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HIGHER SUMMABILITY THEOREMS FROM THE WEIGHTED REVERSE DISCRETE INEQUALITIES

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Motivated by higher integrability theorems due to Muckenhoupt and Gehring, in this paper we establish some related higher summability results for non-increasing sequences, verifying the weighted reverse discrete inequalities. Our main result will be proved by employing the weighted Hardy-type inequality designed and proved for this purpose.

1. INTRODUCTION

A weight w is a non-negative locally integrable function defined on a bounded interval $I \subset \mathbb{R}$. A function w satisfying the condition

$$\frac{1}{|I|}\int_I w(x)dx \le Cw(x), \text{ for all } x \in I, \tag{1}$$

where C is a non-negative constant, is called an $A^1(C)$-weight Muckenhoupt function. In 1972, Muckenhoupt [**19**], proved that if w satisfies (1) with $C > 1$, then for every $p \in \left[1, \frac{C}{C-1}\right)$, the function w belongs to $L^p(I)$ and

$$\frac{1}{|I|}\int_I w^p(x)dx \le \frac{C}{C-p(C-1)}\left(\frac{1}{|I|}\int_I w(x)dx\right)^p. \tag{2}$$

A non-negative measurable weight function w defined on a bounded interval I is an $A^p(C)$-weight, $p > 1$, if there exists a constant $C < \infty$ such that the following

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inequality holds:

$$\left(\frac{1}{|I|}\int_I w(t)dt\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{p-1}}(t)dt\right)^{p-1} \le C. \tag{3}$$

A class of functions satisfying condition (3) is usually called Muckenhoupt $A^p(C)$ class. It is well known that functions belonging to Muckenhoupt classes satisfy a reverse Hölder's inequality. More precisely, if $w \in A^p(C)$, then there exist constants $L > 0$ and $p > q$ such that

$$\left[\int_I w^p(t)dt\right]^{\frac{1}{p}} \le L\left[\int_I w^q(t)dt\right]^{\frac{1}{q}}. \tag{4}$$

For more details about relationship between Muckenhoupt classes and the reverse Hölder's inequality the reader is referred to [**23**]. Inequality (4) shows that there is a natural scale of inclusion for $L^q(I)$ spaces, when the underlying space I has a finite measure $|I|$.

Another important property of a function w satisfying $A^p(C)$-condition (3) is the following assertion obtained by Muckenhoupt in [**19**] (see also [**5**]): if $p > 1$ and $w \in L^{-1/(p-1)}(I)$ satisfies (3), then there exists $\epsilon = \epsilon(p, C) > 0$ such that

$$\left(\frac{1}{|I|}\int_I w(t)dt\right)\left(\frac{1}{|I|}\int_I w^{-\frac{1}{q-1}}(t)dt\right)^{q-1} \le C_{q-\epsilon}, \tag{5}$$

for all $p - \epsilon < q < p$. In other words, Muckenhoupt's result for *self-improving* property asserts that if $w \in A^q(C)$, then there exists $\epsilon > 0$ such that $w \in A^{q-\epsilon}(C_{q-\epsilon})$, and consequently

$$A^q(C) \subset A^{q-\epsilon}(C_{q-\epsilon}). \tag{6}$$

Similarly, Gehring [**7**], extended the result of Muckenhoupt and proved that if there exists a constant $K > 1$ such that

$$\left(\frac{1}{|I|}\int_I w^q(x)dx\right)^{\frac{1}{q}} \le K\frac{1}{|I|}\int_I w(x)dx, \text{ for } q > 1, \tag{7}$$

then w satisfies a higher integrability condition, namely for sufficiently small $\epsilon > 0$ holds the inequality

$$\left(\frac{1}{|I|}\int_I w^{q+\epsilon}(x)dx\right)^{\frac{1}{q+\epsilon}} \le a_{q+\epsilon}\left(\frac{1}{|I|}\int_I w^q(x)dx\right)^{\frac{1}{q}}. \tag{8}$$

It should be noticed here that relations (7) and (8) represent reverse Hölder's inequality (4). Now, if we denote by $G^q(K)$ a class of functions satisfying (7), Gehring's result for *self-improving* property states that if $w \in G^q(K)$, then there exists $\epsilon > 0$ such that $w \in G^{q+\epsilon}(Ka_{q+\epsilon})$, and consequently

$$G^q(K) \subset G^{q+\epsilon}(Ka_{q+\epsilon}). \tag{9}$$

The proof of Gehring's inequality (8) is based on the use of the Calderón-Zygmund decomposition theorem and the scale structure of L^p-spaces.

As proved independently by Muckenhoupt and Gehring in their celebrated papers, these two classes both enjoy the remarkable *self-improving property*: for given couples of constants p, C and q, K, defined as above, there exists limit exponents $p^* = p^*(p, C) < p$ and $q^* = q^*(q, K) > q$, and corresponding constants $C_r = C_r(p, C)$ and $A_s = A_s(q, K)$, such that the following inclusions hold true:

$$w \in A^p(C) \Longrightarrow w \in A^r(C_r), \qquad \forall r \in (p^*, p] \tag{10}$$

$$v \in G^q(K) \Longrightarrow v \in G^s(A_s), \qquad \forall s \in [q, q^*). \tag{11}$$

These properties have been deeply investigated, particularly with respect to the following problems:

(A) to find the value of the limit exponents p^* and q^* for which the self-improving property holds;

(B) to find the best constants A_s and C_r.

In 1992, Bojarski et.al. [**3**], improved the Muckenhoupt inequality (2). The corresponding proof has been established by using the rearrangement of the function on the interval I (see, e.g. [**10**]). In particular, they proved that if w satisfies (1) with $C > 1$, then

$$\frac{1}{|I|}\int_I w^p(t)dt \le \frac{C^{1-p}}{C - p(C-1)}\left(\frac{1}{|I|}\int_I w(t)dt\right)^p, \text{ for } p < \frac{C}{C-1}. \tag{12}$$

Similarly, Nania [**20**], extended the results of Muckenhoupt and Gehring, and proved a higher integrability theorem for non-increasing functions by using the inequality

$$\frac{1}{t}\int_0^t f^q(s)ds \le C f^{q-1}(t)\frac{1}{t}\int_0^t f(s)ds, \text{ for all } t \in (0, T], \tag{13}$$

which holds for the constants $C > 1$ and $q > 1$. In particular, Nania proved that if (13) holds, then for every $p \in [q, q+\varepsilon]$ the function f belongs to $L^p(0, T]$ and

$$\left(\frac{1}{T}\int_0^T f^p(t)dt\right)^{\frac{1}{p}} \le K\left(\frac{1}{T}\int_0^T f(s)ds\right), \tag{14}$$

where $\varepsilon = \frac{q}{\alpha - 1}$, $\alpha = Cq(q-1)$,

$$K = \left[\frac{\alpha^{r+1}}{\alpha - r(\alpha-1)}\right]^{\frac{1}{p}}, \quad \text{and } r = \frac{p}{q}.$$

The inequality (14) has been proved by employing the famous Hardy inequality

$$\int_0^T \left(\frac{1}{t}\int_0^t f(s)ds\right)^p dt \le \left(\frac{p}{p-1}\right)^p \int_0^T f^p(t)dt, \quad p > 1. \tag{15}$$

In 1995, Alzer [**1**] (see also [**21**]), proved a new refinement of Nania's inequality by using the following inequality

$$\begin{aligned}&\int_0^T \left(\frac{1}{t}\int_0^t w(s)ds\right)^p \Delta t + \frac{p}{p-1}T^{1-p}\left(\int_0^T w(t)dt\right)^p \\ &\leq \left(\frac{p}{p-1}\right)^p \int_0^T w^p(t)dt,\end{aligned} \tag{16}$$

which has been proved by Shum in [**22**]. In particular, Alzer proved that if w is a non-negative non-increasing function on $(0, T)$ satisfying (13) for all $t \in (0, T)$, then the function w belongs to $L^p(0, T]$ and

$$\left(\frac{1}{T}\int_0^T w^p(t)dt\right)^{\frac{1}{p}} \leq K_1 \left(\frac{1}{T}\int_0^T w^q(s)ds\right)^{\frac{1}{q}}, \tag{17}$$

holds with a new constant K_1 smaller than K and there exists $\delta > \varepsilon$ such that the inequality (17) holds not only for all $p \in [q, q+\varepsilon]$, but for all $p \in [q, q+\delta]$.

The self-improving property has applications in the study of integrability of the gradient for quasi-conformal mappings, the study of the optimal regularity of solutions to some elliptic PDE's where the L^p solvability of the Dirichlet problem, $\text{div}A(x)\nabla u = 0$ on the unit disc D, with $u_{|\partial D} = \varphi$, can be expressed in terms of G^q conditions on the boundary ∂D for the harmonic measures associated to $A(x)$ (see e.g. [**3, 4, 7, 8, 9, 11, 12, 18, 19, 21**]). On the other hand, we do believe that the reverse discrete inequalities will also play the same important role in various fields of analysis in discrete spaces such as the weighted discrete Sobolev imbedding theorem and regularity theory of variational problems for discrete mappings.

In fact, in the last decade several authors have been interested in finding some discrete results on l^p-analogues for L^p-bounds in harmonic analysis and as a result this subject becomes a topic of ongoing research. In particular, the investigations of regularity of the discrete Hardy-Littlewood maximal operator have been considered recently by some authors and translation of some results from continuous to discrete case has been given, see for example papers [**2, 13, 14, 15, 16**] and references therein.

One of the reasons for this upsurge of interest in a discrete case is also due to the fact that discrete operators may even behave differently from their continuous counterparts as exhibited by a discrete spherical maximal operator (see e.g. [**17**]). On the other hand, there is a big challenge in proving such results in l^p compared by L^p, due to the lack of calculus in a discrete space, since there are no power rules or even chain rules, which are the main tools for proving the corresponding results in a continuous setting.

The main objective of this paper is to overcome these problems by introducing a new approach to develop a study of boundedness and summability of discrete mappings with weights. In particular, our aim in the present paper is to prove higher summability theorems for non-increasing sequences that correspond to integral relations (2) and (8), verifying the weighted reverse discrete inequalities on a discrete space. Recall that $l^p_w(\mathbb{N})$ is the Lebesgue space consisting of all sequences $f : \mathbb{N} \to \mathbb{R}$ with a positive weight $w : \mathbb{N} \to \mathbb{R}$ and a norm $\|f\|_{l^p_w(\mathbb{N})} = \left(\sum_{n=0}^{\infty} |f(n)|^p\, w(n)\right)^{\frac{1}{p}} < \infty$. More precisely, we shall be interested in studying some interesting properties of the mappings of the form

$$\mathcal{M}f(n) := \frac{1}{\Lambda(n)} \sum_{k=0}^{n-1} w(k) f(k), \quad \text{for all} \quad n \in \mathbb{N}, \tag{18}$$

where f and w are non-negative sequences and $\Lambda(n) = \sum_{k=0}^{n-1} w(k)$. As an application of the obtained results, we shall prove higher summability theorems that correspond to (2) and (8).

It should be noticed here that the operator defined by (18) is a generalization of the Muirhead maximal operator, discrete Hardy-Littleewod maximal operator, the Lorentz operator defined on the Lorentz space and the regular Riesz mean. For example, if f is replaced by its non-increasing rearrangement f^* (see e.g. [**10**]) on the right-hand side of (18), we obtain a definition of the weighted Muirhead operator. In particular, if $w(k) = 1$, then relation (18) provides the Muirhead maximal operator. Further, recall that the Lorentz space $l^{p,q}(\mathbb{N})$ is defined by

$$l^{p,q}(\mathbb{N}) = \left\{ a(n);\ \|a(n)\|_{p,q} = \left[\frac{q}{p} \sum_{k=1}^{\infty} \left(k^{\frac{1}{p}} a^*(k) \right)^q \frac{1}{k} \right]^{\frac{1}{q}} < \infty \right\},$$

where $1 < p < \infty$, $1 \le q < \infty$, and where a^* stands for the rearrangement of the sequence a. It is well known that $l^{p,p}(\mathbb{N}) = l^p(\mathbb{N})$. The weighted Hardy-Littlewood maximal operator and the weighted Lorentz operator can also be defined in a similar way as the weighted Muirhead operator, for more details on these operators the reader is referred to monographs [**6, 8**].

The paper is organized as follows: after this introductory part, in Section 2 we give some definitions and auxiliary results necessary for establishing our main theorems. In Section 3 we establish our main results, i.e. we prove higher summability theorems for non-increasing sequences. More precisely, we derive discrete versions of higher integrability relations (2) and (8) due to Muckenhoupt and Gehring. In order to prove the discrete version of relation (8), we shall employ a weighted Hardy-type inequality designed and proved for this purpose.

2. PRELIMINARIES

We start by introducing some definitions and notation that will be used throughout the paper. Let $\mathbb{N}$ stand for the set of non-negative integers, i.e. $\mathbb{N} = \{0, 1, 2, \ldots\}$, and assume that $\mathbb{I}_a^N = \{a, a+1, a+2, \ldots, N\} \subseteq \mathbb{N}$, where a and N are non-negative integers such that $N > a$. Further, let $\mathbb{R}^+$ stand for the set of non-negative real numbers, i.e. $\mathbb{R}^+ = [0, \infty)$. The classical Hölder's inequality asserts that

$$\sum_{n=a}^{N} |u(n)v(n)| \le \left[\sum_{n=a}^{N} |u(n)|^p \right]^{\frac{1}{p}} \left[\sum_{n=a}^{N} |v(n)|^q \right]^{\frac{1}{q}}, \tag{19}$$

where u and v are real sequences defined on $\mathbb{I}_a^N$ and $\frac{1}{p}+\frac{1}{q}=1$, $p>1$. The summation by parts formula reads

$$\sum_{k=0}^{n-1} \Delta u(k)v(k+1) = u(k)v(k)|_{k=0}^{n} - \sum_{k=0}^{n-1} u(k)\Delta v(k), \tag{20}$$

where Δ is a forward difference operator defined by $\Delta u(n) = u(n+1) - u(n)$. In what follows, we shall use the conventions $\sum_{k=a}^{b} y(k) = 0$, whenever $a > b$, and

$$\Delta\left(\sum_{k=a}^{n-1} y(k)\right) = y(n), \quad \sum_{k=a}^{n-1} \Delta y(k) = y(n) - y(a).$$

Throughout this paper, we assume that the sequences in the statements of theorems are non-negative.

For the readers's convenience, we first give a more generalized form of the operator defined by (18). More precisely, for any sequence $f : \mathbb{I}_a^N \to \mathbb{R}^+$, we define the mapping $\mathcal{M}f : \mathbb{I}_a^N \to \mathbb{R}^+$ by

$$\mathcal{M}f(n) := \frac{1}{\Lambda(n)} \sum_{k=a}^{n-1} w(k)f(k), \quad \text{for all} \quad n \in \mathbb{I}_a^N, \tag{21}$$

where $\Lambda(n) = \sum_{k=a}^{n-1} w(k)$. In addition, we also define mappings f^δ, $\mathcal{M}^\delta f$ and $\mathcal{M}[\mathcal{M}^\delta f]^p$ respectively by

$$\begin{aligned} f^\delta(n) &= f(n+1), \\ \mathcal{M}^\delta f(n) &= \mathcal{M}f(n+1) = \frac{1}{\Lambda(n+1)} \sum_{k=a}^{n} w(k)f(k), \\ \mathcal{M}[\mathcal{M}^\delta f]^p(n) &= \frac{1}{\Lambda(n)} \sum_{k=a}^{n-1} w(k) \left(\frac{1}{\Lambda(k+1)} \sum_{s=a}^{k} w(s)f(s)\right)^p, \end{aligned}$$

for all $n \in \mathbb{I}_a^N$. We start by noting some basic properties of the mapping $\mathcal{M}f$, known from the literature, which follow from definition (21). For the reader's convenience, the following two lemmas are given with their short proofs.

Lemma 1. *If $f : \mathbb{I}_a^N \to \mathbb{R}^+$ is a non-increasing sequence, then $\mathcal{M}f \geq f$.*

Proof. Since f is a non-increasing sequence, taking into account definition (21), we have

$$\mathcal{M}f(n) = \frac{1}{\Lambda(n)} \sum_{k=a}^{n-1} w(k)f(k) \geq \frac{1}{\Lambda(n)} \sum_{k=a}^{n-1} w(k)f(n) = f(n),$$

which proves our assertion. □

It should be also noticed here that $\mathcal{M}f$ inherits the non-increasing nature of mapping f.

Lemma 2. *If $f : \mathbb{I}_a^N \to \mathbb{R}^+$ is a non-increasing sequence, then so is $\mathcal{M}f$.*

Proof. Taking into account Lemma 1 and the quotient rule

$$\Delta\left(\frac{x(n)}{y(n)}\right) = \frac{y(n)\Delta x(n) - x(n)\Delta y(n)}{y(n)y(n+1)},$$

with $x(n) = \sum_{k=a}^{n-1} w(k)f(k)$ and $y(n) = \Lambda(n)$, it follows that

$$\Delta\left(\mathcal{M}f(n)\right) = \frac{w(n)}{\Lambda(n+1)}\left[f(n) - \mathcal{M}f(n)\right] \le 0, \text{ for } n \in \mathbb{I}_a^N,$$

which completes the proof. □

Remark 1. *As a consequence of Lemma 1, we notice that if f is non-increasing sequence, then $\mathcal{M}f^q \ge f^q$, for $q > 0$. In addition, by Lemma 2 we see that if f is non-increasing sequence, then so is $\mathcal{M}f^q$, for $q > 0$.*

3. MAIN RESULTS

In this section we prove higher summability theorems that correspond to higher integrability theorems due to Muckenhoupt and Gehring, presented in the Introduction. Our first result is a discrete version of integral relation (2). The following theorem refers to a non-increasing sequence $g : \mathbb{I}_a^N \to \mathbb{R}^+$ satisfying an extra condition.

Theorem 1. *Let $g : \mathbb{I}_a^N \to \mathbb{R}^+$ be a non-increasing sequence such that there exists a constant $\beta > 1$ with $g(n) \le \beta g(n+1)$ for all $n \in \mathbb{I}_a^N$, and let $\mathcal{M}g : \mathbb{I}_a^N \to \mathbb{R}^+$ be defined by (21). If there exists a constant $c \ge 1$ satisfying*

$$\mathcal{M}g(n) \le cg(n), \quad \text{for all } n \in \mathbb{I}_a^N, \tag{22}$$

then the inequality

$$\mathcal{M}g^p(N) \le \frac{p}{A - p(A-1)}\left(\mathcal{M}g(N)\right)^p \tag{23}$$

holds for $p \in \left[1, \frac{A}{A-1}\right)$, where $A = c\beta^p$.

Proof. In this proof we write $G(n) = \sum_{k=a}^{n-1} w(k)g(k)$ for brevity. This implies that $\Delta G(n) = w(n)g(n)$ and therefore,

$$\sum_{n=a}^{N-1} w(n)g^p(n) = \sum_{n=a}^{N-1} g^{p-1}(n)\Delta G(n).$$

Summing by parts the right-hand side of the above relation, it follows that

$$\sum_{n=a}^{N-1} w(n)g^p(n) = g^{p-1}(n)G(n)\Big|_a^N - \sum_{n=a}^{N-1} \Delta g^{p-1}(n)G(n+1). \tag{24}$$

Now, our intention is to find a suitable estimate for $\Delta g^{p-1}(n)$ by considering two cases: The first case appears when $0 \leq p-1 \leq 1$. In this case, we employ an elementary inequality

$$\gamma x^{\gamma-1}(x-y) \leq x^{\gamma} - y^{\gamma} \leq \gamma y^{\gamma-1}(x-y),$$

which holds for $x \geq y > 0$ and $0 \leq \gamma \leq 1$. Considering this inequality with $0 \leq \gamma = p-1 \leq 1$, we obtain the estimate

$$\begin{aligned} -\Delta\left(g^{p-1}(n)\right) &\leq -(p-1)g^{p-2}(n+1)\Delta g(n) \\ &\leq -(p-1)\beta^{p-1}g^{p-2}(n+1)\Delta g(n). \end{aligned} \tag{25}$$

The second case appears when $\gamma = p-1 > 1$. In this case, utilizing the inequality

$$\gamma y^{\gamma-1}(x-y) \leq x^{\gamma} - y^{\gamma} \leq \gamma x^{\gamma-1}(x-y), \tag{26}$$

which holds for $x \geq y > 0$ and $\gamma > 1$, we obtain the estimate

$$-(p-1)g^{p-2}(n+1)\Delta g(n) \leq -\Delta g^{p-1}(n) \leq -(p-1)g^{p-2}(n)\Delta g(n). \tag{27}$$

Now, from the hypothesis $g(n) \leq \beta g(n+1)$, it follows that

$$g^{p-2}(n) \leq \beta^{p-2}g^{p-2}(n+1), \text{ for } p > 2,$$

and consequently,

$$-(p-1)g^{p-2}(n)\Delta g(n) \leq -(p-1)\beta^{p-2}g^{p-2}(n+1)\Delta g(n).$$

In addition, since $\beta > 1$, we have that

$$-\Delta g^{p-1}(n) \leq -(p-1)\beta^{p-1}g^{p-2}(n+1)\Delta g(n). \tag{28}$$

Then, combining (25) and (28), we obtain the estimate

$$-\Delta g^{p-1}(n) \leq -(p-1)\beta^{p-1}g^{p-2}(n+1)\Delta g(n), \text{ for } p \geq 1.$$

Substituting the last inequality in (24), we obtain

$$\begin{aligned} \sum_{n=a}^{N-1} w(n)g^{p}(n) \leq\ & g^{p-1}(n)G(n)\big|_a^N \\ & -(p-1)\beta^{p-2}\sum_{n=a}^{N-1} g^{p-2}(n+1)\Delta g(n)G(n+1). \end{aligned} \tag{29}$$

Moreover, by virtue of (22), since

$$G(n+1) = \Lambda(n+1)\mathcal{M}^{\delta}g(n) \leq c\Lambda(n+1)\mathcal{M}g(n),$$

we see that

$$G(n+1) \le c\Lambda(n+1)g(n), \quad \text{for all} \quad n \in \mathbb{I}_a^N.$$

Now, combining the last inequality with relation (29), and noting that $\Delta g(n) \le 0$, it follows that

$$\sum_{n=a}^{N-1} w(n)g^p(n) \le g^{p-1}(n)G(n)\big|_a^N$$
$$-c(p-1)\beta^{p-1}\sum_{n=a}^{N-1}\Lambda(n+1)g^{p-1}(n+1)g(n)\Delta g(n).$$

In addition, from the hypothesis $g(n) \le \beta g(n+1)$, $\beta > 1$, the previous inequality provides the estimate

$$\sum_{n=a}^{N-1} w(n)g^p(n) \le g^{p-1}(n)G(n)\big|_a^N$$
$$-c(p-1)\beta^{p}\sum_{n=a}^{N-1}\Lambda(n+1)g^{p-1}(n+1)\Delta g(n). \tag{30}$$

On the other hand, since $p > 1$, applying the right inequality in (27), and taking into account that the sequence $g(n)$ is non-increasing, we obtain the inequality

$$-\Delta g^p(n) \ge -pg^{p-1}(n+1)\Delta g(n).$$

Now, by virtue of this relation, the inequality (30) yields the estimate

$$\sum_{n=a}^{N-1} w(n)g^p(n) \le g^{p-1}(n)G(n)\big|_a^N - c(p-1)\frac{\beta^p}{p}\sum_{n=a}^{N-1}\Lambda(n+1)\Delta g^p(n). \tag{31}$$

Summing the last term in (31) by parts, it follows that

$$\sum_{n=a}^{N-1}\Lambda(n+1)\Delta g^p(n) = \Lambda(n)g^p(n)|_a^N - \sum_{n=a}^{N-1} w(n)g^p(n).$$

Now, substituting the above relation in (31), it follows that

$$\sum_{n=a}^{N-1} w(n)g^p(n) \le g^{p-1}(n)G(n)\big|_a^N$$
$$-c(p-1)\frac{\beta^p}{p}\left[\Lambda(n)g^p(n)|_a^N - \sum_{n=a}^{N-1} w(n)g^p(n)\right]$$
$$= g^{p-1}(N)G(N) - c(p-1)\frac{\beta^p}{p}\Lambda(N)g^p(N)$$
$$+c(p-1)\frac{\beta^p}{p}\sum_{n=a}^{N-1} w(n)g^p(n).$$

Moreover, since $p < \frac{A}{A-1}$, we have

$$\frac{p - A(p-1)}{p} \sum_{n=a}^{N-1} w(n)g^p(n) \leq g^{p-1}(N)G(N) - c(p-1)\frac{\beta^p}{p}\Lambda(N)g^p(N),$$

which leads to

$$\sum_{n=a}^{N-1} w(n)g^p(n) \leq \frac{p}{A+p-Ap} g^{p-1}(N)G(N). \tag{32}$$

Now, since g is a non-increasing sequence, we have that

$$g(N) \leq \mathcal{M}g(N) = \frac{1}{\Lambda(N)} \sum_{n=a}^{N-1} w(n)g(n),$$

by virtue of Lemma 1. Finally, since $G(N) = \sum_{n=a}^{N-1} w(n)g(n)$, combining the last inequality and (32), we obtain

$$\sum_{n=a}^{N-1} w(n)g^p(n) \leq \frac{p}{A+p-Ap} \left(\frac{1}{\Lambda(N)} \sum_{n=a}^{N-1} w(n)g(n) \right)^{p-1} \sum_{n=a}^{N-1} w(n)g(n),$$

and therefore

$$\frac{1}{\Lambda(N)} \sum_{n=a}^{N-1} w(n)g^p(n) \leq \frac{p}{A-p(A-1)} \left(\frac{1}{\Lambda(N)} \sum_{n=a}^{N-1} w(n)g(n) \right)^p,$$

which provides desired inequality (23). The proof is now complete. □

The sequence $g : \mathbb{I}_a^N \to \mathbb{R}^+$ satisfying condition $g(n) \leq \beta g(n+1)$, where $\beta > 1$, is usually referred to as a β-increasing sequence.

Remark 2. *It should be noticed here that the inequality (23) also holds when N is replaced by an arbitrary $n \in \mathbb{I}_a^N$.*

Our next result is a new weighted Hardy-type inequality which holds for non-increasing sequences. The following theorem will be a crucial step in establishing a discrete version of integral relation (8). It should be noticed here that the following result represents a discrete version of integral relation (16).

Theorem 2. *Let $f : \mathbb{I}_a^N \to \mathbb{R}^+$ be a non-increasing sequence. If $p > 1$, then the inequality*

$$\mathcal{M}[\mathcal{M}^\delta f]^p(n) + \frac{p}{p-1}[\mathcal{M}f]^p(n) \leq \left(\frac{p}{p-1}\right)^p \mathcal{M}f^p(n) \tag{33}$$

holds for all $n \in \mathbb{I}_a^N$.

Proof. In this proof we write $\mathcal{M}f = F$ for brevity. Since $\Lambda(n)F(n) = \sum_{k=a}^{n-1} w(k)f(k)$, the product rule yields the relation

$$\Lambda(n)\Delta F(n) + w(n)F(n+1) = w(n)f(n). \tag{34}$$

Now, by putting $u(n) = \Lambda(n)$, $v(n) = F^p(n)$, and employing the summation by parts formula (20), it follows that

$$\sum_{k=a}^{n-1} w(k)F^p(k+1) = \Lambda(n)F^p(n) - \sum_{k=a}^{n-1} \Lambda(k)\Delta F^p(k) - \lim_{k\to a^+} u(k)v(k). \tag{35}$$

On the other hand, since F is non-increasing, employing the inequality (26) implies

$$-(F^p(k) - F^p(k+1)) \leq pF^{p-1}(k+1)\Delta F(k),$$

for $p > 1$. This leads to the estimate

$$\begin{aligned}\Delta F^p(k) &= F^p(k+1) - F^p(k) = -(F^p(k) - F^p(k+1)) \\ &\leq -pF^{p-1}(k+1)\,[F(k) - F(k+1)] = pF^{p-1}(k+1)\Delta F(k),\end{aligned}$$

wherefrom we obtain the inequality

$$-\sum_{k=a}^{n-1} \Lambda(k)\Delta F^p(k) \geq -p\sum_{k=a}^{n-1} \Lambda(k)F^{p-1}(k+1)\Delta F(k).$$

Now, substituting the identity (34) in the last inequality, we obtain the following estimate:

$$\begin{aligned}-\sum_{k=a}^{n-1} \Lambda(k)\Delta F^p(k) &\geq -p\sum_{k=a}^{n-1} \Lambda(k)F^{p-1}(k+1)\Delta F(k) \\ &= -p\sum_{k=a}^{n-1} F^{p-1}(k+1)[w(k)f(k) - w(k)F(k+1)] \\ &= -p\sum_{k=a}^{n} w(k)f(k)F^{p-1}(k+1) + p\sum_{k=a}^{n-1} F^p(k+1)w(k).\end{aligned}$$

In addition, combining the last inequality with (35), it follows that

$$\begin{aligned}\sum_{k=a}^{n-1} w(k)F^p(k+1) \geq \Lambda(n)F^p(n) &- p\sum_{k=a}^{n-1} w(k)f(k)F^{p-1}(k+1) \\ &+p\sum_{k=a}^{n} w(k)F^p(k+1) - \lim_{k\to a^+} u(k)v(k).\end{aligned} \tag{36}$$

Now, since

$$\lim_{k\to a^+} u(k)v(k) = \lim_{k\to a^+} \Lambda(k)\left(\frac{1}{\Lambda(k)}\sum_{u=a}^{k-1} w(u)f(u)\right)^p,$$

applying the Hölder inequality with exponents p and $\frac{p}{p-1}$ implies

$$\begin{aligned}\lim_{k\to a^+} u(k)v(k) &= \lim_{k\to a^+}\Lambda(k)\left(\frac{1}{\Lambda(k)}\sum_{u=a}^{k-1} w^{\frac{1}{p}}(u)w^{\frac{p-1}{p}}(u)f(u)\right)^p \\ &\leq \lim_{k\to a^+}\Lambda(k)\left(\frac{1}{\Lambda(k)}\left(\sum_{u=a}^{k-1} w(u)f^p(u)\right)^{\frac{1}{p}}\left(\sum_{u=a}^{k-1} w(u)\right)^{\frac{p-1}{p}}\right)^p \\ &= \lim_{k\to a^+}\Lambda^{1-p}(k)\left(\sum_{u=a}^{k-1} w(u)f^p(u)\right)(\Lambda(k))^{p-1} \\ &= \lim_{k\to a^+}\left(\sum_{u=a}^{k-1} w(u)f^p(u)\right)=0.\end{aligned}$$

Hence, the inequality (36) reduces to

$$(p-1)\sum_{k=a}^{n-1} w(k)F^p(k+1)+\Lambda(n)F^p(n)\leq p\sum_{k=a}^{n-1} w(k)f(k)F^{p-1}(k+1).$$

Furthermore, applying the Hölder inequality with exponents p and $\frac{p}{p-1}$ to the term $\sum_{k=a}^{n-1} w(k)f(k)F^{p-1}(k+1)$, we obtain the inequality

$$\begin{aligned}&(p-1)\sum_{k=a}^{n-1} w(k)F^p(k+1)+\Lambda(n)F^p(n) \\ \leq\ & p\left[\sum_{k=a}^{n-1} w(k)f^p(k)\right]^{\frac{1}{p}}\left[\sum_{k=a}^{n-1} F^p(k+1)\right]^{\frac{p-1}{p}},\end{aligned}$$

which can be rewritten in the following form:

$$\begin{aligned}&\left(\frac{p}{p-1}\right)^p\sum_{k=a}^{n-1} w(k)f^p(k) \\ \geq\ & \left[\left(\sum_{k=a}^{n-1} w(k)F^p(k+1)\right)^{1/p}+\frac{\Lambda(n)F^p(n)/(p-1)}{\left[\sum_{k=a}^{n-1} w(k)F^p(k+1)\right]^{\frac{p-1}{p}}}\right]^p. \qquad (37)\end{aligned}$$

Now, in order to conclude the proof, we shall utilize the inequality

$$(u+v)^p\geq u^p+pu^{p-1}v, \quad \text{where} \quad p>1 \text{ or } p<0. \qquad (38)$$

Recall that this relation is a variant of the well-known Bernoulli inequality and it is valid for all $u\geq 0$ and $u+v\geq 0$, if $p>1$, or for $u>0$ and $u+v>0$, if $p<0$. The equality in (38) holds if and only if $v=0$.

Now, applying the Bernoulli inequality (38) to the right-hand side of (37) yields

$$\begin{aligned}&\left(\frac{p}{p-1}\right)^p\sum_{k=a}^{n-1}w(k)f^p(k)\\ &\geq \sum_{k=a}^{n-1}w(k)F^p(k+1)+\frac{p\Lambda(n)F^p(n)/(p-1)}{\left[\sum\limits_{k=a}^{n-1}w(k)F^p(k+1)\right]^{(p-1)/p}}\\ &\quad\times\left[\sum_{k=a}^{n-1}w(k)F^p(k+1)\right]^{(p-1)/p},\end{aligned}$$

which leads to

$$\frac{1}{\Lambda(n)}\sum_{k=a}^{n-1}F^p(k+1)+\frac{p}{p-1}F^p(n)\leq\left(\frac{p}{p-1}\right)^p\frac{1}{\Lambda(n)}\sum_{k=a}^{n-1}w(k)f^p(k).$$

Finally, taking into account definition of mapping F, we see that the above inequality represents (33), as claimed. The proof is now complete. □

Finally, our intention is to establish a discrete version of the higher integrability condition (8) due to Gehring. To do this, we need some extra conditions on the corresponding sequence and the weight function. Namely, it is easy to see that if $f:\mathbb{I}_a^N\to\mathbb{R}^+$ is a non-increasing sequence and $q>1$, then the inequality

$$f^{q-1}(n)\left[\mathcal{M}f\right](n)\leq\left[\mathcal{M}f^q\right](n), \tag{39}$$

holds for all $n\in\mathbb{I}_a^N$. In the sequel, we shall assume that there exists a constant $C>1$ such that f satisfies the reverse of (39), i.e. it satisfies the inequality

$$\left[\mathcal{M}f^q\right](n)\leq Cf^{q-1}(n)\left[\mathcal{M}^\delta f\right](n),\ \text{for } n\in\mathbb{I}_a^N, \tag{40}$$

and without loss of generality, we shall also assume that there exists a constant $\eta>1$ such that

$$\Lambda(n+1)\leq\eta\Lambda(n),\quad \text{for}\quad n\in\mathbb{I}_a^N. \tag{41}$$

Now, we are able to find an upper bound for the norm of f in $l_w^p(\mathbb{I}_a^N)$ via the norm of f in $l_w^q(\mathbb{I}_a^N)$ when $p>q>1$. In other words, we shall bound $\left[\mathcal{M}f^p\right](n)$ from the above by $\left[\mathcal{M}f^q\right](n)$, for all $n\in\mathbb{I}_a^N$. The proof of the corresponding result relies on Theorems 1 and 2.

Theorem 3. *Let $f:\mathbb{I}_a^N\to\mathbb{R}^+$ be a non-increasing sequence and let there exists a constant $C>1$ such that (40) holds for $q>1$. If the condition (41) is satisfied and $p>q$, then holds the inequality*

$$\left[\mathcal{M}f^p\right](n)\leq K^*\left[\mathcal{M}f^q\right]^{\frac{p}{q}}(n),\qquad n\in\mathbb{I}_a^N, \tag{42}$$

where

$$K^* = \frac{p\lambda^{\frac{p}{q}}}{qB - p(B-1)}, \quad B = \eta^{\frac{p}{q}}\lambda,$$

and

$$\lambda = \left[C^q \left(\frac{q}{q-1}\right)^q - \frac{C^{q-1}q}{q-1}\right]^{\frac{1}{q}}.$$

Proof. From the relation (40), after summing from a to $n-1$, it follows that

$$\sum_{k=a}^{n-1} w(k)\left[\mathcal{M}f^q\right](k) \le C\sum_{k=a}^{n-1} w(k) f^{q-1}(k)\left[\mathcal{M}^\delta f\right](k).$$

Now, applying the Hölder inequality with exponents q and $\frac{q}{q-1}$ to the right-hand side of the previous relation, we obtain the inequality

$$\frac{1}{\Lambda(n)}\sum_{k=a}^{n-1} w(k)\left[\mathcal{M}f^q\right](k) \le C\left[\mathcal{M}f^q\right]^{1-\frac{1}{q}}(n)\left[\mathcal{M}\left[\mathcal{M}^\delta f\right]^q\right]^{\frac{1}{q}}(n). \tag{43}$$

On the other hand, applying relation (33) from Theorem 2, we have

$$\left[\mathcal{M}\left[\mathcal{M}^\delta f\right]^q\right]^{\frac{1}{q}}(n) \le \left[\left(\frac{q}{q-1}\right)^q\left[\mathcal{M}f^q\right](n) - \frac{q}{q-1}[\mathcal{M}f]^q(n)\right]^{\frac{1}{q}}.$$

Combining this inequality and relation (43), we have

$$\begin{aligned} &\left[\mathcal{M}[\mathcal{M}f^q]\right](n) \\ &\quad \le C\left[\mathcal{M}f^q\right]^{1-\frac{1}{q}}(n)\left[\left(\frac{q}{q-1}\right)^q\left[\mathcal{M}f^q\right](n) - \frac{q}{q-1}[\mathcal{M}f]^q(n)\right]^{\frac{1}{q}}. \end{aligned} \tag{44}$$

Utilizing (40), we have

$$\frac{[\mathcal{M}f]^q(n)}{\left[\mathcal{M}f^q\right](n)} \ge \frac{[\mathcal{M}f]^q(n)}{Cf^{q-1}(n)\left[\mathcal{M}^\delta f\right](n)} \ge \frac{1}{C}\left(\frac{[\mathcal{M}f](n)}{f(n)}\right)^{q-1}, \tag{45}$$

since $\left[\mathcal{M}^\delta f\right](n) \le [\mathcal{M}f](n)$, by Lemma 2. In addition, since $[\mathcal{M}f](n) \ge f(n)$ by Lemma 1, the relation (45) implies the inequality

$$[\mathcal{M}f]^q(n) \ge \frac{1}{C}\left[\mathcal{M}f^q\right](n).$$

Combining this inequality and (44), we have

$$\begin{aligned} &\left[\mathcal{M}[\mathcal{M}f^q]\right](n) \\ &\quad \le C\left[\mathcal{M}f^q\right]^{1-\frac{1}{q}}(n)\left[\left(\frac{q}{q-1}\right)^q\left[\mathcal{M}f^q\right](n) - \frac{q}{C(q-1)}\left[\mathcal{M}f^q\right](n)\right]^{\frac{1}{q}}, \end{aligned}$$

that is,

$$\big[\mathcal{M}[\mathcal{M}f^q]\big](n) \le \lambda\,[\mathcal{M}f^q]\,(n), \tag{46}$$

where

$$\lambda = \left[C^q\left(\frac{q}{q-1}\right)^q - \frac{C^{q-1}q}{q-1}\right]^{\frac{1}{q}}.$$

Now, in order to apply Theorem 1 we first show that $\lambda > 1$. First, considering the left inequality in (26) with $x = \frac{q}{q-1}$, $y = 1$ and $\gamma = q$, it follows that

$$\left(\frac{q}{q-1}\right)^q - \frac{q}{q-1} > 1, \qquad \text{for } q > 1.$$

On the other hand, it is easy to see that the function

$$G(t) := t^q\left(\frac{q}{q-1}\right)^q - t^{q-1}\left(\frac{q}{q-1}\right)$$

is strictly increasing on $[1,\infty)$. Therefore, since $C > 1$ it follows that

$$\lambda^p = G(C) > G(1) = \left(\frac{q}{q-1}\right)^q - \frac{q}{q-1} > 1,$$

and consequently, $\lambda > 1$.

On the other hand, taking into account relation (41), it follows that

$$\begin{aligned}[\mathcal{M}f^q]\,(n) &\le \frac{1}{\Lambda(n)}\sum_{k=a}^{n} w(k)f^q(k)\\ &\le \frac{\eta}{\Lambda(n+1)}\sum_{k=a}^{n} w(k)f^q(k) = \eta\big[\mathcal{M}^\delta f^q\big](n).\end{aligned} \tag{47}$$

From (46) and (47) we conclude that the sequence $[\mathcal{M}f^q]\,(n)$ satisfies the assumptions of Theorem 1. More precisely, by putting $g(n) = [\mathcal{M}f^q]\,(n)$, $\beta = \eta$ and $c = \lambda$, the inequality (23) reduces to

$$[\mathcal{M}\,[\mathcal{M}f^q]^r]\,(n) \le K\,[\mathcal{M}\,[\mathcal{M}f^q]]^r\,(n),$$

where $r \in \big[1, \frac{B}{B-1}\big)$, $B = \lambda\eta^r$ and $K = \frac{r}{B-r(B-1)} > 0$. Now, since f^q is a non-increasing sequence, by virtue of Remark 1 we have $f^q \le \mathcal{M}f^q$, and then, utilizing (46) we have

$$[\mathcal{M}f^{rq}]\,(n) \le [\mathcal{M}\,[\mathcal{M}f^q]^r]\,(n) \le K\,[\mathcal{M}\,[\mathcal{M}f^q]]^r\,(n) \le K\lambda^r\,[\mathcal{M}f^q]^r\,(n),$$

that is,

$$[\mathcal{M}f^{rq}]\,(n) \le K\lambda^r\,[\mathcal{M}f^q]^r\,(n).$$

Finally, by putting $r = \frac{p}{q}$ in the last inequality, we obtain relation (42) which proves our assertion. □

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