int_{t_0}^t k^\Delta(t, s) \left[g(s)m(s) + f(s) \left(\frac{p-1}{p} + \frac{m(s)}{p} \right) + h(s) \right] \Delta s, \\
 (41) \quad &= \zeta_2(t)v_1(t) + \xi_2(t), \text{ for all } t \in \mathbb{T}_0.
 \end{aligned}$$

For all $t \in \mathbb{T}_0$, where ζ_2 and ξ_2 are defined in (36) and (37) respectively. From the inequality (41) and by using Lemma 3.4 with $v_1(t_0) = 0$, we get

$$(42) \quad v_1(t) \leq \int_{t_0}^t e_{\zeta_2}(t, \sigma(s)) \xi_2(s) \Delta s, \text{ for all } t \in \mathbb{T}_0.$$

By using (42) in $\omega^p(t) \leq m(t) + n(t)v_1(t)$, we get the required inequality in (35). This completes the proof. \square

Remark 3.11. If we put $\mathbb{T} = \mathbb{R}$, and $h(s) = 0$ in Theorem 3.5, and using the relations (14) and (17), then we obtain [45, Theorem 1 part (a₃)].

Remark 3.12. If we put $\mathbb{T} = \mathbb{Z}$, and $h(s) = 0$ in Theorem 3.5, and using the relations (15) and (18), then we can get the discrete analogue [45, Theorem 3 part (c₃)].

Corollary 3.13. *In Theorem 3.5. Let $c \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ be a nondecreasing function. If*

$$(43) \quad \omega^p(t) \leq c^p(t) + n(t) \int_{t_0}^t k(t, s) \left[g(s)\omega^p(s) + f(s)\omega(s) + h(s) \right] \Delta s,$$

for all $t \in \mathbb{T}_0$, then

$$(44) \quad \omega(t) \leq c(t) \left[1 + n(t) \int_{t_0}^t e_{\zeta_3}(t, \sigma(s)) \xi_3(s) \Delta s \right]^{1/p},$$

for all $t \in \mathbb{T}_0$, where ζ_3 and ξ_3 are defined as follows:

$$\begin{aligned} \zeta_3(t) &= k(\sigma(t), t) \left[n(t) \left(g(t) + \frac{c^{1-p}(t)f(t)}{p} \right) \right] \\ &\quad + \int_{t_0}^t k^\Delta(t, s) n(s) \left[g(s) + \frac{c^{1-p}(s)f(s)}{p} \right] \Delta s, \end{aligned}$$

and

$$\begin{aligned} \xi_3(t) &= k(\sigma(t), t) \left[g(t) + c^{1-p}(t)f(t) + h(t) \right] \\ &\quad + \int_{t_0}^t k^\Delta(t, s) \left[g(s) + c^{1-p}(s)f(s) + c^{-p}(s)h(s) \right] \Delta s, \end{aligned}$$

for all $t \in \mathbb{T}_0$.

Proof. Since $c \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ is nondecreasing function, then from (29), we observe that

$$\left(\frac{\omega(t)}{c(t)} \right)^p \leq 1 + n(t) \int_0^t k(t, s) \left[g(s) \left(\frac{\omega(t)}{c(t)} \right)^p + f(s) c^{1-p}(s) \left(\frac{\omega(t)}{c(t)} \right) + c^{-p}(s) h(s) \right] \Delta s,$$

for all $t \in \mathbb{T}_0$. By applying the inequality given in Theorem 3.5 implies the desired result in (44). This completes the proof. \square

Theorem 3.6. Let $\omega, m, f, n \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$, $k(t, s), k^\Delta(t, s) \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$ and $p > 1$ be a constant and $h \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+, \mathbb{R}^+)$, such that

$$(45) \quad 0 \leq h(t, a) - h(t, b) \leq \delta(t, b)(a - b), \text{ for all } t \in \mathbb{T}_0,$$

and $a \geq 0, b \geq 0$, where $\delta \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+, \mathbb{R}^+)$. If

$$(46) \quad \omega^p(t) \leq m(t) + n(t) \int_{t_0}^t \left[h(s, \omega(s)) + k(t, s) \right] \Delta s,$$

for all $t \in \mathbb{T}_0$, then

$$(47) \quad \omega(t) \leq \left[m(t) + n(t) \int_{t_0}^t e_{\zeta_3}(t, \sigma(s)) \xi_3(s) \Delta s \right]^{1/p},$$

for all $t \in \mathbb{T}_0$, where ζ_3 and ξ_3 are defined as follows:

$$(48) \quad \zeta_3(t) = \delta \left(t, \frac{p-1}{p} + \frac{m(t)}{p} \right) \frac{n(t)}{p},$$

and

$$(49) \quad \xi_3(t) = \alpha^\Delta(t)h\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s)\Delta s,$$

for all $t \in \mathbb{T}_0$.

Proof. Define a function v_2 by

$$(50) \quad v_2(t) = \int_0^t \left(h(s, \omega(s)) + k(t, s) \right) \Delta s, \text{ for all } t \in \mathbb{T}_0.$$

Then as in the proof of previous theorem from (46) we see that the inequalities (25) and (26) hold for v_2 . From (50), (26) and the hypothesis (45). By using Lemma 3.3, we deduce

$$\begin{aligned} v_2^\Delta(t) &= h\left(t, \omega(t)\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s)\Delta s \\ &\leq h\left(t, \frac{p-1}{p} + \frac{m(t)}{p} + \frac{n(t)}{p}v(t)\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s)\Delta s \\ &\leq h\left(t, \frac{p-1}{p} + \frac{m(t)}{p} + \frac{n(t)}{p}v(t)\right) \\ &\quad - h\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) \\ &\quad + h\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s)\Delta s \\ &\leq \delta\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) \frac{n(t)}{p}v(t) \\ &\quad + h\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s)\Delta s, \\ (51) \quad &= \zeta_3(t)v_2(t) + \xi_3(t). \end{aligned}$$

For all $t \in \mathbb{T}_0$, where ζ_3 and ξ_3 are defined in (48) and (49) respectively. From the inequality (51) and by using Lemma 3.4, with $v_2(t_0) = 0$ we get

$$(52) \quad v_2(t) \leq \int_{t_0}^t e_{\zeta_3}(t, \sigma(s))\xi_3(s)\Delta s.$$

For all $t \in \mathbb{T}_0$. From the inequalities (52) in $\omega^p(t) \leq m(t) + n(t)v_2(t)$, the required inequality in (47) follows. This completes the proof. \square

Remark 3.14. If we put $\mathbb{T} = \mathbb{R}$, and $k(t, s) = 0$ in Theorem 3.6, and using the relations (14) and (17), then we obtain [45, Theorem 2 part (b₁)].

Remark 3.15. If we put $\mathbb{T} = \mathbb{Z}$, and $k(t, s) = 0$ in Theorem 3.6, and using the relations (15) and (18), then we obtain [45, Theorem 4 part (d₁)].

Theorem 3.7. Let $\omega, m, n \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$, and $k(t, s), k^\Delta(t, s) \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$, $p > 1$ be a constant, $h \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+, \mathbb{R}^+)$, and $\phi \in \mathcal{C}(\mathbb{T}_0, \mathbb{R}_+)$ be a strictly increasing, with $\phi(t_0) = 0$ such that,

$$(53) \quad 0 \leq h(t, a) - h(t, b) \leq \delta(t, b)\phi^{-1}(a - b),$$

for all $t \in \mathbb{T}_0$, and $a \geq 0, b \geq 0$, where $\delta \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^+, \mathbb{R}^+)$, and

$$(54) \quad \phi^{-1}(ab) \leq \phi^{-1}(a)\phi^{-1}(b),$$

for all $a, b \in \mathbb{T}_0$, where ϕ^{-1} is the inverse function of ϕ . If

$$(55) \quad \omega^p(t) \leq m(t) + n(t)\phi \int_{t_0}^t \left[h(s, \omega(s)) + k(t, s) \right] \Delta s,$$

for all $t \in \mathbb{T}_0$, then

$$(56) \quad \omega(t) \leq m(t) + n(t) \int_{t_0}^t e_{\zeta_5}(t, \sigma(s)) \xi_5(s) \Delta s,$$

for all $t \in \mathbb{T}_0$, where ζ_5 and ξ_5 are defined as follows:

$$(57) \quad \zeta_5(t) = \delta\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) \phi^{-1}\left(\frac{n(t)}{p}\right),$$

and

$$(58) \quad \xi_5(t) = h\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s) \Delta s.$$

Proof. Define a function v_2 as in (50) and following the arguments as in the proof of the previous theorems we see that corresponding to the inequalities (25) and (26), we get

$$(59) \quad \omega^p(t) \leq m(t) + n(t)\phi(v_2(t)),$$

and

$$(60) \quad \omega(t) \leq \frac{p-1}{p} + \frac{a(t)}{p} + \frac{b(t)}{p} \phi(v_2(t)),$$

for all $t \in \mathbb{T}_0$. From (50) and (60) and by using Lemma 3.3, we get

$$(61) \quad \begin{aligned} v_2^\Delta(t) &= \left(h\left(t, \omega(t)\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s) \Delta s \right) \\ &\leq \left[h\left(t, \frac{p-1}{p} + \frac{m(t)}{p} + \frac{n(t)}{p} \phi(v_2(t))\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s) \Delta s \right] \\ &\leq h\left(t, \frac{p-1}{p} + \frac{m(t)}{p} + \frac{n(t)}{p} \phi(v_2(t))\right) \\ &\quad - h\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) \\ &\quad + h\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s) \Delta s. \end{aligned}$$

For all $t \in \mathbb{T}_0$. By using the condition (53) in (61), we get

$$\begin{aligned}
 v_2^\Delta(t) &\leq \delta\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) \phi^{-1}\left(\frac{n(t)}{p} \phi(v_2(t))\right) \\
 &+ h\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s) \Delta s.
 \end{aligned}
 \tag{62}$$

For all $t \in \mathbb{T}_0$. By applying the condition (54) in (62), we obtain

$$\begin{aligned}
 v_2^\Delta(t) &\leq \delta\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) \phi^{-1}\left(\frac{n(t)}{p}\right) v_2(t) \\
 &+ h\left(t, \frac{p-1}{p} + \frac{m(t)}{p}\right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s) \Delta s. \\
 &= \zeta_5(t) v(t) + \xi_5(t).
 \end{aligned}
 \tag{63}$$

For all $t \in \mathbb{T}_0$. where ζ_5 and ξ_5 are defined in (57) and (58) respectively. From the inequality (63), by applying Lemma 3.4, we obtain

$$v_2(t) \leq \int_{t_0}^t e_{\zeta_5}(t, \sigma(s)) \xi_5(s) \Delta s,
 \tag{64}$$

for all $t \in \mathbb{T}_0$. We get the required inequality (56) from (59) and (64). This completes the proof. \square

Remark 3.16. If we put $\mathbb{T} = \mathbb{R}$, $f(s) = 0$ in Theorem 3.7, and using the relations (14) and (17), then we obtain [45, Theorem 1 part (b₂)].

4. APPLICATIONS

In this section, we present some immediate applications of our results. We assume that our physical problem is modeled by the following initial value problems, so we will discuss the boundedness of its solution as follows:

Example 4.17. Consider the following dynamic integral equation:

$$\omega^p(t) = \Xi\left(t, \int_{t_0}^t \Upsilon(s, \omega(s), k(t, s)) \Delta s\right), \quad \omega(t_0) = m(t_0) = 0,
 \tag{65}$$

for all $t \in \mathbb{T}_0$, where $\Xi \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}, \mathbb{R})$ and $\Upsilon \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^2, \mathbb{R})$ satisfying:

$$|\Xi(t, u)| \leq m(t) + n(t)|u|,
 \tag{66}$$

and

$$|\Upsilon(t, x, y)| \leq h(s, |x|) + y,
 \tag{67}$$

where $m, n, h \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$. Then

$$(68) \quad u(t) \leq \left[m(t) + n(t) \int_{t_0}^t e_{\zeta_4}(t, \sigma(s)) \xi_4(s) \Delta s \right]^{1/p},$$

for all $t \in \mathbb{T}_0$, where $p > 1$, ζ_3 , and ξ_3 are defined as in Theorem 3.6. From (65) and by the assumptions (66) and (67), we have

$$(69) \quad \begin{aligned} |\omega^p(t)| &= \left| \Xi \left(t, \int_{t_0}^t \Upsilon(s, \omega(s), k(t, s)) \Delta s \right) \right| \\ &\leq \Xi \left(t, \int_{t_0}^t |\Upsilon(s, \omega(s), k(t, s))| \Delta s \right) \\ &\leq |m(t)| + |n(t)| \int_{t_0}^t \left[|h(s, \omega(s))| + k(t, s) \right] \Delta s, \end{aligned}$$

for all $t \in \mathbb{T}_0$. Now a suitable application of Theorem 3.6 to (69) yields

$$\omega(t) \leq \left[m(t) + n(t) \int_{t_0}^t e_{\zeta_4}(t, \sigma(s)) \xi_4(s) \Delta s \right]^{1/p}.$$

This is the required estimate in (68).

Example 4.18. Consider the following dynamic equation on time scales:

$$(70) \quad (u^p)^\Delta(t) = \Lambda \left(t, u(t), k(t, s) \right), \quad t \in \mathbb{T}_0,$$

with the initial condition $u(t_0) = C^{1/p}$, where $\Lambda \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^3, \mathbb{R})$.

Theorem 4.8. Assume that

$$(71) \quad \left| \Lambda \left(t, u(t), k(t, s) \right) \right| \leq k(t, s) [g(t)u^p(t) + f(t)u(t) + h(t)],$$

where $u, f, g, h, \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$, $k(t, s), k^\Delta(t, s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}^+)$, and $p > 1$. If u is a solution of the dynamic equation (70), then

$$(72) \quad u(t) \leq \left[C + \int_{t_0}^t e_{\zeta_2}(t, \sigma(s)) \xi_2(s) \Delta s \right]^{1/p},$$

for all $t \in \mathbb{T}_0$, where ζ_2 and ξ_2 are defined as follows:

$$\begin{aligned} \zeta_2(t) &= k(\sigma(t), t) \left[g(t) + \frac{1}{p} f(t) \right] \\ &\quad + \int_{t_0}^t k^\Delta(t, s) \left(g(s) + \frac{1}{p} f(s) \right) \Delta s, \end{aligned}$$

and

$$\begin{aligned} \xi_2(t) &= k(\sigma(t), t) \left[g(t)C + f(t) \left(\frac{p-1}{p} \right. \right. \\ &\quad \left. \left. + \frac{C}{p} \right) + h(t) \right] \\ &\quad + \int_{t_0}^t k^\Delta(t, s) \left[g(s)C + f(s) \left(\frac{p-1}{p} + \frac{C}{p} \right) + h(s) \right] \Delta s, \end{aligned}$$

for all $t \in \mathbb{T}_0$.

Proof. Clearly, the solution u of dynamic equation (70) with the initial condition $u(t_0) = C^{1/p}$, satisfies the equivalent dynamic integral equation on time scales

$$(73) \quad u^p(t) = C + \int_{t_0}^t \Lambda(s, u(s), k(t, s)) \Delta s,$$

for all $t \in \mathbb{T}_0$ with the initial condition $u(t_0) = C^{1/p}$. In fact, from (73) and by using the assumption (71), we have

$$\begin{aligned} |u^p(t)| &= \left| C + \int_{t_0}^t \Lambda(s, u(s), k(t, s)) \Delta s \right| \\ &\leq |C| + \int_{t_0}^t \left| \Lambda(s, u(s), k(t, s)) \right| \Delta s \\ (74) \quad &\leq |C| + \int_{t_0}^t k(t, s) \left[g(s)|u^p(s)| + h(s)|u(s)| + h(s) \right], \end{aligned}$$

with the initial condition $u(t_0) = C^{1/p}$. Then a suitable application of Theorem 3.5 (with $m(t) = C$ and $n(t) = 1$) to (74), yields the desired estimate (72) for solutions of dynamic equation (70). We note that, the right hand side of (72) gives us the bound on the solution of (70) in terms of the known quantities. This completes the proof. □

Example 4.19. Consider the following dynamic integral equation on time scales:

$$(75) \quad u^p(t) = \Phi \left(t, u(t), \int_{t_0}^t \Psi(s, u(s)) \Delta s \right), \quad t \in \mathbb{T}_0,$$

where, $\Phi \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}^2, \mathbb{R})$ and $\Psi \in C_{rd}(\mathbb{T}_0 \times \mathbb{R}, \mathbb{R})$.

Theorem 4.9. Assume that

$$(76) \quad \left| \Phi \left(t, u(t), \int_{t_0}^t \Psi(s, u(s)) \Delta s \right) \right| \leq m(t) + n(t) \left| \int_{t_0}^t \Psi(s, u(s)) \right|,$$

and

$$(77) \quad |\Psi(t, u(t))| \leq g(t)u^p(t) + f(t)u(t) + k(t, s),$$

where $u, f, g, m, n, \in C_{rd}(\mathbb{T}_0, \mathbb{R}^+)$, $k(t, s), k^\Delta(t, s) \in C_{rd}(\mathbb{T}_0 \times \mathbb{T}_0, \mathbb{R}^+)$, and $p > 1$. If u is a solution of the dynamic equation (75), then

$$(78) \quad u(t) \leq \left[m(t) + n(t) \int_{t_0}^t e_\zeta(t, \sigma(s)) \xi(s) \Delta s \right]^{1/p},$$

for all $t \in \mathbb{T}_0$, where ζ and Ψ are defined as follows:

$$\zeta(t) = n(t) \left(g(t) + \frac{f(t)}{p} \right),$$

and

$$\xi(t) = g(t)m(t) + f(t) \left(\frac{p-1}{p} + \frac{m(t)}{p} \right) + k(\sigma(t), t) + \int_{t_0}^t k^\Delta(t, s) \Delta s.$$

Proof. In fact, from (75) and by using the assumptions (76) and (77), we have

$$\begin{aligned} |u^p(t)| &= \left| \Phi \left(t, u(t), \int_{t_0}^t \Psi(s, u(s)) \Delta \tau \right) \Delta s \right| \\ &\leq m(t) + n(t) \left| \int_{t_0}^t \Psi(s, u(s)) \Delta s \right| \\ &\leq m(t) + n(t) \int_{t_0}^t |\Psi(s, u(s))| \Delta s \\ (79) \quad &\leq m(t) + n(t) \int_{t_0}^t \left[g(s)|u^p(s)| + f(s)|u(s)| + k(t, s) \right] \Delta s. \end{aligned}$$

Then a suitable application of Theorem 3.4 to (79), yields the desired estimate (78) for solutions of dynamic equation (75). We note that, the right hand side of (78) gives us the bound on the solution of (75) in terms of the known quantities. This completes the proof. \square

5. CONCLUSION

In this work, we established several new Gronwall-Bellman type dynamic inequalities on time scales which provide a very important and powerful handy tool for deriving upper bounds of solutions of certain nonlinear dynamic equations on time scales as we saw in the applications. Furthermore, our results generalize some known inequalities for continuous functions and their corresponding discrete analysis in the literature, as we stated in the previous remarks. Finally, we would like to mention that, the nonlinear dynamic inequalities studied here allow us to study

the stability, boundedness and asymptotic behaviour of the solutions of a class of more general nonlinear dynamic equations.

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