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ON SPECTRAL THEORY OF A K-UNIFORM DIRECTED HYPERGRAPH

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In this paper, we study a k-uniform directed hypergraph in general form and introduce its associated tensors. We present different spectral properties and show that some of them are generalization of the classical results for undirected hypergraphs. The notation of odd-bipartite directed hypergraph are presented and some spectral properties and characterizations of them comparing with ones in undirected hypergraphs are studied. We also introduce power directed hypergraph and cored directed hypergraph and investigate their spectral properties.

1. INTRODUCTION

Directed hypergraphs are deeply used as a successful data structure in modeling the problems arising in computer science [**13**] and operations research, and recently they have found applications in data mining, clustering, association rules [**31**], image processing [**10**] and optical network communications [**21**]. On the other hand, spectral theory of hypergraphs gives useful and important information about them. In 2005 eigenvalues and eigenvectors of a real tensor are defined [**28, 36**]. Qi [**36**] introduced the spectral theory of supersymmetric real tensor. In [**37**] the spectral theory of undirected hypergraphs was presented via eigenvalues and eigenvectors of the adjacency tensor, Laplacian tensor, and signless Laplacian tensor. Recently a number of papers appeared in different aspects such as, spectral hypergraph theory [**7, 8, 15, 23, 24, 26, 32, 34, 40, 41, 46, 53**], eigenvalues [**17, 25, 33, 42, 43, 44, 45, 48, 50**], connectivity [**16, 27**], Laplacian tensor [**1, 18, 20, 35,**

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37, 51], structured tensors related [**5, 9**], special hypergraphs [**2, 19, 22, 38, 49**],hypergraph properties [**3, 11, 14, 29, 30**].

In spite of a lot of researches in the spectral theory of undirected hypergraphs, there is almost blank for spectral directed hypergraph theory. Some of the spectral properties of a special case of k-uniform directed hypergraph, with a tail node and $k-1$ head nodes, were studied in [**46**]. In this paper, we present a generalization of the k-uniform directed hypergraph and introduce its output-adjacency tensor and show that all results in [**46**] are derived from the general form. We present the spectral properties of the generalized directed hypergraphs and extend some classical results of undirected hypergraphs. We also introduce power directed hypergraphs and cored directed hypergraphs and propose some of their spectral properties.

In section 2, we discuss the needed fundamental results of tensors and introduce k-uniform directed hypergraphs in general form with their output-adjacency tensors, Laplacian tensors and signless Laplacian tensors. Section 3 is studied the strongly connected k-uniform directed hypergraph and its associated tensors. We propose some spectral properties of the output-adjacency tensor, Laplacian tensor and signless Laplacian tensor of a general k-uniform directed hypergraph in section 4. In section 5 the notation of odd-bipartite directed hypergraphs and their spectral properties are presented. We also introduce power directed hypergraphs and cored directed hypergraphs in section 6. Finally, section 7 is the conclusion.

2. PRELIMINARIES

We first present some basic definitions of tensors. Then we introduce the general k-uniform directed hypergraph with its adjacency tensor, Laplacian tensor, and signless Laplacian tensor.

2.1 Tensors and some related subjects

A real tensor $\mathcal{T} = (t_{i_1\cdots i_k})$ of order k and dimension n, for integers k≥3 and n≥2, is a multi-dimensional array with entries $t_{i_1\cdots i_k} \in \mathbb{R}$, for $i_j \in [n] := \{1, 2, \cdots, n\}$ and $j \in [k] = \{1, 2, \cdots, k\}$ (see [**36**]).

Definition 1. [**39**]: Let $\mathcal{T}$ be a k order n dimensional tensor and P and Q be $n \times n$ matrices. The tensor $\mathcal{S} = P\mathcal{T}Q^{k-1}$ is a k order n dimensional tensor with the entries

$$s_{i_1\cdots i_k} = \sum_{j_1,\cdots,j_k=1}^{n} t_{j_1\cdots j_k} p_{i_1 j_1} q_{j_2 i_2} \cdots q_{j_k i_k}.$$

Let $x = (x_1, \cdots, x_n)^T \in \mathbb{C}^n$, we write x^k as a k order n dimensional tensor with $(i_1, \cdots, i_k)$-th entry $x_{i_1} x_{i_2} \cdots x_{i_k}$. Then $\mathcal{T}x^{k-1}$ is an n dimensional vector whose ith component is

$$(\mathcal{T}x^{k-1})_i = \sum_{i_2,\cdots,i_k=1}^{n} t_{i i_2\cdots i_k} x_{i_2} \cdots x_{i_k} \qquad \forall i \in [n].$$

The identity tensor of order k and dimension n, $\mathcal{I} = (i_{i_1 \cdots i_k})$, is defined as $i_{i_1 \cdots i_k} = 1$ iff $i_1 = \cdots = i_k \in [n]$ and zero otherwise.

Definition 2. [**6, 36**]: Let $\mathcal{T}$ be a nonzero k order n dimensional tensor. Then $\lambda \in \mathbb{C}$ is called an eigenvalue of $\mathcal{T}$ if the polynomial system $(\lambda\mathcal{I} - \mathcal{T})x^{[k-1]} = 0$ has a nonzero solution $x \in \mathbb{C}^n$, where $x^{[k-1]} = (x_1^{k-1}, \cdots, x_n^{k-1})^T$. In this case x is called an eigenvector of $\mathcal{T}$ corresponding to λ and (λ, x) is called an eigenpair of $\mathcal{T}$.

If $(\lambda, x) \in \mathbb{R} \times \mathbb{R}^n/\{0\}$ then λ is called an H-eigenvalue and x is called an H-eigenvector of $\mathcal{T}$ [**36**].

The set of all eigenvalues of $\mathcal{T}$, denoted by $Spec(\mathcal{T})$, is called the spectrum of $\mathcal{T}$. The H-spectrum of $\mathcal{T}$, denoted by $Hspec(\mathcal{T})$, is defined as follows:

$$Hspec(\mathcal{T}) = \{\lambda \in \mathbb{R} | \lambda \text{ is an H-eigenvalue of } \mathcal{T}\}.$$

The spectral radius of $\mathcal{T}$ is defined as the largest modulus of the eigenvalues of $\mathcal{T}$ and denoted by $\rho(\mathcal{T})$.

Definition 3. [**39**]: Let $\mathcal{T}$ and $\mathcal{S}$ be two k order n dimension tensors. $\mathcal{T}$ and $\mathcal{S}$ are called diagonal similar if there exists a nonsingular diagonal matrix D of order n such that $\mathcal{S} = D^{-(k-1)}\mathcal{T}D^{k-1}$.

2.2 K-uniform directed hypergraph

In this subsection, we present some needed concepts and definitions of directed hypergraphs and then we introduce adjacency tensor of a k-uniform directed hypergraph in general form. The following definition of the k-uniform directed hypergraph was presented in [**10**].

Definition 4. A k-uniform directed hypergraph $\mathcal{H}$ is a pair $= (\mathcal{V}, \mathcal{E})$ where $\mathcal{V} = [n]$ is a set of elements called verteces and $\mathcal{E} = \{\vec{e}_1, \cdots, \vec{e}_m\}$ is the set of arcs. Each $\vec{e}_i$, $(i = 1, \cdots, m)$, is considered as an ordered pair (e_i^+, e_i^-) where e_i^+, e_i^- are two nonempty subsets of $\mathcal{V}$ such that $e_i^+ \cap e_i^- = \phi$, $|e_i^+ \cup e_i^-|$ =k. e_i^+ is called the tail of $\vec{e}_i$ and e_i^- is its head.

Note : we assume that in the k-uniform directed hypergraph for any k vertices there exists at most one arc joining them.
The out-degree of a vertex $j \in \mathcal{V}$ is defined as $d_j^+ = |E_j^+|$, where $E_j^+ = \{\vec{e} \in \mathcal{E} | j \in e^+\}$ and the in-degree of a vertex $j \in \mathcal{V}$ is defined as $d_j^- = |E_j^-|$, where $E_j^- = \{\vec{e} \in \mathcal{E} | j \in e^-\}$. The degree of j is defined as $d_j = d_j^+ + d_j^-$. The hypergraph $\mathcal{H}$ is r-out-regular (or r-in-regular or r-regular, respectively) if for each $j \in \mathcal{V}$, $d_j^+ = r$ (or $d_j^- = r$ or $d_j = r$, respectively).

Let $i, j \in \mathcal{V}$ and $i \neq j$. Two vertices i and j are called weak-connected, if there is a sequence of arcs $\vec{e}_1, \cdots, \vec{e}_l$ such that $i \in e_1^+ \cup e_1^-$, $j \in e_l^+ \cup e_l^-$ and $(e_s^+ \cup e_s^-) \cap (e_{s+1}^+ \cup e_{s+1}^-) \neq \phi$ for all $s \in [l-1]$. Two vertices i and j are called strong-connected, denoted by $i \to j$, if there is a sequence of arcs $\vec{e}_1, \cdots, \vec{e}_l$ such that $i \in e_1^+$, $j \in e_l^-$ and $e_s^- \cap e_{s+1}^+ \neq \phi$ for all $s \in [l-1]$. A directed hypergraph $\mathcal{H}$ is called weak-connected, if every pair of different vertices of $\mathcal{H}$ is weak-connected and $\mathcal{H}$ is called strong-connected, if $i \to j$ and $j \to i$ for all $i, j \in \mathcal{V}$ and $i \neq j$. A directed hypergraph is complete if $\mathcal{E}$ contains of all possible arcs with different number of vertices in their tails.

Now we introduce output-adjacency tensor of a k-uniform directed hypergraph. In [**46**], the authors discussed the case that each arc has only one tail and introduced the adjacency tensor, Laplacian tensor, and signless Laplacian tensor. In this paper we consider the general form of a k-uniform directed hypergraph and present the following definition of its output-adjacency tensor:

Definition 5. The output-adjacency tensor of a k-uniform directed hypergraph $\mathcal{H}$ is the k order n dimensional tensor $\mathcal{A} = (a_{i_1 \cdots i_k})$ whose entries are as follows:

$$a_{i_1 \ldots i_k} = \begin{cases} \frac{1}{(l_{\vec{e}}-1)!(k-l_{\vec{e}})!}, & \text{if } \exists\, \vec{e} = (e^+, e^-) \in \mathcal{E} \text{ s.t } e^+ = \{i_1, \cdots, i_{l_{\vec{e}}}\}, \\ & \qquad\qquad\qquad\qquad\qquad\quad e^- = \{i_{l_{\vec{e}}+1}, \cdots, i_k\} \\ 0, & \text{otherwise.} \end{cases}$$

By Definition 5, it is easy to see that:

$$d_i^+ = \sum_{i_2, \cdots, i_k = 1}^{n} a_{i i_2 \cdots i_k} \qquad \forall i \in \mathcal{V}.$$

That's why we've chosen the name of **output**-adjacency for this tensor.

Similar to [**46**] the degree tensor $\mathcal{D}$ defined as the k order n dimensional diagonal tensor whose diagonal element $d_{i \cdots i}$ is d_i^+, the out-degree of vertex i, for all $i \in [n]$. Also the Laplacian tensor of $\mathcal{H}$ is $\mathcal{L} = \mathcal{D} - \mathcal{A}$ and $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ is the signless Laplacian tensor of $\mathcal{H}$.

Now the following definition of an odd bipartite directed hypergraph is presented (just as undirected hypergraph [**17**]).

Definition 6. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a 4-uniform directed hypergraph. $\mathcal{H}$ is called odd bipartite if k is even and there exists a partitioned of $\mathcal{V}$ so that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$, $\mathcal{V}_1 \neq \phi$ and

$$\forall\, \vec{e} = (e^+, e^-) \in \mathcal{E} \qquad |(e^+ \cup e^-) \cap \mathcal{V}_1| \quad \text{is an odd number.}$$

Example 1. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph, where $\mathcal{V} = [7]$ and $\mathcal{E} = \Big\{(\{1,2,3\},\{4\}), (\{4\},\{5,6,7\})\Big\}$. $\mathcal{H}$ is shown in figure (1).*
The adjacency tensor of $\mathcal{H}$ is $\mathcal{A} = (a_{i_1 i_2 i_3 i_4})$, where $1 \leq i_1, i_2, i_3, i_4 \leq 7$ and we have, $a_{1234} = a_{1324} = a_{2134} = a_{2314} = a_{3124} = a_{3124} = \dfrac{1}{2!1!} = \dfrac{1}{2}$ and $a_{4567} = a_{4576} = a_{4657} = a_{4675} = a_{4756} = a_{4765} = \dfrac{1}{0!3!} = \dfrac{1}{6}$, and the other elements of $\mathcal{A}$ are zero. By Definition 6, $\mathcal{H}$ is odd bipartite in which $\mathcal{V}_1 = \{4\}$.

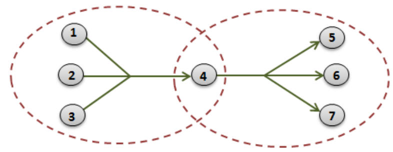

Figure 1: A 4-uniform directed hypergraph

3. A STRONGLY CONNECTED DIRECTED HYPERGRAPH AND ITS ASSOCIATED TENSORS

The notation of weakly irreducible nonnegative tensors was introduced in [**12**].

Definition 7. Let $\mathcal{T} = (t_{i_1 \cdots i_k})$ be a k order n dimension nonnegative tensor and $G(\mathcal{T}) = (V, E(\mathcal{T}))$ be a directed graph, where $V = [n]$ and a directed edge $(i,j) \in E(\mathcal{T})$ if there exists $\{i_2, \cdots, i_k\} \in [n]$ such that $j \in \{i_2, \cdots, i_k\}$ and $t_{ii_2 \ldots i_k} > 0$. Now $\mathcal{T}$ is called weakly irreducible if $G(\mathcal{T})$ is strongly connected.

Let $\mathcal{H}$ be a k-uniform undirected hypergraph, then the output-adjacency of $\mathcal{H}$, $\mathcal{A}$, is weakly irreducible iff $\mathcal{H}$ is connected [**12**]. In the k-uniform directed hypergraph and the special case in which each arc has only one tail, $\mathcal{A}$ is weakly irreducible iff $\mathcal{H}$ is strongly connected, i.e. the strong connectivity of $\mathcal{H}$ is equivalent to strong connectivity of $G(\mathcal{A})$. But we have the following lemma in the general form:

Lemma 1. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph with output-adjacency tensor $\mathcal{A}$. Then, $\mathcal{A}$ is weakly irreducible if $\mathcal{H}$ is strongly connected.*

Proof. Suppose that $\mathcal{H}$ is strongly connected. By Definition 7, it is enough to show that $G(\mathcal{T})$ is strongly connected. Let $i, j \in V$ and $i \neq j$. Since $\mathcal{H}$ is strongly connected, there exist a sequence of vertices and arcs in $\mathcal{H}$ such that:

$$i = j_1 \quad \vec{e}_1 \quad j_2 \quad \vec{e}_2 \quad j_3 \quad \cdots \quad \vec{e}_{q-1} \quad j_q \quad \vec{e}_q \quad j_{q+1} = j$$

where $j_2, \cdots, j_q \in \mathcal{V}$, $\vec{e}_1, \cdots, \vec{e}_q \in \mathcal{E}$ and $j_t \in e_t^+$, $j_{t+1} \in e_t^-$ for all $t = 1, \cdots, q$. On other hand $a_{e_t^+ e_t^-} > 0$ for $t = 1, \cdots, q$, since $\vec{e}_t = (e_t^+, e_t^-) \in \mathcal{E}$. Hence $e_t = (j_t, j_{t+1})$ is a directed edge in $G(\mathcal{A})$, for all $t = 1, \cdots, q$. Therefore there exists a sequence of vertices and directed edges in $G(\mathcal{A})$:

$$i = j_1 \quad e_1 \quad j_2 \quad e_2 \quad j_3 \quad \cdots \quad e_{q-1} \quad j_q \quad e_q \quad j_{q+1} = j$$

i.e. $i \to j$ in $G(\mathcal{A})$. Similarly, it can be proved $j \to i$ in $G(\mathcal{A})$. Thus $G(\mathcal{A})$ is strongly connected and then $\mathcal{A}$ is weakly irreducible. $\square$

Clearly if $\mathcal{A}$ is weakly irreducible then $\mathcal{Q} = \mathcal{D} + \mathcal{A}$ is also weakly irreducible. Note that the Definition 7 is only for nonnegative tensors, however, Qi in [**37**] removed the nonnegativity restriction and present the definition of weakly irreducible tensor in general. By this, if $\mathcal{A}$ is weakly irreducible then $\mathcal{L}$ is weakly irreducible, too. Therefore if $\mathcal{H}$ is strongly connected then $\mathcal{A}$, $\mathcal{Q}$ and $\mathcal{L}$ are weakly irreducible.

4. H-EIGENVALUES OF OUTPUT-ADJACENCY TENSOR, LAPLACIAN TENSOR AND SIGNLESS LAPLACIAN TENSOR OF $\mathcal{H}$

H-eigenvalues of the tensors associated to a k-uniform directed hypergraph are studied in this section. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph with n vertices and $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{Q}$ be the output-adjacency tensor, Laplacian tensor, and signless Laplacian tensor, respectively. We have the following lemma:

Lemma 2. *Let $\mathcal{H}$ be a k-uniform directed hypergraph with n vertices and $\mathcal{A}$ be its output-adjacency tensor. Suppose that $\mathbf{x} \in \mathbb{R}^n$, then we have:*

$$(\mathcal{A}\mathbf{x}^{[k-1]})_i = \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \qquad \forall i \in [n].$$

Proof. Suppose that $i \in [n]$, we have:

$$\begin{aligned}(\mathcal{A}\mathbf{x}^{[k-1]})_i &= \sum_{i_2,\cdots,i_k=1}^{n} a_{ii_2\cdots i_k} x_{i_2}\cdots x_{i_k} \\ &= \sum_{\substack{\vec{e}=(e^+,e^-)\in\mathcal{E} \\ i\in e^+,\ |e^+|=l_{\vec{e}}}} \frac{(l_{\vec{e}}-1)!(k-l_{\vec{e}})!}{(l_{\vec{e}}-1)!(k-l_{\vec{e}})!} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \\ &= \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s.\end{aligned}$$

□

Similarly for $\mathcal{L}$ and $\mathcal{Q}$ we have:

$$(\mathcal{L}\mathbf{x}^{[k-1]})_i = d_i^+ - \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \qquad \forall i \in [n]. \tag{1}$$

and

$$(\mathcal{Q}\mathbf{x}^{[k-1]})_i = d_i^+ + \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \qquad \forall i \in [n]. \tag{2}$$

Now we have the following theorems.

Theorem 1. *Let $\mathcal{H}$ be a k-uniform directed hypergraph with n vertices and $\mathcal{A}$ be its output-adjacency tensor. Then $\lambda = 0$ is an H-eigenvalue of $\mathcal{A}$.*

Proof. Let $\mathbf{x}$ is a vector in $\mathbb{R}^n$ that $|supp(\mathbf{x})| \leq n-2$, where $supp(\mathbf{x}) = \{x_i |\ x_i \neq 0\}$. Then by Lemma 2 we have:

$$(\mathcal{A}\mathbf{x}^{[k-1]})_i = \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s = 0 = 0x_i^{k-1} \qquad \text{for all } i = 1,\cdots,n.$$

The conclusion directly follows from Definition 2. □

Theorem 1 shows that $\{0\} \subset Hspec(\mathcal{A})$. Now we introduce the sufficient condition for $\{0\} = Hspec(\mathcal{A})$.

Theorem 2. *Let $\mathcal{H}$ be a k-uniform directed hypergraph with n vertices and $\mathcal{A}$ be its output-adjacency tensor. If for each $i \in \mathcal{V}$ either $d_i^+ = 0$ or there exist directed paths P_j for $j = 1,\cdots,d = d_i^+$ such that*

$$P_j\ :\ i\ \ \vec{e}_{j_1}\ \ i_{j_1}\ \ \vec{e}_{j_2}\ \ i_{j_2}\ \ \cdots\ \ \vec{e}_{j_{|P_j|}}\ \ i_{j_{|P_j|}} \qquad\qquad j = 1,\cdots,d,$$

and $d^+_{i_{|P_j|}} = 0$ for all $j = 1,\cdots,d$ and

$$E_i^+ = \{\vec{e}_{1_1}, \vec{e}_{2_1}, \cdots, \vec{e}_{d_1}\}.$$

Then $\lambda = 0$ is the only H-eigenvalue of $\mathcal{A}$.

Proof. By Theorem 1 $\lambda = 0$ is an H-eigenvalue of $\mathcal{A}$. Now suppose that $\lambda \neq 0$ is an H-eigenvalue of $\mathcal{A}$ and $\mathbf{x}$ is its associated H-eigenvector. We show that $\mathbf{x} = 0$, that is a contradiction.

Let $i \in \mathcal{V}$ be a vertex and $d_i^+ = d$. Suppose that $d \geq 1$. By assumption there exist directed paths $P_1,\ P_2,\cdots,P_d$ from i to i_j with $d^+_{i_j} = 0$. Let

$$q_i = \max\{|P_j|\ \big|\ j = 1,\cdots,d\}.$$

then $q_i \geq 1$. In the case of $d = 0$ let $q_i = 0$. By induction on q_i we show that $x_i = 0$ for all $i \in \mathcal{V}$.

If $q_i = 0$ then $d_i^+ = 0$. By Definition 2 it is trivial that $x_i = 0$. Now suppose that for all $i \in \mathcal{V}$, if $q_i \leq m$ then $x_i = 0$, where m is a nonnegative integer number. Let $i \in \mathcal{V}$ be a vertex that $q_i = m+1$, we show that $x_i = 0$. By assumption there exist directed paths $P_1,\ P_2,\cdots,P_d$, where $d = d_i^+$, such that

$$P_j\ :\ i\ \ \vec{e}_{j_1}\ \ i_{j_1}\ \ \vec{e}_{j_2}\ \ i_{j_2}\ \ \cdots\ \ \vec{e}_{j_{|P_j|}}\ \ i_{j_{|P_j|}} \qquad\qquad j = 1,\cdots,d.$$

and $d^+_{i_{j_{|P_j|}}} = 0$.

Since each path from vertex i_{j_1} to a vertex with zero out-degree along with $\vec{e}_{j_1}$ is a directed path from vertex i to the same vertex, then

$$q_{i_{j_1}} + 1 \leq q_i = m+1 \quad\Longrightarrow\quad q_{i_{j_1}} \leq m.$$

Then by induction assumption $x_{i_{j_1}} = 0$ for $j = 1, \cdots, d$. Now by Definition 2 and by $E_i^+ = \{\vec{e}_{1_1}, \vec{e}_{2_1}, \cdots, \vec{e}_{d_1}\}$, we have for each $i \in [n]$:

$$(\mathcal{A}\mathbf{x}^{[k-1]})_i = \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s = 0$$

$$\Longrightarrow \quad \lambda x_i^{k-1} = 0 \overset{\lambda \neq 0}{\Longrightarrow} x_i = 0.$$

□

It seems that the above condition is the necessary condition, too. But at this point we are unable to prove it, hence we consider it as a conjecture.

Conjecture 1. *Let $\mathcal{H}$ be a k-uniform directed hypergraph with n vertices and $\mathcal{A}$ be its output-adjacency tensor. $\lambda = 0$ is the only H-eigenvalue of $\mathcal{A}$ if and only if for each $i \in \mathcal{V}$ either $d_i^+ = 0$ or there exist directed paths P_j for $j = 1, \cdots, d = d_i^+$ such that*

$$P_j \ : \ i \ \ \vec{e}_{j_1} \ \ i_{j_1} \ \ \vec{e}_{j_2} \ \ i_{j_2} \ \ \cdots \ \ \vec{e}_{j_{|P_j|}} \ \ i_{j_{|P_j|}} \qquad\qquad j = 1, \cdots, d.$$

and $d^+_{i_{|P_j|}} = 0$ for all $j = 1, \cdots, d$ and

$$E_i^+ = \{\vec{e}_{1_1}, \vec{e}_{2_1}, \cdots, \vec{e}_{d_1}\}.$$

By Theorem 1 $HSpec(\mathcal{A}) \neq \phi$. Next theorem gives bounds on H-eigenvalues of $\mathcal{A}$.

Theorem 3. *Let $\mathcal{H}$ be a k-uniform directed hypergraph with n vertices and $\mathcal{A}$ be its output-adjacency tensor. Suppose that λ is a H-eigenvalue of $\mathcal{A}$, then we have:*

$$-\Delta^+ \leq \lambda \leq \Delta^+,$$

where Δ^+ is the maximum out-degree in $\mathcal{H}$.

Proof. The conclusion follows from Lemma 2 and the proof of Theorem 3.1 in [**46**]. □

The largest H-eigenvalue of tensor $\mathcal{A}$ is denoted by $\lambda(\mathcal{A})$. By Theorem 3 $\lambda(\mathcal{A}) \leq \Delta^+$. The next theorem determines $\lambda(\mathcal{A})$ in a n-vertex k-uniform complete directed hypergraph.

Lemma 3. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a n-vertex k-uniform complete directed hypergraph and $i \in \mathcal{V}$ be an arbitrary vertex, Then $d_i = \binom{n-1}{k-1}$.*

Proof. Since $\mathcal{H}$ is complete then $\mathcal{E}$ contains all possible arcs. Therefore vertex i has common arcs with any $k-1$ vertices that is $\binom{n-1}{k-1}$. Then $d_i = d_i^+ + d_i^- = \binom{n-1}{k-1}$. □

Theorem 4. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a n-vertex k-uniform complete directed hypergraph and $\mathcal{A}$ be its output-adjacency tensor. Then the largest H-eigenvalue of tensor $\mathcal{A}$, $\lambda(\mathcal{A})$, is not greater than $\binom{n-1}{k-1}$.*

Proof. The result follows from Lemma 3 and Theorem 3. □

However similar result holds for directed out-regular hypergraph (Corollary 3.3 in [**46**] together with Lemma 2).

In the sequel, we study the H-eigenvalues of the Laplacian and signless Laplacian tensor of $\mathcal{H}$. In [**46**] some theorems about Laplacian and signless Laplacian spectral properties of a special k-uniform directed hypergraph were presented. On the other hand, by Lemma 2 and by (1) and (2) for a k-uniform directed hypergraph in general form, we have:

$$(\mathcal{A}\mathbf{x}^{[k-1]})_i = \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s \qquad \forall i \in [n],$$

$$(\mathcal{L}\mathbf{x}^{[k-1]})_i = d_i^+ - \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s \qquad \forall i \in [n],$$

$$(\mathcal{Q}\mathbf{x}^{[k-1]})_i = d_i^+ + \sum_{\vec{e} \in E_i^+} \prod_{s \in (e^+ \cup e^-) \setminus \{i\}} x_s \qquad \forall i \in [n].$$

Now by replacing above expressions in the proof of Theorems in [**46**], we have similar results for a k-uniform directed hypergraph in general form, too. In continuation, we present some other theorems in spectral properties of the k-uniform directed hypergraph.

Theorem 5. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed complete hypergraph with $2m$ vertices and in which $d_i^+ = \binom{n-1}{k-1}$ for each $i \in \mathcal{V}$ and let $\mathcal{L}$ be its Laplacian tensor. If k is an odd number then*

$$\lambda(\mathcal{L}) \geq \binom{n-1}{k-1} + \sum_{l=1}^{k-1} (-1)^{l+1} \binom{m-1}{l} \binom{m}{k-1-l}.$$

Proof. Let $\mathbf{x} \in \mathbb{R}^{2m}$ and $x_1 = x_2 = \cdots = x_m = 1$ and $x_{m+1} = x_{m+2} = \cdots = x_{2m} = -1$. We show that $\mathbf{x}$ is an H-eigenvector of $\mathcal{H}$ corresponding to $\lambda = \binom{n-1}{k-1} + \sum_{l=1}^{k-1} (-1)^{l+1} \binom{m-1}{l} \binom{m}{k-1-l}$. By Definition 2 we have:

$$(\mathcal{L}\mathbf{x}^{k-1})_1 = \binom{n-1}{k-1} x_1^{k-1} - \sum_{\substack{\vec{e}=(e^+,e^-) \in E_1^+ \\ \{i_1, \cdots, i_{k-1}\} = e^+ \cup e^- \setminus \{1\}}} x_{i_1} x_{i_2} \cdots x_{i_{k-1}},$$

now since $\mathcal{H}$ is a complete hypergraph then we have:

$$\sum_{\substack{\vec{e}=(e^+,e^-) \in E_1^+ \\ \{i_1, \cdots, i_{k-1}\} = e^+ \cup e^- \setminus \{1\}}} x_{i_1} x_{i_2} \cdots x_{i_{k-1}} = \sum_{1 < i_1 < i_2 < \cdots < i_{k-1} \leq 2m} x_{i_1} x_{i_2} \cdots x_{i_{k-1}}.$$

Now according to the how x_{i_j}s are selected and since k is an odd number we have:

$$\sum_{1<i_1<i_2<\cdots<i_{k-1}\leq 2m} x_{i_1}x_{i_2}\cdots x_{i_{k-1}} = \sum_{l=1}^{k-1}(-1)^l\binom{m-1}{l}\binom{m}{k-1-l}.$$

Therefore

$$(\mathcal{L}\mathbf{x}^{k-1})_1 = \binom{n-1}{k-1} + \sum_{l=1}^{k-1}(-1)^{l+1}\binom{m-1}{l}\binom{m}{k-1-l} = \lambda x_1^{k-1}.$$

It is easy to see that for any $i = 1, 2, \cdots, m$, we have:

$$(\mathcal{L}\mathbf{x}^{k-1})_i = (\mathcal{L}\mathbf{x}^{k-1})_1 = \lambda x_i^{k-1}.$$

We also have for vertex $n = 2m$:

$$\begin{aligned}
(\mathcal{L}\mathbf{x}^{k-1})_n &= \binom{n-1}{k-1}x_n^{k-1} - \sum_{\substack{\vec{e}=(e^+,e^-)\in E_n^+ \\ \{i_1,\cdots,i_{k-1}\}=e^+\cup e^-\setminus\{n\}}} x_{i_1}x_{i_2}\cdots x_{i_{k-1}} \\
&= \binom{n-1}{k-1} - \sum_{1\leq i_1<i_2<\cdots<i_{k-1}<2m} x_{i_1}x_{i_2}\cdots x_{i_{k-1}} \\
&= \binom{n-1}{k-1} + \sum_{l=1}^{k-1}(-1)^{l+1}\binom{m-1}{l}\binom{m}{k-1-l} = \lambda x_n^{k-1}.
\end{aligned}$$

similarly, for any $i = m, m+1, \cdots, 2m$ we have that:

$$(\mathcal{L}\mathbf{x}^{k-1})_i = (\mathcal{L}\mathbf{x}^{k-1})_n = \lambda x_i^{k-1}.$$

□

The next theorem characterizes the extreme weakly connected directed hypergraphs concerning the upper bound of the largest signless Laplacian H-eigenvalue.

Theorem 6. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a weakly connected k-uniform directed hypergraph and $\mathcal{Q}$ be its signless Laplacian tensor. Then $\lambda(\mathcal{Q}) = 2\Delta^+$ if and only if $\mathcal{H}$ is out-regular.*

Proof. Suppose that $\mathcal{H}$ is out-regular. By Corollary (5.3) in [**46**], $\lambda(\mathcal{Q}) = 2\Delta^+$. On the other hand, assume that $\lambda(\mathcal{Q}) = 2\Delta^+$ and $\mathbf{x} \in \mathbb{R}^n$ is its corresponding

H-eigenvector. Let $|x_i| = max\ \{|x_j| \big| j \in [n]\}$. By Definition 2 we have:

$$
\begin{aligned}
&2\Delta^+ x_i^{k-1} = d_i^+ x_i^{k-1} + \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} x_s \\
&\Rightarrow 2\Delta^+ |x_i^{k-1}| \leq d_i^+ |x_i^{k-1}| + \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} |x_s| \\
&\Rightarrow 2\Delta^+ \leq d_i^+ + \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i\}} \frac{|x_s|}{|x_i|} \leq d_i^+ + \sum_{\vec{e}\in E_i^+} 1 = 2d_i^+ \\
&\Rightarrow \Delta^+ \leq d_i^+ \\
&\Rightarrow \Delta^+ = d_i^+.
\end{aligned}
$$

and we must have $|x_i| = |x_j|$ for all $j \in e^+ \cup e^-$, where $\vec{e} = (e^+, e^-) \in E_i^+$. Applying the same argument for all such j, we have that $\Delta^+ = d_j^+$ and $|x_i| = |x_j| = |x_l|$ for all $l \in e^+ \cup e^-$ where $\vec{e} = (e^+, e^-) \in E_j^+$. Since $\mathcal{H}$ is weakly connected, we see that $d_j^+ = \Delta^+$ for all $j \in \mathcal{V}$, then $\mathcal{H}$ is out-regular. □

With a similar discussion as in Theorem 6, we have the following theorem:

Theorem 7. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a weakly connected k-uniform directed hypergraph and $\mathcal{L}$ be its signless Laplacian tensor. If $\lambda(\mathcal{L}) = 2\Delta^+$ then $\mathcal{H}$ is out-regular.*

Suppose that $\mathbf{x}$ is an H-eigenvector of the signless Laplacian of a k-uniform directed hypergraph corresponding to H-eigenvalue λ. The following theorem gives a sufficient condition for equality of some components of $\mathbf{x}$.

Theorem 8. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed hypergraph and $i, j \in \mathcal{V}$ such that $E_i^+ = E_j^+$. Then $d_i^+ = d_j^+ = d$. Now Suppose that $(\lambda, \mathbf{x})$ is a signless Laplacian H-eigenpair of $\mathcal{H}$, such that $\lambda \neq d$. Then $|x_i| = |x_j|$ and if k is odd, then $x_i = x_j$.*

Proof. Clearly $d_i^+ = d_j^+ = d$ By the definition of E_i^+. Now Suppose that $(\lambda, \mathbf{x})$ is a signless Laplacian H-eigenpair of $\mathcal{H}$, such that $\lambda \neq d$. By Definition 2 we have:

$$\lambda x_i^{k-1} = d x_i^{k-1} + x_j \sum_{\vec{e}\in E_i^+} \prod_{s\in(e^+\cup e^-)\setminus\{i,j\}} x_s$$

and

$$\lambda x_j^{k-1} = d x_j^{k-1} + x_i \sum_{\vec{e}\in E_j^+} \prod_{s\in(e^+\cup e^-)\setminus\{i,j\}} x_s.$$

Hence,

$$(\lambda - d) x_i^k = (\lambda - d) x_j^k \quad \xRightarrow{\lambda\neq d} \quad x_i^k = x_j^k.$$

The conclusions follow from the last equality. □

Similar results hold for Laplacian tensor and for output-adjacency tensor.

5. ODD-BIPARTITE K-UNIFORM DIRECTED HYPERGRAPHS AND SOME OF THEIR SPECTRAL PROPERTIES

In this section, we present some theorems about the spectral theory of odd-bipartite k-uniform directed hypergraphs. These theorems generalize the classical results for k-uniform undirected hypergraphs in [**41**].

Theorem 2.1 in [**41**] can be represented for k-uniform directed hypergraphs. The concept of diagonal similarity is used in the proof of this theorem. Shao in [**39**] has proved that similar tensors have the same spectra. Note that the proof is valid for all tensors and not necessarily for symmetric tensors. So we have the following theorem:

Theorem 9. *Let $\mathcal{H}$ be a k-uniform directed hypergraph with n vertices and k be an even number. Suppose that $\mathcal{A}$, $\mathcal{L}$ and $\mathcal{Q}$ be the output-adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of $\mathcal{H}$, respectively. Then the following three statements are equivalent:*

- *There exists some diagonal matrix P of order n with all the diagonal entries ± 1 and $P \neq -I_n$ such that $\mathcal{L} = P^{-(k-1)}\mathcal{Q}P$.*
- *There exists some diagonal matrix P of order n with all the diagonal entries ± 1 and $P \neq -I_n$ such that $\mathcal{A} = -P^{-(k-1)}\mathcal{A}P$.*
- *$\mathcal{H}$ is odd-bipartite.*

Proof. The proof is in the same spirit of and similar to that for Theorem 2.1 in [**41**]. □

In [**12, 4**] the well known Perron-Frobenius Theorem is generalized for nonnegative weakly irreducible tensors:

Lemma 4. *([**12**]): Let $\mathcal{T}$ be a nonnegative tensor. Then*

1. *$\rho(T)$ is an H-eigenvalue of $\mathcal{T}$ with a nonnegative eigenvector. Furthermore, if $\mathcal{T}$ is weakly irreducible, then $\rho(T)$ has a positive eigenvector.*
2. *If λ is an eigenvalue of $\mathcal{T}$ with a positive eigenvector, then $\lambda = \rho(\mathcal{T})$.*

Lemma 5. *([**47**]): Let $\mathcal{T}$ and $\mathcal{S}$ be two k order n dimensional tensors with $|\mathcal{S}| \leq \mathcal{T}$. Then*

1. *$\rho(\mathcal{S}) \leq \rho(\mathcal{T})$.*
2. *Furthermore, if $\mathcal{T}$ is weakly irreducible and $\rho(\mathcal{S}) = \rho(\mathcal{T})$, where $\lambda = \rho(\mathcal{T})e^{i\phi}$ is an eigenvalue of $\mathcal{S}$ with an eigenvector y, then*

i. All the components of y are nonzero.

ii. Let $U = diag(y_1 \setminus |y_1|, , y_n \setminus |y_n|)$ *be a nonsingular diagonal matrix, then we have* $\mathcal{S} = e^{i\phi} U^{-()k-1} \mathcal{T} U$.

As said before, if the k-uniform directed hypergraph $\mathcal{H}$ is strongly connected then $\mathcal{A}, \mathcal{L}$ and $\mathcal{Q}$ are weakly irreducible. Then Lemma 4 and Lemma 5 hold for $\mathcal{L}$ and $\mathcal{Q}$. Therefore according to this, we have the following theorems that generalize Theorem 2.2 and Theorem 2.4 in [**41**], respectively.

Theorem 10. *Let* $\mathcal{H}$ *be a strongly connected* k*-uniform directed hypergraph with* n *vertices and* k *be an even number. Suppose that* $\mathcal{A}$, $\mathcal{L}$ *and* $\mathcal{Q}$ *be the output-adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of* $\mathcal{H}$*, respectively. Then the following three statements are equivalent:*

- $\mathcal{H}$ *is odd-bipartite.*
- *Spec(*$\mathcal{L}$*) = Spec(*$\mathcal{Q}$*) and Hspec(*$\mathcal{L}$*) = Hspec(*$\mathcal{Q}$*).*
- *Hspec(*$\mathcal{L}$*) = Hspec(*$\mathcal{Q}$*).*

Theorem 11. *Let* $\mathcal{H}$ *be a strongly connected* k*-uniform directed hypergraph with* n *vertices and* k *be an even number. Suppose that* $\mathcal{A}$, $\mathcal{L}$ *and* $\mathcal{Q}$ *be the output-adjacency tensor, the Laplacian tensor and the signless Laplacian tensor of* $\mathcal{H}$*, respectively. Then* $\rho(\mathcal{L}) = \rho(\mathcal{Q})$ *if and only if Spec(*$\mathcal{L}$ *) = Spec(*$\mathcal{Q}$*).*

6. CORED DIRECTED HYPERGRAPHS AND POWER DIRECTED HYPERGRAPHS

In this section we introduce two classes of k-uniform directed hypergraphs: 1. Cored directed hypergraphs and 2. Power directed hypergraphs. Hu, Qi and Shao in [**19**] introduced these two classes in undirected hypergraphs and investigated the properties of their Laplacian H-eigenvalues and then Yue et. al in [**52**] studied the properties of their output-adjacency and signless Laplacian H-eigenvalues. We extend their definitions and analyze the spectral properties of power directed hypergraphs and cored directed hypergraphs.

6.1 Cored directed hypergraphs

We begin with the definition of Cored directed hypergraphs.

Definition 8. Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a directed hypergraph. $\mathcal{H}$ is a cored directed hypergraph if there exists in each arc $\vec{e} = (e^+, e^-)$ a vertex $i \in e^+$ such that $d_i^+ = 1$ and $d_i^- = 0$. Such vertex is called core vertex and a vertex with out-degree greater than one is called intersection vertex.

Lemma 6. *Let* $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ *be a cored* k*-uniform directed hypergraph and* $(\lambda, \mathbf{x})$ *be an H-eigenpair of its output-adjacency tensor* $\mathcal{A}$ *and* $\lambda \neq 0$*. If* i *and* j *be two core vertices in arc* $\vec{e}$ *then* $x_i = x_j$ *when* k *is odd and* $|x_i| = |x_j|$ *when* k *is even.*

Proof. Clearly $d_i^+ = d_j^+ = 1$ by Definition 8. Now Suppose that $(\lambda, \mathbf{x})$ is a H-eigenpair of $\mathcal{A}$, such that $\lambda \neq 0$. By Definition 2 we have:

$$\lambda x_i^{k-1} = x_j \prod_{s\in(e^+\cup e^-)\setminus\{i,j\}} x_s$$

and

$$\lambda x_j^{k-1} = x_i \prod_{s\in(e^+\cup e^-)\setminus\{i,j\}} x_s.$$

Hence,

$$\lambda x_i^k = \lambda x_j^k \quad \overset{\lambda\neq 0}{\Longrightarrow} \quad x_i^k = x_j^k.$$

The conclusions follow from the last equality. □

Lemma 7. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a cored k-uniform directed hypergraph and $(\lambda, \mathbf{x})$ be an H-eigenpair of its signless Laplacian tensor $\mathcal{Q}$ and $\lambda \neq 1$. If i and j be two core vertices in arc $\vec{e}$ then $x_i = x_j$ when k is odd and $|x_i| = |x_j|$ when k is even.*

Proof. Clearly $d_i^+ = d_j^+ = 1$, by Definition 8. Now the conclusions follow from the Theorem 8 and since $\lambda \neq 1$. □

With proof similar to the proof of Theorem 8, for Laplacian tensor, we have the following lemmas:

Lemma 8. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a core k-uniform directed hypergraph and $(\lambda, \mathbf{x})$ be an H-eigenpair of its Laplacian tensor $\mathcal{L}$ and $\lambda \neq 1$. If i and j be two cored vertices in arc $\vec{e}$ then $x_i = x_j$ when k is odd and $|x_i| = |x_j|$ when k is even.*

Theorem 12. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a cored k-uniform directed hypergraph and $\mathbf{x}$ be an H-eigenvector of its output-adjacency tensor $\mathcal{A}$ corresponding $\lambda(\mathcal{A})$. If $i_{\vec{e}}$ is a core vertex in arbitrary arc $\vec{e}$, then $\prod_{s\in(e^+\cup e^-)\setminus\{i_{\vec{e}}\}} x_s \geq 0$ when k is odd and $\prod_{s\in(e^+\cup e^-)} x_s \geq 0$ when k is even.*

Proof. By Theorem 1 $\lambda(\mathcal{A}) \geq 0$. By Definition 2 we have:

$$(\mathcal{A}\mathbf{x}^{[k-1]})_{i_{\vec{e}}} = \prod_{s\in(e^+\cup e^-)\setminus\{i_{\vec{e}}\}} x_s = \lambda(\mathcal{A}) x_{i_{\vec{e}}}^{k-1}$$

$$\Longrightarrow \begin{cases} \prod_{s\in(e^+\cup e^-)\setminus\{i_{\vec{e}}\}} x_s \geq 0 & \textit{if } k \textit{ is odd} \\[2ex] \prod_{s\in(e^+\cup e^-)} x_s \geq 0 & \textit{if } k \textit{ is even} \end{cases}$$

□

similarly, for $(\lambda(\mathcal{Q}), \mathbf{x})$ we have:

$$\text{if } i_{\vec{e}} \text{ is a core vertex in } \vec{e} \text{ then} \begin{cases} \prod_{s\in(e^+\cup e^-)\setminus\{i_{\vec{e}}\}} x_s \geq 0 & \text{if } k \text{ is odd} \\ \prod_{s\in(e^+\cup e^-)} x_s \geq 0 & \text{if } k \text{ is even} \end{cases}$$

and for $(\lambda(\mathcal{L}), \mathbf{x})$ we have:

$$\text{if } i_{\vec{e}} \text{ is a core vertex in } \vec{e} \text{ then} \begin{cases} \prod_{s\in(e^+\cup e^-)\setminus\{i_{\vec{e}}\}} x_s \leq 0 & \text{if } k \text{ is odd} \\ \prod_{s\in(e^+\cup e^-)} x_s \leq 0 & \text{if } k \text{ is even} \end{cases}$$

The last expression is the extension of Proposition 3.1 in [**19**].

Theorem 10 gives a necessary and sufficient condition for $Hspec(\mathcal{L}) = Hspec(\mathcal{Q})$ in strongly connected even uniform directed hypergraphs. It's trivial that cored directed hypergraphs are not strongly connected but we can propose the next theorem in which we use the proof of Theorems 2.1 and 2.2 in [**41**]. first the following lemma is presented.

Lemma 9. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a cored directed even uniform hypergraph. Then $\mathcal{H}$ is odd-bipartite.*

Proof. Let $i_{\vec{e}}$ be a core vertex in arc $\vec{e}$ for all arc $\vec{e} \in \mathcal{E}$. Set $\mathcal{V}_1 := \{i_{\vec{e}} \mid \vec{e} \in \mathcal{E}\}$ and $\mathcal{V}_2 := \mathcal{V} \setminus \mathcal{V}_1$. Now it is easy to see that $\mathcal{V} = \mathcal{V}_1 \cup \mathcal{V}_2$ is an odd-bipartition of $\mathcal{H}$. □

Theorem 13. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a cored directed even uniform hypergraph with n vertices and $\mathcal{L}$ and $\mathcal{Q}$ be its Laplacian and signless Laplacian tensor, respectively. Then $Hspec(\mathcal{L}) = Hspec(\mathcal{Q})$.*

Proof. By Theorem 9 there exists some diagonal matrix P of order n with all the diagonal entries ± 1 and $P \neq -I_n$ such that $\mathcal{L} = P^{-(k-1)}\mathcal{Q}P$. Now let $\mathbf{x} \in \mathbb{R}^n$ and $\mathbf{y} = P\mathbf{x}$. Then by $\mathcal{L} = P^{-(k-1)}\mathcal{Q}P$ we have:

$$\mathcal{L}\mathbf{x} = \lambda\mathbf{x}^{[k-1]} \iff P^{-(k-1)}\mathcal{Q}P\mathbf{x} = \lambda\mathbf{x}^{[k-1]}$$
$$\iff \mathcal{Q}\mathbf{y} = \lambda P^{(k-1)}\mathbf{x}^{[k-1]} = \lambda(P\mathbf{x})^{[k-1]} \iff \mathcal{Q}\mathbf{y} = \lambda\mathbf{y}^{[k-1]}$$

Since P is a real nonsingular matrix, the above relation show that λ is an H-eigenvalue of $\mathcal{L}$ if and only if it is an H-eigenvalue of $\mathcal{Q}$. Then we have $Hspec(\mathcal{L}) = Hspec(\mathcal{Q})$. □

Theorem 14. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a cored directed even uniform hypergraph with n vertices and $\mathcal{L}$ and $\mathcal{Q}$ be its Laplacian and signless Laplacian tensor, respectively. For every $\vec{e} \in \mathcal{E}$, let $i_{\vec{e}} \in \vec{e}$ be a core vertex, then we have:*

(1) If $\mathbf{x} \in \mathbb{R}^n$ is a H-eigenvector of $\mathcal{L}$ corresponding to $\lambda(\mathcal{L})$ then $\mathbf{y} \in \mathbb{R}^n$ is a H-eigenvector of $\mathcal{Q}$ corresponding to $\lambda(\mathcal{Q})$ in which $y_{i_{\vec{e}}} = -x_{i_{\vec{e}}}$ for all $\vec{e} \in \mathcal{E}$ and $y_i = x_i$ for the others.

(2) If $\mathbf{x} \in \mathbb{R}^n$ is a H-eigenvector of $\mathcal{Q}$ corresponding to $\lambda(\mathcal{Q})$ then $\mathbf{y} \in \mathbb{R}^n$ is a H-eigenvector of $\mathcal{L}$ corresponding to $\lambda(\mathcal{L})$ in which $y_{i_{\vec{e}}} = -x_{i_{\vec{e}}}$ for all $\vec{e}$ and $y_i = x_i$ for the others.

Proof. By Theorem 13, $\lambda(\mathcal{L}) = \lambda(\mathcal{Q})$. Let P be a diagonal matrix of order n with $p_{ii} = -1$ if $i = i_{\vec{e}}$ for some $\vec{e} \in \mathcal{E}$ and $p_{ii} = 1$ for the others. So P is nonsingular matrix. Now by using the similar proof of the above theorem, we have if $\mathbf{x}$ is a H-eigenvector of $\mathcal{L}$ corresponding to $\lambda(\mathcal{L})$ then $\mathbf{y} = P\mathbf{x}$ is a H-eigenvector of $\mathcal{Q}$ corresponding to $\lambda(\mathcal{Q})$. It is easy to see that $y_{i_{\vec{e}}} = -x_{i_{\vec{e}}}$ for all $\vec{e} \in \mathcal{E}$ and $y_i = x_i$ for the others. Then the results follow from it and the fact that $P^{-1} = P$. □

In the following we study a special cored directed hypergraph.

Definition 9. Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be a cored k-uniform directed hypergraph. We call it directed squid if $\mathcal{V} = \{1, 1_1, 2_1, \cdots, k_1, \cdots, 1_{(k-1)}, 2_{(k-1)}, \cdots, k_{(k-1)}\}$ and the arc set $\mathcal{E} = \{\vec{e}_i \mid i = 0, \cdots, k-1\}$ in which

$$\begin{aligned} &\vec{e}_0 = (\{1\}, \{1_1, 1_2 \cdots, 1_{(k-1)}\}) \\ &\vec{e}_i = (\{1_i\}, \{2_i, 3_i, \cdots, k_i\}) \qquad\qquad i = 1, \cdots, k-1. \end{aligned}$$

By the Definition 9, it's straightforward that $d_1^+ = d_{1_1}^+ = d_{1_2}^+ = \cdots, d_{1_{(k-1)}}^+ = 1$ and $d_i^+ = 0$ otherwise. Then by Theorem 2 we have the following theorem.

Theorem 15. *Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed squid and $\mathcal{A}$ be its output-adjacency tensor, then $Hspec(\mathcal{A}) = \{0\}$.*

The following theorems determine $Hspec(\mathcal{L})$ and $Hspec(\mathcal{Q})$, where $\mathcal{L}$ and $\mathcal{Q}$ are the Laplacian tensor and signless Laplacian tensor of the directed squid $\mathcal{S}$, respectively.

Theorem 16. *Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed squid and $\mathcal{L}$ be its Laplacian tensor, then $Hspec(\mathcal{L}) = \{0, 1\}$.*

Proof. By Proposition 4.1 in [**46**], $0 \in Hspec(\mathcal{L})$. Now let $\mathbf{x}$ is an H-eigenvector of $\mathcal{L}$ corresponding to H-eigenvalue $\lambda \neq 0$. by (1) we have:

$$(1-\lambda)x_1^{k-1} = \prod_{i=1}^{k-1} x_{1_i} \tag{3}$$

$$(1-\lambda)x_{1_i}^{k-1} = \prod_{j=1}^{k} x_{j_i} \qquad i = 1, 2, \cdots, k-1 \tag{4}$$

$$\lambda x_{j_i}^{k-1} = 0 \qquad i = 1, 2, \cdots, k-1, j = 1, 2, \cdots, k. \tag{5}$$

By (5), $x_{j_i} = 0$ for all i, j. By taking it in (4), we have $(1-\lambda)x_{1_i}^{k-1} = 0$. Now three cases are considered:
(**i**) : $x_{1_i} \neq 0$ for $i = 1, 2, \cdots, k-1$, then $\lambda = 1$ and by (3), $\prod_{i=1}^{k-1} x_{1_i} = 0$ that is a contradiction.
(**ii**) : $x_{1_i} = 0$ for $i = 1, 2, \cdots, k-1$, then by (3), $(1-\lambda)x_1^{k-1} = 0$. Thus $\lambda = 1$ and $x_1 \neq 0$.
(**iii**) : $x_{1_i} = 0$ and $x_{1_j} \neq 0$ for some $i, j = 1, 2, \cdots, k-1$. Then $\lambda = 1$ and $x_1 \in \mathbb{R}$.
Therefore $\lambda = 1$ is only nonzero H-eigenvalue of $\mathcal{L}$. □

By similar proof we have the following theorem:

Theorem 17. *Let $\mathcal{S} = (\mathcal{V}, \mathcal{E})$ be a k-uniform directed squid and $\mathcal{Q}$ be its Laplacian tensor, then $Hspec(\mathcal{Q}) = \{0, 1\}$.*

Note that in a k-uniform directed squid $Hspec(\mathcal{L}) = Hspec(\mathcal{Q})$ for all k and by comparing directed squids and undirected squids in [**19**] and [**52**], we find out an obvious difference between their spectral properties.

6.2 Power directed hipergraphs

Definition 10. Let $G = (V, E)$ be a directed graph and $k \geq 3$. The kth power of G, $\mathcal{G}^k = (\mathcal{V}, \mathcal{E})$ is defined as the k-uniform directed hypergraph with the set of arcs

$$\mathcal{E} = \{\vec{e} = (e^+, e^-) \mid e \in E\}$$

where if $e = (i_1^e, i_2^e) \in E$ then $e^+ = \{i_1^e, i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}$ and $e^- = \{i_2^e\}$ and the set of vertices $\mathcal{V} = \left(\bigcup_{e \in E} \{i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}\right) \cup V$

It easy to see that each power directed hypergraph is a cored directed hypergraph but on the contrary, it is not generally correct, for example directed squid which studied in previous subsection.

The next theorem gives some basic results about an ordinary arc in a power directed hypergraph.

Theorem 18. *Let $\mathcal{H} = (\mathcal{V}, \mathcal{E})$ be a power k-uniform directed hypergraph and $\mathbf{x}$ be an H-eigenvector of its output-adjacency tensor, $\mathcal{A}$, corresponding to $\lambda \neq 0$. If $\vec{e} = (e^+, e^-) \in \mathcal{E}$ is an arbitrary arc with $e^+ = \{i_1^e, i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}$ and $e^- = \{i_2^e\}$, then we have:*

(1) If $d^+_{i_1^e} > 1$, $d^+_{i_2^e} \geq 1$ and $x_{i_{e,1}} = \alpha \neq 0$ then $x_{i_1^e} x_{i_2^e} = \lambda\alpha^2$ when k is odd and $x_{i_1^e} x_{i_2^e} = \lambda\alpha^2$ or $-\lambda\alpha^2$ when k is even.

(2) If $d^+_{i_1^e} = 1$, $d^+_{i_2^e} \geq 1$ and $x_{i_{e,1}} = \alpha \neq 0$ then $x_{i_2^e} = \lambda\alpha$ when k is odd and $x_{i_2^e} = \lambda\alpha$ or $-\lambda\alpha$ when k is even.

(3) If $d^+_{i_2^e} = 0$ then $x_j = 0$ for $j \in \{i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}, i_2^e\}$.

Proof. By Lemma 6 $x_{i_{e,j}} = \alpha$ for $j = 2, \cdots, k-2$ when k is odd and $|x_{i_{e,j}}| = \alpha$ for $j = 2, \cdots, k-2$ when k is even.
For (1), by Definition 2 we have:

$$\alpha^{k-3} x_{i_1^e} x_{i_2^e} = \lambda\alpha^{k-1} \qquad \textit{if } k \textit{ is odd}$$

$$\begin{cases} \quad \alpha^{k-3} x_{i_1^e} x_{i_2^e} = \lambda\alpha^{k-1} \\ \quad or \\ - \quad \alpha^{k-3} x_{i_1^e} x_{i_2^e} = \lambda\alpha^{k-1} \end{cases} \qquad \textit{if } k \textit{ is even}$$

the result follows from $\alpha \neq 0$.
For (2), by Lemma 6 $x_{i_1^e} = \alpha$ or $x_{i_1^e} = -\alpha$. By Definition 2 we have:

$$\alpha^{k-2} x_{i_2^e} = \lambda \alpha^{k-1} \qquad \textit{if } k \textit{ is odd}$$

$$\begin{cases} \quad \alpha^{k-2} x_{i_2^e} = \lambda \alpha^{k-1} \\ \quad or & \textit{if } k \textit{ is even} \\ - \quad \alpha^{k-2} x_{i_2^e} = \lambda \alpha^{k-1} \end{cases}$$

the result follows from $\alpha \neq 0$.
For (3), since $d^+_{i_2^e} = 0$ then $x_{i_2^e} = 0$. Thus by Definition 2 and Lemma 6, $x_j = 0$ for $j \in \{i_{e,1}, i_{e,2}, \cdots, i_{e,k-2}\}$. □

In the following we study a special power directed hypergraph, is called directed hyperwheel.

Definition 11. Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a power k-uniform directed hypergraph. We call it directed hyperwheel if $\mathcal{V} = V_0 \cup V_1 \cup \cdots V_d \cup \bar{V}_1 \cup \bar{V}_2 \cup \cdots \cup \bar{V}_d$ is a disjoint partition of $\mathcal{V}$ in which $V_0 = \{1\}$, $V_i = \{1_i, 2_i, \cdots, (k-1)_i\}$ and $\bar{V}_i = \{1^i, 2^i, \cdots, (k-2)^i\}$ for $i = 1, 2, \cdots, d$ and the arc set $\mathcal{E} = \{\vec{e}_i,\ \vec{a}_i |\ i = 1, \cdots, d\}$ in which

$$\begin{aligned} &\vec{e}_i = (\{1, 1_i, \cdots, (k-2)_i\}, \{(k-1)_i\}) && i = 1, \cdots, d \\ &\vec{a}_i = (\{(k-1)_i, 1^i, \cdots, (k-2)^i\}, \{(k-1)_{i+1}\}) && i = 1, \cdots, d-1 \\ &\vec{a}_d = (\{(k-1)_d, 1^d, \cdots, (k-2)^d\}, \{(k-1)_1\}) \end{aligned}$$

By Definition 10 it can be shown easily.

Lemma 10. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a directed k-uniform hyperwheel, then $d^+_1 = d$, $d^+_j = 1$ for $j \neq 1$ and $d^-_{(k-1)_i} = 2$ for $i = 1, \cdots, d$, $d^-_j = 0$ for $i = 1, \cdots, d$ and $j \neq (k-1)_i$.*

In the following theorems the H-spectrum of output-adjacency tensor, Laplacian tensor and signless Laplacian tensor of $\mathcal{W}_d$ are determined.

Theorem 19. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a directed k-uniform hyperwheel with n vertices and $\mathcal{A}$ be its adjacency tensor. Then $Hspec(\mathcal{A}) = \{0, 1\}$ when d and k are odd and $Hspec(\mathcal{A}) = \{0, 1, -1\}$ otherwise.*

Proof. By Theorem 1, $0 \in Hspec(\mathcal{A})$. Now suppose that $\mathbf{x}$ is an H-eigenvector of $\mathcal{A}$ corresponding to H-eigenvalue $\lambda \neq 0$. The proof is divided into two cases, which contain several sub-cases respectively:
1: k is odd.
By Lemma 6 we have:

$$\begin{aligned} &x_{1_i} = x_{2_i} = \cdots = x_{(k-2)_i} = \alpha_i && i = 1, \cdots, d \\ &x_{1^i} = x_{2^i} = \cdots = x_{(k-2)^i} = x_{(k-1)_i} = \beta_i && i = 1, \cdots, d \end{aligned}$$

Now by Definition 2 we have:

$$\lambda x_1^{k-1} = \sum_{i=1}^{d} \beta_i \alpha_i^{k-2} \tag{6}$$

$$\lambda \alpha_i^{k-1} = x_1 \alpha_i^{k-3} \beta_i \qquad i = 1, \cdots, d \tag{7}$$

$$\lambda \beta_i^{k-1} = \beta_i^{k-2} \beta_{i+1} \qquad i = 1, \cdots, d-1 \tag{8}$$

$$\lambda \beta_d^{k-1} = \beta_d^{k-2} \beta_1 \tag{9}$$

By (8) and (9) if $\beta_i = 0$ for some $i = 1, \cdots, d$ then all $\beta_i = 0$ and thus by (7) and (6) $\mathbf{x} = 0$ that is a contradiction. Therefore $\beta_i \neq 0$ for $i = 1, \cdots, d$. Then by (8) and (9)

$$\lambda = \frac{\beta_{i+1}}{\beta_i} = \frac{\beta_1}{\beta_d} \textit{ for } i = 1, \cdots, d-1, \tag{10}$$

then we have:

$$\beta_1 = \lambda^d \beta_1 \quad \Longrightarrow \quad \lambda^d = 1 \quad \Longrightarrow \quad \begin{cases} \lambda = \pm 1 & \textit{if } d \textit{ is even} \\ \lambda = 1 & \textit{if } d \textit{ is odd} \end{cases}$$

2: k is even.
By Lemma 6 we have:

$$|x_{1_i}| = |x_{2_i}| = \cdots = |x_{(k-2)_i}| \qquad i = 1, \cdots, d$$
$$|x_{1^i}| = |x_{2^i}| = \cdots = |x_{(k-2)^i}| = |x_{(k-1)_i}| \qquad i = 1, \cdots, d$$

Now Let $x_{1_i} = \alpha_i$ and $x_{(k-1)_i} = \beta_i$ for $i = 1, \cdots, d$. With a little modification in (6), (7), (8) and (9) and by similar argument in the previous case, $\beta_i \neq 0$ for $i = 1, \cdots, d$. Now we consider two subcases:
(i) d is even. there are two cases:

- $\beta_d = \lambda^{d-1} \beta_1$, then we have:

$$\begin{aligned} \textit{if } \lambda > 0 &\Rightarrow \beta_1 \textit{ and } \beta_d \textit{ have the same sign} \\ &\Rightarrow \beta_1 = \lambda \beta_d \Rightarrow \lambda^d = 1 \Rightarrow \lambda = 1 \\ \textit{if } \lambda < 0 &\Rightarrow \beta_1 \textit{ and } \beta_d \textit{ have different signs} \\ &\Rightarrow \beta_1 = \lambda \beta_d \Rightarrow \lambda^d = 1 \Rightarrow \lambda = -1 \end{aligned}$$

- $\beta_d = -\lambda^{d-1} \beta_1$, then we have:

$$\begin{aligned} \textit{if } \lambda < 0 &\Rightarrow \beta_1 \textit{ and } \beta_d \textit{ have the same sign} \\ &\Rightarrow \beta_1 = -\lambda \beta_d \Rightarrow \lambda^d = 1 \Rightarrow \lambda = -1 \\ \textit{if } \lambda > 0 &\Rightarrow \beta_1 \textit{ and } \beta_d \textit{ have different signs} \\ &\Rightarrow \beta_1 = -\lambda \beta_d \Rightarrow \lambda^d = 1 \Rightarrow \lambda = 1 \end{aligned}$$

(ii) d is odd. there are two cases:

- $\beta_d = \lambda^{d-1}\beta_1$, then β_1 and β_d have the same sign and we have:

$$\begin{aligned} &\textit{if } \lambda > 0 \ \Rightarrow\ \beta_1 = \lambda\beta_d \ \Rightarrow\ \lambda^d = 1 \ \Rightarrow\ \lambda = 1 \\ &\textit{if } \lambda < 0 \ \Rightarrow\ \beta_1 = -\lambda\beta_d \ \Rightarrow\ \lambda^d = -1 \ \Rightarrow\ \lambda = -1 \end{aligned}$$

- $\beta_d = -\lambda^{d-1}\beta_1$, then β_1 and β_d have different signs and we have:

$$\begin{aligned} &\textit{if } \lambda < 0 \ \Rightarrow\ \beta_1 = \lambda\beta_d \ \Rightarrow\ \lambda^d = -1 \ \Rightarrow\ \lambda = -1 \\ &\textit{if } \lambda > 0 \ \Rightarrow\ \beta_1 = -\lambda\beta_d \ \Rightarrow\ \lambda^d = 1 \ \Rightarrow\ \lambda = 1 \end{aligned}$$

□

Theorem 20. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$, n and $\mathcal{A}$ be as above and k be an odd number. Suppose that $\mathbf{x}$ is an H-eigenvector of $\mathcal{A}$ corresponding to H-eigenvalue 1 such that $x_i \neq 0$ for $i = 1, 2, \cdots, n$. Then $x_1 = \alpha$, $x_{j_i} = \pm\sqrt{\alpha}$ for $j = 1, \cdots, k-2$ and $i = 1, \cdots, d$ and $x_i = 1$, otherwise, where $\alpha = \sqrt[k]{d^2}$.*

Proof. Suppose that α_i and β_i for $i = 1, \cdots, d$ are as in the proof of Theorem 19 and all $\alpha_i \neq 0$. By (7), $\alpha_i^{k-1} = x_1\alpha_i^{k-3}\beta_i$ for $i = 1, \cdots, d$, then we have:

$$\begin{aligned} &\overset{\alpha_i \neq 0}{\Rightarrow} \alpha_i^2 = x_1\beta_i \ \Rightarrow\ x_1 = \frac{\alpha_i^2}{\beta_i} \quad i = 1, \cdots, d \\ &\Rightarrow\ \frac{\alpha_1^2}{\beta_i} = \frac{\alpha_2^2}{\beta_i} = \cdots = \frac{\alpha_d^2}{\beta_i} \end{aligned} \tag{11}$$

By (10) we can let $\beta_i = 1$ for $i = 1, \cdots, d$ and then let $\alpha_1^2 = \alpha_2^2 = \cdots = \alpha_d^2 = \alpha$, then $\alpha_i = \pm\sqrt{\alpha}$ for $i = 1, \cdots, d$. On other hand by (6), $x_1^{k-1} = \sum_{i=1}^{d} \beta_i\alpha_i^{k-2}$ and by (11), $x_1 = \alpha$. By taking it, we have:

$$\alpha^{k-1} = d\alpha^{(\frac{k-2}{2})} \ \Rightarrow\ \alpha^k = d^2 \ \Rightarrow\ \alpha = \sqrt[k]{d^2}.$$

□

Theorem 21. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a directed k-uniform hyperwheel and $\mathcal{L}$ be its Laplacian tensor. Then $Hspec(\mathcal{L}) = \{0, 1, d\}$ when d and k are odd and $Hspec(\mathcal{A}) = \{0, 1, 2, d\}$ otherwise.*

Proof. By Theorem 1, $1, d \in Hspec(\mathcal{L})$. Now suppose that $\mathbf{x}$ is an H-eigenvector of $\mathcal{L}$ corresponding to H-eigenvalue $\lambda \neq 1, d$. The proof is divided into two cases, which contain several sub-cases respectively:

1: k is odd.
By Lemma 8 we have:

$$x_{1_i} = x_{2_i} = \cdots = x_{(k-2)_i} = \alpha_i \qquad i = 1, \cdots, d$$
$$x_{1^i} = x_{2^i} = \cdots = x_{(k-2)^i} = x_{(k-1)_i} = \beta_i \qquad i = 1, \cdots, d$$

Now by Definition 2 we have:

$$(1-\lambda)x_1^{k-1} = \sum_{i=1}^{d} \beta_i \alpha_i^{k-2} \tag{12}$$
$$(1-\lambda)\alpha_i^{k-1} = x_1 \alpha_i^{k-3} \beta_i \qquad i = 1, \cdots, d \tag{13}$$
$$(1-\lambda)\beta_i^{k-1} = \beta_i^{k-2} \beta_{i+1} \qquad i = 1, \cdots, d-1 \tag{14}$$
$$(1-\lambda)\beta_d^{k-1} = \beta_d^{k-2} \beta_1 \tag{15}$$

By (14) and (15) if $\beta_i = 0$ for some $i = 1, \cdots, d$ then all $\beta_i = 0$ and thus by (13) and (12) $\mathbf{x} = 0$ that is a contradiction. Therefore $\beta_i \neq 0$ for $i = 1, \cdots, d$. Then by (14) and (15), $(1-\lambda) = \frac{\beta_{i+1}}{\beta_i} = \frac{\beta_1}{\beta_d}$ for $i = 1, \cdots, d-1$, then we have:

$$\beta_1 = (1-\lambda)^d \beta_1 \quad \Longrightarrow \quad (1-\lambda)^d = 1 \quad \Longrightarrow \quad \begin{cases} \lambda = 0, 2 & \textit{if } d \textit{ is even} \\ \lambda = 0 & \textit{if } d \textit{ is odd} \end{cases}$$

2: k is even.
By Lemma 8 we have:

$$|x_{1_i}| = |x_{2_i}| = \cdots = |x_{(k-2)_i}| \qquad i = 1, \cdots, d$$
$$|x_{1^i}| = |x_{2^i}| = \cdots = |x_{(k-2)^i}| = |x_{(k-1)_i}| \qquad i = 1, \cdots, d$$

Now, let $x_{1_i} = \alpha_i$ and $x_{(k-1)_i} = \beta_i$ for $i = 1, \cdots, d$. With a little modification in (12), (13), (14) and (15) and by similar argument in the previous case, $\beta_i \neq 0$ for $i = 1, \cdots, d$. Now we consider two subcases:
(i) d is even. there are two cases:

- $\beta_d = (1-\lambda)^{d-1}\beta_1$, then we have:

$$\begin{aligned} \textit{if } \lambda < 1 \;&\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have the same sign} \\ &\Rightarrow\; \beta_1 = (1-\lambda)\beta_d \;\Rightarrow\; (1-\lambda)^d = 1 \;\Rightarrow\; \lambda = 0 \\ \textit{if } \lambda > 1 \;&\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have different signs} \\ &\Rightarrow\; \beta_1 = (1-\lambda)\beta_d \;\Rightarrow\; (1-\lambda)^d = 1 \;\Rightarrow\; \lambda = 2 \end{aligned}$$

- $\beta_d = -(1-\lambda)^{d-1}\beta_1$, then we have:

$$\begin{aligned} \textit{if } \lambda > 1 \;&\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have the same sign} \\ &\Rightarrow\; \beta_1 = -(1-\lambda)\beta_d \;\Rightarrow\; (1-\lambda)^d = 1 \;\Rightarrow\; \lambda = 2 \\ \textit{if } \lambda < 1 \;&\Rightarrow\; \beta_1 \textit{ and } \beta_d \textit{ have different signs} \\ &\Rightarrow\; \beta_1 = -(1-\lambda)\beta_d \;\Rightarrow\; (1-\lambda)^d = 1 \;\Rightarrow\; \lambda = 0 \end{aligned}$$

(ii) d is odd. there are two cases:

- $\beta_d = (1-\lambda)^{d-1}\beta_1$, then β_1 and β_d have the same sign and we have:

$$\begin{aligned} &\textit{if } \lambda < 1 \;\Rightarrow\; \beta_1 = (1-\lambda)\beta_d \;\Rightarrow\; (1-\lambda)^d = 1 \;\Rightarrow\; \lambda = 0 \\ &\textit{if } \lambda > 1 \;\Rightarrow\; \beta_1 = -(1-\lambda)\beta_d \;\Rightarrow\; (1-\lambda)^d = -1 \;\Rightarrow\; \lambda = 2 \end{aligned}$$

- $\beta_d = -(1-\lambda^{d-1})\beta_1$, then β_1 and β_d have different signs and we have:

$$\begin{aligned} &\textit{if } \lambda > 1 \;\Rightarrow\; \beta_1 = (1-\lambda)\beta_d \;\Rightarrow\; (1-\lambda)^d = -1 \;\Rightarrow\; \lambda = 2 \\ &\textit{if } \lambda < 1 \;\Rightarrow\; \beta_1 = -(1-\lambda)\beta_d \;\Rightarrow\; (1-\lambda)^d = 1 \;\Rightarrow\; \lambda = 0 \end{aligned}$$

□

By similar proof we have the following theorem:

Theorem 22. *Let $\mathcal{W}_d = (\mathcal{V}, \mathcal{E})$ be a directed k-uniform hyperwheel and $\mathcal{Q}$ be its signless Laplacian tensor. Then $Hspec(\mathcal{Q}) = \{1, 2, d\}$ when d and k are odd and $Hspec(\mathcal{A}) = \{0, 1, 2, d\}$ otherwise.*

7. CONCLUSION

In this paper we consider a k-uniform directed hypergraph in general form and introduce its output-adjacency tensor, Laplacian tensor and signless Laplacian tensor. Then we propose theorems in spectral theory of k-uniform directed hypergraphs that some of them are generalizations of the classical results for undirected hypergraphs. Cored directed hypergraphs and power directed hypergraphs are introduced and presented some their spectral properties.

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