

THEORY OF DISCRETE MUCKENHOUP T WEIGHTS AND DISCRETE RUBIO DE FRANCIA EXTRAPOLATION THEOREMS

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In this paper, we will prove a discrete Rubio De Francia extrapolation theorem in the theory of discrete \mathcal{A}^p –Muckenhoupt weights for which the discrete Hardy-Littlewood maximal operator is bounded on $\ell_w^p(\mathbb{Z}_+)$. The results will be proved by employing the self-improving property of the discrete \mathcal{A}^p –Muckenhoupt weights and the Marcinkiewicz Interpolation Theorem.

1. INTRODUCTION

The necessary and sufficient condition for the boundedness of a series of classical operators in the weighted spaces L_w^p is the A^p –condition of Muckenhoupt for the weighted functions. The proof of the boundedness of operators is based precisely on the application of the self-improving properties of A^p weights, see [8], [18] and [20]. A weight w , which is a nonnegative locally integrable function, is in the Muckenhoupt class $A^p(\mathbb{R}^n)$ for $p > 1$ if

$$(1) \quad \sup_Q \left(\frac{1}{|Q|} \int_Q w(t) dt \right) \left(\frac{1}{|Q|} \int_Q w^{-\frac{1}{p-1}}(t) dt \right)^{p-1} \leq C,$$

where the supremum is taken over all cubes Q in \mathbb{R}^n . The smallest possible constant C is called the A^p –constant of w and is denoted by $[A^p(w)]$. For $p = 1$, we say that w is in $A^1(\mathbb{R}^n)$ if $Mw(x) \leq Cw(x)$, where M is Hardy-Littlewood maximal operator, i.e,

$$(Mw)(x) = \sup_{x \in Q} \frac{1}{|Q|} \int_Q w(t) dt.$$

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The smallest possible C is called the A^1 -constant and is denoted by $[A^1(w)]$. In what follows, we write simply A^p instead of $A^p(\mathbb{R}^n)$. Muckenhoupt in [20] proved for $1 < p < \infty$ that the maximal Hardy-Littlewood operator is bounded on the space $L_w^p(\mathbb{R}_+)$, which is defined by

$$L_w^p(\mathbb{R}_+) = \left\{ f : \|f\|_{L_w^p(\mathbb{R}_+)} := \left(\int_0^\infty |f(t)|^p w(t) dt \right)^{1/p} < \infty \right\},$$

if and only if $w \in A^p$. Hunt, Muckenhoupt and Wheeden proved in [18] that the A^p condition also characterizes the boundedness of the Hilbert transform

$$Tf(x) = p.v. \int_{\mathbb{R}} \frac{f(y)}{x-y} dy, \text{ in } L_w^p.$$

Coifman and Fefferman [8] extended the A^p theory to general Calderón-Zygmund operators. The work of these authors found their reasons due to the fact that M and T play crucial roles in classical Harmonic Analysis. Another important property of the A^p Muckenhoupt weights is the extrapolation theorem due to Rubio de Francia, that has been announced in [21] and a detailed proof has been given in [22]. Since then, many results concerning this topic have been considered by several authors (see [9, 10, 11, 12, 13, 14, 15, 16, 17]). In the following, we present the celebrated extrapolation theorem due to Rubio de Francia.

Theorem 1. *Let T be a sublinear operator defined on a class of measurable functions in \mathbb{R}^n . Suppose for some p_0 , with $1 \leq p_0 < \infty$, and every weight $w \in A^p$, T satisfies the inequality*

$$\int_{\mathbb{R}^n} |Tf(x)|^{p_0} w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^{p_0} w(x) dx,$$

for every f , where C depends only on A^{p_0} -constant of w . Then for every p with $1 < p < \infty$, and every $w \in A^p$, the operator satisfies the inequality

$$\int_{\mathbb{R}^n} |Tf(x)|^p w(x) dx \leq C \int_{\mathbb{R}^n} |f(x)|^p w(x) dx,$$

for every f , where C depends only on A^p -constant of w .

During the past few years there has been renewed interest in the area of discrete harmonic analysis and then it becomes an active field of research. For example, the study of discrete l^p analogues for L^p -results has been considered by some authors, see for example [3, 4, 5, 7, 19, 23, 24, 25, 26, 27, 28, 29, 30, 31] and the references cited therein. This began with an observation of M. Riesz in his work on the Hilbert transform in 1928 that was carried over in the work of Calderón and Zygmund on singular integrals in 1952.

Although the discrete Hardy Littlewood and Hilbert transforms are less involved in analysis theory than their continuous versions, the discrete operators turn

out to be a powerful and indispensable tool for experimental science and engineering as in signal processing. Recent investigations on discrete functional spaces make up clear the need of considering and solving problems related to the discrete versions of M and T analogue to continuous ones like Theorem 1 for instance.

Whereas some results from Euclidean harmonic analysis admit an obvious variant in the discrete setting, some others do not. It is well known that passage from integral operators to their discrete analogues is not trivial (see, e.g., [1, 2, 6]) and each of these two settings requires its own techniques.

We can divide the methods that widely employed in the literature when proving results in the discrete setting to: The “imitation” method where the classical techniques are adapted to the discrete setting and, the “implication” method where the results are directly obtained from the continuous (classical) results. The main challenge in such studies is that there are no general methods in studying problems in discrete spaces due to the lack of the discrete calculus compared with the calculus for functions. For example, there are no power rules for differences and summations. In fact to find Δn^α when $0 < \alpha < 1$ is different from the cases when $\alpha < 0$ and $\alpha > 1$. Also to find the exact value $\sum n^\beta$ is not an easy task and depends also on the value of β . The results in general are inequalities instead of equalities. The summation by parts

$$\sum_{k=1}^N u(n) \Delta v(n) = u(k)v(k)|_{k=1}^{N+1} - \sum_{k=1}^N \Delta u(n)v(n+1),$$

which is non-homogenous, i.e., contains the term $v(n+1)$ in the right hand side, also let the proofs are not easy and need a modified technique different from the continuous cases to get the desired results.

The natural questions which arise now are the following:

Q_1). *Is it possible to prove Theorems 1 for sequences in the discrete space $\ell_w^p(\mathbb{Z}_+)$?*

Q_2). *Is it possible to prove Theorems 1 for sequences in the discrete space $\ell_w^p(\mathbb{Z}_+^d)$?*

Our aim in paper is to give an affirmative answer to the first question which leads to the development of the applications of discrete Muckenhoupt weights on discrete harmonic analysis. The results will be proved in Section 2 which is organized as follows: First, we will present the basic definitions and prove some basic properties of the discrete Muckenhoupt weights. Next, we will prove that the discrete Hardy-Littlewood maximal operator is bounded on ℓ_w^p in connection with the discrete \mathcal{A}^p -weights. Finally, we will prove our main discrete extrapolation theorem. To the best of the author’s knowledge the discrete extrapolation theorem is essentially new.

2. MAIN RESULTS

Throughout the paper, we will assume that the sequences in the statements of theorems that follow are assumed to be nonnegative sequences defined on \mathbb{Z}_+ . We shall denote by C the universal constant depending only on p, p_0 but independent on w and C might be not be the same in all the instances. We write $A \lesssim B$ if there exists a universal constant C such that $A \leq CB$ and $A \approx B$ if $A \lesssim B$ and $B \lesssim A$. Before, we present the main results, we present the necessary background in discrete harmonic analysis used in the remainder of this paper. Throughout the paper, we assume that $1 < p < \infty$. A discrete weight on $\mathbb{Z}_+ = \{1, 2, \dots\}$ is a sequence $w = \{w(n)\}_{n=1}^{\infty}$ of nonnegative real numbers. Most often w will appear in the role of the weight in ℓ_w^p -estimates, i.e., we shall consider the norm

$$\|f\|_{\ell_w^p(\mathbb{Z}_+)} := \left(\sum_{n=1}^{\infty} |f(n)|^p w(n) \right)^{1/p} < \infty.$$

By an interval J , we mean a finite subset of \mathbb{Z}_+ consisting of consecutive integers, i.e. $J = \{a, a+1, \dots, a+n\}$, $a, n \in \mathbb{Z}_+$, and $|J|$ stands for its cardinality. A discrete weight w is said to belong to the discrete Muckenhoupt class $\mathcal{A}^1(A)$ on \mathbb{Z}_+ , for $p > 1$ and $A > 1$, if the inequality

$$\mathcal{M}w \leq Aw,$$

holds for every bounded subinterval $J \subset \mathbb{Z}_+$ and $|J|$ is the cardinality of the set J , where

$$[\mathcal{M}w](k) = \sup_{k \in J} \frac{1}{|J|} \sum_J w, \text{ for } k \in \mathbb{Z}_+$$

where $[\mathcal{M}f]$ denotes the k^{th} term of a sequence and the supremum is taken over all bounded intervals $J \subset \mathbb{Z}_+$. The operator $w \rightarrow \mathcal{M}w$ is called the Hardy-Littlewood maximal operator. A discrete weight w is said to belong to the discrete Muckenhoupt class $\mathcal{A}^2(A)$ on \mathbb{Z}_+ for $p > 1$ and $A > 1$, if the inequality

$$\left(\frac{1}{|J|} \sum_J w \right) \left(\frac{1}{|J|} \sum_J w^{-1} \right) \leq A,$$

holds for every subinterval $J \subset \mathbb{Z}_+$. A discrete nonnegative sequence w is said to belong to the discrete Muckenhoupt class $\mathcal{A}^p(A)$ on \mathbb{Z}_+ for $p > 1$ and $A > 1$ if the inequality

$$(2) \quad \left(\frac{1}{|J|} \sum_J w \right) \left(\frac{1}{|J|} \sum_J w^{\frac{-1}{p-1}} \right)^{p-1} \leq A,$$

holds for every bounded subinterval interval $J \subset \mathbb{Z}_+$. For a given exponent $p > 1$, we define the \mathcal{A}^p -norm of the discrete weight w by the following quantity

$$[\mathcal{A}^p(w)] := \sup_{J \subset \mathbb{I}} \left(\frac{1}{|J|} \sum_J w \right) \left(\frac{1}{|J|} \sum_J w^{\frac{-1}{p-1}} \right)^{p-1},$$

where the supremum is taken over all bounded intervals $J \subset \mathbb{Z}_+$. When we fix a constant $A > 1$ the couple of real numbers (p, A) defines the \mathcal{A}^p discrete Muckenhoupt class $\mathcal{A}^p(A)$:

$$w \in \mathcal{A}^p(A) \iff [\mathcal{A}^p(w)] \leq A,$$

and we will refer to A as the \mathcal{A}^p -constant of the class. Note that by Hölder's inequality $[\mathcal{A}^p(w)] \geq 1$ for all $1 < p < \infty$ and that the following inclusion is true:

$$\text{if } 1 < p \leq q < \infty, \text{ then } \mathcal{A}^p \subset \mathcal{A}^q \text{ and } [\mathcal{A}^q(w)] \leq [\mathcal{A}^p(w)].$$

It is evident from the condition (2) that w and $w^{-1/(p-1)}$ must be summable and consequently almost every where finite. In particular, $w > 0$ almost everywhere.

Now, we present some properties of the discrete Muckenhoupt weights which are adapted from [26, 28] and needed in the proofs.

Theorem 2. *Let w be a nonnegative weight and p and q be positive real numbers. The following properties hold:*

$$(1). \ w \in \mathcal{A}^p \text{ if and only if } w^{1-p'} \in \mathcal{A}^{p'}, \text{ with } [\mathcal{A}^{p'}(w^{1-p'})] = [\mathcal{A}^p(w)]^{p'-1}$$

where p' is the conjugate of p ,

$$(2). \ \mathcal{A}^p \subset \mathcal{A}^q \text{ for all } 1 < p \leq q, \text{ and } [\mathcal{A}^q(w)] \leq [\mathcal{A}^p(w)],$$

$$(3). \ \text{if } w \in \mathcal{A}^p \text{ then } w^\alpha \in \mathcal{A}_p, \text{ for } 0 \leq \alpha \leq 1, \text{ with } [\mathcal{A}^p(w^\alpha)] = [\mathcal{A}^p(w)]^\alpha,$$

$$(4). \ \text{If } w_1, w_2 \in \mathcal{A}^p, \text{ then } w_1^\alpha w_2^{1-\alpha} \in \mathcal{A}^p, \ 0 \leq \alpha \leq 1, \text{ with a constant}$$

$$[\mathcal{A}^p(w_1^\alpha w_2^{1-\alpha})] = [\mathcal{A}^p(w_1)]^\alpha [\mathcal{A}^p(w_2)]^{1-\alpha}.$$

Lemma 1. *If $q > 1$, $A > 1$ and v is a nonnegative weight belongs to $\mathcal{A}^q(A)$, then $v \in \mathcal{A}^p(A_1)$ for $p \in (p_0, q]$ where $p_0 > 1$ is a unique positive root of the algebraic equation*

$$\frac{q-x}{q-1} (Ax)^{\frac{1}{p-1}} = 1.$$

Remark 1. *This result proves that if $v \in \mathcal{A}^q(A)$ then there exists an $\epsilon > 0$ and a constant $A_1 = A_1(p, A)$ such that $v \in \mathcal{A}^{q-\epsilon}(A_1)$, (self-improving property) and thus*

$$(3) \quad \mathcal{A}^q(A) \subset \mathcal{A}^{q-\epsilon}(A_1).$$

The discrete nonnegative sequences (w, v) belongs to the discrete Muckenhoupt class $\mathcal{A}^p(A)$ on \mathbb{Z}_+ for $p > 1$ and $A > 1$ if the inequality

$$(4) \quad \left(\frac{1}{|J|} \sum_J w \right) \left(\frac{1}{|J|} \sum_J v^{1-p^*} \right)^{p-1} \leq A,$$

holds for every bounded subinterval interval $J \subset \mathbb{I}$, where p^* is the conjugate exponent of p . This is more convenient, when we dealing with a double of weights.

From (4), we see that if $(w, v) \in \mathcal{A}^p$, then $(v^{1-p^*}, w^{1-p^*}) \in \mathcal{A}^{p^*}$. This is obtained from (4), by using (v^{1-p^*}, w^{1-p^*}) instead of (w, v) to get that

$$\left(\frac{1}{|J|} \sum_J v^{1-p^*} \right)^{p^*-1} \left(\frac{1}{|J|} \sum_J w \right)^{(p-1)(p^*-1)} \leq C.$$

Since $(p-1)(p^*-1) = 1$, we have that

$$\left(\frac{1}{|J|} \sum_J w \right) \left(\frac{1}{|J|} \sum_J v^{1-p^*} \right)^{p^*-1} \leq C,$$

which proves that $(v^{1-p^*}, w^{1-p^*}) \in \mathcal{A}^{p^*}$. Also as in the Property (3) of Theorem 2, we see that if $(w, v) \in \mathcal{A}^p$, then $(w^\delta, v^\delta) \in \mathcal{A}^q$ for $0 < \delta < 1$ such that $(q-1)/(p-1) = \delta$. Note that $1 < q < p$. Since

$$(v^{1-p^*}, w^{1-p^*}) \in \mathcal{A}^{p^*}, \text{ and } q^* > p^*,$$

we have by Property (2) of Theorem 2 that $(v^{1-p^*}, w^{1-p^*}) \in \mathcal{A}^{q^*}$. We use this property again and infer that

$$(w^{(1-p^*)(1-q)}, v^{(1-p^*)(1-q)}) \in \mathcal{A}^q,$$

that is $(w^\delta, v^\delta) \in \mathcal{A}^q$. We summarize the above results for the two weights in the following Lemma.

Lemma 2. *Let $(w, v) \in \mathcal{A}^p$ for $p > 1$. Then*

- 1). $(v^{1-p^*}, w^{1-p^*}) \in \mathcal{A}^{p^*}$, $1/p + 1/p^* = 1$.
- 2). $(w^\delta, v^\delta) \in \mathcal{A}^q$, for $0 < \delta < 1$ such that $(q-1)/(p-1) = \delta$.

In order to proceed, we need to define the relevant concepts. For a nonnegative sequence f , the Hardy-Littlewood maximal sequence $\mathcal{M}f$ of a nonnegative f is defined by

$$[\mathcal{M}f](k) = \sup_{k \in J} \frac{1}{|J|} \sum_{s \in J} f(s), \quad k \in \mathbb{Z}_+,$$

where $[\mathcal{M}f](k)$ denotes the k^{th} term of a sequence and the supremum is taken over all bounded intervals $J \subset \mathbb{Z}_+$. The operator $f \rightarrow \mathcal{M}f$ is called the Hardy-Littlewood maximal operator. In [2] the authors proved that if $f \in \ell^p(\mathbb{Z}_+)$ then $\mathcal{M}f \in \ell^p(\mathbb{Z}_+)$ and there exists a constant $C > 0$ such that $\|\mathcal{M}f\|_{\ell^p(\mathbb{Z}_+)} \leq C \|f\|_{\ell^p(\mathbb{Z}_+)}$. In [17] the authors proved that if f is bounded on $\ell^p(\mathbb{Z}_+)$ then $\mathcal{M}f$ is bounded on $\ell^\infty(\mathbb{Z}_+)$. One of our aims in this paper is to prove that the operator $\mathcal{M}f$ is bounded on $\ell_w^p(\mathbb{Z}_+)$. To do this, we need to study the boundedness of the weighted maximal Hardy-Littlewood operator

$$(5) \quad [\mathcal{M}^w(f)](k) = \sup_{k \in J} \frac{1}{\Lambda(J)} \sum_{s \in J} w(s)f(s), \text{ for } k \in \mathbb{Z}_+,$$

where $\Lambda(J) = \sum_{s \in J} w(s)$. The operator $f \rightarrow \mathcal{M}^w f$ is called the weighted Hardy-Littlewood maximal operator with a weight w . The Hardy operator associated to the operator (5) is given by

$$(6) \quad \mathcal{H}^w f = \frac{1}{\Lambda(J)} \sum_{s \in J} f(s)w(s), \text{ and } \Lambda(J) = \sum_{s \in J} w(s).$$

The following lemma which proves the boundedness of the operator (5) plays a crucial role in establishing our main results. Without loss of generality in the proof of the following, we fix an interval $J \subset \mathbb{Z}_+$ of the form $J = \{1, 2, \dots, N\}$, for $k, N \in \mathbb{Z}_+$.

Lemma 3. *Let $r > 1$ and f be a nonnegative weight defined on \mathbb{Z}_+ . If $r > 1$, then there exists a constant $C > 0$ such that*

$$(7) \quad \|\mathcal{M}^w f\|_{\ell_r^w(\mathbb{Z}_+)} \leq C \|f\|_{\ell_r^w(\mathbb{Z}_+)},$$

holds, where $\mathcal{M}^w f$ be defined as in (5).

Proof. From the definition of $\mathcal{M}^w(f)$, we have that

$$\mathcal{M}^w(f)(k) \geq \frac{1}{\Lambda(k)} \sum_{s=1}^k w(s)f(s), \text{ for } 1 \leq k \leq N,$$

where $\Lambda(k) = \sum_{s=1}^k w(s)$, and there exists a constant $C \geq 1$, such that

$$\mathcal{M}^w(f)(k) \leq \frac{C}{\Lambda(k)} \sum_{s=1}^k w(s)f(s).$$

Now, by using the definition of $\mathcal{H}^w f$ (see (6)), we get that

$$(8) \quad \mathcal{H}^w f(k) \leq \mathcal{M}^w(f)(k) \leq C\mathcal{H}^w f(k).$$

By setting $A(k) = \sum_{s=1}^k \lambda(s)f(s)$, we see that $\Delta A(k-1) = w(k)f(k)$, and from the definition of $\mathcal{H}^w f$, we also have that

$$\Lambda(k)\mathcal{H}^w(f)(k) = A(k).$$

Thus

$$(9) \quad \begin{aligned} & w(k) (\mathcal{H}^w f(k))^r - \frac{r}{r-1} w(k)f(k) (\mathcal{H}^w f(k))^{r-1} \\ &= w(k) (\mathcal{H}^w f(k))^r - \frac{r}{r-1} [A(k) - A(k-1)] (\mathcal{H}^w f(k))^{r-1} \\ &= w(k) (\mathcal{H}^w f(k))^r - \frac{r}{r-1} A(k) (\mathcal{H}^w f(k))^{r-1} \\ &\quad + \frac{r}{r-1} A(k-1) (\mathcal{H}^w f(k))^{r-1} \\ &= w(k) (\mathcal{H}^w f(k))^r + \frac{r}{(r-1)} \Lambda(k-1) \mathcal{H}^w(k-1) (\mathcal{H}^w f(k))^{r-1} \\ &\quad - \frac{r}{(r-1)} \Lambda(k) (\mathcal{H}^w f(k))^r. \end{aligned}$$

By applying the Young inequality

$$ab \leq \frac{a^r}{r} + \frac{b^q}{q}, \quad a, b > 0, \quad \frac{1}{r} + \frac{1}{q} = 1, \quad r, q > 0,$$

with $q = (r/r - 1)$, $a = \mathcal{H}^w f(k - 1)$ and $b = (\mathcal{H}^w f(k))^{r-1}$, we have

$$\frac{r}{(r-1)} (\mathcal{H}^w f(k))^{r-1} \mathcal{H}^w f(k-1) \leq (\mathcal{H}^w f(k))^r + \frac{1}{(r-1)} (\mathcal{H}^w f(k-1))^r.$$

This and (9) imply (noting that $w(k) = \Lambda(k) - \Lambda(k-1)$), that

$$\begin{aligned} & w(k) (\mathcal{H}^w f(k))^r - \frac{r}{r-1} w(k) f(k) (\mathcal{H}^w f(k))^{r-1} \\ & \leq w(k) (\mathcal{H}^w f(k))^r + \Lambda(k-1) (\mathcal{H}^w f(k))^r \\ & \quad + \frac{1}{r-1} \Lambda(k-1) (\mathcal{H}^w f(k))^r - \frac{r}{r-1} \Lambda(k) (\mathcal{H}^w f(k))^{r-1} \\ & = \Lambda(k) (\mathcal{H}^w f(k))^r - \Lambda(k-1) (\mathcal{H}^w f(k))^r \\ & \quad + \Lambda(k-1) (\mathcal{H}^w f(k))^r + \frac{\Lambda(k-1)}{(r-1)} (\mathcal{H}^w f(k-1))^r - \frac{r\Lambda(k) (\mathcal{H}^w f(k))^r}{(r-1)} \\ & = \Lambda(k) (\mathcal{H}^w f(k))^r \left[1 - \frac{r}{(r-1)} \right] + \frac{\Lambda(k-1)}{r-1} (\mathcal{H}^w f(k-1))^r. \end{aligned}$$

So that

$$\begin{aligned} & w(k) (\mathcal{H}^w f(k))^r - \frac{r}{r-1} w(k) f(k) (\mathcal{H}^w f(k))^{r-1} \\ & \leq \Lambda(k) (\mathcal{H}^w f(k))^r \left[\frac{-1}{(r-1)} \right] + \left[\frac{\Lambda(k-1)}{r-1} (\mathcal{H}^w f(k-1))^r \right] \\ & = \frac{-1}{(r-1)} [\Lambda(k) (\mathcal{H}^w f(k))^r - \Lambda(k-1) (\mathcal{H}^w f(k-1))^r] \\ & = \frac{-1}{(r-1)} \Delta \Lambda(k-1) (\mathcal{H}^w f(k-1))^r. \end{aligned}$$

Hence

$$\begin{aligned} & \sum_{k=1}^n w(k) (\mathcal{H}^w f(k))^r - \frac{r}{r-1} \sum_{k=1}^n w(k) f(k) (\mathcal{H}^w f(k))^{r-1} \\ & \leq \frac{-1}{(r-1)} \sum_{k=1}^n \Delta \Lambda(k-1) (\mathcal{H}^w f(k-1))^r = \frac{-1}{(r-1)} \Lambda(n) (\mathcal{H}^w f(n))^r \leq 0. \end{aligned}$$

This implies by using (8), that

$$\sum_{k=1}^n w(k) (\mathcal{M}^w f(k))^r \leq C \sum_{k=1}^n w(k) f(k) (\mathcal{M}^w f(k))^{r-1}.$$

Making n tends to infinity and applying Hölder's inequality, we get that

$$\begin{aligned} & \sum_{k=1}^{\infty} w(k) (\mathcal{M}^w f(k))^r \\ & \leq C \left(\sum_{k=1}^{\infty} w(k) f^r(k) \right)^{1/r} \left(\sum_{k=1}^{\infty} w(k) (\mathcal{M}^w f(k))^r \right)^{\frac{r-1}{r}}. \end{aligned}$$

Dividing by the last term on the right hand side, we get the desired inequality (7). The proof is complete \square

Now, we prove the boundedness of discrete Hardy-Littlewood maximal operator $\mathcal{M}(f)$ which shows the explicit dependence of boundedness on \mathcal{A}^p condition.

Theorem 3. *If $p > 1$ then $\mathcal{M}f$ is bounded in $\ell_w^p(\mathbb{Z}_+)$ if and only if $w \in \mathcal{A}^p$ and the following inequality*

$$(10) \quad \|\mathcal{M}f\|_{\ell_w^p(\mathbb{Z}_+)} \leq C \|f\|_{\ell_w^p(\mathbb{Z}_+)},$$

holds, where the constant C depends on $[\mathcal{A}^p(w)]$ constant of w .

Proof. Necessity: That is (10) implies \mathcal{A}^p is similar to the proof given in [28]. To prove that $w \in \mathcal{A}^p$ implies (10), we first note that $w \in \mathcal{A}^p$ implies, by replacing f by $(fw^{1/p})w^{-1/p}$ in the the Hardy operator $\mathcal{H}^w f(n)$, that

$$\begin{aligned} (11) \quad \frac{1}{|J|} \sum_{k \in J} f(k) &= \frac{1}{|J|} \sum_{k \in J} (f(k)w^{1/p}) w^{-1/p} \\ &\leq \frac{1}{|J|} \left(\sum_{k \in J} f^p(k)w(k) \right)^{1/p} \left(\sum_{k \in J} w^{-1/(p-1)} \right)^{\frac{p-1}{p}} \\ &\leq C \left(\frac{1}{\Lambda(J)} \sum_{k \in J} f^p(k)w(k) \right)^{1/p}, \end{aligned}$$

Now, taking the supremum in the last inequality, we have that

$$(12) \quad (\mathcal{M}f(k)) \leq C \mathcal{M}^w (f^p(k))^{1/p},$$

where

$$\mathcal{M}^w (f^p(k)) = \sup_{k \in J} \frac{1}{\Lambda(J)} \sum_{s \in J} w(s) f^p(s).$$

Now Lemma 3 yields, with $f = f^p$, that

$$\sum_{k=1}^{\infty} w(k) (\mathcal{M}^w f^p(k))^r \leq C_r \sum_{k=1}^{\infty} f^{pr}(k)w(k).$$

Since the above inequality is valid only for $r > 1$, so we can take $r = p_1/p > 1$, so that

$$(13) \quad \sum_{k=1}^{\infty} (\mathcal{M}^w f^p(k))^{p_1/p} w(k) \leq C_{p_1} \sum_{k=1}^{\infty} f^{p_1}(k) w(k).$$

By raising both sides in (12) with some exponent $p_1 > p$, we have that

$$(14) \quad \sum_{k=1}^{\infty} (\mathcal{M}f(k))^{p_1} w(k) \leq C \sum_{k=1}^{\infty} (\mathcal{M}^w f^p(k))^{\frac{p_1}{p}} w(k).$$

By combining (13) and (14), we have that the weight w which satisfies \mathcal{A}^p condition also satisfies the following weighted inequality (since $r = p_1/p > 1$)

$$\sum_{k=1}^{\infty} (\mathcal{M}f(k))^{p_1} w(k) \leq C_{p_1} \sum_{k=1}^{\infty} f^{p_1}(k) w(k), \text{ for every } p_1 > p,$$

whenever $w \in \mathcal{A}^p$. Finally, it only remains to recall the backward propagation property (3) that is the weight w also satisfies $\mathcal{A}^{p-\epsilon}$ condition (with $p_1 = p + \epsilon$) which gives the desired result (10). To clarify this last point, instead of (11), we restart the proof with the following (since $w(n) \in \mathcal{A}^{p-\epsilon}$)

$$\frac{1}{|J|} \sum_{k \in J} f(k) = \frac{1}{|J|} \sum_{k \in J} f(k) w^{\frac{1}{p-\epsilon}}(k) w^{\frac{-1}{p-\epsilon}}(k),$$

such that $p-\epsilon \geq 1$. Then by applying Hölder's inequality and $\mathcal{A}^{p-\epsilon}$ condition which already satisfied by the weight w as mentioned before. Till we reach the inequality (14) which is satisfied now with "new" $p_1 = p - \epsilon + \epsilon = p$, to get that

$$\sum_{k=1}^{\infty} (\mathcal{M}f(k))^p w(k) \leq C \sum_{k=1}^{\infty} w(k) f^p(k),$$

which is the required inequality (10). The proof is complete. \square

There is a weak type version of the above theorem which also depends on the self-improving property of \mathcal{A}^p -weights. Before, we sketch the proof, let us formulate some definitions of weak and strong boundedness. The power set of \mathbb{Z}_+ will be denoted by $A = 2^{\mathbb{Z}_+}$, and $\mu =$ a counting measure. The distribution sequence of any real sequence $\{f(n)\}_{n \geq 1}$ is defined by

$$D_f(\lambda) = \mu\{n \in \mathbb{Z}^+ : |f(n)| > \lambda\}, \text{ for } \lambda > 0,$$

with respect to counting measure μ . Recall that the weak type (1,1) estimate of \mathcal{M} for any $\lambda > 0$

$$|\{k : \mathcal{M}(f) w > \lambda\}| \leq \frac{C}{\lambda} \sum_{k=1}^{\infty} |f(k)| w(k).$$

For $1 < p < \infty$, the operator $\mathcal{M} : \ell_w^p(\mathbb{Z}_+) \rightarrow \ell_w^p(\mathbb{Z}_+)$ is of weak type (p, p) with respect to the weight w if the following weak type inequality

$$(15) \quad |\{k : \mathcal{M}(f)w > \lambda\}| \leq \frac{C}{\lambda^p} \|f\|_{\ell_w^p(\mathbb{Z}_+)}^p, \text{ for all } \lambda > 0,$$

holds. Now, we give a brief proof of weak boundedness of Theorem 3. The first part when (10) holds is similar to the proof given above. For the reverse direction, if $w \in \mathcal{A}^p$, we claim that $\mathcal{M}f$ satisfies a weak (p, p) condition, i.e. (15). Recall the weak type $(1, 1)$ estimate of $\mathcal{M}^w f$, for any $\lambda > 0$,

$$|\{k : \mathcal{M}^w(f) > \lambda\}| \leq \frac{C}{\lambda} \sum_{k=1}^{\infty} |f(k)| w(k).$$

Replace f with f^p and λ with λ^p , we get that

$$|\{k : \mathcal{M}^w(f^p) > \lambda^p\}| \leq \frac{C}{\lambda^p} \sum_{k=1}^{\infty} f^p(k) w(k).$$

In the given proof above, we have showed that if $w \in \mathcal{A}^p$, then (see (12) $(\mathcal{M}f(k))^p \leq c\mathcal{M}^w(f^p(k))$), where

$$\mathcal{M}^w(f^p(k)) = \sup_{k \in J} \frac{1}{\Lambda(J)} \sum_{k \in J} w(s) f^p(s).$$

This implies $|\{k : \mathcal{M}(f)w > \lambda^p\}| \subset |\{k : \mathcal{M}^w(f^p) > \lambda^p\}|$, and thus the weak type estimate of $\mathcal{M}(f)$ holds on $\ell_w^p(\mathbb{Z}_+)$. Then from the backward propagation property (3) $w \in \mathcal{A}^{p_1}$ for some $p_1 < p$ and satisfy the weak type estimate on $\ell_w^{p_1}(\mathbb{Z}_+)$. Now, by using the fact that $\mathcal{M}(f)$ is bounded on $\ell_w^\infty(\mathbb{Z}_+)$, we have by applying Marcinkiewicz Interpolation Theorem that $\mathcal{M}(f)$ is bounded on $\ell_w^p(\mathbb{Z}_+)$.

Theorem 4. *Let $1 < p < \infty$, $w \in \mathcal{A}^p$, $0 < \gamma \leq 1$. For a nonnegative sequence g , define*

$$Gg = \left(\mathcal{M}(g^{1/\gamma} w) w^{-1} \right)^\gamma,$$

where \mathcal{M} stands for Hardy-Littlewood maximal operator. Then

i). $g \rightarrow G$ is a bounded operator in $\ell_w^{p^*/\gamma}(\mathbb{Z}_+)$, where p^* is exponent conjugate to p .

ii). $(gw, Gw) \in \mathcal{A}^{(p-\gamma p)/p^*}$.

Besides, the operator norm in (i) and the constant for the pair of weights in (ii) depends only on $[\mathcal{A}^p(w)]$ constant of w .

Proof. From the definition of G , we see that

$$\begin{aligned}
\|G\|_{\ell_w^{p^*/\gamma}}^{p^*/\gamma} &: = \sum_{n=1}^{\infty} (G(n))^{p^*/\gamma} w(n) \\
&= \sum_{n=1}^{\infty} \left((\mathcal{M}(g^{1/\gamma}(n)w(n))w^{-1}(n))^{\gamma} \right)^{p^*/\gamma} w(n) \\
&= \sum_{n=1}^{\infty} \left(\mathcal{M}(g^{1/\gamma}(n)w(n)) \right)^{p^*} w^{-p^*}(n)w(n) \\
(16) \quad &= \sum_{n=1}^{\infty} \left(\mathcal{M}(g^{1/\gamma}(n)w(n)) \right)^{p^*} w^{1-p^*}(n).
\end{aligned}$$

From the Property (1) of Theorem 2 and since $w^{1-p^*}(n) \in \mathcal{A}^{p^*}$, we have from Theorem 3, that

$$\sum_{n=1}^{\infty} \left(\mathcal{M}(g^{1/\gamma}(n)w(n)) \right)^{p^*} w^{1-p^*}(n) \leq C \sum_{n=1}^{\infty} g^{p^*/\gamma}(n)w(n).$$

This and (16) give us

$$\|G\|_{\ell_w^{p^*/\gamma}}^{p^*/\gamma} \leq C \|g\|_{\ell_w^{p^*/\gamma}}^{p^*/\gamma},$$

which proves the statement (i).

(JJ). Let $p_0 = p - \gamma p/p^*$. Clearly $p_0 \geq 1$, since $p_0 - 1 = (1 - \gamma)p/p^* \geq 0$. If $\gamma = 1$, then $p_0 = 1$ and it is immediate that $(gw, Gw) \in \mathcal{A}^1$, because $G = \mathcal{M}(gw)w^{-1}$ and then $Gw = \mathcal{M}(gw)$. Let $0 < \gamma < 1$. We have to prove that

$$(17) \quad \left(\frac{1}{|J|} \sum_{k \in J} gw \right) \left(\frac{1}{|J|} \sum_{k \in J} (Gw)^{-1/(p_0-1)} \right)^{(p_0-1)} \leq C,$$

with a constant C independent of the interval J where $|J|$ is the cardinality of the set $J \subset \mathbb{Z}_+$. From the definition of \mathcal{M} , we have that

$$\frac{1}{|J|} \sum_{k \in J} g^{1/\gamma} w \leq \mathcal{M} \left(g^{1/\gamma} w \right),$$

and then

$$(18) \quad \left(\mathcal{M} \left(g^{1/\gamma} w \right) \right)^{-\gamma p^*/((p-1)\gamma)} \leq \left(\frac{1}{|J|} \sum_{k \in J} g^{1/\gamma} w \right)^{-\gamma p^*/((p-1)\gamma)}.$$

The left hand side of the inequality (17) now reads

$$\begin{aligned}
(19) \quad & \left(\frac{1}{|J|} \sum_{k \in J} gw \right) \left(\frac{1}{|J|} \sum_{k \in J} (Gw)^{-1/(p_0-1)} \right)^{(p_0-1)} \\
& \leq \left(\frac{1}{|J|} \sum_{k \in J} gw \right) \left(\frac{1}{|J|} \sum_{k \in J} \left(\mathcal{M}(g^{1/\gamma} w) \right)^{-\frac{\gamma p^*}{(p(1-\gamma))}} w^{\frac{\gamma p^*}{(p(1-\gamma))}} w^{-\frac{p^*}{(1-\gamma)p}} \right)^{(1-\gamma)p/p^*} \\
& = \left(\frac{1}{|J|} \sum_{k \in J} gw \right) \left(\frac{1}{|J|} \sum_{k \in J} \left(\mathcal{M}(g^{1/\gamma} w) \right)^{-\frac{\gamma p^*}{(p(1-\gamma))}} w^{-p^*/p} \right)^{(1-\gamma)p/p^*} \\
& = \left(\frac{1}{|J|} \sum_{k \in J} gw \right) \left(\frac{1}{|J|} \sum_{k \in J} \left(\mathcal{M}(g^{1/\gamma} w) \right)^{-\gamma p^*/(p(1-\gamma))} w^{-1/(p-1)} \right)^{(1-\gamma)p/p^*}.
\end{aligned}$$

By using the inequality (18) in the last line in (19), we see that

$$\begin{aligned}
(20) \quad & \left(\frac{1}{|J|} \sum_{k \in J} gw \right) \left(\frac{1}{|J|} \sum_{k \in J} \left(\mathcal{M}(g^{1/\gamma} w) \right)^{-\gamma p^*/(p(1-\gamma))} w^{-1/(p-1)} \right)^{(1-\gamma)p/p^*} \\
& \leq \left(\frac{1}{|J|} \sum_{k \in J} gw \right) \left(\frac{1}{|J|} \sum_{k \in J} \left(\frac{1}{|J|} \sum_{k \in J} g^{1/\gamma} w \right)^{-\gamma p^*/(p(1-\gamma))} w^{-1/(p-1)} \right)^{(1-\gamma)p/p^*}.
\end{aligned}$$

By applying Holder's inequality with $1/\gamma$ and $(1/(1-\gamma))$, we have that the last line in (20) is bounded by

$$\begin{aligned}
& \left(\frac{1}{|J|} \sum_{k \in J} g^{1/\gamma} w \right)^\gamma \left(\frac{1}{|J|} \sum_{k \in J} w \right)^{1-\gamma} \left(\frac{1}{|J|} \sum_{k \in J} g^{1/\gamma} w \right)^{-\gamma} \\
& \quad \times \left(\frac{1}{|J|} \sum_{k \in J} w^{-1/(p-1)} \right)^{(p-1)(1-\gamma)} \\
(21) \quad & = \left[\left(\frac{1}{|J|} \sum_{k \in J} w \right) \left(\frac{1}{|J|} \sum_{k \in J} w^{-1/(p-1)} \right)^{(p-1)} \right]^{1-\gamma} = C.
\end{aligned}$$

By combining (19) and (21), we have that

$$\left(\frac{1}{|J|} \sum_{k \in J} gw \right) \left(\frac{1}{|J|} \sum_{k \in J} (Gw)^{-1/(p_0-1)} \right)^{(p_0-1)} \leq C,$$

which proves (17) where $C = [\mathcal{A}^p(w)]^{1-\gamma}$ and $[\mathcal{A}^p(w)]$ is the \mathcal{A}^p constant for w . The proof is complete. \square

Remark 2. If we set $p - \gamma p/p^* = p_0 < p$, we shall have $1 - \gamma/p^* = p_0/p < 1$ or in other words $p^*/\gamma = (p/p_0)^*$. With this change, Theorem 4 can be written as follows.

Lemma 4. Let $1 < p_0 < p$ and $w \in \mathcal{A}^p$. Then for every nonnegative sequence $g \in \ell_w^{(p/p_0)^*}(\mathbb{Z}_+)$, there is $G \geq g$ such that

$$\|G\|_{\ell_w^{(p/p_0)^*}} \leq C \|g\|_{\ell_w^{(p/p_0)^*}},$$

and $(gw, Gw) \in \mathcal{A}^{p_0}$, where C and \mathcal{A}^{p_0} -constant depending only on the \mathcal{A}^p -constant of w .

Our next result is striking application of the relation between the boundedness of the Hardy-Littlewood maximal operator and the \mathcal{A}_p -weights. In what follows we write simply ℓ_w^s instead of $\ell_w^s(\mathbb{Z}_+)$.

Lemma 5. 1). Let $1 < p_0 < p$ and $w \in \mathcal{A}^p$. Then for every nonnegative sequence $g \in \ell_w^{(p/p_0)^*}(\mathbb{Z}_+)$, there is $G \geq g$ such that

$$(22) \quad \|G\|_{\ell_w^{(p/p_0)^*}} \leq C \|g\|_{\ell_w^{(p/p_0)^*}},$$

and $Gw \in \mathcal{A}^{p_0}$, where C and $[\mathcal{A}^{p_0}(Gw)]$ -constant for Gw depending only on the \mathcal{A}^p -constant for w .

2). Let $1 < p < p_0$ and $w \in \mathcal{A}^p$. Then for every nonnegative sequence $g \in \ell_w^{(p/(p_0-p))}(\mathbb{Z}_+)$, there is $G \geq g$ such that

$$\|G\|_{\ell_w^{(p/(p_0-p))}} \leq C \|g\|_{\ell_w^{(p/(p_0-p))}},$$

and $G^{-1}w \in \mathcal{A}^{p_0}$, where C and $[\mathcal{A}^{p_0}(G^{-1}w)]$ -constant for $G^{-1}w$ depending only on the \mathcal{A}^p -constant for w .

Proof. 1). The inequality (22) follows from Lemma 4 and $(gw, Gw) \in \mathcal{A}^{p_0}$, where \mathcal{A}^{p_0} -constant depending only on the \mathcal{A}^p -constant of w and given by $[\mathcal{A}^p(w)]^{1-\gamma}$. Now, we prove that $Gw \in \mathcal{A}^{p_0}$. Let $g_0 = g$. Then from Lemma 4 there exists $g_1 \geq g$ such that

$$\|g_1\|_{\ell_w^{(p/p_0)^*}} \leq C \|g\|_{\ell_w^{(p/p_0)^*}}.$$

From Lemma 1, since $w \in \mathcal{A}^p$, then $w \in \mathcal{A}^{p_0}$ for $1 \leq p_0 < p$ and thus by Theorem 3, we have that

$$\sum_{k=1}^{\infty} (\mathcal{M}(f(k)))^{p_0} w(k) \leq C \sum_{k=1}^{\infty} (f(k))^{p_0} w(k),$$

and then the inequality

$$\begin{aligned} \sum_{k=1}^{\infty} (\mathcal{M}(f(k)))^{p_0} g_0(k)w(k) &\leq C \sum_{k=1}^{\infty} (f(k))^{p_0} g_0(k)w(k) \\ &\leq C \sum_{k=1}^{\infty} (f(k))^{p_0} g_1(k)w(k), \end{aligned}$$

holds for every nonnegative sequence f with a constant C depending only on the \mathcal{A}^p -constant for w . Now, we proceed by induction. We can use g_1 in place of g_0 and continue in this way. In general given g_j , we obtain $g_{j+1} \geq g_j$ such that

$$\|g_{j+1}\|_{\ell_w^{(p/p_0)^*}} \leq C \|g_j\|_{\ell_w^{(p/p_0)^*}} < \dots \leq C^{j+1} \|g\|_{\ell_w^{(p/p_0)^*}},$$

and the inequality

$$(23) \quad \sum_{k=1}^{\infty} (\mathcal{M}(f(k)))^{p_0} g_j(k)w(k) \leq C \sum_{k=1}^{\infty} (f(k))^{p_0} g_{j+1}(k)w(k),$$

holds for every nonnegative sequence f with a constant C depending only on the \mathcal{A}^p -constant for w . Now, we consider $G = \sum_{j=0}^{\infty} (C+1)^{-j} g_j$ where C is the same as above. Since

$$(C+1)^{-j} \|g_j\|_{\ell_w^{(p/p_0)^*}} \leq (C/C+1)^j \|g\|_{\ell_w^{(p/p_0)^*}},$$

we see the series G converges in $\ell_w^{(p/p_0)^*}$ and we get $G \geq g$ with

$$\|G\|_{\ell_w^{(p/p_0)^*}} \leq (C+1) \|g\|_{\ell_w^{(p/p_0)^*}},$$

and also adding in j the inequality (23), we have that

$$\sum_{k=1}^{\infty} (\mathcal{M}(f(k)))^{p_0} G(k)w(k) \leq C(C+1) \sum_{k=1}^{\infty} (f(k))^{p_0} G(k)w(k),$$

for every nonnegative sequence f with a constant C depending only on the \mathcal{A}^p -constant for w . Then $Gw \in \mathcal{A}^{p_0}$ and its $[\mathcal{A}^{p_0}(Gw)]$ constant depends only on the \mathcal{A}^p -constant of w .

2). Now we consider the case when $1 < p < p_0$ and $w \in \mathcal{A}^p$. This is the same, see Property (1) in Theorem 2, as saying that $1 < p_0^* < p^*$ and $u = w^{-1/(p-1)} \in \mathcal{A}^{p^*}$. Therefore, we can apply Part (1) above for every $h \in \ell_u^{(p^*/p_0^*)^*}(\mathbb{Z}_+)$ and get that there exists $H \geq h$ such that

$$\|H\|_{\ell_u^{(p^*/p_0^*)^*}} \leq (C+1) \|h\|_{\ell_u^{(p^*/p_0^*)^*}},$$

and $Hu \in \mathcal{A}^{p_0^*}$ with C and $\mathcal{A}^{p_0^*}$ -constant for Hu depending only the \mathcal{A}^p -constant for w . But $(p^*/p_0^*)^* = (p_0-1)p/(p_0-p)$, so that $h \in \ell_w^{(p^*/p_0^*)^*}(\mathbb{Z}_+)$ if and only if

$$h^{p_0-1} w^{-(p_0-p)/(p-1)} \in \ell_w^{(p/(p_0-p))}(\mathbb{Z}_+).$$

Also $Hu \in \mathcal{A}^{p_0^*}$ if and only if

$$(Hu)^{-1/(p_0^*-1)} = [H^{p_0-1} w^{-(p_0-p)/(p-1)}]^{-1} w \in \mathcal{A}^{p_0}.$$

Thus if $g \in \ell_w^{(p/(p_0-p))}(\mathbb{Z}_+)$, we just write $g = h^{p_0-1} w^{-(p_0-p)/(p-1)}$ with $h \in \ell_u^{(p^*/p_0^*)^*}(\mathbb{Z}_+)$ obtain the corresponding H and then define $G = H^{p_0-1} w^{-(p_0-p)/(p-1)}$. This proves the Part (2). The proof is complete. \square

For an operator T bounded from a Banach space X into itself ($T \in \mathcal{B}(X)$), we will denote by $\|T\|_X$ the standard operator norm of T defined by $\sup_{\|f\|_X \leq 1} \|Tf\|_X$. In the following, theorem, which is the discrete result of the celebrated extrapolation theorem of Rubio de Francia, we will suppose that for each $B > 1$, there exists an increasing function φ_r on $(0, \infty)$ such that we have $\|T\|_{\ell_w^r} \leq \varphi_r(B)$ for all $w \in \mathcal{A}^r$ with a constant $[\mathcal{A}^r(w)] \leq B$.

Theorem 5. *Let φ be an increasing function on $(0, \infty)$ and assume that T be a sublinear operator defined on \mathbb{Z}_+ . Suppose for some p_0 , with $1 \leq p_0 < \infty$, and every weight $w \in \mathcal{A}^{p_0}$, $T \in \mathcal{B}(\ell_w^{p_0})$ and satisfies the inequality*

$$(24) \quad \sum_{k=1}^{\infty} |Tf(k)|^{p_0} w(k) \leq \varphi[\mathcal{A}^{p_0}(w)] \sum_{k=1}^{\infty} |f(k)|^{p_0} w(k).$$

for every f , where C depends only on \mathcal{A}^{p_0} -constant of w . Then for every p with $1 < p < \infty$, and every $w \in \mathcal{A}^p$, $T \in \mathcal{B}(\ell_w^p)$ and satisfies the inequality

$$(25) \quad \sum_{k=1}^{\infty} (|Tf(k)|)^p w(k) \leq \varphi[\mathcal{A}^p(w)] \sum_{k=1}^{\infty} |f(k)|^p w(k),$$

for every f , where C depends only on \mathcal{A}^p -constant of w .

Proof. Case 1). Assume that $1 \leq p_0 < p$, $w \in \mathcal{A}^p$ and $1/s = 1 - (p_0/p)$, i.e. $s^* = p/p_0$ and so that $s = (p/p_0)^*$. Then

$$(26) \quad \begin{aligned} \left(\sum_{k=1}^{\infty} (Tf(k))^p w(k) \right)^{p_0/p} &= \| |Tf(k)|^{p_0} \|_{\ell_w^{s^*}} \\ &= \sup_{g>0, \|g\|_{\ell_w^{(p/p_0)^*}}=1} \sum_{k=1}^{\infty} (|Tf(k)|)^{p_0} g(k) w(k). \end{aligned}$$

For each g , by the first case of Lemma 5, there exists $G \in \ell_w^{(p/p_0)^*}$ such that $g \leq G$, and

$$\|G\|_{\ell_w^s} = \|G\|_{\ell_w^{(p/p_0)^*}} \leq C \|g\|_{\ell_w^{(p/p_0)^*}} \leq C.$$

Then $Gw \in \mathcal{A}^{p_0}$ and its $[\mathcal{A}^{p_0}(Gw)]$ constant depends only on the \mathcal{A}^p -constant of

w that is $[\mathcal{A}^{p_0}(Gw)] \leq C[\mathcal{A}^p(w)]$. Then

$$\begin{aligned}
& \sum_{k=1}^{\infty} (|Tf(k)|)^{p_0} g(k)w(k) \leq \sum_{k=1}^{\infty} (|Tf(k)|)^{p_0} G(k)w(k) \leq \|T\|_{\ell_w^{p_0}}^{p_0} \|f\|_{\ell_w^{p_0}}^{p_0} \\
& \leq \|T\|_{\ell_w^{p_0}}^{p_0} \sum_{k=1}^{\infty} (|f(k)|)^{p_0} G(k)w^{p_0/p}(k)w^{1-\frac{p_0}{p}}(k) \\
& \leq \|T\|_{\ell_w^{p_0}}^{p_0} \left(\sum_{k=1}^{\infty} (|f(k)|)^p w(k) \right)^{1/(p/p_0)} \left(\sum_{k=1}^{\infty} G^{(p/p_0)^*}(k)w(k) \right)^{1/(p/p_0)^*} \\
& \leq \|T\|_{\ell_w^{p_0}}^{p_0} \|G\|_{\ell_w^{(p/p_0)^*}} \left(\sum_{k=1}^{\infty} (|f(k)|)^p w(k) \right)^{p_0/p} \\
(27) \quad & = \|T\|_{\ell_w^{p_0}}^{p_0} \|G\|_{\ell_w^{(p/p_0)^*}} (\|f\|_{\ell_w^p})^{p_0}.
\end{aligned}$$

Now, we use the hypotheses, $\|T\|_{\ell_w^{p_0}} \leq \varphi_{p_0}[\mathcal{A}^{p_0}(wG)]$ and the fact that φ_{p_0} is increasing function and $[\mathcal{A}^{p_0}(wG)] \leq C[\mathcal{A}^p(w)]$ (see Lemma 5), we have from (27) that

$$(28) \quad \sum_{k=1}^{\infty} (|Tf(k)|)^{p_0} g(k)w(k) \leq \varphi_p([\mathcal{A}^p(w)]) \left(\sum_{k=1}^{\infty} (|f(k)|)^p w(k) \right)^{p_0/p}$$

Taking the supremum over all g , and combining (26) with (28), we obtain that

$$\left(\sum_{k=1}^{\infty} (|Tf(k)|)^p w(k) \right)^{p_0/p} \leq \varphi_p([\mathcal{A}^p(w)]) \left(\sum_{k=1}^{\infty} (|f(k)|)^p w(k) \right)^{p_0/p},$$

which is the desired inequality (25).

Case 2). Assume that $1 < p < p_0$ and $s = p/(p_0 - p)$. For $f \in \ell_w^p$ define $g = |f|^{p_0-p}$. Then $g \in \ell_w^s$ and $\|g\|_{\ell_w^s} = \|f\|_{\ell_w^p}^{p_0-p}$. By Lemma 5 (2) there is a sequence G such that $g < G$ such that

$$\|G\|_{\ell_w^s} \leq C \|g\|_{\ell_w^s} = C \|f\|_{\ell_w^p}^{p_0-p},$$

and $G^{-1}w \in \mathcal{A}^{p_0}$, where C and $[\mathcal{A}^{p_0}(G^{-1}w)]$ –constant for $G^{-1}w$ depending only on the \mathcal{A}^p –constant for w , that is

$$[\mathcal{A}^{p_0}(G^{-1}w)] \leq C[\mathcal{A}^p(w)].$$

Now, by applying Hölder's inequality with $q = p_0/p$ and $q^* = p_0/(p_0 - p)$ and

noting that $q^*/q = s$, we have that

$$\begin{aligned} \|Tf(k)\|_{\ell_w^{p_0}}^{p_0} &\leq \|G\|_{\ell_w^s} \sum_{k=1}^{\infty} (|Tf(k)|)^{p_0} G^{-1}(k)w(k) \\ &\leq C \|f\|_{\ell_w^{p_0-p}}^{p_0-p} \varphi_{p_0} [\mathcal{A}^{p_0}(G^{-1}w)] \sum_{k=1}^{\infty} (f(k))^{p_0} f^{p-p_0}(k)w(k) \\ &\leq C \|f\|_{\ell_w^{p_0-p}}^{p_0-p} \varphi_p [\mathcal{A}^p(w)] \sum_{k=1}^{\infty} f^p(k)w(k), \end{aligned}$$

which is the desired inequality (25). The proof is complete. \square

Remark 3. We note that there is a weak version of Theorem 5, in which one uses for (24) the inequality

$$(29) \quad \sum_{|k: Tf(k) > \lambda|} w(k) \leq C \lambda^{-p_0} \sum_{k=1}^{\infty} |f(k)|^{p_0} w(k), \text{ for } \lambda > 0,$$

and also change the inequality (25) in the inclusion to

$$(30) \quad \sum_{|k: Tf(k) > \lambda|} w(k) \leq C \lambda^{-p} \sum_{k=1}^{\infty} |f(k)|^p w(k), \text{ for } \lambda > 0.$$

The proof is as follows. For $\lambda > 0$, let $E_\lambda = \{k : |Tf(k)| > \lambda\}$ and for E_λ , $w(E_\lambda) = \sum_{k \in E_\lambda} w(k)$ and χ_{E_λ} will denote the characteristic function of E_λ .

Case 1). Assume that $1 \leq p_0 < p$, $w \in \mathcal{A}^p$ and $1/s = 1 - (p_0/p)$, i.e. $s^* = p/p_0$ and so that $s = (p/p_0)^*$. Then for $\lambda > 0$,

$$\lambda^{p_0} (w(E_\lambda))^{p_0/p} = \lambda^{p_0} \|\chi_{E_\lambda}\|_{\ell_w^{p/p_0}}^{p_0} = \lambda^{p_0} \sum_{k=1}^{\infty} \chi_{E_\lambda}(k) g(k) w(k),$$

for some $g \geq 0$, $\|g\|_{\ell_w^{(p/p_0)^*}} = 1$. Associate with g a sequence G as in Lemma 5 part (1). Then

$$\begin{aligned} \lambda^{p_0} (w(E_\lambda))^{p_0/p} &\leq \lambda^{p_0} \sum_{k=1}^{\infty} \chi_{E_\lambda}(k) G(k) w(k) \leq C \sum_{k=1}^{\infty} f^{p_0} G(k) w(k) \\ &\leq C \|f\|_{\ell_w^{p_0}}^{p_0}. \end{aligned}$$

Then, we have proved that $w(E_\lambda) \leq C \lambda^{-p} \|f\|_{\ell_w^p}^p$, which is the inequality (30).

Case 2). Assume that $1 < p < p_0$ and $s = p/(p_0 - p)$. For $f \in \ell_w^p$ define $g = |f|^{p_0-p}$. Then $g \in \ell_w^s$ and $\|g\|_{\ell_w^s} = \|f\|_{\ell_w^p}^{p_0-p}$. Associate with g the sequence G as in Lemma 5 part (2) such that $g < G$ and

$$\|G\|_{\ell_w^s} \leq C \|g\|_{\ell_w^s} = C \|f\|_{\ell_w^p}^{p_0-p},$$

and $G^{-1}w \in \mathcal{A}^{p_0}$. Then for $\lambda > 0$,

$$\begin{aligned} \lambda^{p_0} (w(E_\lambda))^{p_0/p} &= \lambda^{p_0} \|\chi_{E_\lambda}\|_{\ell_w^{p_0/p_0}} \leq \lambda^{p_0} \sum_{k=1}^{\infty} \chi_{E_\lambda}(k) G^{-1}(k) w(k) \\ &\leq C \sum_{k=1}^{\infty} f^{p_0}(k) G^{-1}(k) w(k) \leq C \sum_{k=1}^{\infty} f^{p_0}(k) g^{-1}(k) w(k) \\ &= C \|f\|_{\ell_w^{p_0}}^{p_0}, \end{aligned}$$

which is the inequality (30) as before.

The discrete version of Rubio de Francia iteration algorithm that can be used in the proof of the main results instead of the operator \mathcal{M} will be proved in the next theorem. The details will be left to the interested reader.

Theorem 6. Fix $1 < p < \infty$ and $w \in \mathcal{A}^p$. For any non-negative sequence $f \in \ell^p(w)$, define

$$(31) \quad \Pi f(n) = \sum_{l=0}^{\infty} \frac{\mathcal{M}^l f(n)}{2^l \|\mathcal{M}\|_{\ell^p(w)}^l},$$

where for $l > 0$, $\mathcal{M}^l f = \mathcal{M} \circ \dots \circ \mathcal{M} f$ denotes l iterations of the maximal operator and $\mathcal{M}^0 f = f$. Then:

- (1). $f(n) \leq \Pi f(n)$,
- (2). $\|\Pi f\|_{\ell^p(w)} \leq 2 \|f\|_{\ell^p(w)}$,
- (3). $\Pi f \in \mathcal{A}^1$ and $[\Pi f]_{\mathcal{A}^1} \leq 2 \|\mathcal{M}\|_{\ell^p(w)}$.

Proof. (1). It is sufficient to consider the first term in (31) for $l = 0$.

(2). By applying Minkowski's inequality to the norm of (31), we obtain that

$$\|\Pi f\|_{\ell^p(w)} \leq \sum_{l=0}^{\infty} \frac{\|\mathcal{M}^l f\|_{\ell^p(w)}}{2^l \|\mathcal{M}\|_{\ell^p(w)}^l} \leq \sum_{l=0}^{\infty} 2^{-l} \|f\|_{\ell^p(w)} = 2 \|f\|_{\ell^p(w)},$$

where we have used the boundedness property of the discrete maximal operator $\mathcal{M}f$ on the weighted discrete Lebesgue space $\ell^p(w)$.

(3). Since the maximal operator is subadditive, we get that

$$\begin{aligned} \mathcal{M}(\Pi f)(n) &= \mathcal{M} \left(\sum_{l=0}^{\infty} \frac{\mathcal{M}^l f(n)}{2^l \|\mathcal{M}\|_{\ell^p(w)}^l} \right) \leq \sum_{l=0}^{\infty} \frac{\mathcal{M}^{l+1} f(n)}{2^l \|\mathcal{M}\|_{\ell^p(w)}^l} \\ &\leq 2 \|\mathcal{M}\|_{\ell^p(w)} \Pi f(n). \end{aligned}$$

The proof is complete. \square

Remark 4. Our results are also can be applied for the weights associated to the other discrete maximal operators. In fact the technique works for all of them since the description of the discrete classes \mathcal{A}^p is formally the same.

Remark 5. *In our technique, we have used the self-improving property of the A^p -weights which allows us to strengthen the weak results so that the strong results become corollaries. Indeed, we can get the strong inequalities (24) and (25) from the weak type inequalities (29) and (30). We just need to use the self-improving property of A^p and then apply Marcinkiewicz Interpolation Theorem and obtain the strong estimates.*

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