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NEW CLASSES OF RECURRENCE RELATIONS INVOLVING HYPERBOLIC FUNCTIONS, SPECIAL NUMBERS AND POLYNOMIALS

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By using the calculus of finite differences methods and the umbral calculus, we construct recurrence relations for a new class of special numbers. Using this recurrence relation, we define generating functions for this class of special numbers and also new classes of special polynomials. We investigate some properties of these generating functions. By using these generating functions with their functional equations, we obtain many new and interesting identities and relations related to these classes of special numbers and polynomials, the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Stirling numbers. Finally, some derivative formulas and integral formulas for these classes of special numbers and polynomials are given. In general, this article includes results that have the potential to be used in areas such as discrete mathematics, combinatorics analysis and their applications.

1. INTRODUCTION

Recurrence relations have many applications in discrete mathematics and related areas. For instance, a recurrence relation of any sequence involving special numbers and polynomials can be used to construct the generating function of that sequence. Recurrence relations are really the fundamental basis of generating functions for special numbers and polynomials. Especially, usage of the recurrence

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relations is very convenient not only to find the generating functions of the relevant families of special numbers or polynomials, but also to calculate the numerical values of the given these numbers or polynomials.

The main motivation of this article is to introduce a new class of special numbers, which is denoted by $\mathcal{Y}_n(k,a)$, by the following recurrence relation constituted with the aid of the finite differences methods and umbral calculus as follows:

$$(1)\qquad (\mathcal{Y}(k,a)+k+1)^n-(\mathcal{Y}(k,a)-k-1)^n+(\mathcal{Y}(k,a)+1)^n-(\mathcal{Y}(k,a)-1)^n = \begin{cases} 0 & \text{if} \quad n\neq 1, \\ a & \text{if} \quad n=1. \end{cases}$$

where, after expansion, each index of $\mathcal{Y}^n(k,a)$ is to be replaced by the corresponding $a\in\mathbb{R}$, $n\in\mathbb{N}_0=\{0,1,2,\ldots\}$, and $k\in\mathbb{Z}$ as in the usual umbral convention method.

Using (1), three values of the numbers $\mathcal{Y}_n(k,a)$ are given as follows:

$$\begin{aligned} \mathcal{Y}_0(k,a) &= \frac{a}{2(k+2)}, \\ \mathcal{Y}_1(k,a) &= 0, \\ \mathcal{Y}_2(k,a) &= -\frac{a((k+1)^3+1)}{6(k+2)^2}. \end{aligned}$$

Here are some notations, definitions and relations that will be used throughout this paper:

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of integers, the set of rational numbers, the set of real numbers and the set of complex numbers, respectively. Moreover, the falling factorial is given by

$$(x)_n=\begin{cases} x(x-1)(x-2)\ldots(x-n+1) & \text{if} \quad n\in\mathbb{N}, \\ 1 & \text{if} \quad n=0, \end{cases}$$

where $x\in\mathbb{R}$.

The Bernoulli numbers and polynomials of higher order are defined respectively by

$$(2)\qquad F_B(w,n)=\left(\frac{w}{e^w-1}\right)^n=\sum_{k=0}^{\infty}B_k^{(n)}\frac{w^k}{k!}$$

and

$$(3)\qquad G_B(w,x,n)=F_B(w,n)e^{wx}=\sum_{k=0}^{\infty}B_k^{(n)}(x)\frac{w^k}{k!},$$

where $n\in\mathbb{Z}$. For $n=0$, we have

$$B_k^{(0)}(x)=x^k$$

and

$$B_k^{(0)} = \begin{cases} 1 & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k > 0. \end{cases}$$

By using (2) and (3), we have

$$B_k^{(n)}(x) = \sum_{v=0}^{k} \binom{k}{v} x^v B_{k-v}^{(n)},$$

where $n, k \in \mathbb{N}_0$ (*cf.* [**3**], [**4**], [**6**], [**7**], [**18**], [**19**], [**20**], [**21**], [**22**]).

Using (2), we also have the following well-known recurrence relation for the numbers $B_k^{(n)}$:

$$B_k^{(n)} = \left(1 - \frac{k}{n-1}\right) B_k^{(n-1)} - k B_{k-1}^{(n-1)},$$

where $n, k \in \mathbb{N} \setminus \{1\}$ and

$$B_k^{(k+1)} = (-1)^k k!, \tag{4}$$

where $k \in \mathbb{N}_0$ (*cf.* [**3**]).

Substituting $n = 1$ into (2) and (3), one has the generating functions for the Bernoulli numbers and polynomials.

The Euler numbers and polynomials of higher order are defined respectively by

$$F_E(w, n) = \left(\frac{2}{e^w + 1}\right)^n = \sum_{k=0}^{\infty} E_k^{(n)} \frac{w^k}{k!} \tag{5}$$

and

$$H_E(w, x, n) = F_E(w, n) e^{wx} = \sum_{k=0}^{\infty} E_k^{(n)}(x) \frac{w^k}{k!} \tag{6}$$

where $n \in \mathbb{Z}$ (*cf.* [**3**], [**4**], [**6**], [**7**], [**18**], [**19**], [**20**], [**21**], [**22**]). For $n = 0$, we have

$$E_k^{(0)}(x) = x^k$$

and

$$E_k^{(0)} = \begin{cases} 1 & \text{if} \quad k = 0, \\ 0 & \text{if} \quad k > 0. \end{cases}$$

Substituting $n = 1$ into (5) and (6), one has the generating functions for the Euler numbers and polynomials.

The Euler numbers have many applications in combinatorics and partition theory. Therefore, in recent years, numerous articles and books have been published on combinatoric interpretations and applications of the Euler numbers. Here, we give Sundaram's [**14**] and Stanley's [**23**] combinatoric comments and applications about these numbers related to the tangent numbers $E_{2n+1}^{\ast}$ defined by

$$\tan(w)=\sum_{n=0}^{\infty}E_{2n+1}^{\ast}\frac{w^{2n+1}}{(2n+1)!},$$

(*cf.* [**4**], [**6**], [**8**], [**14**], [**15**], [**22**], [**23**]) where

$$\begin{aligned}E_{2n+1}^{\ast} &= (-1)^{n+1}2^{2n+1}E_{2n+1}\\ &= (-1)^{n}\frac{2^{2n+1}(2^{2n+2}-1)}{n+1}B_{2n+2}\\ &= \lim_{w\to 0}\frac{d^{2n+1}}{dw^{2n+1}}\left\{\tan(w)\right\}\end{aligned}$$

(*cf.* [**1**], [**8**], [**15**], [**19**]).

Let Π_n denote the lattice of set partitions of an n-element set ordered by refinement; the partition with n nonempty blocks each of size 1 is the minimum element, and the partition with exactly one nonempty block is the maximum element. The tangent number $E_{2n+1}^{\ast}$ counts the number of alternating permutations in S_{2n-1}, that is, it is the number of permutations $\sigma\in S_{2n+1}$ such that $\sigma(1)>\sigma(2)<\sigma(3)>\cdots<\sigma(2n+1)$. The tangent number is also an Euler number of the second kind, which are also defined by the following generating function:

$$\tan(w)+\sec(w)=\sum_{n=0}^{\infty}e_n\frac{w^n}{n!}$$

(*cf.* [**23**]). Here the numbers e_{2n} are so-called the $n-$th secant number with $e_0=0$. Sundaram [**14**] showed that these numbers are related to the permutation representation of S_n on the maximal chains of the full partition lattice Π_n.

The Stirling numbers of the first and the second kinds have also many applications in combinatorics and partition theory. We now give generating functions for these numbers.

The Stirling numbers of the second kind, $S_2(n,k)$, are defined by

$$F_S(w,k)=\frac{\left(e^w-1\right)^k}{k!}=\sum_{n=0}^{\infty}S_2(n,k)\frac{w^n}{n!}, \tag{7}$$

and

$$x^n=\sum_{k=0}^{n}S_2(n,k)(x)_k,$$

where $k\in\mathbb{N}_0$ (*cf.* [**4**], [**6**], [**7**], [**19**], [**21**], [**22**]).

By using (7), we have

$$S_2(n,k)=\frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}j^n,$$

where $n, k \in \mathbb{N}_0$. If $k > n$ or $k < 0$, we have $\binom{n}{k} = 0$, so

$$S_2(n,k) = 0$$

(*cf.* [**4**], [**6**], [**7**], [**16**], [**17**], [**21**], [**22**], [**24**]).

In the theory of combinatorics, the Stirling number of the second kind are also known as the Stirling partition numbers. It is well-known that the numbers $S_2(n,k)$ correspond to the number of ways to partition a set of n objects into k non-empty subsets. These numbers have many applications in combinatorics and the theory of partitions (*cf.* [**9**, p. 244]).

The Stirling numbers of the first kind $S_1(n,k)$ are defined means of the following generating function:

$$F_{S1}(w,k) = \frac{\left(\log(1+w)\right)^k}{k!} = \sum_{n=0}^{\infty} S_1(n,k)\frac{w^n}{n!}. \tag{8}$$

These numbers are also defined by

$$(x)_n = \sum_{k=0}^{n} S_1(n,k)x^k, \tag{9}$$

and $S_1(n,k) = 0$ if $k > n$ (*cf.* [**4**], [**6**], [**7**], [**16**], [**17**], [**19**], [**21**], [**22**], [**24**]).

In the theory of combinatorics, the Stirling number of the second kind are also known as the Stirling cycle numbers. The Stirling number of the first kind is drived from the study of permutations. These numbers count permutations according to their number of cycles (counting fixed points as cycles of length one). These numbers are known as inverses of one another when viewed as triangular matrices (*cf.* [**24**, p. 34]).

2. GENERATING FUNCTIONS FOR NEW CLASSES OF SPECIAL NUMBERS DERIVED RECURRENCE RELATIONS

In this section, with the help of (1), we construct generating function for the numbers $\mathcal{Y}_n(k,a)$. We investigate some properties of this generating function and these numbers.

Theorem 1. *Let $k \in \mathbb{Z}$ and $a \in \mathbb{R}$ (or $\mathbb{C}$). Then we have*

$$F_{\mathcal{Y}}(w,k,a) = \sum_{n=0}^{\infty} \mathcal{Y}_n(k,a)\frac{w^n}{n!}, \tag{10}$$

and

$$F_{\mathcal{Y}}(w,k,a) = \frac{aw}{e^{(k+1)w} - e^{-(k+1)w} + e^{w} - e^{-w}}. \tag{11}$$

Proof. Using (1), we get

$$\sum_{n=0}^{\infty}\left[(\mathcal{Y}(k,a)+k+1)^n-(\mathcal{Y}(k,a)-k-1)^n + (\mathcal{Y}(k,a)+1)^n-(\mathcal{Y}(k,a)-1)^n\right]\frac{w^n}{n!}=aw.$$

Applying the umbral calculus convention to the above equation, after some algebraic operations, we obtain

$$\begin{aligned} aw &= e^{(k+1)w}\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{w^n}{n!}-e^{-(k+1)w}\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{w^n}{n!} \\ &\quad +e^{w}\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{w^n}{n!}-e^{-w}\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{w^n}{n!}. \end{aligned}$$

After some calculations, we get the desired result. □

Some properties of generating function $F_{\mathcal{Y}}(w,k,a)$ are given as follows:

$$\begin{aligned} F_{\mathcal{Y}}(w,k,a) &= \frac{aw}{2\left(\sinh\left((k+1)w\right)+\sinh(w)\right)} \\ &= \frac{aw}{4\sinh\left(\frac{(k+2)w}{2}\right)\cosh\left(\frac{kw}{2}\right)}. \end{aligned}$$

Combining the above equation with the well-known infinite product formulas for the hyperbolic sine function $\sinh(w)$ and hyperbolic cosine function $\cosh(w)$ (*cf.* [**2, 5**]):

$$\sinh(w)=w\prod_{n=1}^{\infty}\left(1+\left(\frac{w}{n\pi}\right)^2\right),$$

and

$$\cosh(w)=\prod_{n=1}^{\infty}\left(1+\left(\frac{2w}{(2n-1)\pi}\right)^2\right),$$

we get infinite product formula for the function $F_{\mathcal{Y}}(w,k,a)$ as follows:

$$F_{\mathcal{Y}}(w,k,a)=\frac{a}{2(k+2)}\left(\prod_{n=1}^{\infty}\left(1+\frac{(k+2)^2w^2}{4n^2\pi^2}\right)\left(1+\frac{w^2k^2}{(2n-1)^2\pi^2}\right)\right)^{-1}. \tag{12}$$

Remark 1. *By using the same method in* [**2**] *and applying the Weierstrass factorization theorem to* (12)*, one may obtain many results associated with the Bernoulli numbers and polynomials, the Euler numbers and polynomials, the Riemann zeta function, the Laplace distribution, the Moment generating function, the characteristic function, the orthogonal polynomials, the Hermite polynomials, three-term recurrence relation and Hankel determinant.*

By using (11), we get

$$F_{\mathcal{Y}}(w,k,a)=\frac{awe^{(k+1)w}}{\left(e^{(k+2)w}-1\right)\left(e^{kw}+1\right)}. \tag{13}$$

Therefore

$$\begin{aligned}F_{\mathcal{Y}}(w,-2,a) &= \infty,\\ F_{\mathcal{Y}}\left(\frac{\pi i}{k},k,a\right) &= \infty,\end{aligned}$$

and

$$F_{\mathcal{Y}}\left(\frac{2\pi i}{k+2},k,a\right) = \infty.$$

In order to give convergence property of the function $F_{\mathcal{Y}}(w,k,a)$ in (10), by using (13), we assume that

$$\left\{w:|w|<\frac{\pi}{|k|}\right\}\cap\left\{w:|w|<\frac{2\pi}{|k+2|}\right\}.$$

By using (13), some properties of the numbers $\mathcal{Y}_n(k,a)$, involving the Bernoulli polynomials, the Bernoulli numbers and the Euler numbers, are given as follows:

Substituting $k=0$ and $a=2$ into (13), we have

$$\begin{aligned}\mathcal{Y}_n(0,2) &= 2^{n-1}B_n\left(\frac{1}{2}\right)\\ &= \left(1-2^{n-1}\right)B_n\\ &= \frac{1}{2}\sum_{j=0}^{n}\binom{n}{j}B_j(1)E_{n-j}.\end{aligned}$$

Setting $k=1$ and $a=2$ into (13), we have

$$\begin{aligned}\mathcal{Y}_n(1,2) &= \sum_{j=0}^{n}\binom{n}{j}3^{n-j-1}B_{n-j}\left(\frac{2}{3}\right)E_j\\ &= \sum_{j=0}^{n}\binom{n}{j}3^{n-j-1}B_{n-j}E_j(2).\end{aligned}$$

Putting $k=-1$ and $a=2$ into (13), we have

$$\begin{aligned}\mathcal{Y}_n(-1,2) &= 2^nB_n\left(\frac{1}{2}\right)\\ &= 2\left(1-2^{n-1}\right)B_n\\ &= 2\mathcal{Y}_n(0,2).\end{aligned}$$

Setting $k=1$ and $a=2$ into (13), we also get the following corollary:

Corollary 1. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} \binom{n}{j} B_{n-j}\left(2\right) E_j = \mathcal{Y}_n(1,2) + \sum_{j=0}^{n} \binom{n}{j} \left(1+2^{n-j}\right) \mathcal{Y}_j(1,2).$$

By using (1) and (10), we arrive at the following theorem:

Theorem 2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{j=0}^{n} \left(1-(-1)^{n-j}\right) \binom{n}{j} \left((k+1)^{n-j}+1\right) \mathcal{Y}_j(k,a) = \begin{cases} 0 & \text{if} \quad n \neq 1, \\ a & \text{if} \quad n = 1. \end{cases}$$

Theorem 3. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{Y}_n(k,a) = \frac{a}{2(k+2)} \sum_{v=0}^{n} \binom{n}{v} k^{n-v} (k+2)^v B_v E_{n-v} \left(\frac{k+1}{k}\right). \tag{14}$$

Proof. Combining (3) and (6) with (13), we get the following functional equation:

$$F_{\mathcal{Y}}(w,k,a) = \frac{a}{2(k+2)} G_B((k+2)w,0,1) H_E\left(kw, \frac{k+1}{k}, 1\right).$$

By using the above functional equation, we obtain

$$\sum_{n=0}^{\infty} \mathcal{Y}_n(k,a) \frac{w^n}{n!} = \frac{a}{2(k+2)} \sum_{n=0}^{\infty} (k+2)^n B_n \frac{w^n}{n!} \sum_{n=0}^{\infty} k^n E_n \left(\frac{k+1}{k}\right) \frac{w^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathcal{Y}_n(k,a) \frac{w^n}{n!} = \frac{a}{2(k+2)} \sum_{n=0}^{\infty} \sum_{v=0}^{n} \binom{n}{v} k^{n-v} (k+2)^v B_v E_{n-v} \left(\frac{k+1}{k}\right) \frac{w^n}{n!}.$$

Comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we get the desired result. □

Theorem 4. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{Y}_n(k,a) = \frac{a}{2(k+2)} \sum_{v=0}^{n} \binom{n}{v} k^{n-v} (k+2)^v E_{n-v} B_v \left(\frac{k+1}{k+2}\right). \tag{15}$$

Proof. Combining (3) and (6) with (13), we get the following functional equation:

$$F_{\mathcal{Y}}(w,k,a) = \frac{a}{2(k+2)} H_E(kw,0,1) G_B\left(\left(k+2\right)w, \frac{k+1}{k+2}, 1\right).$$

By using the above functional equation, we obtain

$$\sum_{n=0}^{\infty} \mathcal{Y}_n(k,a) \frac{w^n}{n!} = \frac{a}{2(k+2)} \sum_{n=0}^{\infty} \left(k+2\right)^n B_n \left(\frac{k+1}{k+2}\right) \frac{w^n}{n!} \sum_{n=0}^{\infty} k^n E_n \frac{w^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathcal{Y}_n(k,a)\frac{w^n}{n!} = \frac{a}{2(k+2)}\sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v}k^{n-v}(k+2)^v E_{n-v}B_v\left(\frac{k+1}{k+2}\right)\frac{w^n}{n!}.$$

Comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we get the desired result. □

Theorem 5. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_n\left(1-\frac{1}{k+2}\right) = \frac{1}{a(k+2)^{n-1}}\left(\mathcal{Y}_n(k,a)+\sum_{v=0}^{n}\binom{n}{v}k^{n-v}\mathcal{Y}_v(k,a)\right). \tag{16}$$

Proof. Combining (3) with (13), we get the following functional equation:

$$\frac{k+2}{a}\left(e^{kw}+1\right)F_{\mathcal{Y}}(w,k,a) = G_B\left((k+2)w, \frac{k+1}{k+2}, 1\right). \tag{17}$$

By using the above functional equation, we obtain

$$\begin{aligned} &\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{w^n}{n!}+\sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v}k^{n-v}\mathcal{Y}_v(k,a)\frac{w^n}{n!} \\ = &\frac{a}{k+2}\sum_{n=0}^{\infty}(k+2)^n B_n\left(\frac{k+1}{k+2}\right)\frac{w^n}{n!}. \end{aligned}$$

Comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we get the desired result. □

Combining the following well-known identity (*cf.* [**1, 4, 6, 9, 11, 22**]):

$$B_n(1-x) = (-1)^n B_n(x)$$

with (16), we arrive at the following result:

Corollary 2. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{Y}_n(k,a)+\sum_{v=0}^{n}\binom{n}{v}k^{n-v}\mathcal{Y}_v(k,a) = (-1)^n a(k+2)^{n-1}B_n\left(\frac{1}{k+2}\right).$$

By using (17), we also get the following result:

Corollary 3. *Let $n \in \mathbb{N}_0$. Then we have*

$$B_n\left(\frac{k+1}{k+2}\right) = \frac{2}{a(k+2)^{n-1}}\sum_{v=0}^{n}\binom{n}{v}k^{n-v}\mathcal{Y}_v(k,a)E_{n-v}^{(-1)}.$$

Theorem 6. *Let $n \in \mathbb{N}_0$. Then we have*

$$E_n\left(1+\frac{1}{k}\right)=\frac{2}{a(n+1)k^n}\sum_{v=0}^{n}\binom{n+1}{v}(k+2)^{n+1-v}\mathcal{Y}_v(k,a). \tag{18}$$

Proof. Combining (3) with (13), we get the following functional equation:

$$\left(e^{(k+2)w}-1\right)F_{\mathcal{Y}}(w,k,a)=\frac{aw}{2}H_E\left(kw,\frac{k+1}{k},1\right). \tag{19}$$

By using the above functional equation, we obtain

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{v=0}^{n}\binom{n}{v}(k+2)^{n-v}\mathcal{Y}_v(k,a)\frac{w^n}{n!}-\sum_{n=0}^{\infty}\mathcal{Y}_n(k,a)\frac{w^n}{n!}\\ =&\ \frac{aw}{2}\sum_{n=0}^{\infty}k^nE_n\left(\frac{k+1}{k}\right)\frac{w^n}{n!}.\end{aligned}$$

After some elementary calculations, comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we get the desired result. □

Combining the following well-known identity (*cf.* [**1, 4, 6, 9, 11**]):

$$E_n(1+x)=2x^n-E_n(x)$$

with (18), we arrive at the following result:

Corollary 4. *Let $n \in \mathbb{N}_0$. Then we have*

$$\sum_{v=0}^{n}\binom{n+1}{v}(k+2)^{n+1-v}\mathcal{Y}_v(k,a)=\frac{a(n+1)k^n}{2}\left(\frac{2}{k^n}-E_n\left(\frac{1}{k}\right)\right).$$

3. GENERATING FUNCTIONS FOR NEW CLASSES OF SPECIAL POLYNOMIALS DERIVED RECURRENCE RELATIONS

In this section, we define two new class of special polynomials. We investigate some properties of these polynomials.

We define the polynomials $Q_n(x,k,a)$ by following generating functions:

$$K_{\mathcal{Y}}(w,x,k,a)=e^{wx}F_{\mathcal{Y}}(w,k,a)=\sum_{n=0}^{\infty}Q_n(x,k,a)\frac{w^n}{n!}, \tag{20}$$

and

$$K_{\mathcal{Y}}(w,x,k,a)=\frac{awe^{(k+1+x)w}}{\left(e^{(k+2)w}-1\right)\left(e^{kw}+1\right)}=\sum_{n=0}^{\infty}Q_n(x,k,a)\frac{w^n}{n!}. \tag{21}$$

By using (20), we get the following theorem:

Theorem 7. *Let $n \in \mathbb{N}_0$. Then we have*

$$Q_n(x,k,a) = \sum_{v=0}^{n} \binom{n}{v} \mathcal{Y}_{n-v}(k,a) x^v. \tag{22}$$

We also define the polynomials $\mathcal{P}_n(x,k,a)$ by the following generating function:

$$H_{\mathcal{Y}}(w,x,k,a) = (1+w)^x F_{\mathcal{Y}}(w,k,a) = \sum_{n=0}^{\infty} \mathcal{P}_n(x,k,a) \frac{w^n}{n!}. \tag{23}$$

By using (23) with the application of the Binomial theorem on the function $(1+w)^x$ when $|w| < 1$, we obtain

$$\sum_{n=0}^{\infty} \mathcal{P}_n(x,k,a) \frac{w^n}{n!} = \sum_{n=0}^{\infty} (x)_n \frac{w^n}{n!} \sum_{n=0}^{\infty} \mathcal{Y}_n(k,a) \frac{w^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty} \mathcal{P}_n(x,k,a) \frac{w^n}{n!} = \sum_{n=0}^{\infty} \sum_{v=0}^{n} \binom{n}{v} \mathcal{Y}_{n-v}(k,a) \, (x)_v \, \frac{w^n}{n!}.$$

Thus, comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we get the following formula for the polynomials $\mathcal{P}_n(x,k,a)$:

Theorem 8. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{P}_n(x,k,a) = \sum_{v=0}^{n} \binom{n}{v} \mathcal{Y}_{n-v}(k,a) \, (x)_v \, . \tag{24}$$

Theorem 9. *Let $n \in \mathbb{N}_0$. Then we have*

$$\mathcal{P}_n(x,k,a) = \sum_{v=0}^{n} \sum_{j=0}^{v} \binom{n}{v} \mathcal{Y}_{n-v}(k,a) S_1(v,j) x^j. \tag{25}$$

Proof. By using (23), we obtain

$$H_{\mathcal{Y}}(w,x,k,a) = e^{x \log(1+w)} F_{\mathcal{Y}}(w,k,a) = \sum_{n=0}^{\infty} \mathcal{P}_n(x,k,a) \frac{w^n}{n!}.$$

From the above equation, we get

$$\sum_{n=0}^{\infty} \mathcal{P}_n(x,k,a) \frac{w^n}{n!} = \sum_{n=0}^{\infty} x^n F_{S1}(w,n) \sum_{n=0}^{\infty} \mathcal{Y}_n(k,a) \frac{w^n}{n!}.$$

Hence, combining the above equation with (8), we obtain

$$\sum_{n=0}^{\infty}\mathcal{P}_n(x,k,a)\frac{w^n}{n!}=\sum_{n=0}^{\infty}\sum_{v=0}^{n}\sum_{j=0}^{v}\binom{n}{v}\mathcal{Y}_{n-v}(k,a)S_1(v,j)x^j\frac{w^n}{n!}.$$

Now, comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we arrive at the equation (25). □

Remark 2. *Observe that combining* (24) *with* (9) *also yields* (25).

Remark 3. *Note that curve families and splines containing the polynomials $Q_n(x,k,a)$ and $\mathcal{P}_n(x,k,a)$ may be defined and the analysis and applications of these curves and splines may be investigated.*

3.1. Reccurence relations for the polynomials $Q_n(x,k,a)$ and $\mathcal{P}_n(x,k,a)$

By using the following partial differential equation, a reccurence relation for the polynomial $Q_n(x,k,a)$ is derived:

$$\begin{aligned}\frac{\partial}{\partial w}\left\{K_{\mathcal{Y}}(w,x,k,a)\right\} &= \left(x+\frac{1}{w}\right)K_{\mathcal{Y}}(w,x,k,a)\\ &\quad -\frac{2((k+1)\cosh((k+1)w)+\cosh(w))}{aw}K_{\mathcal{Y}}^2(w,x,k,a).\end{aligned}$$

Since

$$\cosh(w)=\sum_{n=0}^{\infty}\frac{w^{2n}}{(2n)!},$$

we also get

$$\begin{aligned}\sum_{n=0}^{\infty}nQ_n(x,k,a)\frac{w^n}{n!} &= x\sum_{n=0}^{\infty}nQ_{n-1}(x,k,a)\frac{w^n}{n!}+\sum_{n=0}^{\infty}Q_n(x,k,a)\frac{w^n}{n!}\\ &\quad -\frac{2(k+1)}{a}\sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n}{2j}(k+1)^{2j}Q_{n-2j}^{(2)}(x,k,a)\frac{w^n}{n!}\\ &\quad -\frac{2}{a}\sum_{n=0}^{\infty}\sum_{j=0}^{\left[\frac{n}{2}\right]}\binom{n}{2j}Q_{n-2j}^{(2)}(x,k,a)\frac{w^n}{n!},\end{aligned}$$

where

$$Q_n^{(2)}(x,k,a)=\sum_{v=0}^{n}\binom{n}{v}Q_v(x,k,a)Q_{n-v}(x,k,a).$$

After some elementary calculations in the above equation, comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we arrive at the following recurrence relation for the polynomial $Q_n(x,k,a)$:

Theorem 10. *Let $n \in \mathbb{N}$ with $n \neq 1$. Then we have*

$$\begin{aligned} Q_n(x,k,a) &= \frac{n}{n-1} x Q_{n-1}(x,k,a) \\ &\quad + \frac{2}{a(1-n)} \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} Q^{(2)}_{n-2j}(x,k,a) \left((k+1)^{2j+1}+1\right). \end{aligned}$$

Corollary 5. *Let $n \in \mathbb{N}$ with $n \neq 1$. Then we have*

$$\begin{aligned} Q_n(x,k,a) &= \frac{n}{n-1} x Q_{n-1}(x,k,a) + \frac{2}{a(1-n)} \sum_{j=0}^{\left[\frac{n}{2}\right]} \sum_{l=0}^{n-2j} \binom{n}{2j} \binom{n-2j}{l} \\ &\quad \times Q_l(x,k,a) Q_{n-2j-l}(x,k,a) \left((k+1)^{2j+1}+1\right). \end{aligned}$$

Combining the following well-known identity

$$e^{wx} = \cosh(wx) + \sinh(wx)$$

with (20), we have

$$(\cosh(wx) + \sinh(wx)) F_{\mathcal{Y}}(w,k,a) = \sum_{n=0}^{\infty} Q_n(x,k,a) \frac{w^n}{n!}.$$

By using the above equation and (10), we get

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n(x,k,a) \frac{w^n}{n!} &= \sum_{n=0}^{\infty} x^{2n} \frac{w^{2n}}{(2n)!} \sum_{n=0}^{\infty} \mathcal{Y}_n(k,a) \frac{w^n}{n!} \\ &\quad + \sum_{n=0}^{\infty} x^{2n+1} \frac{w^{2n+1}}{(2n+1)!} \sum_{n=0}^{\infty} \mathcal{Y}_n(k,a) \frac{w^n}{n!}. \end{aligned}$$

Therefore

$$\begin{aligned} \sum_{n=0}^{\infty} Q_n(x,k,a) \frac{w^n}{n!} &= \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} x^{2j} \frac{1}{(2j)!} \mathcal{Y}_{n-2j}(k,a) \frac{w^n}{(n-2j)!} \\ &\quad + w \sum_{n=0}^{\infty} \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} \frac{1}{2j+1} \mathcal{Y}_{n-2j}(k,a) \frac{x^{2j+1} w^n}{n!}. \end{aligned}$$

After some elementary calculations, comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 11. *Let $n \in \mathbb{N}$. Then we have*

$$Q_n(x,k,a) = \sum_{j=0}^{\left[\frac{n}{2}\right]} \binom{n}{2j} \mathcal{Y}_{n-2j}(k,a) x^{2j} + n \sum_{j=0}^{\left[\frac{n-1}{2}\right]} \binom{n-1}{2j} \frac{\mathcal{Y}_{n-2j-1}(k,a)}{2j+1} x^{2j+1}.$$

By using (21), the following identities of the polynomials $Q_n(x,k,a)$ are given as follows:

Corollary 6. *Let $n \in \mathbb{N}_0$. Then we have*

$$Q_n(x,k,a) = \frac{a}{2}\sum_{v=0}^{n}\binom{n}{v}(k+2)^{n-1-v}k^v E_v B_{n-v}\left(\frac{k+1+x}{k+2}\right).$$

Corollary 7. *Let $n \in \mathbb{N}_0$. Then we have*

$$Q_n(x,k,a) = \frac{a}{2}\sum_{v=0}^{n}\binom{n}{v}(k+2)^{v-1}k^{n-v} B_v E_{n-v}\left(\frac{k+1+x}{k}\right).$$

3.2. Derivative formulas for the polynomials $Q_n(x,k,a)$ and $\mathcal{P}_n(x,k,a)$:

Here, we compute the derivative of equations (22), (24), and (25) with respect to x to derive the following derivative formulas for the polynomials $Q_n(x,k,a)$ and $\mathcal{P}_n(x,k,a)$.

By using the following partial differential equation:

$$\frac{\partial}{\partial x}\left\{K_{\mathcal{Y}}(w,x,k,a)\right\} = wK_{\mathcal{Y}}(w,x,k,a),$$

we get

$$\sum_{n=0}^{\infty}\frac{d}{dx}\left\{Q_n(x,k,a)\right\}\frac{w^n}{n!} = w\sum_{n=0}^{\infty}Q_n(x,k,a)\frac{w^n}{n!}.$$

After some elementary calculations, comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 12. *Let $n \in \mathbb{N}$. Then we have*

$$\frac{d}{dx}\left\{Q_n(x,k,a)\right\} = nQ_{n-1}(x,k,a).$$

By using (23), we now give some formulas for $\frac{d}{dx}\left\{\mathcal{P}_n(x,k,a)\right\}$:

Theorem 13. *Let $n \in \mathbb{N}_0$. Then we have*

$$\frac{d}{dx}\left\{\mathcal{P}_n(x,k,a)\right\} = \sum_{v=0}^{n}\sum_{j=0}^{v}\binom{n}{v}\mathcal{Y}_{n-v}(k,a)S_1(v,j)jx^{j-1}.$$

Using the following well-known derivative formula for the falling factorial polynomial $(x)_n$ (*cf.* [**12**]):

$$\frac{d}{dx}\left\{(x)_n\right\} = (x)_n\sum_{j=0}^{n-1}\frac{1}{x-j},$$

with

$$\frac{d}{dx}\left\{(x)_0\right\}=0,$$

we have the following derivative formula for the polynomials $\mathcal{P}_n(x,k,a)$:

Theorem 14. *Let $n\in\mathbb{N}$. Then we have*

$$\frac{d}{dx}\left\{\mathcal{P}_n(x,k,a)\right\}=\sum_{v=1}^{n}\binom{n}{v}\mathcal{Y}_{n-v}(k,a)\,(x)_v\sum_{j=0}^{v-1}\frac{1}{x-j}.$$

By using the following partial differential equation:

$$\frac{\partial}{\partial x}\left\{H_{\mathcal{Y}}(w,x,k,a)\right\}=H_{\mathcal{Y}}(w,x,k,a)\log(1+w),$$

with help of (8), we obtain

$$\sum_{n=0}^{\infty}\frac{d}{dx}\left\{\mathcal{P}_n(x,k,a)\right\}\frac{w^n}{n!}=\sum_{n=0}^{\infty}S_1(n,1)\frac{w^n}{n!}\sum_{n=0}^{\infty}\mathcal{P}_n(x,k,a)\frac{w^n}{n!}.$$

Therefore

$$\sum_{n=0}^{\infty}\frac{d}{dx}\left\{\mathcal{P}_n(x,k,a)\right\}\frac{w^n}{n!}=\sum_{n=0}^{\infty}\sum_{j=0}^{n}\binom{n}{j}S_1(j,1)\mathcal{P}_{n-j}(x,k,a)\frac{w^n}{n!}.$$

After some elementary calculations, comparing the coefficients of $\frac{w^n}{n!}$ on both sides of the above equation, we arrive at the following theorem:

Theorem 15. *Let $n\in\mathbb{N}_0$. Then we have*

$$\frac{d}{dx}\left\{\mathcal{P}_n(x,k,a)\right\}=\sum_{j=0}^{n}\binom{n}{j}S_1(j,1)\mathcal{P}_{n-j}(x,k,a).$$

Since

$$S_1(j,1)=(-1)^{j+1}(j-1)!$$

(*cf.* [**4, 10, 22**]), we get the following corollary:

Corollary 8. *Let $n\in\mathbb{N}$. Then we have*

$$\frac{d}{dx}\left\{\mathcal{P}_n(x,k,a)\right\}=\sum_{j=1}^{n}(-1)^{j+1}\binom{n}{j}(j-1)!\mathcal{P}_{n-j}(x,k,a). \tag{26}$$

Combining (26) with (4), we arrive at the following corollary:

Corollary 9. *Let $n\in\mathbb{N}$. Then we have*

$$\frac{d}{dx}\left\{\mathcal{P}_n(x,k,a)\right\}=\sum_{j=1}^{n}\binom{n}{j}B_{j-1}^{(j)}\mathcal{P}_{n-j}(x,k,a).$$

3.3. Integral formulas for the polynomials $Q_n(x,k,a)$ and $\mathcal{P}_n(x,k,a)$

In this section, we derive some integral formulas for the polynomials $Q_n(x,k,a)$ and $\mathcal{P}_n(x,k,a)$.

Integrating the equations (22), (24) and (25) with respect to x from 0 to 1, we obtain some integral formulas, for the polynomials $Q_n(x,k,a)$ and $\mathcal{P}_n(x,k,a)$, which are respectively given by the following theorems:

Theorem 16. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_0^1 Q_n(x,k,a)dx = \sum_{v=0}^{n} \binom{n}{v} \frac{\mathcal{Y}_{n-v}(k,a)}{1+v}.$$

Theorem 17. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_0^1 \mathcal{P}_n(x,k,a)dx = \sum_{v=0}^{n} \binom{n}{v} \mathcal{Y}_{n-v}(k,a) b_v(0),$$

*where $b_v(0)$ denotes the Bernoulli numbers of the second kind given by the following integral representation (cf. [**13**]):*

$$b_l(0) = \int_0^1 (x)_l \, dx.$$

Theorem 18. *Let $n \in \mathbb{N}_0$. Then we have*

$$\int_0^1 \mathcal{P}_n(x,k,a)dx = \sum_{v=0}^{n} \sum_{j=0}^{v} \binom{n}{v} \frac{\mathcal{Y}_{n-v}(k,a) S_1(v,j)}{j+1}.$$

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