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A NOTE ON SHARPENING OF A THEOREM OF ANKENY AND RIVLIN

Aseem Dalal and N. K. Govil

Let $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ be a polynomial of degree n, $M(p,R) := \max_{|z|=R\geq 0} |p(z)|$, and $M(p,1) := \|p\|$. Then according to a well-known result of Ankeny and Rivlin, we have for $R \geq 1$, $M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\|$. This inequality has been sharpened among others by Govil, who proved that for $R \geq 1$,

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right)\|p\| - \frac{n}{2}\left(\frac{\|p\|^2-4|a_n|^2}{\|p\|}\right)\left\{\frac{(R-1)\|p\|}{\|p\|+2|a_n|} - \ln\left(1+\frac{(R-1)\|p\|}{\|p\|+2|a_n|}\right)\right\}.$$

In this paper, we sharpen the above inequality of Govil, which in turn sharpens inequality of Ankeny and Rivlin. We present our result in terms of the LerchPhi function $\Phi(z,s,a)$, implemented in Wolfram's MATHEMATICA as LerchPhi [z, s, a], which can be evaluated to arbitrary numerical precision, and is suitable for both symbolic and numerical manipulations. Also, we present an example and by using MATLAB show that for some polynomials the improvement in bound can be considerably significant.

1. INTRODUCTION

To answer a question raised by chemist Menedleev, A. A. Markov [**13**] proved the following result which is known as Markov's Theorem.

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Theorem 1. *Let* $p(x) = \sum_{j=0}^{n} a_j x^j$ *be an algebraic polynomial of degree* n *such that* $|p(x)| \leq 1$ *for* $x \in [-1, 1]$*. Then*

$$|p'(x)| \leq n^2, \qquad x \in [-1, 1]$$

The inequality is sharp. Equality holds only for polynomials $p(x) = \alpha T_n(x)$*, where* α *is a complex number such that* $|\alpha| = 1$*, and*

$$T_n(x) = \cos(n \cos^{-1} x) = 2^{n-1} \prod_{j=1}^{n} \left[x - \cos((j - \frac{1}{2})\pi/n)\right]$$

is the n^{th} *degree Tchebycheff polynomial of the first kind. It can be easily verified that* $|T_n(x)| \leq 1$ *for* $x \in [-1, 1]$ *and* $|T'_n(1)| = n^2$*.*

It is natural to look for an analogue of above inequality for $|p^{(k)}(x)|$, where $1 \leq k \leq n$. On iterating, Markov's Theorem yields $|p^{(k)}(x)| \leq n^{2k} L$, if $|p(x)| \leq L$ for $x \in [-1, 1]$. This result is not sharp, and the sharp inequality was given by V. A. Markov [**14**], brother of A. A. Markov, who proved

Theorem 2. *Let* $p(x) = \sum_{j=0}^{n} a_j x^j$ *be an algebraic polynomial of degree* n *with real coefficients such that* $|p(x)| \leq 1$ *for* $x \in [-1, 1]$*. Then*

$$|p^{(k)}(x)| \leq \frac{(n^2 - 1^2)(n^2 - 2^2) \cdots (n^2 - (k-1)^2)}{1 \cdot 3 \cdots (2k-1)}, \qquad x \in [-1, 1].$$

The inequality is sharp, and the equality holds again only for $p(x) = T_n(x)$*, where* $T_n(x) = \cos(n \cos^{-1} x)$ *is the Chebyschev polynomial of degree* n*.*

Several years later, around 1926, Serge Bernstein needed the analogue of the above Theorem 1 of A. A. Markov for polynomials in the complex domain and proved the following.

Theorem 3. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a complex polynomial of degree at most* n*. Then*

$$\max_{|z|=1} |p'(z)| \leq n \max_{|z|=1} |p(z)|.$$

The inequality is best possible and equality holds only for polynomials of the form $p(z) = \alpha z^n$*,* $\alpha \neq 0$ *being a complex number.*

The above theorem is, in fact, a special case of a more general result due to M. Riesz [**22**] for trigonometric polynomials.

In 1945, S. Bernstein observed the following result, which in fact is a simple consequence of the maximum modulus principle (see [**22**, volume 1, p. 137] or [**18**, Problem 269, p. 158]) This inequality is also known as the Bernstein's Inequality.

Theorem 4. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n*. Then for* $R \geq 1$,

$$\max_{|z|=R} |p(z)| \leq R^n \max_{|z|=1} |p(z)|.$$

Equality holds for $p(z) = \alpha z^n$, α *being a complex number.*

It was proved by Bernstein himself that Theorem 3 can be obtained from Theorem 4. However, it was not known if Theorem 4 can also be obtained from Theorem 3, and this has been shown by Govil, Qazi and Rahman [**11**]. Thus both the Theorems 3 and 4 are equivalent in the sense that anyone can be obtained from the other.

For the sharpening of Theorem 3 and Theorem 4, we refer the reader to the papers of Frappier, Rahman and Ruscheweyh [**7**] and of Sharma and Singh [**23**]. The following result of Duffin and Schaeffer [**5**] is an anlogue of Theorem 4 where the maximum modulus is taken on an ellipse rather than on a circle.

Theorem 5. *Let* $p(z) = \sum_{j=0}^{n} a_j z^j$ *be a polynomial of degree* n *such that* $p(z)$ *is real for real* z *and* $\max_{-1 \leq x \leq 1} |p(x)| \leq 1$*. Then for* $R > 1$,

$$\max_{z \in \mathcal{E_R}} |p(z)| \leq \frac{R^n + R^{-n}}{2},$$

where $\mathcal{E_R} := \left\{ z = x + iy : \frac{x^2}{\left(\frac{R+R^{-1}}{2}\right)^2} + \frac{y^2}{\left(\frac{R-R^{-1}}{2}\right)^2} = 1 \right\}$.

In the following result due to Frappier and Rahman [**6**], in the above result the hypothesis on the polynomial $p(z)$ that it is real for real z has been dropped.

Theorem 6. *If* $p(z) = \sum_{j=0}^{n} a_j z^j$ *is a polynomial of degree* n *such that* $\max_{-1 \leq x \leq 1} |p(x)| \leq 1$*, then for* $R > 1$*, we have*

$$\max_{z \in \mathcal{E_R}} |p(z)| \leq \frac{R^n}{2} + \frac{5 + \sqrt{17}}{4} R^{n-2},$$

where $\mathcal{E_R}$ *is the same as above, in Theorem 5.*

Inequalities of Markov and Bernstein-type have been fundamental for the proof of many inverse theorems in polynomial approximation theory. For example, Telyakovskii [**24**] (see Milovanović, Mitrinović and Rassias [**15**]) writes:
"Among those that are fundamental in approximation theory are the extremal problems connected with the inequalities for the derivatives of polynomials ... . The

use of inequalities of this kind is a fundamental method in proofs of inverse problems of approximation theory. Frequently further progress in inverse theorems has depended on first obtaining a corresponding generalization or analogue of Markov's and Bernstein's inequality."

Several papers, research monographs and books have been published in this area (see Boas [**2**], Borwein and Erdelyi [**3**], Frappier, Rahman and Ruscheweyh [**7**], Gardner, Govil and Weems [**8**], Govil [**9, 10**], Govil, Qazi and Rahman [**11**], Lorentz, Golitshek and Makovoz [**12**], Milovanović, Mitrinović and Rassias [**15**], Mitrinović, Pecaric and Fink [**17**], Milovanović and Rassia [**16**], Rahman and Schmeisser [**19, 20**], Rassias, Srivastava, and Yanushauskas [**21**], Sharma and Singh [**23**] and Telyakovskii [**24**]).

For polynomials of degree n not vanishing in the interior of the unit circle, Ankeny and Rivlin [**1**] sharpened inequality in Theorem 3, by proving

Theorem 7. *If $p(z)$ is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \left(\frac{R^n+1}{2}\right) \|p\|, \quad R \geq 1.$$

The above inequality is sharp and equality holds for polynomials having all their zeros on the unit circle.

A refinement of the above inequality was given by Govil [**9**], who proved

Theorem 8. *If $p(z)$ is a polynomial of degree n, and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$,*

$$M(p,R) \leq \left(\tfrac{R^n+1}{2}\right) \|p\| - \tfrac{n}{2} \left(\tfrac{\|p\|^2 - 4|a_n|^2}{\|p\|}\right) \left\{ \tfrac{(R-1)\|p\|}{\|p\|+2|a_n|} - \ln\left(1 + \tfrac{(R-1)\|p\|}{\|p\|+2|a_n|}\right) \right\}.$$

The result is best possible and the equality holds for $p(z) = (\lambda + \mu z^n)$, where λ and μ are complex numbers with $|\lambda| = |\mu|$.

In an attempt to sharpen the above Theorem 8 , Dalal and Govil [**4**] recently proved the following result which sharpens Theorem 8, and so in turn Theorem 7 due to Ankeny and Rivlin [**1**].

Theorem 9. *If $p(z)$ is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$ and any N, $1 \leq N \leq n$,*

$$M(p,R) \leq \frac{(R^n+1)}{2} \|p\| - \frac{n}{2} \|p\| \left(1 - \frac{2|a_n|}{\|p\|}\right) h(N),$$

where

$$h(1) = (R-1) - \left(1 + \frac{2|a_n|}{\|p\|}\right) \ln\left(1 + \frac{(R-1)\|p\|}{\|p\| + 2|a_n|}\right),$$

and

$$h(N) = \left(\frac{R^N - 1}{N}\right) + \sum_{k=1}^{N-1} (-1)^k \left(\frac{R^{N-k}-1}{N-k}\right) \left(\frac{2|a_n|}{\|p\|} + 1\right) \left(\frac{2|a_n|}{\|p\|}\right)^{k-1}$$

$$+ (-1)^N \left(\frac{2\,|a_n|}{\|p\|} + 1\right) \left(\frac{2\,|a_n|}{\|p\|}\right)^{N-1} \ln \left(1 + \frac{(R-1)\,\|p\|}{\|p\| + 2|a_n|}\right),\ for\ N \geq 2.$$

It has been shown in Dalal and Govil [**4**, Lemma 3.7] that for $R \geq 1$ and $N \geq 1$, we have

$$h(N) = \int_1^R \frac{(r-1)r^{N-1}}{(r+a)} dr.$$

Therefore, $h(N)$ is a nonnegative and increasing function of N, implying that the above Theorem 9 always sharpens Theorem 7 of Ankeny and Rivlin. Also, it is easy to verify that for $N = 1$, the Theorem 9 gives the Theorem 8 of Govil, and for $N > 1$, it sharpens Theorem 8.

The Lerch function $z \mapsto \Phi(z, s, a)$, is defined by

$$\Phi(z, s, a) = \sum_{\nu=0}^{\infty} \frac{z^\nu}{(\nu + a)^s},$$

and its integral representation is given by

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt, \tag{1}$$

for $z < 1,\ \Re\,(s) > 0$, and $\Re\,(a) > 0$.
The Legendre chi function is its special case, and is given by

$$\chi_n(z) = 2^{-n} z \Phi(z^2, n, 1/2).$$

The Dirichlet eta function is its special case, that is given by

$$\eta(s) = \Phi(-1, s, 1),$$

and the Riemann zeta function is its special case, given by

$$\zeta(s) = \Phi(1, s, 1).$$

The function $\Phi(z, s, a)$ is implemented in Wolfram's MATHEMATICA as LerchPhi [z, s, a], and is suitable for both symbolic and numerical manipulations. Also, LerchPhi can be evaluated to arbitrary numerical precision.

In this paper, we observe that an alternate expression of $h(N)$ in Theorem 9 can be given in terms of Lerch function $z \mapsto \Phi(z, s, a)$, which can be of considerable help while computing the bound. In this regard we prove the following

Theorem 10. *If* $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ *is a polynomial of degree* n *and* $p(z) \neq 0$ *for* $|z| < 1$*, then for* $R \geq 1$ *and any* $N,\ 1 \leq N \leq n$*, we have*

$$M(p, R) \leq \frac{(R^n + 1)}{2} \|p\| - \frac{n}{2} \|p\| \left(1 - \frac{2\,|a_n|}{\|p\|}\right) h(N),$$

where $h(N)$, as defined in Theorem 9, is given by

$$h(N) = \sum_{\nu=0}^{1} \frac{(-1)^{\nu+1}}{a} \left\{ R^{N+\nu} \Phi\left(\frac{-R}{a}, 1, N+\nu\right) - \Phi\left(\frac{-1}{a}, 1, N+\nu\right) \right\}. \tag{2}$$

Since by Dalal and Govil [**4**, Lemma 3.7], the function $h(N)$, $1 \leq N \leq n$ is an increasing function of N, value of $h(N)$ will therefore be maximum at $n = N$. Therefore, if we see in Theorem 9 the bound for $M(p, R)$ as a function of N, then bound for $M(p, R)$ will be minimum for that value of N for which $h(N)$ is maximum, which is the case for $N = n$. Therefore, the following Corollary can be considered the best possible result that can be obtained from Theorem 10

Corollary 11. *If $p(z) = \sum_{\nu=0}^{n} a_\nu z^\nu$ is a polynomial of degree n and $p(z) \neq 0$ for $|z| < 1$, then for $R \geq 1$, we have*

$$M(p, R) \leq \frac{(R^n + 1)}{2} \|p\| - \frac{n}{2} \|p\| \left(1 - \frac{2|a_n|}{\|p\|}\right) h(n),$$

where

$$h(n) = \sum_{\nu=0}^{1} \frac{(-1)^{\nu+1}}{a} \left\{ R^{n+\nu} \Phi\left(\frac{-R}{a}, 1, n+\nu\right) - \Phi\left(\frac{-1}{a}, 1, n+\nu\right) \right\}.$$

As the LerchPhi Function is already implemented in various mathematical and engineering tools, the above Corollary will be useful to obtain an estimate for $M(p, R)$.

2. PROOF OF THE THEOREM

Proof. In view of Theorem 9, to prove Theorem 10 we only need to prove (2), and for this we use the integral representation (1) of Lerch function $z \mapsto \Phi(z, s, a)$, defined by

$$\Phi(z, s, a) = \frac{1}{\Gamma(s)} \int_0^\infty \frac{t^{s-1} e^{-at}}{1 - ze^{-t}} dt \quad \left(= \sum_{\nu=0}^{\infty} \frac{z^\nu}{(\nu + a)^s} \right).$$

If

$$I(q) = \int_0^q \frac{(t-1)t^{N-1}}{t+a} dt,$$

where $a = 2|a_n|/\|p\|$, then

$$h(N) = \int_1^R \frac{(t-1)t^{N-1}}{t+a} dt = \int_0^R \frac{(t-1)t^{N-1}}{t+a} dt - \int_0^1 \frac{(t-1)t^{N-1}}{t+a} dt$$

$$= I(R) - I(1). \tag{3}$$

Making the substitution $t = qe^{-\zeta}$, gives

$$\begin{aligned} I(q) &= \int_0^{\infty} \frac{q^{(N+1)}e^{-\zeta(N+1)}}{(qe^{-\zeta}+a)}d\zeta - \int_0^{\infty} \frac{q^N e^{-N\zeta}}{(qe^{-\zeta}+a)}d\zeta \\ &= \frac{q^{N+1}}{a}\int_0^{\infty} \frac{e^{-\zeta(N+1)}}{1+\frac{q}{a}e^{-\zeta}}d\zeta - \frac{q^N}{a}\int_0^{\infty} \frac{e^{-N\zeta}}{1+\frac{q}{a}e^{-\zeta}}d\zeta \\ &= \frac{q^{N+1}}{a}\Phi\left(\frac{-q}{a},1,N+1\right) - \frac{q^N}{a}\Phi\left(\frac{-q}{a},1,N\right). \end{aligned}$$

Therefore

$$I(R) = \frac{R^{N+1}}{a}\Phi\left(\frac{-R}{a},1,N+1\right) - \frac{R^N}{a}\Phi\left(\frac{-R}{a},1,N\right), \tag{4}$$

and

$$I(1) = \frac{1}{a}\Phi\left(\frac{-1}{a},1,N+1\right) - \frac{1}{a}\Phi\left(\frac{-1}{a},1,N\right). \tag{5}$$

Now on combining (3), (4) and (5), we get

$$h(N) = \frac{R^{N+1}}{a}\Phi\left(\frac{-R}{a},1,N+1\right) - \frac{R^N}{a}\Phi\left(\frac{-R}{a},1,N\right) -$$

$$\frac{1}{a}\Phi\left(\frac{-1}{a},1,N+1\right) + \frac{1}{a}\Phi\left(\frac{-1}{a},1,N\right),$$

$$= \sum_{\nu=0}^{1} \frac{(-1)^{\nu+1}}{a}\left\{R^{N+\nu}\Phi\left(\frac{-R}{a},1,N+\nu\right) - \Phi\left(\frac{-1}{a},1,N+\nu\right)\right\},$$

where $a = \dfrac{2|a_n|}{\|p\|}$. □

3. COMPUTATION

For computation purposes, this section uses MATLAB for implementation of LerchPhi function, and for this we consider the polynomial $p(z) = (z-2)^4$. Then, $p(z) \neq 0$ for $|z| < 1$ and $\|p\| = 81$. If we take $R = 3$, then $M(p,3) = 625$, and as is easy to see that by using Theorem 7, we get

$$M(p,3) \leq 3321,$$

by using Theorem 8, we get

$$M(p,3) \leq 3180.245,$$

while if we use Corollary 11 of Theorem 10, we get

$$M(p,3) \leq 1548.349,$$

which is smaller than both the bounds, obtained by using Theorems 7 and 8. Thus the Corollary 11 of Theorem 10 gives an improvement over both the Theorems 7 and 8, and as is easy to verify the bound obtained by Corollary 11 is in fact about 46.6% of the bound obtained from Theorem 7, and about 48.7% of the bound obtained from Theorem 8.

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Aseem Dalal
Department of Mathematics,
Indian Institute of Technology Delhi,
Delhi-110016, India
E-mail: *aseem.dalal@gmail.com*

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Narendra K. Govil
Department of Mathematics and Statistics,
Auburn University,
Auburn, AL 36849-5108, USA
E-mail: *govilnk@auburn.edu*